

Contraction and optimality properties of an adaptive Legendre-Galerkin method: the multi-dimensional case

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Abstract

We analyze the theoretical properties of an adaptive Legendre-Galerkin method in the multidimensional case. After the recent investigations for Fourier-Galerkin methods in a periodic box and for Legendre-Galerkin methods in the one dimensional setting, the present study represents a further step towards a mathematically rigorous understanding of adaptive spectral/ hp discretizations of elliptic boundary-value problems. The main contribution of the paper is a careful construction of a multidimensional Riesz basis in H^1 , based on a quasi-orthonormalization procedure. This allows us to design an adaptive algorithm, to prove its convergence by a contraction argument, and to discuss its optimality properties (in the sense of non-linear approximation theory) in certain sparsity classes of Gevrey type.

1 Introduction

The use of adaptivity in numerical modelling and simulation has now become a standard in Engineering and industrial applications. Although the practice goes back to the 70's, the mathematical understanding of the convergence and optimality properties of adaptive algorithms for approximating the solution of multidimensional PDEs is rather recent. For linear elliptic problems the first convergence results of adaptive h -type finite element methods (h -AFEM) have been proved by Dörfler [16] and Morin, Nochetto, and Siebert [26]. On the other hand, the first convergence rates were derived for wavelets in any dimensions by Cohen, Dahmen, and DeVore [14], and for h -AFEM by Binev, Dahmen, and DeVore [6] for the two-dimensional case and Stevenson [29] for any dimensions. The most general results for h -AFEM are those contained in Cascón, Kreuzer, Nochetto, and Siebert [13] for any dimensions and L^2 data, and in Cohen, DeVore, and Nochetto [15] for two-dimensional case and H^{-1} data. The key result of this theory is that wavelets and h -AFEM are capable of guaranteeing convergence rates coherent with those dictated by the (best N -term) approximation classes where the solution and data belong. However, in the above wavelet and FEM contexts, convergence rates are limited by the

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approximation power of the method, which is finite and related to the polynomial degree of the basis functions or the number of their vanishing moments, as well as the *sparsity* of the solution and the data. The latter is always measured in an *algebraic* approximation class, i.e., the best N -term approximation error decays at least as a power of N^{-1} . We refer to the surveys [27] by Nochetto, Siebert and Veiser for AFEM and [30] by Stevenson for adaptive wavelets.

For adaptive methods with infinite approximation power (such as spectral or spectral-element methods, and hp -type finite element methods), the state of the art is less developed. Although the numerical implementation started long time ago and has led to the design of very sophisticated and efficient adaptive hp algorithms (see, e.g., [25]; see also [12] and the references therein), very little is known on their theoretical properties. In particular, after the pioneering work [20] focussed on the approximation of specific types of functions, some rigorous convergence results for the hp adaptive solution of elliptic problems have been obtained only recently in [28, 17, 7]. However, these studies do not address any optimality analysis.

A first step in this direction has been accomplished in [10] by considering adaptive spectral Fourier-Galerkin methods in a periodic box in \mathbb{R}^d , $d \geq 1$, which represent the simplest instance of infinite-order methods yet providing a very important conceptual benchmark. The contraction and the optimal cardinality properties of various algorithms are presented therein; in the analysis, suitable nonlinear approximation classes (also termed sparsity classes) are involved, namely the already mentioned algebraic classes and the newly introduced *exponential* classes corresponding to a (sub-)exponential decay of the best N -term approximation error. The latter classes, of Gevrey type, are natural to describe situations that motivate the use of high-order methods.

A second step towards the study of optimality for high-order methods has been performed in [9] where the method and the results contained in [10] have been extended to a non-periodic setting in one dimension. Such a setting is the closest to the periodic one, since an H^1 -orthonormal basis is readily available (the so-called Babuška-Shen basis formed by the primitives of the Legendre polynomials); together with the associated dual basis, it allows one to represent the norm of a function or a functional (e.g., the residual associated to the approximate solution) as an ℓ_2 -type norm of the vector of its expansion coefficients. Furthermore, the use of an orthonormal basis allows the efficient implementation of the greedy and coarsening procedures required by the adaptive algorithm. Indeed, the study of the optimality properties performed in [10, 9] relies on a careful analysis of the relation between the sparsity class of a function and the sparsity class of its image through the differential operator. This analysis is based on the observation that the stiffness matrix associated to a differential operator with smooth coefficients exhibits a quasi-sparse behavior, i.e., an exponential decay of its entries as one goes away from the diagonal. The discrepancy between the sparsity classes of the residual and the exact solution suggests the introduction of a coarsening step that guarantees the optimality of the computed approximation at the end of each adaptive iteration.

The present paper deals with adaptive Legendre-Galerkin methods in a tensorial domain in \mathbb{R}^d , $d > 1$, for elliptic equations submitted to Dirichlet boundary conditions. This poses additional difficulties with respect to the one dimensional case, considered in [9]. In particular, the crucial issue is represented by the H^1 -stability properties of the multidimensional Legendre polynomials. Unfortunately, the natural basis, formed by tensor products of one-dimensional basis functions, is not H^1 -orthogonal (because, unlike the Fourier basis, the one-dimensional Babuška-Shen basis is not simultaneously orthogonal in L^2 and H^1) and not even a Riesz basis. This suggests searching for a Riesz basis in H^1 , still remaining closely related to the tensorial BS basis in order to take advantage of the properties of Legendre polynomials. The main idea developed in this paper is to start with a Gram-Schmidt (GS) orthogonalization of the latter basis,

but then apply a controlled thresholding procedure that discards the smallest contributions from the linear combinations generated by GS: in other words, we devise a quasi-orthonormalization technique. A fundamental ingredient for rigorously controlling this procedure is the construction of sharp estimates on the decay of the GS coefficients, which in turn involve the decay of the entries of the inverse stiffness matrix for the Laplacian [11]. With such a new basis, results comparable to those of [10, 9] can be established. In particular, they rely on the exponential decay of the entries of the stiffness matrix, when the differential operators has smooth (analytic) coefficients, and on the repeated application of a coarsening stage in the adaptive algorithm.

The results of the present paper can be easily extended to cover the case of adaptive “ p -type” spectral element methods, i.e., when the domain is decomposed in a fixed number of (images of) tensorial elements, and adaptivity concerns the choice of the expansion functions in each element. Furthermore, some of the ideas and methods here introduced could influence the design of adaptive hp -type algorithms, as far as the phase of “ p -enrichment” within the elements is concerned. With this respect, we remark that very recently, the optimality properties of an hp -adaptive finite element method have been obtained in [8], employing the pioneering results on hp -tree approximation of [5, 4].

The outline of the paper is as follows. In Section 2 we detail the construction of our multidimensional Riesz basis and provide the reader with both theoretical results and quantitative insight, the latter concerning in particular the compression properties of the thresholding procedure. In Section 3 we introduce the algebraic representation of an elliptic differential problem in terms of the above Riesz basis and discuss the exponential decay properties of the entries of the corresponding stiffness matrix. Finally, in Section 4 we present our adaptive Legendre algorithm (FPC-ADLEG) and prove its contraction and optimality properties.

Throughout the paper, $A \lesssim B$ means $A \leq cB$ for some constant $c > 0$ independent of the relevant parameters in the inequality; $A \simeq B$ means $B \lesssim A \lesssim B$.

2 Modal bases in H_0^1 and norm representations

We start with the one-dimensional case. Set $I = (-1, 1)$ and let $L_k(x)$, $k \geq 0$, stand for the k -th Legendre orthogonal polynomial in I , which satisfies $\deg L_k = k$, $L_k(1) = 1$ and

$$\int_I L_k(x) L_m(x) dx = \frac{2}{2k+1} \delta_{km}, \quad m \geq 0. \quad (2.1)$$

The natural modal basis in $H_0^1(I)$ is the *Babuška-Shen basis* (BS basis), whose elements are defined as

$$\eta_k(x) = \sqrt{\frac{2k-1}{2}} \int_x^1 L_{k-1}(s) ds = \frac{1}{\sqrt{4k-2}} (L_{k-2}(x) - L_k(x)), \quad k \geq 2. \quad (2.2)$$

The basis elements satisfy $\deg \eta_k = k$ and

$$(\eta_k, \eta_m)_{H_0^1(I)} = \int_I \eta'_k(x) \eta'_m(x) dx = \delta_{km}, \quad k, m \geq 2, \quad (2.3)$$

i.e., they form an orthonormal basis for the $H_0^1(I)$ -inner product. Equivalently, the (semi-infinite) stiffness matrix S_η of the Babuška-Shen basis with respect to this inner product is the

identity matrix I . On the other hand, one has

$$(\eta_k, \eta_m)_{L^2(I)} = \begin{cases} \frac{2}{(2k-3)(2k+1)} & \text{if } m = k, \\ -\frac{1}{(2k+1)\sqrt{(2k-1)(2k+3)}} & \text{if } m = k+2, \\ 0 & \text{elsewhere.} \end{cases} \quad \text{for } k \geq m, \quad (2.4)$$

which means that the mass matrix M_η is pentadiagonal with only three non-zero entries per row. (Since even and odd modes are mutually orthogonal, the mass matrix could be equivalently represented by a couple of tridiagonal matrices, each one collecting the inner products of all modes with equal parity.)

Any $v \in H_0^1(I)$ can be expanded in terms of the Babuška-Shen basis, as $v = \sum_{k=2}^{\infty} \hat{v}_k \eta_k$ with $\hat{v}_k = (v, \eta_k)_{H_0^1(I)}$ and its $H_0^1(I)$ -norm can be expressed, according to the Parseval identity, as

$$\|v\|_{H_0^1(I)}^2 = \sum_{k=2}^{\infty} |\hat{v}_k|^2 = \hat{v}^T \hat{v}, \quad (2.5)$$

where the vector $\hat{v} = (\hat{v}_k)$ collects the coefficients of v . The $L^2(I)$ -norm of v is given by

$$\|v\|_{L^2(I)}^2 = \hat{v}^T M_\eta \hat{v}. \quad (2.6)$$

Correspondingly, any element $f \in H^{-1}(I)$ can be expanded along the *dual Babuška-Shen basis*, whose elements η_k^* , $k \geq 2$, are defined by the conditions $\langle \eta_k^*, v \rangle = \hat{v}_k \forall v \in H_0^1(I)$, precisely one has $f = \sum_{k=2}^{\infty} \hat{f}_k \eta_k^*$ with $\hat{f}_k = \langle f, \eta_k \rangle$, and its $H^{-1}(I)$ -norm can be expressed, according to the Parseval identity, as

$$\|f\|_{H^{-1}(I)}^2 = \sum_{k=2}^{\infty} |\hat{f}_k|^2. \quad (2.7)$$

Summarizing, we see that the one-dimensional Legendre case is perfectly similar, from the point of view of expansions and norm representations, to the Fourier case [10]. The situation changes significantly in higher dimensions. For the sake of simplicity, we confine ourselves to the case of dimension $d = 2$, since higher dimensions pose no conceptual difficulties but require a larger computational effort in the numerical experiments.

Let us set $\Omega = (-1, 1)^2$ and let us consider in $H_0^1(\Omega)$ the *tensorized Babuška-Shen basis*, whose elements are defined as

$$\eta_k(x) = \eta_{k_1}(x_1) \eta_{k_2}(x_2), \quad k_1, k_2 \geq 2, \quad (2.8)$$

where we set $k = (k_1, k_2)$ and $x = (x_1, x_2)$; indices vary in $\mathcal{K} = \{k \in \mathbb{N}^2 : k_i \geq 2 \text{ for } i = 1, 2\}$.

The tensorized BS basis is no longer orthogonal, since

$$(\eta_k, \eta_m)_{H_0^1(\Omega)} = (\eta_{k_1}, \eta_{m_1})_{H_0^1(I)} (\eta_{k_2}, \eta_{m_2})_{L^2(I)} + (\eta_{k_1}, \eta_{m_1})_{L^2(I)} (\eta_{k_2}, \eta_{m_2})_{H_0^1(I)}, \quad (2.9)$$

hence, by (2.3) and (2.4), we have $(\eta_k, \eta_m)_{H_0^1(\Omega)} \neq 0$ if and only if $k_1 = m_1$ and $k_2 - m_2 \in \{-2, 0, 2\}$, or $k_2 = m_2$ and $k_1 - m_1 \in \{-2, 0, 2\}$.

In the sequel, this basis and the corresponding index set \mathcal{K} will be ordered by increasing total degree $k_{\text{tot}} = k_1 + k_2$ and, for the same total degree, by increasing values of k_1 (this will be referred to as the “A” ordering, see Fig. 1(a)). We will use the following notational convention:

$$m < k \quad \text{means that } \eta_m \text{ precedes } \eta_k \text{ in this ordering.}$$

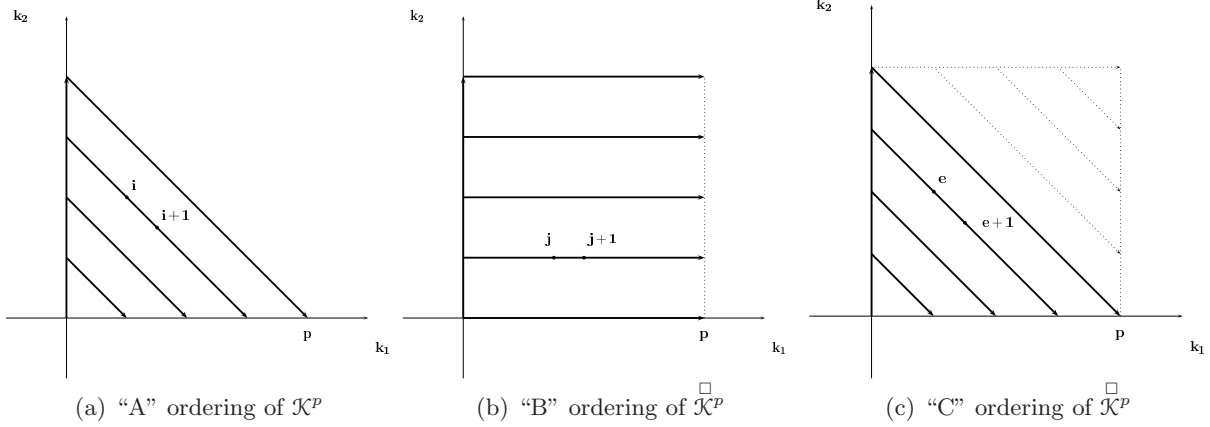


Figure 1: Orderings of index sets.

With this ordering, let us denote again by S_η the stiffness matrix of the tensorized Babuška-Shen basis with respect to the $H_0^1(\Omega)$ -inner product. The matrix S_η is infinite-dimensional.

In the sequel, we will also need finite dimensional matrices defined as follows. For fixed $p \geq 2$ define the set

$$\mathcal{K}^p := \{k \in \mathcal{K} : k_1 + k_2 \leq p\}, \quad (2.10)$$

with cardinality $\text{card}(\mathcal{K}^p) \simeq p^2$, which identifies the basis functions of total degree not greater than p , ordered as above. The corresponding stiffness matrix will be denoted by S_η^p , which is a truncated version of S_η (upper-left section); its sparsity pattern is shown in Fig. 2(a). Let us also introduce the index set

$$\mathcal{K}^{\square p} := \{k \in \mathcal{K} : k_i \leq p \text{ for } i = 1, 2\} \quad (2.11)$$

whose elements, on the contrary, are ordered lexicographically, i.e., by increasing values of k_2 and, for the same k_2 , by increasing values of k_1 (this will be referred to as the "B" ordering, see Fig. 1(b)). The corresponding stiffness matrix will be denoted by $S_\eta^{\square p}$. Recalling (2.9) it is the sum of two Kronecker products, i.e., $S_\eta^{\square p} = M_p \otimes I_p + I_p \otimes M_p$, where M_p is the one-dimensional mass matrix truncated at order p and I_p is the identity matrix of the same size. We observe that the position of an index $k \in \mathcal{K}^p$ does depend on p because of the lexicographical ordering; on the contrary, the position of $k \in \mathcal{K}^p$ is independent of p because it coincides with the position in the infinite dimensional index set \mathcal{K} . We note for further reference that

$$\mathcal{K}^{\tilde{p}} \subset \mathcal{K}^p \subset \mathcal{K}^{\square p},$$

where \tilde{p} is the integer part of $p/2$. This implies, thanks to a Rayleigh quotient argument, that

$$\lambda_{\min}(\mathcal{K}^{\tilde{p}}) \geq \lambda_{\min}(S_\eta^p) \geq \lambda_{\min}(\mathcal{K}^{\square p}) \quad \text{and} \quad \lambda_{\max}(\mathcal{K}^{\tilde{p}}) \leq \lambda_{\max}(S_\eta^p) \leq \lambda_{\max}(\mathcal{K}^{\square p}). \quad (2.12)$$

2.1 Orthonormalization and quasi-orthonormalization

Given any $v \in H_0^1(\Omega)$, let us expand it as $v = \sum_{k \in \mathcal{K}} \hat{v}_k \eta_k$ and let \hat{v} be the vector collecting its coefficients \hat{v}_k . Obviously, we cannot have a Parseval representation of the $H_0^1(\Omega)$ -norm of v as in (2.5), since the basis is not orthonormal. However, we would be happy to have just

$$\|v\|_{H_0^1(\Omega)}^2 = \hat{v}^T S_\eta \hat{v} \simeq \hat{v}^T \Delta_\eta \hat{v} = \sum_{k \in \mathcal{K}} |\hat{v}_k|^2 d_k, \quad (2.13)$$

for some diagonal matrix Δ_η . Taking as \hat{v} each vector of the canonical basis, this should imply

$$s_{kk} \simeq d_k \quad \forall k \in \mathcal{K},$$

i.e., Δ_η should be uniformly spectrally equivalent to $D_\eta := \text{diag } S_\eta$. Unfortunately, the eigenvalues of the generalized eigenvalue problem $S_\eta w = \lambda D_\eta w$ are not uniformly bounded away from 0 and $+\infty$. These eigenvalues are indeed the eigenvalues of the matrix $\tilde{S}_\eta = D_\eta^{-1/2} S_\eta D_\eta^{-1/2}$ which is the stiffness matrix of the H_0^1 -normalized BS basis. In particular, if we consider the finite dimensional matrices S_η^p , it is known [24, Proposition 5] that their largest eigenvalues are uniformly bounded, but the smallest eigenvalues tend to 0 as p^{-2} . Due to (2.12) the same results hold for the matrices S_η^p . In conclusion, there is no hope to have (2.13) with the tensorized BS basis, and a new basis has to be sought. \square

2.1.1 Orthonormalization

We have pursued the idea of orthonormalizing the BS basis, since, as shown above, many inner-products between its functions are indeed zero. To this end, we first observe that the BS basis functions can be grouped in four families depending on their parity in each of the two variables (recall that the univariate BS basis functions are alternately even and odd); according to (2.9) functions belonging to different families are mutually H_0^1 -orthogonal. Consequently, after reordering of rows and columns, the matrix S_η is block-diagonal with four blocks S_η^{++} , S_η^{+-} , S_η^{-+} and S_η^{--} corresponding to all combinations of even (+) or odd (−) one-dimensional basis functions in each direction. In addition, it is convenient to deal with H_0^1 -normalized basis functions, which lead to blocks $\tilde{S}_\eta^{\pm\pm}$. For notational simplicity, from now on any normalized block $\tilde{S}_\eta^{\pm\pm}$ will be again denoted by S_η and the corresponding basis functions will still be indicated by η_k for k belonging to an index set again denoted by \mathcal{K} . Since the four blocks behave in an equivalent way, in the following numerical results will be given only for the even-even case.

As a first step we resort to the modified Gram-Schmidt algorithm (see e.g. [19]), which allows one to build a sequence of functions

$$\Phi_k = \sum_{m \leq k} g_{mk} \eta_m, \quad (2.14)$$

such that $g_{kk} \neq 0$ and

$$(\Phi_k, \Phi_m)_{H_0^1(\Omega)} = \delta_{km} \quad \forall k, m \in \mathcal{K}.$$

We will refer to the collection $\Phi := \{\Phi_k : k \in \mathcal{K}\}$ as the *orthonormal Babuška-Shen basis* (OBS basis) of the above chosen parity; obviously, the associated stiffness matrix S_Φ with respect to the $H_0^1(\Omega)$ -inner product is the identity matrix. Equivalently, if $G = (g_{mk})$ is the upper triangular

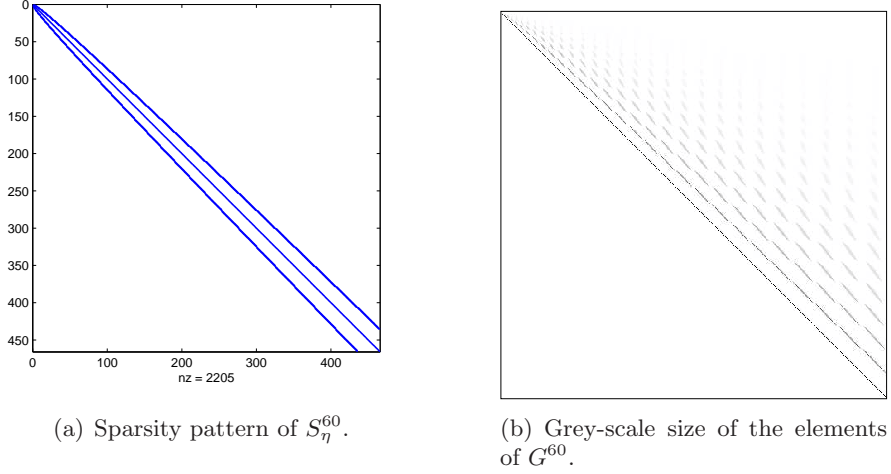


Figure 2: Sparsity patterns

matrix which collects the coefficients generated by the modified Gram-Schmidt algorithm above, one has

$$G^T S_\eta G = S_\Phi = I, \quad (2.15)$$

which shows that $L := G^{-T}$ is the lower-triangular Cholesky factor of S_η . It is important to notice that for any finite dimensional section S_η^p of S_η a similar relation holds, namely

$$(G^p)^T S_\eta^p G^p = S_{\Phi^p} = I^p,$$

where G^p is the upper-left section of the infinite dimensional matrix G with the same size as S_η^p and $\Phi^p := \{\Phi_k : k \in \mathcal{K}^p\}$. This is an obvious consequence of the structure of the Gram-Schmidt algorithm and the fact that the ordering of the basis functions in \mathcal{K}^p is the same as in \mathcal{K} . On the contrary, due to the different orderings of the basis functions in \mathcal{K}^p and \mathcal{K} , the GS algorithm applied to the matrix S_η^p gives rise to a matrix \tilde{G}^p which cannot be obtained by simply truncating the infinite dimensional G . \square

Unfortunately, unlike S_η , which is very sparse, the upper triangular matrix G is full. However, the elements of G exhibit nice decay features which are exemplified in Figure 2(b) again for $p = 60$. The intensity of grey indicates that the entries of G decay to zero moving away from the main diagonal, with different rates depending on the column. Indeed, recalling formula (2.14), it is meaningful to monitor the decay of the elements g_{mk} of G that belong to a given column k for the row index m decreasing from k to 1. Figures 3(left) and 3(right) are representative of two extreme behaviors. Each plot in the figures represents the elements of a column of G starting from the the main diagonal and moving towards the first row. Figure 3(left) refers to columns 98, 221, 338 which exhibit a “slow” decay. This behavior is typical of those columns associated to an index $k \in \mathcal{K}$ with k_1 close to k_2 . On the other hand, Figure 3(right) refers to columns 105, 231, 351 which exhibit a “fast” decay, a typical behavior of those columns associated to an index $k \in \mathcal{K}$ for which k_1 and k_2 are very different from each other.

A theoretical upper bound for the elements of G can be obtained applying Theorem 4.1 in [3]. It ensures that if A is an SPD banded matrix with bandwidth b such that $\max_i a_{ii} = 1$, and

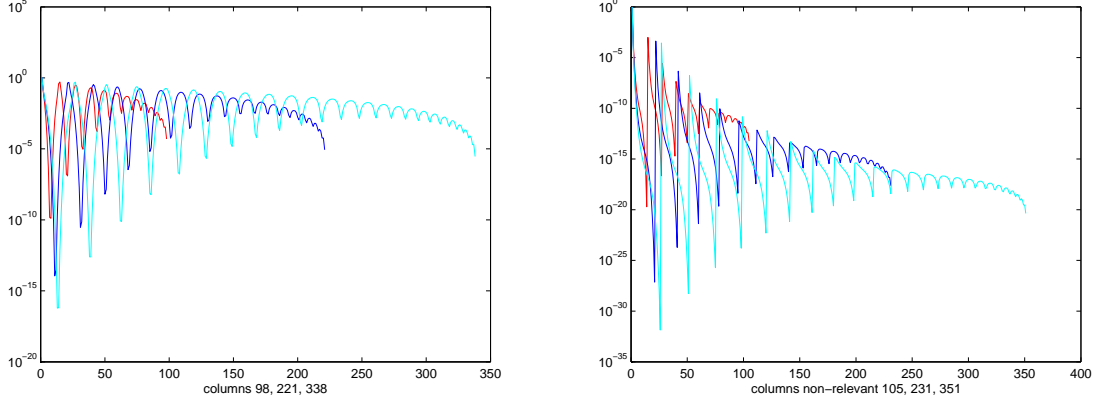


Figure 3: Semi-logarithmic plot of some “slow” decaying columns (left) and “fast” decaying columns (right) of G^{60} .

if $A = LL^T$ is its Cholesky factorization, then the entries of $G = L^{-T}$ obey an exponentially decaying bound away from the main diagonal, precisely

$$|g_{ij}| \leq \frac{2}{\lambda_{\min}} \rho^{j-i}, \quad \rho = \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{2/b}, \quad (2.16)$$

where $\kappa = \lambda_{\max}/\lambda_{\min}$ is the condition number of A . Note however that the observed decay of g_{ij} is far from being monotonic (see Figures 3(left) and 3(right)).

This oscillatory behavior can indeed be explained by resorting to the recent results presented in [11], as we are now going to detail. In what follows, for the ease of presentation, let “A”, “B”, and “C”, resp., refer to the index orderings depicted in Fig. 1(a), 1(b), and 1(c), resp.; the latter is a different ordering of the set \mathcal{K}^p , which coincides with the A-ordering on the subset $\mathcal{K}^p \subset \mathcal{K}^p$. Furthermore, we drop the dependence on p when using matrices. As the scheme C is an expansion of the scheme A, the associated stiffness matrix \check{S}_η has the form

$$\check{S}_\eta = \begin{bmatrix} S_\eta & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}$$

where S_η is the stiffness matrix associated with the scheme A.

Let $S_\eta = LL^T$ and let $\check{L}\check{L}^T = \check{S}_{22} := S_{22} - S_{12}^T S_\eta^{-1} S_{12}$. It holds that

$$\check{S}_\eta = \begin{bmatrix} L & 0 \\ S_{12}^T L^{-T} & \check{L} \end{bmatrix} \begin{bmatrix} L^T & L^{-1} S_{12} \\ 0 & \check{L}^T \end{bmatrix} =: \check{L}\check{L}^T,$$

with \check{L} banded with bandwidth m . Since all elements of \check{S}_η are less than one, we have

$$|(\check{L})_{ij}| \leq 1. \quad (2.17)$$

Taking the inverse of the previous factorization, we have

$$\check{S}_\eta^{-1} = \check{L}^{-T} \check{L}^{-1} =: \check{G} \check{G}^T$$

with

$$\check{G} = \begin{bmatrix} L^{-T} & -L^{-T}L^{-1}S_{12}\tilde{L}^{-T} \\ 0 & \tilde{L}^{-T} \end{bmatrix}.$$

This relation tells us that the elements of $G := L^{-T}$ we are interested in, can be read off from the upper left block of the factor \check{G} of \check{S}_η^{-1} .

Now, we recall that

$$\check{S}_\eta = P \overset{\square}{S}_\eta P^T, \quad \check{S}_\eta^{-1} = P \overset{\square}{S}_\eta^{-1} P^T,$$

where $\overset{\square}{S}_\eta$ is the stiffness matrix obtained with a lexicographic order (scheme B), and P is a permutation matrix. It is worth observing that \check{S}_η corresponds to the reverse Cuthill-McKee reordering of $\overset{\square}{S}_\eta$ (see, e.g., [18]).

Denoting by $\pi(u)$ the permutation of the index u defined by P , that is $e_u^T P = e_{\pi(u)}^T$, we have

$$(\check{S}_\eta^{-1})_{uv} = (\overset{\square}{S}_\eta^{-1})_{\pi(u), \pi(v)}. \quad (2.18)$$

For every index u in the diagonal ordering “C”, $\pi(u)$ is the associated index in the lexicographical ordering “B”. Next lemma details the construction of the map π .

Lemma 2.1. *Let $n := p - 1$, d with $1 \leq d \leq 2n - 1$ and e with $1 \leq e \leq \min(d, n) - \max(0, d - n)$. For*

$$u = \sum_{s=1}^{d-1} (\min(s, n) - \max(0, s - n)) + e, \quad (2.19)$$

it holds

$$\pi(u) = n(\min(d, n) - e) + \max(0, d - n) + e. \quad (2.20)$$

Proof. The parameter d with $1 \leq d \leq 2n - 1$ is the index numbering the diagonals in the diagonal ordering “C”. The index $d = 1$ corresponds to the first diagonal made of a single element (lower-left corner of the square in Figure 1(c)), while $d = n$ is associated to the main diagonal and $d = 2n - 1$ to the last diagonal (upper-right corner of the square in Figure 1(c)). The parameter e with $1 \leq e \leq \min(d, n) - \max(0, d - n)$ is the index numbering the elements on the d -th diagonal (from upper-left to lower-right). Note that on the d -th diagonal there are exactly $\min(d, n) - \max(0, d - n)$ elements. Let $(e, d) = (e(u), d(u))$ be such that $u = \sum_{s=1}^{d-1} (\min(s, n) - \max(0, s - n)) + e$, i.e., the element u is associated to the e -th element on the d -th diagonal. Then, straightforward calculations show that $\pi(u) = n(\min(d, n) - e) + \max(0, d - n) + e$. \square

Let u, v be two indices in the diagonal ordering “C” and $(e, d) = (e(u), d(u))$ and $(f, g) = (f(v), g(v))$ be such that equation (2.19) holds. We preliminary want to estimate (2.18). For $\overset{\square}{S}_\eta^p$ we know that the following result holds.

Proposition 2.1. ([11, Proposition 2.7]). *Let $\alpha = n(m - 1) + \ell$ and $\beta = n(j - 1) + i$ for proper choices of i, j, ℓ, m .*

1. *If $\ell = i$ or $m = j$ then there exists a positive constant $\gamma_1 = \gamma_1(\kappa(\overset{\square}{S}_\eta))$ such that*

$$|(\overset{\square}{S}_\eta^{-1})_{\alpha, \beta}| \leq \gamma_1 \frac{1}{\sqrt{\mathbf{n}_1}} \quad (2.21)$$

with $\mathbf{n}_1 = \mathbf{n}_1(\alpha, \beta) = |\ell - i| + |m - j| - 1$.

2. If $\ell \neq i$ and $m \neq j$ then there exists a positive constant $\gamma_2 = \gamma_2(\kappa(\overset{\square}{S}_\eta))$ such that

$$|(\overset{\square}{S}_\eta^{-1})_{\alpha,\beta}| \leq \gamma_2 \frac{1}{\sqrt{\mathbf{n}_2}} \quad (2.22)$$

with $\mathbf{n}_2 = \mathbf{n}_2(\alpha, \beta) = |\ell - i| + |m - j| - 2$.

Using (2.20) and Proposition 2.1 we have the following result on the entry decay of the matrix $(\check{S}_\eta)^{-1}$.

Corollary 2.1. *Let u, v be indexes in the C -ordering such that the following holds*

$$u = \sum_{s=1}^{d-1} (\min(s, n) - \max(0, s - n)) + e, \quad v = \sum_{s=1}^{g-1} (\min(s, n) - \max(0, s - n)) + f,$$

for proper choices of the pairs (d, e) and (g, f) . Consider (2.22) for $\pi(u), \pi(v)$ such that

$$\pi(u) = n(\min(d, n) - e) + \max(0, d - n) + e, \quad \pi(v) = n(\min(g, n) - f) + \max(0, g - n) + f.$$

Let i, j, ℓ, m such that $\pi(u) = n(m - 1) + \ell$ and $\pi(v) = n(j - 1) + i$ then it holds

1. If $\ell = i$ or $m = j$ there exists a positive constant $\gamma_1 = \gamma_1(\kappa(\overset{\square}{S}_\eta))$ such that

$$|(\overset{\square}{S}_\eta^{-1})_{\pi(u), \pi(v)}| \leq \gamma_1 \frac{1}{\sqrt{\mathbf{n}_1}} \quad (2.23)$$

with $\mathbf{n}_1 = \mathbf{n}_1(\pi(u), \pi(v)) = |\ell - i| + |m - j| - 1$.

2. If $\ell \neq i$ and $m \neq j$ then there exists a positive constant $\gamma_2 = \gamma_2(\kappa(\overset{\square}{S}_\eta))$ such that

$$|(\overset{\square}{S}_\eta^{-1})_{\pi(u), \pi(v)}| \leq \gamma_2 \frac{1}{\sqrt{\mathbf{n}_2}} \quad (2.24)$$

with $\mathbf{n}_2 = \mathbf{n}_2(\pi(u), \pi(v)) = |\ell - i| + |m - j| - 2$.

Proof. It is sufficient to employ Proposition 2.1 with $\alpha = \pi(u)$ and $\beta = \pi(v)$. \square

Now we are ready to estimate the entries of $\check{G} = \check{S}_\eta^{-1} \check{L}$. Using Corollary 2.1 we have the following estimate.

Proposition 2.2. *Under the assumptions of Corollary 2.1, for $\mathbf{n}(\cdot, \cdot) = \mathbf{n}_i(\cdot, \cdot)$, $i = 1, 2$ depending on the values of the indexes u, v , it holds*

$$|(\check{G})_{u,v}| \leq \gamma \frac{\tilde{p}}{\sqrt{\mathbf{n}(\pi(u), \pi(v^*))}} \quad (2.25)$$

where \tilde{p} is the integer part of $p/2$ and $v^* = \operatorname{argmin}_{v \leq w \leq v + \tilde{p} - 1} \mathbf{n}(\pi(u), \pi(w))$.

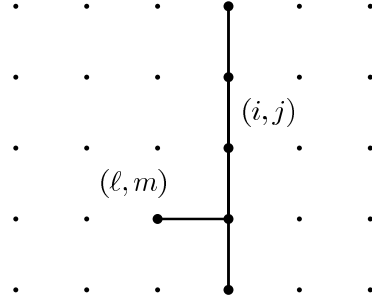


Figure 4: An example of the non-monotone behavior of $\mathbf{n}_i(\pi(u), \pi(v))$. We fix the pair (ℓ, m) (associated to $\pi(u)$) and vary (i, j) (associated to $\pi(v)$). Moving vertically (bottom-up) the quantity $|\ell - i| + |m - j| - 1$ first decreases and then increases.

Proof. It is enough to proceed as in [3, Theorem 4.1], with \tilde{p} corresponding to the matrix bandwidth, to get the result. Indeed, using $\check{G} = (\check{S}_\eta)^{-1} \check{L}$ together with (2.18), (2.23) and (2.24) it holds

$$\begin{aligned} |(\check{G})_{u,v}| &\leq \sum_{w=v}^{v+b-1} |(\check{S}_\eta^{-1})_{u,w}| |(\check{L})_{w,v}| = \sum_{w=v}^{v+b-1} |(\check{S}_\eta^{-1})_{\pi(u), \pi(w)}| |(\check{L})_{w,v}| \\ &\leq \gamma \frac{b}{\sqrt{\mathbf{n}(\pi(u), \pi(v^*))}} \end{aligned} \quad (2.26)$$

where we employ (2.17) and set $v^* = \operatorname{argmin}_{v \leq w \leq v+b-1} \mathbf{n}(\pi(u), \pi(w))$. \square

Let us briefly comment on the estimate (2.25). We first consider the term $\mathbf{n}_i(\pi(u), \pi(v))$ where, for the sake of exposition, we fix the index $\pi(u)$ and vary the index $\pi(v)$ in a given range of values. This is equivalent to fixing the pair (ℓ, m) (associated to $\pi(u)$) and varying the pair (i, j) (associated to $\pi(v)$) in a given bounded subset \mathcal{B} of \mathbb{N}^2 . As the quantity $\mathbf{n}_i(\pi(u), \pi(v)) = |\ell - i| + |m - j| - i$ represents a (modified) ℓ^1 -distance between the points (ℓ, m) and (i, j) , it follows that varying this latter point can yield (depending on the choice of \mathcal{B}) a non-monotone behavior for $\mathbf{n}_i(\pi(u), \pi(v))$ (see Figure 4). One can conclude similarly on the non-monotone behavior of the denominator $\mathbf{n}_i(\pi(u), \pi(v^*))$ in (2.25), hence justifying the observed oscillatory behavior of the entries of \check{G} (and hence of G^p).

2.1.2 Quasi-orthonormalization

The above documented features suggest us to invoke procedures to wipe-out from G a large portion of non-zero entries, without significantly modifying the properties of the OBS basis. A realistic approach will lead us to modify only an upper-left section of the infinite-dimensional matrix G (corresponding to a certain maximal polynomial degree) and leave the rest of G unchanged. We will consider and compare various strategies for compressing the chosen section of that matrix.

In all cases we will use the following notation: G_t will indicate the matrix obtained from G by setting to zero a certain finite set of entries, $E := G_t - G$ will be the matrix measuring the truncation quality. We assume $\operatorname{diag}(G_t) = \operatorname{diag}(G)$, so that $\operatorname{diag}(E) = 0$. Finally, we introduce the matrix

$$S_\phi = G_t^T S_\eta G_t \quad (2.27)$$

which we interpret as the stiffness matrix associated to the modified BS basis defined in analogy to (2.14) as

$$\phi_k = \sum_{m \in \mathcal{M}_t(k)} g_{mk} \eta_m \quad (2.28)$$

where $\mathcal{M}_t(k) = \{m : E_{mk} = 0\}$. This forms a new basis in $H_0^1(\Omega)$ (it is a basis since $k \in \mathcal{M}_t(k)$ and $g_{kk} \neq 0$), that will be termed a *nearly-orthonormal Babuška-Shen basis* (NOBS basis).

Let $D_\phi = \text{diag } S_\phi$. We want to find a strategy to build G_t such that the eigenvalues λ of the problem

$$S_\phi x = \lambda D_\phi x \quad (2.29)$$

are close to one and bounded from above and away from 0 independently of the polynomial degree. To this end the following result provides a sharp limitation on the eigenvalues in terms of the error matrix E . In the following, we employ the matrix norm induced by the Euclidean norm for vectors.

Proposition 2.3. *Let $L := G^{-T}$ be the lower-triangular Cholesky factor of S_η and $E = G_t - G$. Assume that $\|L^T E\| < 1$. Then the eigenvalues λ of (2.29) satisfy*

$$\frac{(1 - \|L^T E\|)^2}{1 + \max_i \|(L^T E)_{:,i}\|^2} \leq \lambda \leq (1 + \|L^T E D_\phi^{-\frac{1}{2}}\|)^2.$$

Proof. We recall that $S_\eta = LL^T$ and $G_t = G + E$ from which it follows

$$S_\phi = (G^T + E^T)S_\eta(G + E) = (I + L^T E)^T(I + L^T E).$$

For $x \neq 0$, we write $S_\phi x = \lambda D_\phi x$. Multiplying by x^T we get $x^T S_\phi x = \lambda x^T D_\phi x$ with

$$x^T D_\phi x \leq (1 + \max_i \|(L^T E)_{:,i}\|^2) x^T x \quad (2.30)$$

and, denoting by $\sigma_{\min}(A)$ the smallest singular value of A ,

$$x^T S_\phi x \geq (\sigma_{\min}((I + L^T E)))^2 \geq (1 - \|L^T E\|)^2. \quad (2.31)$$

This gives the lower bound. In order to prove the upper bound we proceed as follows. For $y^T y = 1$, we have

$$y^T D_\phi^{-\frac{1}{2}} S_\phi D_\phi^{-\frac{1}{2}} y \leq (\sigma_{\max}((I + L^T E) D_\phi^{-\frac{1}{2}}))^2 \leq (1 + \|L^T E D_\phi^{-\frac{1}{2}}\|)^2.$$

□

Corollary 2.2. *Assume that $\|L^T E\| < 1$. Then the eigenvalues λ of (2.29) satisfy*

$$\frac{(1 - \|L^T E\|)^2}{1 + \|(L^T E)\|^2} \leq \lambda \leq \frac{1}{(1 - \|(L^T E)\|)^2}.$$

Proof. Observing that $\|(L^T E)_{:,i}\| = \|L^T E e_i\| \leq \|L^T E\|$, being e_i the i -th element of the canonical basis, yields the lower bound. On the other hand, taking $x = e_i$ in (2.31), gives

$$\|L^T E D_\phi^{-\frac{1}{2}}\| \leq \frac{\|L^T E\|}{1 - \|L^T E\|}$$

from which the upper bound easily follows.

□

Remark 2.1. If $\sigma_{\min}(D_\phi^{-\frac{1}{2}}) \geq \|L^T E D_\phi^{-\frac{1}{2}}\|$, the lower bound can be sharpened as follows

$$\lambda \geq (\sigma_{\min}((I + L^T E)D_\phi^{-\frac{1}{2}}))^2 \geq (\sigma_{\min}(D_\phi^{-\frac{1}{2}}) - \|L^T E D_\phi^{-\frac{1}{2}}\|)^2$$

with $\sigma_{\min}(D_\phi^{-\frac{1}{2}}) = \min_i (D_\phi^{-\frac{1}{2}})_{ii}$.

Remark 2.2. It is easy to prove that $L^T E$ is a banded matrix whose bandwidth depends on the bandwidth of L^T and on the bandwidth ℓ of G_t . In particular, for $i \neq j$ we have

$$0 = (L^T G)_{ij} = (L^T G_t + L^T E)_{ij} = (L^T G_t)_{ij} + (L^T E)_{ij},$$

or, equivalently, $|(L^T E)_{ij}| = |(L^T G_t)_{ij}|$. Hence, if i, j are such that $(L^T G_t)_{ij}$ is nonzero away from the nonzero band of $L^T G_t$, then the corresponding element of $L^T E$ is zero as well. Moreover, one can prove that for $j - \ell < i$ we have $(L^T E)_{ij} = 0$.

2.2 Compressing the sections G^p

Hereafter, we indicate how to efficiently build compressed versions G_t^p of the sections G^p of the matrix G for increasing values of the maximal polynomial degree p . One of the crucial quantities in the subsequent discussion will be the *compression ratio*

$$r = \frac{\text{nnz}(G_t^p)}{\text{nnz}(G^p)},$$

where $\text{nnz}(A)$ denotes the number of non-zero elements of the matrix A .

Proposition 2.3 suggests to build G_t^p in such a way that

$$\|L^T E\| \leq \text{tol}_G \quad (2.32)$$

is fulfilled, once a tolerance $\text{tol}_G < 1$ has been fixed. This rigorously guarantees the achievement of our target, namely that all eigenvalues of (2.29) are bounded with their reciprocals independently of the polynomial degree. Note that both L^T and E are infinite dimensional matrices; however, the elements of E are certainly zero out of a finite-dimensional section, since we modify G only within a section G^p . As a consequence, the quantity $\|L^T E\|$ is indeed computable, by considering the corresponding finite dimensional section of L^T .

In order to fulfill (2.32), the decay estimate (2.16) suggests to proceed “diagonal-wise”, namely to build G_t^p from G^p by initially retaining its main diagonal and subsequently adding the ℓ -th diagonal, for $\ell = 1, 2, \dots$ until condition (2.32) is satisfied. Figure 5, obtained with the choice $\text{tol}_G = 0.5$, illustrates the typical output of this strategy: the number of activated diagonals (left) and the compression ratio (right) are reported as functions of the polynomial degree p . A close inspection reveals that both quantities stabilize around constant values. This implies that the number of nonzero entries of G_t^p needed to ensure (2.32) by this strategy grows significantly with p ; note in particular the large value of r (only slightly less than 50%), a clear indication of the low efficiency of the procedure.

However, Proposition 2.2 comes in our help, as it indicates that a more sophisticated compression strategy should be applied, than simply neglecting the farthest diagonals from the main diagonal. Thus, we are led to compressing G^p via a thresholding procedure; precisely, recalling formula (2.14), the section G_t^p is obtained by neglecting those entries of G^p for which

$$\frac{|g_{km}|}{g_{kk}} < t, \quad (2.33)$$

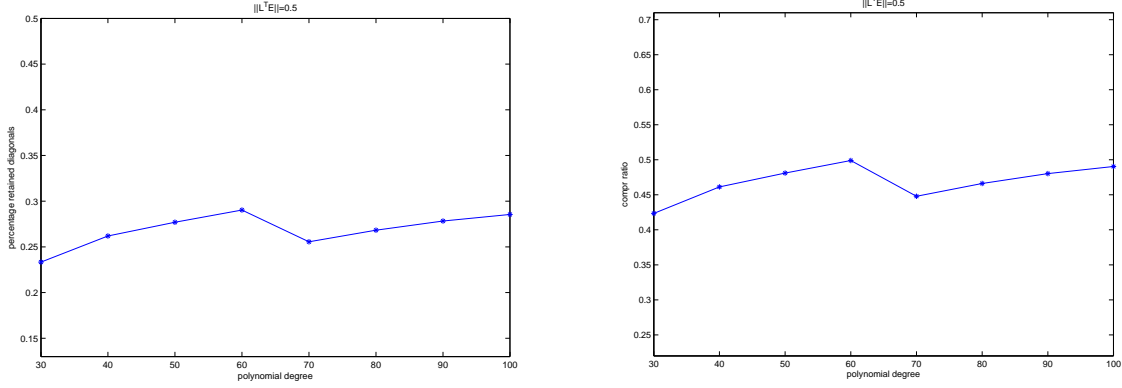


Figure 5: Building G_t^p by adding subsequent diagonals of G^p : percentage of retained diagonals of G^p (left) and compression ratio r of G_t^p (right), versus the polynomial degree

where $t \in (0, 1)$ is the thresholding parameter. The value of t is implicitly defined by the condition that (2.32) be satisfied with $\|L^T E\|$ as close as possible to tol_G . A simple bisection procedure allows one to identify such a nearly-optimal value of t .

Figures 6 and 7, again obtained with $tol_G = 0.5$, illustrate typical outputs of this thresholding strategy. In particular, Fig. 6 (left) shows that the thresholding parameter t , identified by the bisection procedure, decays as the polynomial degree increases. This behavior seems unavoidable in order to reach our final target; indeed, numerical experiments (not reported here) clearly indicate that if we fix the thresholding parameter and vary p , not only the quantity $\|L^T E\|$ eventually becomes larger than any fixed $tol_G < 1$, but the smallest eigenvalue of the resulting stiffness matrix S_ϕ decays to 0. The observed behavior of t implies that entries of G that were set to 0 in G_t^p for a lower value of p , may subsequently be included in G_t^p for higher values of p . This phenomenon is well-documented in Fig. 6 (right); we fixed one of the “slow decaying” columns of G , precisely column 98 already considered in Figure 3 (left), and we counted the number of nonzero elements in that column of G_t^p : the growth with p is apparent. This indicates that there are no (upper left) sectors of G_t^p that remain unchanged while furtherly increasing p ; the construction of G_t^p is “global” and may involve all relevant columns. Note, however, that for the same range of p as in Fig. 6 (right), we observed that the number of nonzero elements of the “fast decaying” column 105 of G_t^p remains fixed to 1.

Although we cannot expect the number of non-zero entries in G_t^p to be proportional to the dimension of the matrix, Fig. 7 (left) shows that the compression ratio is decaying, but at a very slow rate with p and, more importantly, it is more than one order of magnitude smaller than the compression rate guaranteed by the “diagonal-wise” strategy (compare with Fig. 5 (right)). One example of compressed matrix G_t^p produced in this manner (for $p = 100$) is shown in Fig. 7 (right). In addition, the minimal and maximal eigenvalues of the resulting stiffness matrices $S_\phi = S_\phi^p$ exhibit a very moderate deviation from the optimal value 1; this is documented in Fig. 8, which provides a quantitative insight of the upper and lower bounds guaranteed by Corollary 2.2.

In conclusion, the thresholding strategy here discussed appears to guarantee the achievement of our target with a good efficiency for all values of the polynomial degree p relevant in practical implementations.

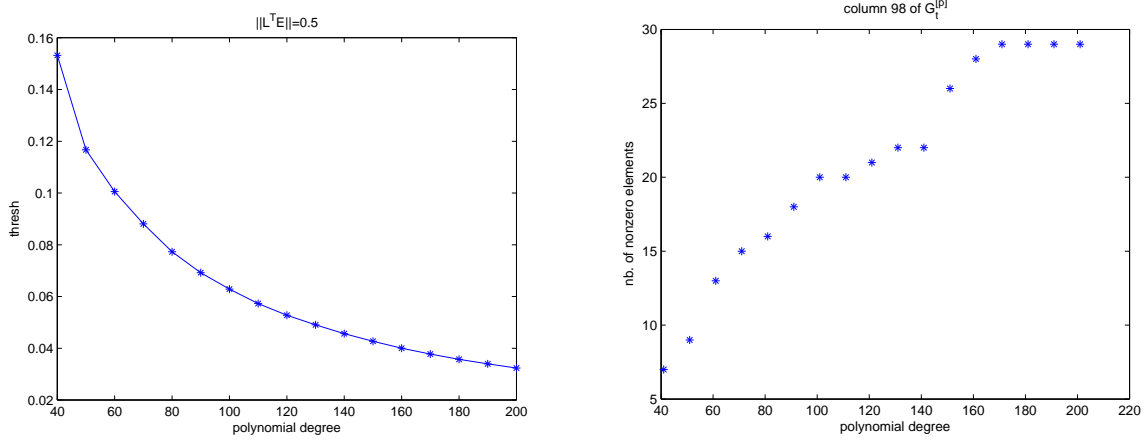


Figure 6: Building G_t^p by a thresholding procedure: thresholding parameter t (left) and number of nonzero elements of a “slow” decaying column of G_t (right), versus the polynomial degree

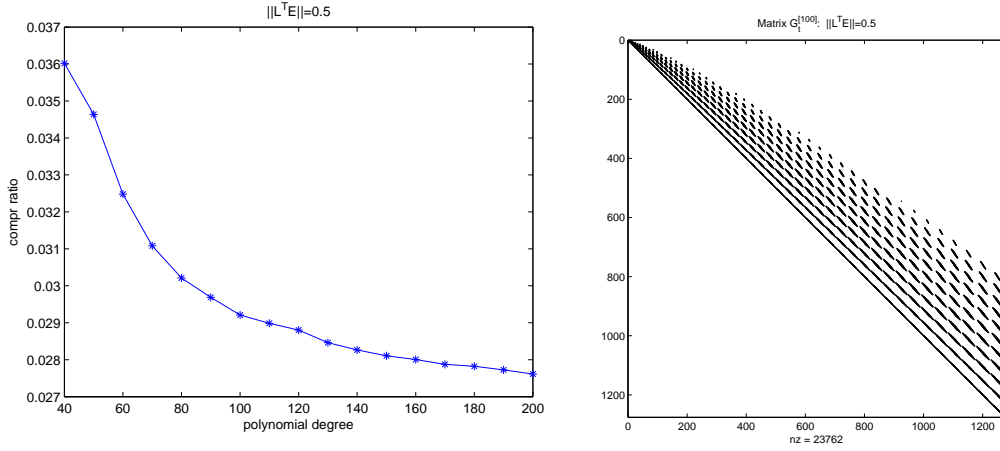


Figure 7: Building G_t^p by a thresholding procedure: compression ratio r of G_t^p versus the polynomial degree (left), sparsity pattern of the matrix G_t^{100} (right)

2.3 Norm representation

From now on, we assume that we work in $H_0^1(\Omega)$ with a Riesz basis $\phi = \{\phi_k : k \in \mathcal{K}\}$ given by (2.28), where the matrix G_t is built according to the thresholding strategy presented above, with a fixed value of tolerance $tol_G < 1$ and a fixed polynomial degree p_{\max} ; precisely, the upper-left section $G^{p_{\max}}$ of G is replaced by $G_t^{p_{\max}}$ while the rest of G is unchanged. Obviously, dealing with such a basis is computationally efficient only if the adaptive algorithm will reach the prescribed accuracy by activating only basis functions having polynomial degree $p \leq p_{\max}$. We will implicitly make this assumption in the sequel.

Thus, if $v \in H_0^1(\Omega)$ admits the expansion $v = \sum_{k \in \mathcal{K}} \hat{v}_k \phi_k$ and \hat{v} denotes the vector collecting

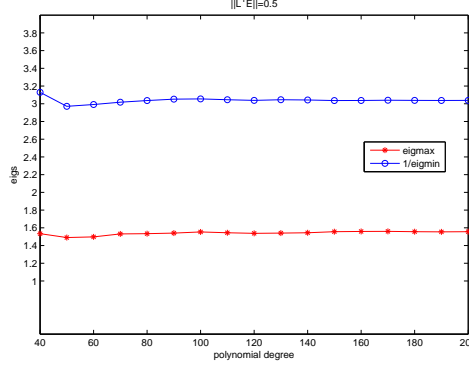


Figure 8: Extreme eigenvalues of S_ϕ versus the maximum polynomial degree.

its coefficients \hat{v}_k , we have

$$\|v\|_{H_0^1(\Omega)}^2 = \hat{v}^T S_\phi \hat{v} \simeq \hat{v}^T D_\phi \hat{v} = \sum_{k \in \mathcal{K}} |\hat{v}_k|^2 d_k =: \|v\|_\phi^2, \quad (2.34)$$

where $D_\phi := \text{diag } S_\phi$ is the diagonal matrix with diagonal elements $d_k = (S_\phi)_{k,k}$. Correspondingly, any element $f \in H^{-1}(\Omega)$ can be expanded along the *dual nearly-orthonormal Babuška-Shen basis* $\phi^* = \{\phi_k^*\}$ as $f = \sum_{k \in \mathcal{K}} \hat{f}_k \phi_k^*$, with $\hat{f}_k = \langle f, \phi_k \rangle$, yielding the dual norm representation

$$\|f\|_{H^{-1}(\Omega)}^2 \simeq \sum_{k \in \mathcal{K}} |\hat{f}_k|^2 d_k^{-1} =: \|v\|_{\phi^*}^2. \quad (2.35)$$

Note that each coefficient \hat{f}_k can be efficiently computed from the values of f on the elements of the standard BS basis, via (2.28):

$$\hat{f}_k = \sum_{m \in \mathcal{M}_t(k)} g_{km} \langle f, \eta_m \rangle. \quad (2.36)$$

Notation. For future references, we introduce the constants of the norm equivalence in (2.34), i.e., we assume that the constants β_* and β^* are such that

$$\beta_* \|v\|_{H_0^1(\Omega)} \leq \|v\|_\phi \leq \beta^* \|v\|_{H_0^1(\Omega)} \quad \forall v \in H_0^1(\Omega), \quad (2.37)$$

which implies

$$\frac{1}{\beta^*} \|f\|_{H^{-1}(\Omega)} \leq \|f\|_{\phi^*} \leq \frac{1}{\beta_*} \|f\|_{H^{-1}(\Omega)} \quad \forall f \in H^{-1}(\Omega). \quad (2.38)$$

Moreover, the ϕ -norm of any $v \in H_0^1(\Omega)$ is equivalent to the ℓ^2 -norm of the vector \hat{v} of its coefficients, i.e., there exist two constants $0 < d_* < d^*$ such that

$$d_* \|\hat{v}\|_{\ell^2}^2 \leq \|v\|_\phi^2 \leq d^* \|\hat{v}\|_{\ell^2}^2 \quad \forall v \in H_0^1(\Omega). \quad (2.39)$$

Indeed, employing (2.30) and (2.31) (under our assumption $\|L^T E\| < 1$) it is immediate to prove that it holds

$$d_* := (1 - \|L^T E\|)^2 \leq d_k \leq 1 + \|L^T E\|^2 =: d^* \quad \forall k \in \mathcal{K}.$$

3 The differential problem and its algebraic representation

We now consider the elliptic problem

$$\begin{cases} \mathcal{L}u = -\nabla \cdot (\nu \nabla u) + \sigma u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

where ν and σ are sufficiently smooth real coefficients satisfying $0 < \nu_* \leq \nu(x) \leq \nu^* < \infty$ and $0 \leq \sigma(x) \leq \sigma^* < \infty$ in Ω ; let us set $\alpha_* = \nu_*$ and $\alpha^* = \max(\nu^*, \sigma^*)$. Assuming $f \in H^{-1}(\Omega)$, we formulate this problem variationally as

$$u \in V \quad : \quad a(u, v) = \langle f, v \rangle \quad \forall v \in V, \quad (3.2)$$

where $a(u, v) = \int_{\Omega} \nu \nabla u \cdot \nabla v + \int_{\Omega} \sigma uv$. We denote by $\|v\| = \sqrt{a(v, v)}$ the energy norm of any $v \in V$, which satisfies

$$\sqrt{\alpha_*} \|v\|_{H_0^1(\Omega)} \leq \|v\| \leq \sqrt{\alpha^*} \|v\|_{H_0^1(\Omega)}. \quad (3.3)$$

Let us identify the solution $u = \sum_k \hat{u}_k \phi_k$ of Problem (3.2) with the vector $u = (\hat{u}_k)_{k \in \mathcal{K}}$ of its nearly-orthonormal Babuška-Shen (NOBS) coefficients. Similarly, let us identify the right-hand side f with the vector $f = (\hat{f}_\ell)_{\ell \in \mathcal{K}}$ of its dual NOBS coefficients. Finally, let us introduce the semi-infinite, symmetric and positive-definite stiffness matrix

$$A_\phi = (a_{\ell k}^\phi)_{\ell, k \in \mathcal{K}} \quad \text{with} \quad a_{\ell k}^\phi = a(\phi_k, \phi_\ell). \quad (3.4)$$

Then, Problem (3.2) can be equivalently written as

$$A_\phi u = f, \quad (3.5)$$

where, thanks to the previous assumptions and the norm equivalences (2.37)-(2.39), A_ϕ defines a bounded invertible operator in $\ell^2(\mathcal{K})$.

The rest of this section will be devoted to prove that if the operator coefficients ν and σ are real analytic in a neighborhood of $\bar{\Omega} = [0, 1]^2 \subset \mathbb{C} \times \mathbb{C}$ (which implies that the rate of decay of their Legendre coefficients is exponential), then the stiffness matrix A_ϕ belongs to a certain exponential class, i.e., its entries exponentially decay away from the diagonal. This property will be crucial in studying the optimality properties of the subsequent adaptive algorithm. After introducing the classes of exponentially decaying matrices and some useful properties related to them, we will obtain the claimed result by using the relation

$$A_\phi = G_t^T A_\eta G_t \quad (3.6)$$

together with suitable exponential decay properties of the factors A_η and G_t .

Definition 3.1 (Regularity classes for A). *A matrix A is said to belong to the exponential class $\mathcal{D}_e(\gamma)$ if there exists a constant $c_\gamma > 0$ such that its elements satisfy*

$$|a_{mn}| \leq c_\gamma e^{-\gamma \|m-n\|_{\ell^1}} \quad m, n \in \mathcal{K}. \quad (3.7)$$

Property 3.1 (Inverse of A). *If $A \in \mathcal{D}_e(\gamma)$ and it is invertible then $A^{-1} \in \mathcal{D}_e(\bar{\gamma})$, for some $\bar{\gamma} \in (0, \gamma]$*

Proof. See [22, Proposition 2]. \square

For any integer $J \geq 0$, let A_J denote the following symmetric truncation of the matrix A

$$(A_J)_{\ell k} = \begin{cases} a_{\ell k} & \text{if } \|\ell - k\|_{\ell^1} \leq J, \\ 0 & \text{elsewhere.} \end{cases} \quad (3.8)$$

Then, we have the following well-known results (see, e.g., [10, Property 2.4]).

Property 3.2 (Truncation). *If $A \in \mathcal{D}_e(\gamma)$ then there exists a constant C_A such that*

$$\|A - A_J\| \leq \psi_A(J, \gamma) := C_A e^{-\gamma J}$$

for all $J \geq 0$. Consequently, under the assumptions of Property 3.1, one has

$$\|A^{-1} - (A^{-1})_J\| \leq \psi_{A^{-1}}(J, \bar{\gamma}) \quad (3.9)$$

where we let $\bar{\gamma}$ be defined in Property 3.1.

We now state the basic assumption on the coefficients of the operator \mathcal{L} .

Assumption 3.1. *Let $\nu(x) = \sum_{k \in \mathcal{K}} \nu_k L_k(x)$ and $\sigma(x) = \sum_{k \in \mathcal{K}} \sigma_k L_k(x)$, resp., be the multidimensional Legendre expansions of the operator coefficients ν and σ , resp. (with $L_k(x) := L_{k_1}(x_1)L_{k_2}(x_2)$). There exist $\gamma > 0$ and a positive constant C_γ only depending on γ such that*

$$|\nu_k|, |\sigma_k| \leq C_\gamma e^{-\gamma \|k\|_{\ell^1}} \quad \forall k \in \mathcal{K}.$$

Lemma 3.1 (Exponential decay of A_η). *Let $\{\eta_k\}_{k \in \mathcal{K}}$ be the tensorized BS basis functions and $A_\eta = (a_{\ell k}^\eta)_{\ell, k \in \mathcal{K}}$ with $a_{\ell k}^\eta = a(\eta_k, \eta_\ell)$ be the stiffness matrix associated to the operator \mathcal{L} . Under Assumption 3.1, it holds*

$$|a_{mn}^\eta| \leq C e^{-\gamma \|n-m\|_{\ell^1}} \quad \forall n, m \in \mathcal{K}, \quad (3.10)$$

where C is a constant only depending on η .

Proof. Due to the highly technical nature of the proof, we postpone it to the Appendix. \square

The proof of the exponential decay of G_t relies on the following intermediate result.

Lemma 3.2 (Exponential decay of S_η^{-1}). *Let S_η be the stiffness matrix of the tensorized Babuška-Shen basis with respect to the $H_0^1(\Omega)$ -inner product. Then there exists $\hat{\gamma} > 0$ such that $S_\eta^{-1} \in \mathcal{D}_e(\hat{\gamma})$.*

Proof. We preliminary observe that, thanks to (2.9), S_η is a banded matrix with half bandwidth equal to 2, once we endow the index set \mathcal{K} with the ℓ^1 -metric, i.e., $(S_\eta)_{mk} = 0$ if $\|k - m\|_{\ell^1} > 2$. To conclude it is sufficient to apply Property 3.1 to S_η (which is banded, thus trivially with exponential decay). \square

Lemma 3.3 (Exponential decay of G and G_t). *Let G be the semi-infinite matrix satisfying equation (2.15). Then there exists $\tilde{\gamma} > 0$ such that $G \in \mathcal{D}_e(\tilde{\gamma})$. Moreover, let G_t denote the matrix obtained from G by setting to zero a certain finite (or infinite) set of entries. Then G_t belongs to the same exponential class of G , i.e. $G_t \in \mathcal{D}_e(\tilde{\gamma})$.*

Proof. We proceed extending the idea of [3, Theorem 4.1] to the present infinite-dimensional setting (see also [21, 23] for similar results). Recall that S_η has been normalized so that $(S_\eta)_{i,i} = 1$ for all $i \in \mathcal{K}$. Here $S_\eta = LL^T$ is the Cholesky factorization of S_η (with L lower triangular infinite dimensional matrix). Using the fact that S_η is a banded matrix (with unitary band) once the index set \mathcal{K} is endowed with the ℓ^1 metric and noticing that it holds $G = L^{-T}$ we get

$$G_{ij} = \sum_{k=j}^{j+1} (S_\eta^{-1})_{ik} \ell_{kj} \quad i, j \in \mathcal{K}. \quad (3.11)$$

Employing the exponential decay of S_η^{-1} (see Lemma 3.2) and recalling that the normalization of S_η implies $|\ell_{ij}| \leq 1$ for all $i, j \in \mathcal{K}$, we obtain

$$\begin{aligned} |G_{ij}| &\leq \sum_{k=j}^{j+1} |(S_\eta^{-1})_{ik}| |\ell_{kj}| \leq C \sum_{k=j}^{j+1} e^{-\hat{\gamma} \|i-k\|_{\ell^1}} \\ &\leq C \sum_{q \geq \|i-j\|_{\ell^1}} \text{card}(K(q)) e^{-\hat{\gamma} q} \end{aligned} \quad (3.12)$$

where $K(q) := \{k \in [j, j+1] : \|i-k\|_{\ell^1} = q\} \subset \mathcal{K}$. Note that as the index subset $K(q)$ inherits the ordering of \mathcal{K} (see Fig. 1(a)), the notion of interval $[j, j+1]$ entering into the definition of $K(q)$ has to be intended consistently with this ordering. It is immediate to verify that it holds $\text{card}(K(q)) \leq \tilde{C}q$ for a positive constant \tilde{C} independent of q . Then it follows, for some $\tilde{\gamma} < \hat{\gamma}$.

$$|G_{ij}| \leq C' \sum_{q \geq \|i-j\|_{\ell^1}} e^{-\hat{\gamma} q} \leq C'' e^{-\tilde{\gamma} \|i-j\|_{\ell^1}} \quad i, j \in \mathcal{K}. \quad (3.13)$$

The second part of the theorem is immediate. \square

Now we are ready to establish the main result of this section.

Proposition 3.1. *Under Assumption 3.1 there exists $\gamma_{\mathcal{L}} \in (0, \gamma]$ such that $A_\phi \in \mathcal{D}_e(\gamma_{\mathcal{L}})$.*

Proof. Recall the expression of A_ϕ given in (3.6). Employing [22, Proposition 1] together with Lemmas 3.3 and 3.1 we deduce that the product $A_\phi = G_t^T A_\eta G_t$ of exponentially decaying semi-infinite matrices is exponentially decaying with a lower decay rate. \square

4 An adaptive algorithm with contraction properties

Given any finite index set $\Lambda \subset \mathcal{K}$, we define the subspace $V_\Lambda = \text{span}\{\phi_k \mid k \in \Lambda\}$ of $V = H_0^1(\Omega)$; we set $|\Lambda| = \text{card } \Lambda$, so that $\dim V_\Lambda = |\Lambda|$. We set $\text{supp } v := \Lambda$ if $v = \sum_{k \in \Lambda} \hat{v}_k \phi_k$ with all $\hat{v}_k \neq 0$. If $v \in V$ admits the expansion $v = \sum_{k \in \mathcal{K}} \hat{v}_k \phi_k$, then we define its projection $P_\Lambda v$ upon V_Λ by setting $P_\Lambda v = \sum_{k \in \Lambda} \hat{v}_k \phi_k$. Similarly, we define the subspace $V_\Lambda^* = \text{span}\{\phi_k^* \mid k \in \Lambda\}$ of $V' = H^{-1}(\Omega)$; if v admits an expansion $f = \sum_{k \in \mathcal{K}} \hat{f}_k \phi_k^*$, then we define its projection $P_\Lambda^* f$ upon V_Λ^* by setting $P_\Lambda^* f = \sum_{k \in \Lambda} \hat{f}_k \phi_k^*$.

Given any finite $\Lambda \subset \mathcal{K}$, the Galerkin approximation of (3.2) is defined as

$$u_\Lambda \in V_\Lambda \quad : \quad a(u_\Lambda, v_\Lambda) = \langle f, v_\Lambda \rangle \quad \forall v_\Lambda \in V_\Lambda. \quad (4.1)$$

For any $w \in V_\Lambda$, we define the residual $r(w) \in V'$ as

$$r(w) = f - \mathcal{L}w = \sum_{k \in \mathcal{K}} \hat{r}_k(w) \phi_k^*, \quad \text{where} \quad \hat{r}_k(w) = \langle f - \mathcal{L}w, \phi_k \rangle = \langle f, \phi_k \rangle - a(w, \phi_k).$$

Then, the previous definition of u_Λ is equivalent to the condition $P_\Lambda^* r(u_\Lambda) = 0$, i.e., $\hat{r}_k(u_\Lambda) = 0$ for every $k \in \Lambda$. By the continuity and coercivity of the bilinear form, one has

$$\frac{1}{\alpha_*} \|r(u_\Lambda)\|_{H^{-1}(\Omega)} \leq \|u - u_\Lambda\|_{H_0^1(\Omega)} \leq \frac{1}{\alpha_*} \|r(u_\Lambda)\|_{H^{-1}(\Omega)}, \quad (4.2)$$

which in view of (2.38) can be rephrased as

$$\frac{\beta_*}{\alpha_*} \|r(u_\Lambda)\|_{\phi^*} \leq \|u - u_\Lambda\|_{H_0^1(\Omega)} \leq \frac{\beta^*}{\alpha_*} \|r(u_\Lambda)\|_{\phi^*}. \quad (4.3)$$

The norm of the residual is hardly computable in practice, since in general the residual $r(u_\Lambda)$ contains infinitely many coefficients. Therefore, we introduce an approximation $\tilde{r}(u_\Lambda)$ of such residual with finite expansion (indexed in some finite set $\tilde{\Lambda} \subset \mathcal{K}$). More precisely, we assume there exists a feasible algorithm which for any fixed parameter $0 < \delta < 1$ and for any v with a finite expansion builds an approximation $\tilde{r}(v)$ of $r(v)$ such that the following crucial inequality holds :

$$\|r(v) - \tilde{r}(v)\|_{\phi^*} \leq \delta \|\tilde{r}(v)\|_{\phi^*}. \quad (4.4)$$

This implies $(1 - \delta) \|\tilde{r}(v)\|_{\phi^*} \leq \|r(v)\|_{\phi^*} \leq (1 + \delta) \|\tilde{r}(v)\|_{\phi^*}$ so that, by (4.3), we obtain

$$(1 - \delta) \frac{\beta_*}{\alpha_*} \|\tilde{r}(u_\Lambda)\|_{\phi^*} \leq \|u - u_\Lambda\|_{H_0^1(\Omega)} \leq (1 + \delta) \frac{\beta^*}{\alpha_*} \|\tilde{r}(u_\Lambda)\|_{\phi^*}. \quad (4.5)$$

Therefore, we are led to use as a posteriori error estimator the quantity

$$\text{Est}(u_\Lambda) = \|\tilde{r}(u_\Lambda)\|_{\phi^*} = \left(\sum_{k \in \tilde{\Lambda}} |(\tilde{r})_k| \right)^{1/2} \quad (4.6)$$

where $(\tilde{r})_k$ are the coefficients of $\tilde{r}(u_\Lambda)$ with respect to the dual basis ϕ^* . In order to adaptively increase the accuracy of the approximation of the Galerkin solution, we apply a Dörfler-marking (or bulk-chasing) strategy. Precisely, given $\theta \in (0, 1)$ we look for a minimal set $\Lambda^* \subset \mathcal{K}$ such that

$$\text{Est}(u_\Lambda; \Lambda^*) := \|P_{\Lambda^*}^* \tilde{r}(u_\Lambda)\|_{\phi^*} \geq \theta \text{Est}(u_\Lambda),$$

which is equivalent to $\|\tilde{r}(u_\Lambda) - P_{\Lambda^*}^* \tilde{r}(u_\Lambda)\|_{\phi^*} \leq \sqrt{1 - \theta^2} \|\tilde{r}(u_\Lambda)\|_{\phi^*}$. A set Λ^* of minimal cardinality can be immediately determined rearranging the coefficients $(\tilde{r})_k$ in non-increasing order of modulus. Next, exploiting the exponential decay of the entries of the stiffness matrix inverse (see Proposition 3.1 and Property 3.1), we enrich the set Λ^* by considering its neighborhood of some radius J depending on θ and the constants in (3.9). This will guarantee that the convergence of our algorithm can be made arbitrarily fast by choosing θ sufficiently close to 1. As a final ingredient, we introduce a coarsening procedure which removes the negligible components of the Galerkin solution at the expense of a controlled increase of the approximation error. This stage is crucial to guarantee the optimality (in a sense made precise later on) of the approximate solution produced by our adaptive algorithm.

4.1 FPC-ADLEG: a feasible adaptive algorithm

We now introduce the following procedures, by which we build our adaptive algorithm.

- $u_\Lambda := \mathbf{GAL}(\Lambda)$
Given a finite subset $\Lambda \subset \mathcal{K}$, the output $u_\Lambda \in V_\Lambda$ is the solution of the Galerkin problem (4.1) relative to Λ .
- $\tilde{r} := \mathbf{F-RES}(v_\Lambda, \delta)$
Given $\delta \in (0, 1)$ and a function $v_\Lambda \in V_\Lambda$ for some finite index set Λ , the module builds an approximate residual $\tilde{r}(v)$ such that (4.4) holds. This is accomplished by building suitable finite approximations of the image $\mathcal{L}v_\Lambda$ and of the right-hand side f (see [10], Sect. 3.2 for further details). Employing a feasible residual allows us to work with a finite set of dual basis functions ϕ_k^* (i.e., with the index k belonging to a finite dimensional subset of \mathcal{K}), or, equivalently, to involve in the expression of the residual only an upper-left (finite) section of the infinite-dimensional matrix G_t and not the whole G_t . As already mentioned, we assume that the polynomial degrees of the basis functions activated during the adaptive algorithm (see below the sets $\partial\Lambda_n$ produced by the Dörfler modulus) never exceed a given maximum degree p_{\max} . This, in turn, is equivalent to assuming that the value of p_{\max} , which determines the construction of $G^{p_{\max}}$ via the Gram-Schmidt procedure, is chosen so large that all the generated upper-left (finite) sections G_t^p are contained in $G^{p_{\max}}$. Clearly, such a choice of p_{\max} is related to the value of the parameter δ and to the tolerance tol employed to stop the algorithm. The possibility of adaptively increasing p_{\max} during the algorithm in order to fulfil this requirement will be considered elsewhere.
- $\Lambda^* := \mathbf{DÖRFLER}(r, \theta)$
Given $\theta \in (0, 1)$ and an element $r \in V'$ having a finite expansion, the output $\Lambda^* \subset \mathcal{K}$ is a finite set of minimal cardinality such that the following inequality holds:

$$\|P_{\Lambda^*}^* r\|_{\phi^*} \geq \theta \|r\|_{\phi^*} . \quad (4.7)$$

- $\Lambda^* := \mathbf{ENRICH}(\Lambda, J)$
Given an integer $J \geq 0$ and a finite set $\Lambda \subset \mathcal{K}$, the output is the set

$$\Lambda^* := \{k \in \mathcal{K} : \text{there exists } \ell \in \Lambda \text{ such that } \|k - \ell\|_{\ell^1} \leq J\} .$$

Note that since the procedure adds a 2-dimensional ball of radius J in the ℓ^1 -metric around each point of Λ , the cardinality of the new set Λ^* can be estimated as $|\Lambda^*| \lesssim 2J^2 |\Lambda|$.

- $\Lambda^* := \mathbf{E-DÖRFLER}(r, \theta)$
Given $\theta \in (0, 1)$ and an element $r \in H^{-1}(I)$ with finite expansion, the output $\Lambda^* \subset \mathcal{K}$ is defined by the sequence

$$\tilde{\Lambda} := \mathbf{DÖRFLER}(r, \theta), \quad \Lambda^* := \mathbf{ENRICH}(\tilde{\Lambda}, J_\theta) \quad (4.8)$$

where J_θ is the smallest integer for which $\psi_{A^{-1}}(J_\theta, \bar{\gamma}) = C_{A^{-1}} e^{-\bar{\gamma} J_\theta} \leq \frac{\beta_*^2}{d_*} \sqrt{\frac{1-\theta^2}{\alpha_* \alpha^*}}$ (recall Property 3.2).

- $\Lambda := \mathbf{COARSE}(w, \epsilon)$

Given a function $w \in V_{\Lambda^*}$ for some finite index set Λ^* , and an accuracy $\epsilon > 0$ which is known to satisfy $\|u - w\|_{H_0^1(\Omega)} \leq \epsilon$, the output $\Lambda \subseteq \Lambda^*$ is a set of minimal cardinality such that

$$\|w - P_{\Lambda} w\|_{\phi} \leq 2\beta_* \epsilon. \quad (4.9)$$

We are now ready to introduce our adaptive algorithm, which we call Feasible Predictor-Corrector ADaptive LEGendre method (**FPC-ADLEG**). Given a tolerance $tol \in [0, 1]$, a marking parameter $\theta \in (0, 1)$ and a feasibility parameter $0 < \delta < \sqrt{1 - \theta^2}$, **FPC-ADLEG** reads as follows.

Algorithm FPC-ADLEG(θ, δ, tol)

Set $u_0 := 0, \Lambda_0 := \emptyset, n = -1$

$\tilde{r}_0 := \mathbf{F-RES}(u_0, \delta)$

do

$n \leftarrow n + 1$

$\partial\Lambda_n := \mathbf{E-DÖRFLER}(\tilde{r}_n, \theta)$

$\hat{\Lambda}_{n+1} := \Lambda_n \cup \partial\Lambda_n$

$\hat{u}_{n+1} := \mathbf{GAL}(\hat{\Lambda}_{n+1})$

$\Lambda_{n+1} := \mathbf{COARSE}\left(\hat{u}_{n+1}, 3\frac{\beta^*}{\alpha_*}\sqrt{1 - \theta^2}\|\tilde{r}_n\|_{\phi^*}\right)$

$u_{n+1} := \mathbf{GAL}(\Lambda_{n+1})$

$\tilde{r}_{n+1} := \mathbf{F-RES}(u_{n+1}, \delta)$

while $\|\tilde{r}_{n+1}\|_{\phi^*} > \frac{tol}{1+\delta}$

Theorem 4.1 (contraction property of **FPC-ADLEG**). *Setting $\rho := 9\frac{\alpha^*}{\alpha_*}\frac{\beta^*}{\beta_*}\frac{\sqrt{1-\theta^2}}{1-\delta}$ then the errors $u - u_n$ generated for $n \geq 0$ by the algorithm satisfy the inequalities*

$$\|u - u_{n+1}\|_{H_0^1(\Omega)} \leq \rho \|u - u_n\|_{H_0^1(\Omega)}. \quad (4.10)$$

Therefore, if θ is chosen in such a way that $\rho < 1$, for any $tol > 0$ the algorithm terminates in a finite number of iterations, whereas for $tol = 0$ the sequence u_n converges to u in $H_0^1(\Omega)$ as $n \rightarrow \infty$. \square

Proof. The contraction factor ρ can be estimated following the same guidelines used in [10, Sect. 3.3] for getting formula (3.23) therein; obviously, here we use eqns. (4.3) and (4.5). From (4.10) it is standard to deduce the bounds $\|u - u_n\|_{H_0^1(\Omega)} \leq \rho^n \|u - u_0\|_{H_0^1(\Omega)}$ for any $n \geq 0$, which imply the convergence of the algorithm for $tol = 0$ and the finite termination property for $tol > 0$ using the left-hand side estimate in (4.5). \square

Note that each iteration of **PC-ADLEG** can be viewed as a predictor step followed by a corrector step. The predictor step guarantees an arbitrarily large error reduction (by suitably enriching the output set from the Dörfler procedure) at the expense of possibly activating an unnecessarily large number of basis functions. The coarsening procedure acts as a corrector step which removes the negligible components of the output of the predictor step. The quantitative description of this mechanism will be possible after we introduce suitable sparsity classes.

4.2 Nonlinear approximation and sparsity classes

Given any $v \in V$ we define its *best N -term approximation error* as

$$E_N(v) = \inf_{\Lambda \subset \mathcal{K}, |\Lambda|=N} \|v - P_\Lambda v\|_\phi.$$

We will be interested in classifying functions according to the decay law of their best N -term approximations, as $N \rightarrow \infty$, i.e., according to the “sparsity” of their expansions along the NOBS basis. In particular, we will consider the following exponential class.

Definition 4.1 (Exponential class of functions). *For $\gamma > 0$ and $0 < q \leq 2$, we denote by $\mathcal{A}_G^{\gamma,q}$ the set defined as*

$$\mathcal{A}_G^{\gamma,q} := \left\{ v \in V : \|v\|_{\mathcal{A}_G^{\gamma,q}} := \sup_{N \geq 0} E_N(v) \exp(\gamma(N/2)^{q/2}) < +\infty \right\}.$$

For functions v in $\mathcal{A}_G^{\gamma,q}$ one can estimate the minimal cardinality of a set Λ such that $\|v - P_\Lambda v\|_\phi \lesssim \varepsilon$ as follows

$$|\Lambda| \leq \frac{2}{\gamma^{2/q}} \left(\log \frac{\|v\|_{\mathcal{A}_G^{\gamma,q}}}{\varepsilon} \right)^{2/t} + 1. \quad (4.11)$$

We note that the class of functions that are analytic in an ellipsoid containing in its interior the set $\bar{\Omega}$ belongs to $\mathcal{A}_G^{\gamma,1}$. More generally, functions that are not analytic in $\bar{\Omega}$ but possess a certain Gevrey regularity belong to $\mathcal{A}_G^{\gamma,q}$ for some $0 < q < 1$ (see [9, 2] for more details).

Let us assume that the solution u to (3.1) belongs to some $\mathcal{A}_G^{\gamma,q}$. The optimality of an algorithm for approximating u is defined as the capability of constructing, for any $\epsilon > 0$, a finite dimensional approximation u_Λ satisfying $\|u - u_\Lambda\|_{H_0^1(\Omega)} \leq \epsilon$ with the cardinality of $\Lambda := \text{supp } u_\Lambda$ bounded as in (4.11), possibly up to some additive constant. A further optimality requirement concerns the cardinality of the supports of all the intermediate functions introduced by the algorithm in order to compute u_Λ : these cardinalities should all be proportional to $|\Lambda|$, with a proportionality constant independent of ϵ .

For the analysis of the optimality of our algorithm it is important to investigate the sparsity class of the image $\mathcal{L}v$ for the operator \mathcal{L} defined in (3.1), when the function v belongs to the sparsity class $\mathcal{A}_G^{\gamma,q}$. The proof is omitted, as it is similar to the one of [10, Proposition 5.2].

Proposition 4.1 (Continuity of \mathcal{L} in $\mathcal{A}_G^{\gamma,q}$). *Let the differential operator \mathcal{L} be such that the corresponding stiffness matrix satisfies $A_\phi \in \mathcal{D}_\epsilon(\gamma_\mathcal{L})$ for some constant $\gamma_\mathcal{L} > 0$ (recall Proposition 3.1). Assume that $v \in \mathcal{A}_G^{\gamma,q}$ for some $\gamma > 0$ and $q \in (0, 2]$. Let one of the two following set of conditions be satisfied.*

- (a) *If the matrix A_ϕ is banded with $2m + 1$ non-zero diagonals, let us set*

$$\bar{\gamma} = \frac{\gamma}{(2m + 1)^{q/2}}, \quad \bar{q} = q.$$

- (b) *If the matrix A_ϕ is dense, but the constants $\gamma_\mathcal{L}$ and γ satisfy the inequality $\gamma < 2^{q/2}\gamma_\mathcal{L}$, let us set*

$$\bar{\gamma} = \zeta(q)\gamma, \quad \bar{q} = \frac{q}{1 + q},$$

where we define

$$\zeta(q) := \left(\frac{1 + q}{8 \cdot 2^q} \right)^{\frac{q}{2(1+q)}}. \quad (4.12)$$

Then, one has $\mathcal{L}v \in \mathcal{A}_G^{\tilde{\gamma}, \bar{q}}$, with

$$\|\mathcal{L}v\|_{\mathcal{A}_G^{\tilde{\gamma}, \bar{q}}} \lesssim \|v\|_{\mathcal{A}_G^{\gamma, q}}. \quad (4.13)$$

This result indicates that the residual is expected to belong to a less favorable sparsity class than the one of the solution.

We are ready to relate the cardinality of the finite expansions activated by the algorithm **FPC-ADLEG** to the sparsity class of the solution.

Theorem 4.2 (Cardinalities in **FPC-ADLEG**). *Suppose that $u \in \mathcal{A}_G^{\gamma, q}$, for some $\gamma > 0$ and $q \in (0, 2]$. Then, there exists a constant $C > 1$ independent of θ such that*

$$|\text{supp } u_n| \leq \frac{2}{\gamma^{2/q}} \left(\log \frac{\|u\|_{\mathcal{A}_G^{\gamma, q}}}{\|u - u_n\|_{H_0^1(\Omega)}} + \log C \right)^{2/q} + 1, \quad \forall n \geq 0. \quad (4.14)$$

If, in addition, the assumptions of Proposition 4.1 are satisfied, then there exists a constant $C_* > 1$ proportional to $(1 - \theta^2)^{-1/2}$ such that the feasible residual $\tilde{r}(u_n)$ satisfies

$$|\text{supp } \tilde{r}(u_n)| \leq \frac{2}{\tilde{\gamma}^{2/\bar{q}}} \left(\log \frac{\|u\|_{\mathcal{A}_G^{\gamma, q}}}{\|u - u_n\|_{H_0^1(\Omega)}} + \log C_* \right)^{2/\bar{q}} + 1 \quad \forall n \geq 0, \quad (4.15)$$

where $\tilde{\gamma} = 2^{-\bar{q}/2}\gamma$ and $\tilde{\gamma} \leq \gamma$, $\bar{q} \leq q$ are the parameters defined in Proposition 4.1. Moreover, the intermediate Galerkin solution \hat{u}_{n+1} computed in the predictor step satisfies

$$|\text{supp } \hat{u}_{n+1}| \leq \frac{2}{\hat{\gamma}^{d/\bar{q}}} \left(\log \frac{\|u\|_{\mathcal{A}_G^{\gamma, q}}}{\|u - u_n\|_{H_0^1(\Omega)}} + \log C_* \right)^{2/\bar{q}} + 1, \quad \forall n \geq 0, \quad (4.16)$$

where $\hat{\gamma}$ is defined by the relation $\hat{\gamma}^{-2/\bar{q}} = \gamma^{-2/q} + 2J_\theta^2 \gamma^{-2/\bar{q}}$ with J_θ introduced in **E-DÖRFLER**.

Proof. The results can be established following the guidelines used in [10] for the Fourier case: see, in particular, the proof of Theorem 8.1 for establishing (4.14) and the proof of Theorem 8.3 for establishing (4.15)-(4.16). We omit the details since the essential ingredients are the same as in the quoted reference. \square

The theorem indicates that the predictor step is driven by the sparsity class of the residual, which in view of Proposition 4.1 may be worse than the sparsity class of the exact solution; therefore, optimality with respect to the latter class is not guaranteed. On the other hand, the corrector step brings the cardinality close to the optimal one determined by the sparsity class of the solution (compare (4.14) to (4.11) in which $v = u$ and $\varepsilon = \|u - u_n\|_{H_0^1(\Omega)}$).

Appendix

Proof of Lemma 3.1. In the following we will make extensive use of the following property of the product of univariate Legendre polynomials (see, e.g., [1]):

$$L_m(x_i)L_n(x_i) = \sum_{r=0}^{\min(m,n)} A_{m,n}^r L_{m+n-2r}(x_i) \quad i = 1, 2 \quad (4.17)$$

with

$$A_{m,n}^r := \frac{A_{m-r}A_rA_{n-r}}{A_{n+m-r}} \frac{2n+2m-4r+1}{2n+2m-2r+1}$$

and

$$A_0 := 1, \quad A_m := \frac{1 \cdot 3 \cdot 5 \dots (2m-1)}{m!} = \frac{(2m)!}{2^m(m!)^2}.$$

Moreover we recall the following asymptotic estimates (see [9]):

- Case $0 < r < \min(m, n)$:

$$\frac{A_{m-r}A_rA_{n-r}}{A_{n+m-r}} \sim \frac{1}{\pi} \frac{\sqrt{n+m-r}}{\sqrt{m-r}\sqrt{n-r}\sqrt{r}}; \quad (4.18)$$

- Case $r = 0$:

$$\frac{A_mA_n}{A_{n+m}} \sim \frac{1}{\sqrt{\pi}} \frac{\sqrt{n+m}}{\sqrt{nm}}; \quad (4.19)$$

- Case $r = \min(m, n)$ and $m \neq n$:

$$\frac{A_{\min(m,n)}A_{|m-n|}}{A_{\max(m,n)}} \sim \frac{1}{\sqrt{\pi}} \frac{\sqrt{\max(m,n)}}{\sqrt{\min(m,n)}\sqrt{|m-n|}}. \quad (4.20)$$

When $m = n$ it is sufficient to use $A_0 = 1$ to get $\frac{A_mA_0}{A_m} = 1$.

We begin from the following expression:

$$a_{mn}^\eta = \int_{\Omega} \nu(x) \nabla \eta_m(x) \nabla \eta_n(x) dx + \int_{\Omega} \sigma(x) \eta_m(x) \eta_n(x) dx =: a_{mn}^{(1)} + a_{mn}^{(0)}. \quad (4.21)$$

We first estimate $a_{mn}^{(1)}$. Let $m = (m_1, m_2)$ and $n = (n_1, n_2)$ then using the notation $\nu := \nu(x_1, x_2)$ and the relation $\eta'_k(x_i) = -\sqrt{k-1/2}L_{k-1}(x_i)$, $i = 1, 2$, we have

$$\begin{aligned} a_{mn}^{(1)} &= \int_{\Omega} \nu \eta'_{m_1}(x_1) \eta'_{n_1}(x_1) \eta_{m_2}(x_2) \eta_{n_2}(x_2) dx_1 dx_2 + \int_{\Omega} \nu \eta_{m_1}(x_1) \eta_{n_1}(x_1) \eta'_{m_2}(x_2) \eta'_{n_2}(x_2) dx_1 dx_2 \\ &= B_{m,n}^1 \int_{\Omega} \nu L_{m_1-1}(x_1) L_{n_1-1}(x_1) [L_{m_2-2}(x_2) - L_{m_2}(x_2)] [L_{n_2-2}(x_2) - L_{n_2}(x_2)] dx_1 dx_2 \\ &\quad + B_{m,n}^2 \int_{\Omega} \nu [L_{m_1-2}(x_1) - L_{m_1}(x_1)] [L_{n_1-2}(x_1) - L_{n_1}(x_1)] L_{m_2-1}(x_2) L_{n_2-1}(x_2) dx_1 dx_2 \\ &=: J_1 + J_2 \end{aligned} \quad (4.22)$$

where we set

$$B_{m,n}^1 := \frac{\sqrt{m_1-1/2}\sqrt{n_1-1/2}}{\sqrt{4m_2-2}\sqrt{4n_2-2}} \quad B_{m,n}^2 := \frac{\sqrt{m_2-1/2}\sqrt{n_2-1/2}}{\sqrt{4m_1-2}\sqrt{4n_1-2}}.$$

Let us focus on the first term J_1 . Straightforward calculations yield

$$\begin{aligned}
J_1 &= B_{m,n}^1 \left\{ \int_{\Omega} \nu L_{m_1-1}(x_1) L_{n_1-1}(x_1) L_{m_2-2}(x_2) L_{n_2-2}(x_2) dx_1 dx_2 \right. \\
&\quad - \int_{\Omega} \nu L_{m_1-1}(x_1) L_{n_1-1}(x_1) L_{m_2}(x_2) L_{n_2-2}(x_2) dx_1 dx_2 \\
&\quad - \int_{\Omega} \nu L_{m_1-1}(x_1) L_{n_1-1}(x_1) L_{m_2-2}(x_2) L_{n_2}(x_2) dx_1 dx_2 \\
&\quad \left. + \int_{\Omega} \nu L_{m_1-1}(x_1) L_{n_1-1}(x_1) L_{m_2}(x_2) L_{n_2}(x_2) dx_1 dx_2 \right\} \\
&=: J_1^1 + J_1^2 + J_1^3 + J_1^4.
\end{aligned}$$

For the ease of presentation we only show how to estimate J_1^1 as the other terms can be worked out similarly. Employing (4.17) we obtain

$$\begin{aligned}
J_1^1 &= B_{m,n}^1 \int_{\Omega} \nu \sum_{r_1=0}^{\min(m_1-1, n_1-1)} A_{m_1-1, n_1-1}^{r_1} L_{m_1+n_1-2-2r_1}(x_1) \\
&\quad \sum_{r_2=0}^{\min(m_2-2, n_2-2)} A_{m_2-2, n_2-2}^{r_2} L_{m_2+n_2-4-2r_2}(x_2) dx_1 dx_2 \\
&= B_{m,n}^1 \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} A_{m_1-1, n_1-1}^{r_1} A_{m_2-2, n_2-2}^{r_2} \int_{\Omega} \nu L_{m_1+n_1-2-2r_1} L_{m_2+n_2-4-2r_2} dx_1 dx_2.
\end{aligned}$$

Using the multidimensional Legendre expansion $\nu(x) = \sum_{k \in \mathcal{K}} \nu_k L_k(x)$ we obtain

$$J_1^1 = 4B_{m,n}^1 \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \frac{A_{m_1-1, n_1-1}^{r_1} A_{m_2-2, n_2-2}^{r_2} \nu_{m_1+n_1-2-2r_1, m_2+n_2-4-2r_2}}{[2(m_1+n_1-2-2r_1)+1][2(m_2+n_2-4-2r_2)+1]}.$$

We now employ the asymptotic estimates (4.18)-(4.20) to bound the terms $A_{m_1-2, n_1-2}^{r_1}$ and $A_{m_2-2, n_2-2}^{r_2}$. Accordingly, we need to distinguish among several cases depending on the combination of the values assumed by r_1 and r_2 . However, for the ease of reading, we only consider the case $0 < r_1 < \min(m_1-2, n_1-2)$ and $0 < r_2 < \min(m_2-2, n_2-2)$ as the other ones can be treated similarly. In this case, (4.18) yields

$$A_{m_1-2, n_1-2}^{r_1} \simeq \frac{1}{\pi} \frac{\sqrt{n_1+m_1-4-r_1}}{\sqrt{m_1-2-r_1}\sqrt{n_1-2-r_1}\sqrt{r_1}} \frac{2(n_1-2)+2(m_1-2)-4r_1+1}{2(n_1-2)+2(m_1-2)-2r_1+1}. \quad (4.23)$$

A similar estimate holds also for $A_{m_2-2, n_2-2}^{r_2}$. Thus we have

$$\begin{aligned}
& \frac{B_{m,n}^1 A_{m_1-1, n_1-1}^{r_1} A_{m_2-2, n_2-2}^{r_2}}{[2(m_1 + n_1 - 2 - 2r_1) + 1][2(m_2 + n_2 - 4 - 2r_2) + 1]} \simeq \\
& \simeq \frac{\sqrt{2m_1-1}\sqrt{2n_1-1}}{\sqrt{2m_2-1}\sqrt{2n_2-1}} \frac{\sqrt{n_2+m_2-4-r_2}\sqrt{n_1+m_1-4-r_1}}{\sqrt{m_2-2-r_2}\sqrt{n_2-2-2r_2}\sqrt{r_2}\sqrt{m_2-2-r_2}\sqrt{n_2-2-2r_2}\sqrt{r_2}} \\
& \quad \frac{1}{(2m_2+2n_2-2r_2-7)(2m_1+2n_1-2r_1-3)} \\
& \simeq \frac{\sqrt{m_1 n_1}}{\sqrt{m_1-1-r_1}\sqrt{n_1-1}\sqrt{r_1}} \frac{1}{\sqrt{n_1+m_1-2-r_1}} \frac{1}{\sqrt{n_2+m_2-4-r_2}} \frac{1}{\sqrt{2m_2-1}\sqrt{2n_2-1}} \\
& \quad \frac{1}{\sqrt{m_2-2-r_2}\sqrt{n_2-2-2r_2}\sqrt{r_2}} \\
& \simeq \frac{1}{\sqrt{\min(m_1+n_1-2, |m_1-n_1|)}} \frac{1}{\sqrt{n_2+m_2-4-r_2}} \frac{1}{\sqrt{2m_2-1}\sqrt{2n_2-1}} \\
& \quad \frac{1}{\sqrt{m_2-2-r_2}\sqrt{n_2-2-2r_2}\sqrt{r_2}} \\
& \lesssim 1.
\end{aligned}$$

Thus we have

$$J_1^1 \lesssim \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-4-2r_2}. \quad (4.24)$$

Similar estimates can be obtained for the terms J_1^2, \dots, J_1^4 yielding

$$\begin{aligned}
J_1 & \lesssim \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-4-2r_2} \\
& + \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2, n_2-2)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-2-2r_2} \\
& + \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-2-2r_2} \\
& + \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2, n_2)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-2r_2}.
\end{aligned}$$

Analogously, we can prove the following estimate for the term J_2

$$\begin{aligned}
J_2 &\lesssim \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-1, n_2-1)} \nu_{m_1+n_1-4-2r_1, m_2+n_2-2-2r_2} \\
&+ \sum_{r_1=0}^{\min(m_1, n_1-2)} \sum_{r_2=0}^{\min(m_2-1, n_2-1)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-2-2r_2} \\
&+ \sum_{r_1=0}^{\min(m_1-2, n_1)} \sum_{r_2=0}^{\min(m_2-1, n_2-1)} \nu_{m_1+n_1-2-2r_1, m_2+n_2-2-2r_2} \\
&+ \sum_{r_1=0}^{\min(m_1, n_1)} \sum_{r_2=0}^{\min(m_2-1, n_2-1)} \nu_{m_1+n_1-2r_1, m_2+n_2-2-2r_2}.
\end{aligned}$$

Assuming $|\nu_k| \leq C_\eta e^{-\gamma\|k\|_{\ell^1}}$ for every $k \in \mathcal{K}$ and employing the above estimates for J_1 and J_2 , we obtain

$$\begin{aligned}
|a_{mn}^{(1)}| &\lesssim e^{-\gamma(|m_1-n_1|+|m_2-n_2|)} \left\{ \sum_{r_1=0}^{\min(m_1-1, n_1-1)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} e^{-2\gamma(\min(m_1-1, n_1-1)-r_1)} e^{-2\gamma(\min(m_2-2, n_2-2)-r_2)} \right. \\
&\quad \left. + \dots + \sum_{r_1=0}^{\min(m_1, n_1)} \sum_{r_2=0}^{\min(m_2-1, n_2-1)} e^{-2\gamma(\min(m_1, n_1)-r_1)} e^{-2\gamma(\min(m_2-1, n_2-1)-r_2)} \right\} \\
&\lesssim e^{-\gamma(|m_1-n_1|+|m_2-n_2|)} = C e^{-\gamma\|m-n\|_{\ell^1}}.
\end{aligned}$$

We now estimate $a_{mn}^{(0)}$. Let $m = (m_1, m_2)$ and $n = (n_1, n_2)$ then recalling (2.2) and using the notation $\sigma := \sigma(x_1, x_2)$ we have

$$\begin{aligned}
a_{mn}^{(0)} &= \int_{\Omega} \sigma \eta_{m_1}(x_1) \eta_{n_1}(x_1) \eta_{m_2}(x_2) \eta_{n_2}(x_2) dx_1 dx_2 \\
&= C_m^n \int_{\Omega} \sigma [(L_{m_1-2} - L_{m_1})(L_{n_1-2} - L_{n_1})](x_1) [(L_{m_2-2} - L_{m_2})(L_{n_2-2} - L_{n_2})](x_2) dx_1 dx_2 \\
&= C_m^n \int_{\Omega} \sigma [L_{m_1-2} L_{n_1-2} - L_{m_1-2} L_{n_1} - L_{m_1} L_{n_1-2} + L_{m_1} L_{n_1}](x_1) [L_{m_2-2} L_{n_2-2} + \\
&\quad - L_{m_2-2} L_{n_2} - L_{m_2} L_{n_2-2} + L_{m_2} L_{n_2}](x_2) dx_1 dx_2
\end{aligned}$$

with $C_m^n := \frac{1}{\sqrt{(4m_1-1)(4m_2-1)(4n_1-1)(4n_2-1)}}$. Now employing (4.17) we obtain

$$\begin{aligned}
a_{mn}^{(0)} &= C_m^n \int_{\Omega} \sigma \left[\sum_{r_1=0}^{\min(m_1-2, n_1-2)} A_{m_1-2, n_1-2}^{r_1} L_{m_1+n_1-4-2r_1} - \sum_{r_1=0}^{\min(m_1-2, n_1)} A_{m_1-2, n_1}^{r_1} L_{m_1+n_1-2-2r_1} \right. \\
&\quad \left. - \sum_{r_1=0}^{\min(m_1, n_1-2)} A_{m_1, n_1-2}^{r_1} L_{m_1+n_1-2-2r_1} + \sum_{r_1=0}^{\min(m_1, n_1)} A_{m_1, n_1}^{r_1} L_{m_1+n_1-2r_1} \right] (x_1) \\
&\quad \left[\sum_{r_2=0}^{\min(m_2-2, n_2-2)} A_{m_2-2, n_2-2}^{r_2} L_{m_2+n_2-4-2r_2} - \sum_{r_2=0}^{\min(m_2-2, n_2)} A_{m_2-2, n_2}^{r_2} L_{m_2+n_2-2-2r_2} \right. \\
&\quad \left. - \sum_{r_2=0}^{\min(m_2, n_2-2)} A_{m_2, n_2-2}^{r_2} L_{m_2+n_2-2-2r_2} + \sum_{r_2=0}^{\min(m_2, n_2)} A_{m_2, n_2}^{r_2} L_{m_2+n_2-2r_2} \right] (x_2) dx_1 dx_2. \\
&= I_1 + \dots + I_{16}.
\end{aligned}$$

We now need to estimate I_1, \dots, I_{16} . To simplify the exposition, we only show how to estimate I_1 , as the other terms can be treated similarly.

Using the multidimensional Legendre expansion $\sigma(x) = \sum_{k \in \mathcal{K}} \sigma_k L_k(x)$ together with (2.1) we get

$$\begin{aligned}
I_1 &= C_m^n \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} A_{m_1-2, n_1-2}^{r_1} A_{m_2-2, n_2-2}^{r_2} \int_{\Omega} \sigma L_{m_1+n_1-4-2r_1} L_{m_2+n_2-4-2r_2} dx_1 dx_2 \\
&= C_m^n \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \frac{A_{m_1-2, n_1-2}^{r_1} A_{m_2-2, n_2-2}^{r_2} \sigma_{m_1+n_1-4-2r_1, m_2+n_2-4-2r_2}}{[2(m_1+n_1-4-2r_1)+1][2(m_2+n_2-4-2r_2)+1]}.
\end{aligned}$$

We now employ the asymptotic estimates (4.18)-(4.20) to bound the terms $A_{m_1-2, n_1-2}^{r_1}$ and $A_{m_2-2, n_2-2}^{r_2}$. Accordingly, we need to distinguish among several cases depending on the combination of the values assumed by r_1 and r_2 . However, for the ease of reading, we only consider the case $0 < r_1 < \min(m_1-2, n_1-2)$ and $0 < r_2 < \min(m_2-2, n_2-2)$ as the other ones can be treated similarly. In this case, (4.18) yields

$$A_{m_1-2, n_1-2}^{r_1} \simeq \frac{1}{\pi} \frac{\sqrt{n_1+m_1-4-r_1}}{\sqrt{m_1-2-r_1} \sqrt{n_1-2-r_1} \sqrt{r_1}} \frac{2(n_1-2)+2(m_1-2)-4r_1+1}{2(n_1-2)+2(m_1-2)-2r_1+1}. \quad (4.25)$$

A similar estimate holds also for $A_{m_2-2, n_2-2}^{r_2}$. Hence, we have

$$\frac{C_m^n A_{m_1-2, n_1-2}^{r_1} A_{m_2-2, n_2-2}^{r_2}}{[2(m_1+n_1-4-2r_1)+1][2(m_2+n_2-4-2r_2)+1]} \lesssim 1. \quad (4.26)$$

Employing (4.19) and (4.20) yields similar estimates for the cases $r_1 = 0, \min(m_1-2, n_1-2)$ and $r_2 = 0, \min(m_2-2, n_2-2)$. In conclusion, we get

$$I_1 \lesssim \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \sigma_{m_1+n_1-4-2r_1, m_2+n_2-4-2r_2}. \quad (4.27)$$

Similar estimates can be obtained for I_2, \dots, I_{16} thus yielding

$$\begin{aligned} a_{mn}^{(0)} &\lesssim \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} \sigma_{m_1+n_1-4-2r_1, m_2+n_2-4-2r_2} \\ &\quad + \dots + \sum_{r_1=0}^{\min(m_1, n_1)} \sum_{r_2=0}^{\min(m_2, n_2)} \sigma_{m_1+n_1-2r_1, m_2+n_2-2r_2}. \end{aligned}$$

Assuming $|\sigma_k| \leq C_\eta e^{-\gamma\|k\|_{\ell^1}}$ for every $k \in \mathcal{K}$ and employing the above estimate, we obtain

$$\begin{aligned} |a_{mn}^{(0)}| &\lesssim e^{-\gamma(|m_1-n_1|+|m_2-n_2|)} \left\{ \sum_{r_1=0}^{\min(m_1-2, n_1-2)} \sum_{r_2=0}^{\min(m_2-2, n_2-2)} e^{-2\gamma(\min(m_1-2, n_1-2)-r_1)} e^{-2\gamma(\min(m_2-2, n_2-2)-r_2)} \right. \\ &\quad \left. + \dots + \sum_{r_1=0}^{\min(m_1, n_1)} \sum_{r_2=0}^{\min(m_2, n_2)} e^{-2\gamma(\min(m_1, n_1)-r_1)} e^{-2\gamma(\min(m_2, n_2)-r_2)} \right\} \\ &\lesssim e^{-\gamma(|m_1-n_1|+|m_2-n_2|)} = C e^{-\gamma\|m-n\|_{\ell^1}}. \end{aligned}$$

This concludes the proof. \square

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