

# A WEAK GALERKIN FINITE ELEMENT METHOD FOR THE MAXWELL EQUATIONS

LIN MU\*, JUNPING WANG<sup>†</sup>, XIU YE<sup>‡</sup>, AND SHANGYOU ZHANG<sup>§</sup>

**Abstract.** This paper introduces a numerical scheme for time harmonic Maxwell's equations by using weak Galerkin (WG) finite element methods. The WG finite element method is based on two operators: discrete weak curl and discrete weak gradient, with appropriately defined stabilizations that enforce a weak continuity of the approximating functions. This WG method is highly flexible by allowing the use of discontinuous approximating functions on arbitrary shape of polyhedra and, at the same time, is parameter free. Optimal-order of convergence is established for the weak Galerkin approximations in various discrete norms which are either  $H^1$ -like or  $L^2$  and  $L^2$ -like. An effective implementation of the WG method is developed through variable reduction by following a Schur-complement approach, yielding a system of linear equations involving unknowns associated with element boundaries only. Numerical results are presented to confirm the theory of convergence.

**Key words.** weak Galerkin, finite element methods, weak curl, weak gradient, Maxwell equations, polyhedral meshes

**AMS subject classifications.** Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

**1. Introduction.** In this paper, we are concerned with new developments of numerical methods for the time-harmonic Maxwell equations in a heterogeneous medium  $\Omega \subset \mathbb{R}^3$ . The model problem seeks unknown functions  $\mathbf{u}$  and  $p$  satisfying

$$(1.1) \quad \nabla \times (\mu \nabla \times \mathbf{u}) - \epsilon \nabla p = \mathbf{f}_1 \quad \text{in } \Omega,$$

$$(1.2) \quad \nabla \cdot (\epsilon \mathbf{u}) = g_1 \quad \text{in } \Omega,$$

$$(1.3) \quad \mathbf{u} \times \mathbf{n} = \phi \quad \text{on } \partial\Omega,$$

$$(1.4) \quad p = 0 \quad \text{on } \partial\Omega,$$

where the coefficients  $\mu > 0$  and  $\epsilon > 0$  are the magnetic permeability and the electric permittivity of the medium, respectively.

A weak formulation for (1.1)-(1.4) seeks  $(\mathbf{u}, p) \in H(\text{curl}; \Omega) \times H_0^1(\Omega)$  such that  $\mathbf{u} \times \mathbf{n} = \phi$  on  $\partial\Omega$  and

$$(1.5) \quad (\nu \nabla \times \mathbf{u}, \nabla \times \mathbf{v}) - (\mathbf{v}, \nabla p) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in H_0(\text{curl}; \Omega)$$

$$(1.6) \quad (\mathbf{u}, \nabla q) = -(g, q), \quad \forall q \in H_0^1(\Omega),$$

where  $\nu = \mu/\epsilon$ ,  $\mathbf{f} = \mathbf{f}_1/\epsilon$  and  $g = g_1/\epsilon$ .

The Maxwell equations have been studied extensively in literature by using various numerical methodologies including  $H(\text{curl}; \Omega)$ -conforming edge element approaches

---

\*Department of Mathematics, Michigan State University, East Lansing, MI 48824 (linmu@msu.edu)

<sup>†</sup>Division of Mathematical Sciences, National Science Foundation, Arlington, VA 22230, USA, [jwang@nsf.gov](mailto:jwang@nsf.gov). The research of Wang was supported by the NSF IR/D program, while working at the Foundation. However, any opinion, finding, and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

<sup>‡</sup>Department of Mathematics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA, [xye@ualr.edu](mailto:xye@ualr.edu). This author was supported in part by the National Science Foundation under Grant No. DMS-1115097.

<sup>§</sup>Department of Mathematical Sciences, University of Delaware, Newark, DE 19716, USA, ([szhang@udel.edu](mailto:szhang@udel.edu))

[1, 7, 8, 10, 11] and discontinuous Galerkin methods [2, 3, 4, 5, 12, 13]. Particularly in [6], a mixed DG formulation for the problem (1.1)-(1.4) was introduced and analyzed. In this DG formulation, both  $\mathbf{u}$  and  $p$  are approximated by piecewise  $[P_k(T)]^3$  and  $P_k(T)$  functions if  $T$  is a tetrahedron and by piecewise  $[Q_k(T)]^3$  and  $Q_k(T)$  if  $T$  is a parallelepiped, where  $P_k(T)$  denotes the set of polynomials of total degree  $k$  and  $Q_k(T)$  the set of polynomials of degree  $k$  in each variable.

The weak Galerkin (WG) finite element method refers to a general finite element technique for partial differential equations where differential operators are approximated as discrete distributions or discrete weak derivatives. The method was first introduced in [15, 16] for second order elliptic equations, and was later extended to other partial differential equations including the Stokes equations [17] and the biharmonic equation [9, 14]. The current research indicates that the concept of discrete weak differential operators offers a new paradigm in numerical methods for partial differential equations.

In this paper, we apply the idea of weak Galerkin to the problem (1.1)-(1.4). In essence, this procedure shall introduce a discrete curl operator, which shall be combined with the discrete weak gradient as introduced in [15] to yield a finite element scheme for the Maxwell equations. In this WG method, two types of weak functions are used:  $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in [P_s(T)]^3 \times [P_t(e)]^3$  and  $p_h = \{p_0, p_b\} \in P_\ell(T) \times P_\iota(e)$ , with  $\mathbf{u}_h = \mathbf{u}_0$  and  $p_h = p_0$  inside of each element and  $\mathbf{u}_h = \mathbf{u}_b$  and  $p_h = p_b$  on the boundary of the element. Error estimates of optimal order are established for the WG approximations in appropriate norms for the case of  $s = t = k$  and  $\ell = k - 1$ ,  $\iota = k$  with  $k \geq 1$ . For the case of  $s = t = k$  and  $\ell = \iota = k - 1$ , only numerical experiments are conducted to illustrate the performance of the corresponding WG finite element scheme; a theoretical study of this WG method is left to interested readers for an investigation.

The use of weak functions and weak derivatives makes the WG method highly flexible on the construction of finite element functions on partitions with arbitrary polygons or polyhedrons. Compared with the DG method in [6], our WG methods make use of additional variables  $\mathbf{u}_b$  and  $p_b$  defined on the boundary of the elements. However, the variables  $\mathbf{u}_0$  and  $p_0$  defined on each element can be eliminated through a local process/computation, yielding a system of linear equations involving only the variables  $\mathbf{u}_b$  and  $p_b$ . Consequently, the WG method has much less number of globally coupled unknowns than DG methods. In addition, the weak Galerkin finite element method is parameter independent in its stability and convergence.

The paper is organized as follows. In Section 2, we introduce some basic notations. In Section 3, we discuss some discrete weak differential operators, particularly a discrete weak curl. Section 4 is devoted to a presentation of the weak Galerkin finite element scheme for the problem (1.5)-(1.6). In Section 5, we derive an error equation for the WG finite element approximation. In Section 6, we introduce two types of  $L^2$  projection operators and derive some estimates for them. Sections 7 and 8 are devoted to an error analysis for the WG finite element approximations. In Section 9, we discuss an efficient implementation method by using variable reductions/elimination. Finally in Section 10, we present some numerical results that verify the theory established in the previous sections.

**2. Preliminaries and Notations.** Let  $D$  be any open bounded domain with Lipschitz continuous boundary in  $\mathbb{R}^3$ . We use the standard definition for the Sobolev space  $H^s(D)$  and their associated inner products  $(\cdot, \cdot)_{s,D}$ , norms  $\|\cdot\|_{s,D}$ , and seminorms  $|\cdot|_{s,D}$  for any  $s \geq 0$ . For example, for any integer  $s \geq 0$ , the seminorm  $|\cdot|_{s,D}$

is given by

$$|v|_{s,D} = \left( \sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}}$$

with the usual notation

$$\alpha = (\alpha_1, \dots, \alpha_d), \quad |\alpha| = \alpha_1 + \dots + \alpha_d, \quad \partial^\alpha = \prod_{j=1}^3 \partial_{x_j}^{\alpha_j}.$$

The Sobolev norm  $\|\cdot\|_{m,D}$  is given by

$$\|v\|_{m,D} = \left( \sum_{j=0}^m |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space  $H^0(D)$  coincides with  $L^2(D)$ , for which the norm and the inner product are denoted by  $\|\cdot\|_D$  and  $(\cdot, \cdot)_D$ , respectively. When  $D = \Omega$ , we shall drop the subscript  $D$  in the norm and inner product notation.

The space  $H(\text{curl}; D)$  is defined as the set of vector-valued functions on  $D$  which, together with their curl, are square integrable; i.e.,

$$H(\text{curl}; D) = \{ \mathbf{v} : \mathbf{v} \in [L^2(D)]^3, \nabla \times \mathbf{v} \in [L^2(D)]^3 \}.$$

**3. Weak Derivatives.** The two differential operators used in (1.5) and (1.6) are curl and gradient operators. The goal of this section is to introduce an analogy of the curl and gradient operator, called weak curl and weak gradient operators, when the applied functions are discontinuous.

**3.1. Weak gradient and discrete weak gradient.** The concept of weak gradient and its discrete analogue was introduced in [15]. This subsection is presented for the sake of completeness of presentation.

Let  $K$  be any polyhedral domain with boundary  $\partial K$ . A weak function on the region  $K$  refers to a function  $v = \{v_0, v_b\}$  such that  $v_0 \in L^2(K)$  and  $v_b \in L^2(\partial K)$ . The first component  $v_0$  can be understood as the value of  $v$  in  $K$ , and the second component  $v_b$  represents  $v$  on the boundary of  $K$ . Note that  $v_b$  may not necessarily be related to the trace of  $\mathbf{v}_0$  on  $\partial K$  should a trace be well-defined. Denote by  $\mathcal{W}(K)$  the space of weak functions on  $K$ ; i.e.,

$$(3.1) \quad \mathcal{W}(K) := \{v = \{v_0, v_b\} : v_0 \in L^2(K), v_b \in L^2(\partial K)\}.$$

The weak gradient operator is defined as follows.

**DEFINITION 3.1.** (*Weak Gradient*) *The dual of  $L^2(K)$  can be identified with itself by using the standard  $L^2$  inner product as the action of linear functionals. With a similar interpretation, for any  $v \in \mathcal{W}(K)$ , the weak gradient of  $v$  is defined as a linear functional  $\nabla_w v$  in the dual space of  $[H^1(K)]^3$  whose action on each  $q \in [H^1(K)]^3$  is given by*

$$(3.2) \quad (\nabla_w v, q)_K := -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial K},$$

where  $\mathbf{n}$  is the outward normal direction to  $\partial K$ ,  $(v_0, \nabla \cdot q)_K = \int_K v_0 (\nabla \cdot q) dK$  is the  $L^2$  inner product of  $v_0$  and  $\nabla \cdot q$ , and  $\langle v_b, q \cdot \mathbf{n} \rangle_{\partial K}$  is the  $L^2$  inner product of  $q \cdot \mathbf{n}$  and  $v_b$  in  $L^2(\partial K)$ .

The Sobolev space  $H^1(K)$  can be embedded into the space  $\mathcal{W}(K)$  by an inclusion map  $i_{\mathcal{W}} : H^1(K) \rightarrow \mathcal{W}(K)$  defined as follows

$$i_{\mathcal{W}}(\phi) = \{\phi|_K, \phi|_{\partial K}\}, \quad \phi \in H^1(K).$$

With the help of the inclusion map  $i_{\mathcal{W}}$ , the Sobolev space  $H^1(K)$  can be viewed as a subspace of  $\mathcal{W}(K)$  by identifying each  $\phi \in H^1(K)$  with  $i_{\mathcal{W}}(\phi)$ .

Let  $P_r(K)$  be the set of polynomials on  $K$  with degree no more than  $r$ .

**DEFINITION 3.2.** (*Discrete Weak Gradient*) The discrete weak gradient operator, denoted by  $\nabla_{w,r,K}$ , is defined as the unique polynomial  $(\nabla_{w,r,K}v) \in [P_r(K)]^3$  satisfying the following equation

$$(3.3) \quad (\nabla_{w,r,K}v, q)_K = -(v_0, \nabla \cdot q)_K + \langle v_b, q \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall q \in [P_r(K)]^d.$$

**3.2. Weak curl and discrete weak curl.** To define weak curl, we require weak functions  $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$  such that  $\mathbf{v}_0 \in [L^2(K)]^3$  and  $\mathbf{v}_b \times \mathbf{n} \in [L^2(\partial K)]^3$ . The first component  $\mathbf{v}_0$  can be understood as the value of  $\mathbf{v}$  in  $K$ . The second component  $\mathbf{v}_b$  represents the value of  $\mathbf{v}$  on the boundary of  $K$ .

Denote by  $\mathcal{V}(K)$  the space of vector-valued weak functions on  $K$ ; i.e.,

$$(3.4) \quad \mathcal{V}(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \mathbf{v}_0 \in [L^2(K)]^3, \mathbf{v}_b \times \mathbf{n} \in [L^2(\partial K)]^3\}.$$

Then, we define a weak curl operator as follows.

**DEFINITION 3.3.** (*Weak Curl*) The dual of  $[L^2(K)]^3$  can be identified with itself by using the standard  $L^2$  inner product as the action of linear functionals. With a similar interpretation, for any  $\mathbf{v} \in \mathcal{V}(K)$ , the weak curl of  $\mathbf{v}$  is defined as a linear functional  $\nabla_w \times \mathbf{v}$  in the dual space of  $[H^1(K)]^3$  whose action on each  $\varphi \in [H^1(K)]^3$  is given by

$$(3.5) \quad (\nabla_w \times \mathbf{v}, \varphi)_K := (\mathbf{v}_0, \nabla \times \varphi)_K + \langle \mathbf{v}_b \times \mathbf{n}, \varphi \rangle_{\partial K},$$

where  $\mathbf{n}$  is the outward normal direction to  $\partial K$ ,  $(\mathbf{v}_0, \nabla \times \varphi)_K = \int_K \mathbf{v}_0 \cdot \nabla \times \varphi dK$  is the  $L^2$  inner product of  $\mathbf{v}_0$  and  $\nabla \times \varphi$ , and  $\langle \mathbf{v}_b \times \mathbf{n}, \varphi \rangle_{\partial K}$  is the inner product in  $L^2(\partial K)$ .

The Sobolev space  $[H^1(K)]^3$  can be embedded into the space  $\mathcal{V}(K)$  by an inclusion map  $i_{\mathcal{V}} : [H^1(K)]^3 \rightarrow \mathcal{V}(K)$  defined as follows

$$i_{\mathcal{V}}(\phi) = \{\phi|_K, \phi|_{\partial K}\}, \quad \phi \in [H^1(K)]^3.$$

Let  $K$  be any polyhedral domain with boundary  $\partial K$ . For each face  $e \in \partial K$ , let  $\mathbf{t}_1$  and  $\mathbf{t}_2$  be two assigned unit vectors on the face  $e$  such that  $\mathbf{t}_1$ ,  $\mathbf{t}_2$  and  $\mathbf{n}$  are orthogonal each other. Thus, we have  $\mathbf{v}_b|_e = v_1 \mathbf{t}_1 + v_2 \mathbf{t}_2 + v_n \mathbf{n}$ . Define  $\bar{\mathbf{v}}_b = v_1 \mathbf{t}_1 + v_2 \mathbf{t}_2$ . Obviously,  $\bar{\mathbf{v}}_b \times \mathbf{n} = \mathbf{v}_b \times \mathbf{n}$ . Since the quantity of interest is not  $\mathbf{v}_b$  but  $\mathbf{v}_b \times \mathbf{n}$ , we will let  $\mathbf{v}_b = \bar{\mathbf{v}}_b$  in order to reduce the number of the unknowns.

**DEFINITION 3.4.** (*Discrete Weak Curl*) For a given  $K$ , a discrete weak curl operator, denoted by  $\nabla_{w,r,K} \times$ , is defined as the unique polynomial  $(\nabla_{w,r,K} \times \mathbf{v}) \in [P_r(K)]^3$  that satisfies the following equation

$$(3.6) \quad (\nabla_{w,r,K} \times \mathbf{v}, \varphi)_K := (\mathbf{v}_0, \nabla \times \varphi)_K + \langle \mathbf{v}_b \times \mathbf{n}, \varphi \rangle_{\partial K}, \quad \forall \varphi \in [P_r(K)]^3.$$

**4. Numerical Algorithms.** Let  $\mathcal{T}_h$  be a partition of the domain  $\Omega$  with mesh size  $h$  that consists of polyhedra of arbitrary shape. Assume that the partition  $\mathcal{T}_h$  is shape regular in the sense as defined in [16]; i.e.  $\mathcal{T}_h$  satisfies a set of conditions given in [16]. Denote by  $\mathcal{E}_h$  the set of all faces in  $\mathcal{T}_h$ , and let  $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$  be the set of all interior faces.

Let  $e \in \mathcal{E}_h^0$  be shared by two elements  $T_1$  and  $T_2$ . Let  $\mathbf{t}_1$  and  $\mathbf{t}_2$  be two tangential unit vectors on face  $e \in \mathcal{E}_h$ . For  $k \geq 1$ , define a weak Galerkin finite element spaces associated with  $\mathcal{T}_h$  as

$$(4.1) \quad V_h = \left\{ \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b = v_1 \mathbf{t}_1 + v_2 \mathbf{t}_2\} : \mathbf{v}_0|_T \in [P_k(T)]^3, \right. \\ \left. v_1, v_2 \in P_k(e), e \subset \partial T \right\},$$

and

$$(4.2) \quad W_h = \left\{ w = \{w_0, w_b\} : \{w_0, w_b\}|_T \in P_{k-1}(T) \times P_k(e), e \subset \partial T, \right. \\ \left. w_b = 0 \text{ on } \partial\Omega \right\}.$$

We also introduce the following subspace of  $V_h$ ,

$$V_{h,0} = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_h, \mathbf{v}_b \times \mathbf{n}|_e = 0, e \subset \partial\Omega\}.$$

The discrete weak gradient  $\nabla_{w,k-1}$  and the discrete weak curl  $\nabla_{w,k} \times$  on the finite element spaces  $W_h$  and  $V_h$  can be computed by using (3.3) and (3.6) on each element  $T$  respectively; i.e.,

$$(\nabla_{w,k} v)|_T = \nabla_{w,k,T}(v|_T), \quad \forall v \in W_h \\ (\nabla_{w,k-1} \times \mathbf{v})|_T = \nabla_{w,k-1,T} \times (\mathbf{v}|_T), \quad \forall \mathbf{v} \in V_h.$$

For simplicity of notation, from now on we shall drop the subscript  $k$  in  $\nabla_{w,k}$  and  $k-1$  in  $\nabla_{w,k-1} \times$  for the discrete weak gradient and the discrete weak curl.

Corresponding to the bilinear forms in (1.5)-(1.6), we introduce the following bilinear forms:

$$(\nu \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{w})_h = \sum_{T \in \mathcal{T}_h} (\nu \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{w})_T \\ (\mathbf{v}, \nabla_w q)_h = \sum_{T \in \mathcal{T}_h} (\nabla_w q, \mathbf{v})_T.$$

Furthermore, we stabilize the first one by adding an appropriate stabilization term as follows:

$$(4.3) \quad a(\mathbf{v}, \mathbf{w}) = (\nu \nabla_w \times \mathbf{v}, \nabla_w \times \mathbf{w})_h + s_1(\mathbf{v}, \mathbf{w}),$$

where

$$(4.4) \quad s_1(\mathbf{v}, \mathbf{w}) = \sum_{T \in \mathcal{T}_h} h^{-1} \langle (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}, (\mathbf{w}_0 - \mathbf{w}_b) \times \mathbf{n} \rangle_{\partial T}.$$

For simplicity of notation, we introduce the following notation

$$(4.5) \quad b(\mathbf{v}, q) = (\mathbf{v}_0, \nabla_w q)_h$$

and a second stabilization term

$$(4.6) \quad s_2(p, q) = \sum_{T \in \mathcal{T}_h} h \langle p_0 - p_b, q_0 - q_b \rangle_{\partial T}.$$

WEAK GALERKIN ALGORITHM 1. Find  $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$  and  $p_h = \{p_0, p_b\} \in W_h$  satisfying  $\mathbf{u}_b \times \mathbf{n} = Q_b \phi$  on  $\partial\Omega$  and

$$(4.7) \quad a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0), \quad \forall \mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_{h,0},$$

$$(4.8) \quad b(\mathbf{u}_h, q) + s_2(p_h, q) = -(g, q_0), \quad \forall q = \{q_0, q_b\} \in W_h,$$

where  $Q_b \phi$  is an approximation of the boundary value in the polynomial space  $[P_k(\partial T \cap \partial\Omega)]^3$ . For simplicity, one may take  $Q_b \phi$  as the standard  $L^2$  projection of the boundary value  $\phi$  on each boundary segment.

LEMMA 4.1. The weak Galerkin finite element algorithm (4.7)-(4.8) has a unique solution.

*Proof.* It suffices to show that zero is the only solution of (4.7)-(4.8) if  $\mathbf{f} = 0$ ,  $\phi = 0$ , and  $g = 0$ . To this end, assume that the homogeneous conditions are given. Take  $\mathbf{v} = \mathbf{u}_h$  and  $q = p_h$  in (4.7)-(4.8). By adding the two resulting equations, we obtain

$$\begin{aligned} (\nu \nabla_w \times \mathbf{u}_h, \nabla_w \times \mathbf{u}_h)_h + \sum_{T \in \mathcal{T}_h} h^{-1} \langle (\mathbf{u}_0 - \mathbf{u}_b) \times \mathbf{n}, (\mathbf{u}_0 - \mathbf{u}_b) \times \mathbf{n} \rangle_{\partial T} \\ + \sum_{T \in \mathcal{T}_h} h \langle p_0 - p_b, p_0 - p_b \rangle_{\partial T} = 0, \end{aligned}$$

which implies  $\nabla_w \times \mathbf{u}_h = 0$  on each  $T$ ,  $\mathbf{u}_0 \times \mathbf{n} = \mathbf{u}_b \times \mathbf{n}$  and  $p_0 = p_b$  on  $\partial T$ . Note that the boundary condition implies  $\mathbf{u}_b \times \mathbf{n} = 0$  on each  $e \subset \partial\Omega$ . Then, it follows from (3.6) and the integration by parts that for any  $\mathbf{v} \in [P_{k-1}(T)]^3$

$$\begin{aligned} 0 &= (\nabla_w \times \mathbf{u}_h, \mathbf{v})_T \\ &= (\mathbf{u}_0, \nabla \times \mathbf{v})_T + \langle \mathbf{u}_b \times \mathbf{n}, \mathbf{v} \rangle_{\partial T} \\ &= (\nabla \times \mathbf{u}_0, \mathbf{v})_T + \langle (\mathbf{u}_b - \mathbf{u}_0) \times \mathbf{n}, \mathbf{v} \rangle_{\partial T} \\ &= (\nabla \times \mathbf{u}_0, \mathbf{v})_T, \end{aligned}$$

which gives  $\nabla \times \mathbf{u}_0 = 0$  on each  $T \in \mathcal{T}_h$ . Using (4.8), (3.3) and the integration by parts, we have

$$0 = \sum_{T \in \mathcal{T}_h} (\mathbf{u}_0, \nabla_w q)_T = - \sum_{T \in \mathcal{T}_h} (\nabla \cdot \mathbf{u}_0, q_0)_T + \sum_{T \in \mathcal{T}_h} \langle \mathbf{u}_0 \cdot \mathbf{n}, q_b \rangle_{\partial T}.$$

Letting  $q_0 = \nabla \cdot \mathbf{u}_0$  and  $q_b = 0$  in the above equation yield  $\nabla \cdot \mathbf{u}_0 = 0$  on each  $T \in \mathcal{T}_h$ . Next, by letting  $q_0 = 0$  and  $q_b$  be the jump of  $\mathbf{u}_0 \cdot \mathbf{n}$  on each interior face  $e$ , we conclude that  $\mathbf{u}_0$  is continuous across each interior face  $e$  in the normal direction.

Note that  $\nabla \times \mathbf{u}_0 = 0$ . Thus, there exists a potential function  $\phi$  such that  $\mathbf{u}_0 = \nabla \phi$  on  $\Omega$ . It follows from  $\nabla \cdot \mathbf{u}_0 = 0$  and the fact that  $\mathbf{u}_0 \cdot \mathbf{n}$  is continuous that  $\Delta \phi = 0$  is strongly satisfied in  $\Omega$ . The boundary condition of (1.3) implies that  $\mathbf{u}_0 \times \mathbf{n} = \nabla \phi \times \mathbf{n} = 0$  on  $\partial\Omega$ . Therefore,  $\phi$  must be a constant on  $\partial\Omega$ . The uniqueness of the solution of the Laplace equation implies that  $\phi = \text{const}$  is the only solution of  $\Delta \phi = 0$  if  $\Omega$  is simply connected. Then we must have  $\mathbf{u}_0 = \nabla \phi = 0$ . Since  $\mathbf{u}_b \times \mathbf{n} = \mathbf{u}_0 \times \mathbf{n} = 0$ , we have  $\mathbf{u}_b = 0$ .

Since  $\mathbf{u}_h = 0$ , we then have  $b(\mathbf{v}, p_h) = 0$  for any  $\mathbf{v} \in V_{h,0}$ . It follows from the definition of  $b(\cdot, \cdot)$  and  $\nabla_w$  that

$$(4.9) \quad \begin{aligned} 0 &= b(\mathbf{v}, p_h) = (\mathbf{v}_0, \nabla_w p_h)_h \\ &= - \sum_{T \in \mathcal{T}_h} (\nabla \cdot \mathbf{v}_0, p_0)_T + \sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_0 \cdot \mathbf{n}, p_b \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} (\mathbf{v}_0, \nabla p_0)_T, \end{aligned}$$

where we have used the fact that  $p_0 = p_b$  on  $\partial T$ . Letting  $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} = \{\nabla p_0, 0\}$  in (4.9) gives  $\nabla p_0 = 0$  on each  $T \in \mathcal{T}_h$ , i.e.  $p_0$  is a constant on  $T \in \mathcal{T}_h$ . Using the facts  $p_0 = p_b$  and  $p_b = 0$  on  $\partial\Omega$ , we obtain  $p_h = 0$ .  $\square$

**5. Error Equations.** For each element  $T \in \mathcal{T}_h$ , denote by  $\mathbf{Q}_0$  and  $Q_0$  the  $L^2$  projections onto  $[P_k(T)]^3$  and  $P_{k-1}(T)$  respectively. Let  $Q_b$  be the  $L^2$  projection onto  $P_k(e)$ . Then we can define two projections onto the finite element space  $V_h$  and  $W_h$  such that on each element  $T$ ,

$$\mathbf{Q}_h \mathbf{v} = \{\mathbf{Q}_0 \mathbf{v}, Q_b \mathbf{v}\} = Q_b(v_1) \mathbf{t}_1 + Q_b(v_2) \mathbf{t}_2, \quad Q_h q = \{Q_0 q, Q_b q\}.$$

In addition, denote by  $\mathbb{Q}_h$  the local  $L^2$  projection onto  $[P_{k-1}(T)]^3$ . The projection operators  $\mathbb{Q}_h$ ,  $Q_h$  and  $\mathbf{Q}_h$  have some useful properties as stated in the following Lemma.

LEMMA 5.1. *Let  $\mathbf{Q}_h = \{\mathbf{Q}_0, Q_b\}$  and  $Q_h = \{Q_0, Q_b\}$  be the projection operators onto the finite element spaces  $V_h$  and  $W_h$  respectively. Then, we have*

$$(5.1) \quad \nabla_w \times (\mathbf{Q}_h \mathbf{u}) = \mathbb{Q}_h(\nabla \times \mathbf{u}) \quad \forall \mathbf{u} \in H(\text{curl}; \Omega)$$

and

$$(5.2) \quad \nabla_w(Q_h q) = \mathbf{Q}_0(\nabla q) \quad \forall q \in H^1(\Omega).$$

*Proof.* Using (3.6), the integration by parts, and the definition of  $\mathbf{Q}_h$  and  $\mathbb{Q}_h$ , we have

$$\begin{aligned} (\nabla_w \times (\mathbf{Q}_h \mathbf{u}), \mathbf{w})_T &= (\mathbf{Q}_0 \mathbf{u}, \nabla \times \mathbf{w})_T + \langle (Q_b \mathbf{u}) \times \mathbf{n}, \mathbf{w} \rangle_{\partial T} \\ &= (\mathbf{u}, \nabla \times \mathbf{w})_T + \langle \mathbf{u} \times \mathbf{n}, \mathbf{w} \rangle_{\partial T} \\ &= (\nabla \times \mathbf{u}, \mathbf{w})_T = (\mathbb{Q}_h(\nabla \times \mathbf{u}), \mathbf{w})_T \end{aligned}$$

for any  $\mathbf{w} \in [P_{k-1}(T)]^3$ . This implies that (5.1) holds true.

As to (5.2), we use the definition of  $Q_h$  and the discrete gradient operator  $\nabla_w$  to obtain

$$\begin{aligned} (\nabla_w(Q_h p), \mathbf{v})_T &= -(Q_0 p, \nabla \cdot \mathbf{v})_T + \langle Q_b p, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} \\ &= -(p, \nabla \cdot \mathbf{v})_T + \langle p, \mathbf{v} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla p, \mathbf{v})_T = (\mathbf{Q}_0(\nabla p), \mathbf{v})_T \end{aligned}$$

for all  $\mathbf{v} \in [P_k(T)]^3$ , which verifies the desired relation (5.2).  $\square$

Define two error functions as follows

$$(5.3) \quad \mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = \{\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0, Q_b \mathbf{u} - \mathbf{u}_b\},$$

$$(5.4) \quad \varepsilon_h = \{\varepsilon_0, \varepsilon_b\} = \{Q_0 p - p_0, Q_b p - p_b\}.$$

The rest of this section is to derive some equations that the above error functions must satisfy. For simplicity of analysis, we assume that the coefficient  $\nu$  in (1.5) is a piecewise constant function with respect to the finite element partition  $\mathcal{T}_h$ .

LEMMA 5.2. *Let  $(\mathbf{u}_h; p_h)$  be the WG finite element solution arising from (4.7) and (4.8), and  $(\mathbf{e}_h; \epsilon_h)$  be the error between the WG finite element solution and the  $L^2$  projection of the exact solution as defined in (5.3)-(5.4). Then, the following equations are satisfied*

$$(5.5) \quad a(\mathbf{e}_h, \mathbf{v}) - b(\mathbf{v}, \epsilon_h) = \varphi_{\mathbf{u}}(\mathbf{v}) \quad \forall \mathbf{v} \in V_{h,0},$$

$$(5.6) \quad b(\mathbf{e}_h, q) + s_2(\epsilon_h, q) = \phi_{\mathbf{u},p}(q) \quad \forall q \in W_h,$$

where

$$(5.7) \quad \varphi_{\mathbf{u}}(\mathbf{v}) = s_1(\mathbf{Q}_h \mathbf{u}, \mathbf{v}) - l_1(\mathbf{u}, \mathbf{v}),$$

$$(5.8) \quad \phi_{\mathbf{u},p}(q) = s_2(\mathbf{Q}_h p, q) + l_2(\mathbf{u}, q),$$

and

$$(5.9) \quad l_1(\mathbf{u}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} \langle (I - \mathbf{Q}_h) \nabla \times \mathbf{u}, \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n} \rangle_{\partial T}$$

$$(5.10) \quad l_2(\mathbf{u}, q) = \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, (\mathbf{u} - \mathbf{Q}_0 \mathbf{u}) \cdot \mathbf{n} \rangle_{\partial T}.$$

*Proof.* Using (5.1), (3.6), and the integration by parts we have

$$(5.11) \quad \begin{aligned} & (\nu \nabla_w \times (\mathbf{Q}_h \mathbf{u}), \nabla_w \times \mathbf{v})_T \\ &= (\nu \mathbf{Q}_h (\nabla \times \mathbf{u}), \nabla_w \times \mathbf{v})_T \\ &= (\nu \mathbf{v}_0, \nabla \times \mathbf{Q}_h (\nabla \times \mathbf{u}))_T + \langle \nu \mathbf{v}_b \times \mathbf{n}, \mathbf{Q}_h (\nabla \times \mathbf{u}) \rangle_{\partial T} \\ &= (\nu \nabla \times \mathbf{v}_0, \mathbf{Q}_h (\nabla \times \mathbf{u}))_T + \langle \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, \mathbf{Q}_h (\nabla \times \mathbf{u}) \rangle_{\partial T} \\ &= (\nu \nabla \times \mathbf{u}, \nabla \times \mathbf{v}_0)_T + \langle \mathbf{Q}_h (\nabla \times \mathbf{u}), \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

It follows from (5.2) that

$$(5.12) \quad (\nabla_w (\mathbf{Q}_h p), \mathbf{v}_0)_T = (\mathbf{Q}_0 \nabla p, \mathbf{v}_0)_T = (\nabla p, \mathbf{v}_0)_T.$$

Next, using the definition of  $\nabla_w$  and  $\mathbf{Q}_0$ , we obtain

$$(5.13) \quad \begin{aligned} (\mathbf{Q}_0 \mathbf{u}, \nabla_w q)_T &= -(q_0, \nabla \cdot (\mathbf{Q}_0 \mathbf{u}))_T + \langle q_b, \mathbf{Q}_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T} \\ &= (\nabla q_0, \mathbf{u})_T - \langle q_0 - q_b, \mathbf{Q}_0 \mathbf{u} \cdot \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Testing (1.1) by  $\mathbf{v}_0$  with  $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_{h,0}$  gives

$$(5.14) \quad (\nabla \times (\nu \nabla \times \mathbf{u}), \mathbf{v}_0) - (\nabla p, \mathbf{v}_0) = (\mathbf{f}, \mathbf{v}_0).$$

It follows from the integration by parts that

$$(\nabla \times (\nu \nabla \times \mathbf{u}), \mathbf{v}_0) = \sum_{T \in \mathcal{T}_h} (\nu \nabla \times \mathbf{u}, \nabla \times \mathbf{v}_0)_T + \sum_{T \in \mathcal{T}_h} \langle \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n}, \nabla \times \mathbf{u} \rangle_{\partial T},$$

where we use the fact that  $\sum_{T \in \mathcal{T}_h} \langle \mathbf{v}_b \times \mathbf{n}, \nu \nabla \times \mathbf{u} \rangle_{\partial T} = 0$ . Using (5.11) and the equation above, we have

$$(5.15) \quad \begin{aligned} (\nabla \times (\nu \nabla \times \mathbf{u}), \mathbf{v}_0) &= (\nu \nabla_w \times (\mathbf{Q}_h \mathbf{u}), \nabla_w \times \mathbf{v})_h \\ &\quad + \sum_{T \in \mathcal{T}_h} \langle (I - \mathbf{Q}_h) \nabla \times \mathbf{u}, \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n} \rangle_{\partial T}. \end{aligned}$$

Substituting (5.12) and (5.15) into (5.14) yields

$$(\nu \nabla_w \times (\mathbf{Q}_h \mathbf{u}), \nabla_w \times \mathbf{v})_h - (\nabla_w Q_h p, \mathbf{v}_0)_h = (\mathbf{f}, \mathbf{v}_0) - l_1(\mathbf{v}, \mathbf{u}).$$

Adding  $s_1(\mathbf{Q}_h \mathbf{u}, \mathbf{v})$  to both sides of the equation above gives

$$(5.16) \quad a(\mathbf{Q}_h \mathbf{u}, \mathbf{v}) - b(\mathbf{v}, Q_h p) = (\mathbf{f}, \mathbf{v}_0) + \varphi_{\mathbf{u}}(\mathbf{v}).$$

To derive a second equation, we test equation (1.2) by  $q_0$  with  $q = \{q_0, q_b\} \in W_h$  and then use the integration by parts to obtain

$$(5.17) \quad - \sum_{T \in \mathcal{T}_h} (\mathbf{u}, \nabla q_0)_T + \sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_0 - q_b \rangle_{\partial T} = (g, q_0),$$

where we have used the fact  $\sum_{T \in \mathcal{T}_h} \langle \mathbf{u} \cdot \mathbf{n}, q_b \rangle_{\partial T} = 0$ . Combining (5.13) with (5.17) gives

$$\sum_{T \in \mathcal{T}_h} (\mathbf{Q}_0 \mathbf{u}, \nabla_w q)_T = -(g, q_0) + l_2(\mathbf{u}, q).$$

Adding  $s_2(Q_h p, q)$  to both sides of the equation above gives

$$(5.18) \quad b(\mathbf{Q}_h \mathbf{u}, q) + s_2(Q_h p, q) = -(g, q_0) + \phi_{\mathbf{u}, p}(q).$$

Finally, the differences of (5.16) and (4.7), (5.18) and (4.8) yield the error equations (5.5) and (5.6), respectively.  $\square$

**6. Preparation for Error Estimates.** For  $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_{h,0}$ , define  $\|\mathbf{v}\|$  as follows

$$(6.1) \quad \|\mathbf{v}\|^2 = a(\mathbf{v}, \mathbf{v}) = \sum_{T \in \mathcal{T}_h} \nu \|\nabla_w \times \mathbf{v}\|_T^2 + \sum_{T \in \mathcal{T}_h} h^{-1} \|(\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}\|_{\partial T}^2.$$

It is clear that  $\|\cdot\|$  defines merely a semi-norm for the linear space  $V_{h,0}$ . A norm can be derived from the semi-norm  $\|\mathbf{v}\|$  by adding two more terms given as follows

$$(6.2) \quad \|\mathbf{v}\|_1 = \|\mathbf{v}\| + \left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{v}_0\|_T^2 \right)^{\frac{1}{2}} + \left( \sum_{e \in \mathcal{E}_h^0} h^{-1} \|[\mathbf{v}_0 \cdot \mathbf{n}]\|_e^2 \right)^{\frac{1}{2}},$$

where  $[\mathbf{v}_0 \cdot \mathbf{n}]$  is the jump of the function  $\mathbf{v}_0$  at each edge/face in the normal direction. The proof of Lemma 4.1 can be employed to verify that  $\|\cdot\|_1$  is indeed a norm in  $V_{h,0}$ . For convenience, we also use the following notation:

$$(6.3) \quad |\mathbf{v}|_{1,h} := \left( \sum_{T \in \mathcal{T}_h} h^{-1} \|(\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n}\|_{\partial T}^2 \right)^{1/2}.$$

The linear space  $W_h$  can be equipped with the following norm

$$\|q\|_0 = |q|_{0,h} + h \|\nabla q\|_{0,h},$$

where

$$|q|_{0,h}^2 = \sum_{T \in \mathcal{T}_h} h \|q_0 - q_b\|_{\partial T}^2$$

and

$$(6.4) \quad \|\nabla q\|_{0,h} = \left( \sum_{T \in \mathcal{T}_h} \|\nabla q_0\|_T^2 \right)^{\frac{1}{2}}$$

for any  $q \in W_h$ .

The following Lemma provides some approximation estimates for the projections  $\mathbf{Q}_h$ ,  $Q_h$ , and  $Q_h$ .

LEMMA 6.1. *Let  $\mathcal{T}_h$  be a WG shape regular partition of  $\Omega$ ,  $\mathbf{w} \in [H^{t+1}(\Omega)]^3$ ,  $\rho \in H^t(\Omega)$ , and  $0 \leq t \leq k$ . Then, for  $0 \leq s \leq 1$ , we have*

$$(6.5) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\mathbf{w} - \mathbf{Q}_0 \mathbf{w}\|_{s,T}^2 \leq Ch^{2(t+1)} \|\mathbf{w}\|_{t+1}^2,$$

$$(6.6) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\nabla \times \mathbf{w} - Q_h(\nabla \times \mathbf{w})\|_{s,T}^2 \leq Ch^{2t} \|\mathbf{w}\|_{t+1}^2,$$

$$(6.7) \quad \sum_{T \in \mathcal{T}_h} h_T^{2s} \|\rho - Q_0 \rho\|_{s,T}^2 \leq Ch^{2t} \|\rho\|_t^2.$$

Since the mesh  $\mathcal{T}_h$  is assumed to be very general, the proof of Lemma 6.1 is rather technical and can be found in [16].

Let  $K$  be an element with  $e$  as a face. For any function  $g \in H^1(K)$ , the following trace inequality has been proved for arbitrary polyhedra  $K$  in [16].

$$(6.8) \quad \|g\|_e^2 \leq C (h_K^{-1} \|g\|_K^2 + h_K \|\nabla g\|_K^2).$$

In particular, if  $\xi$  is a polynomial on  $K$ , then the standard inverse inequality can be applied to yield

$$(6.9) \quad \|\xi\|_e^2 \leq Ch_K^{-1} \|\xi\|_K^2.$$

Using (6.8) and Lemma 6.1, we can prove the following result.

LEMMA 6.2. *Let  $\mathbf{w} \in [H^{t+1}(\Omega)]^3$  and  $p \in H^t(\Omega)$  and  $\mathbf{v} \in V_h$  with  $\frac{1}{2} < t \leq k$ . Then*

$$(6.10) \quad |s_1(\mathbf{Q}_h \mathbf{w}, \mathbf{v})| + |l_1(\mathbf{w}, \mathbf{v})| \leq Ch^t \|\mathbf{w}\|_{t+1} |\mathbf{v}|_{1,h},$$

$$(6.11) \quad |s_2(Q_h p, q)| + |l_2(\mathbf{w}, q)| \leq Ch^t (\|\mathbf{w}\|_{t+1} + \|p\|_t) |q|_{0,h},$$

where  $l_1(\mathbf{w}, \mathbf{v})$  and  $l_2(\mathbf{w}, q)$  are defined in (5.9) and (5.10).

*Proof.* Using the definition of  $Q_b$ , (6.8) and (6.5), we have

$$\begin{aligned} |s_1(\mathbf{Q}_h \mathbf{w}, \mathbf{v})| &= \left| \sum_{T \in \mathcal{T}_h} h^{-1} \langle (\mathbf{Q}_0 \mathbf{w} - Q_b \mathbf{w}) \times \mathbf{n}, (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} \right| \\ &= \left| \sum_{T \in \mathcal{T}_h} h^{-1} \langle (\mathbf{Q}_0 \mathbf{w} - \mathbf{w}) \times \mathbf{n}, (\mathbf{v}_0 - \mathbf{v}_b) \times \mathbf{n} \rangle_{\partial T} \right| \\ &\leq \left( \sum_{T \in \mathcal{T}_h} (h^{-2} \|\mathbf{Q}_0 \mathbf{w} - \mathbf{w}\|_T^2 + \|\nabla(\mathbf{Q}_0 \mathbf{w} - \mathbf{w})\|_T^2) \right)^{1/2} |\mathbf{v}|_{1,h} \\ &\leq Ch^t \|\mathbf{w}\|_{t+1} |\mathbf{v}|_{1,h}. \end{aligned}$$

Similarly, we have from (6.8) and (6.6) that

$$\begin{aligned}
|l_1(\mathbf{v}, \mathbf{w})| &\equiv \left| \sum_{T \in \mathcal{T}_h} \langle (I - \mathbb{Q}_h) \nabla \times \mathbf{w}, \nu(\mathbf{v}_b - \mathbf{v}_0) \times \mathbf{n} \rangle_{\partial T} \right| \\
&\leq \left( \sum_{T \in \mathcal{T}_h} h \|(I - \mathbb{Q}_h) \nabla \times \mathbf{w}\|_{\partial T}^2 \right)^{1/2} |\mathbf{v}|_{1,h} \\
&\leq Ch^t \|\mathbf{w}\|_{t+1} |\mathbf{v}|_{1,h}.
\end{aligned}$$

This completes the proof of (6.10).

As to (6.11), note that

$$\begin{aligned}
|s_2(Q_h p, q)| &= \left| \sum_{T \in \mathcal{T}_h} h \langle Q_0 p - Q_b p, q_0 - q_b \rangle_{\partial T} \right| \\
&\leq \sum_{T \in \mathcal{T}_h} h |\langle Q_0 p - p, q_0 - q_b \rangle_{\partial T}| \\
&\leq Ch^t \|p\|_t |q|_{0,h}.
\end{aligned}$$

It follows from (6.8) and (6.7) that

$$\begin{aligned}
|l_2(\mathbf{w}, q)| &= \left| \sum_{T \in \mathcal{T}_h} \langle q_0 - q_b, (\mathbf{w} - \mathbf{Q}_0 \mathbf{w}) \cdot \mathbf{n} \rangle_{\partial T} \right| \\
&\leq \left( \sum_{T \in \mathcal{T}_h} h^{-1} \|\mathbf{w} - \mathbf{Q}_0 \mathbf{w}\|_{\partial T}^2 \right)^{1/2} \left( \sum_{T \in \mathcal{T}_h} h \|q_0 - q_b\|_{\partial T}^2 \right)^{1/2} \\
&\leq Ch^t \|\mathbf{w}\|_{t+1} |q|_{0,h}.
\end{aligned}$$

Combining the above two estimates leads to the inequality (6.11). This completes the proof of the lemma.  $\square$

**7. Error Estimates.** The objective of this section is to establish some optimal order error estimates for  $\mathbf{u}_h$  and  $p_h$  in certain discrete norms. We start with a modified *inf-sup* condition commonly used for analyzing saddle point problem.

LEMMA 7.1. *For any  $q = \{q_0, q_b\} \in W_h$ , there exist a  $\mathbf{v}_q = h^2 \{\nabla q_0, 0\} \in V_{h,0}$  such that*

$$(7.1) \quad b(\mathbf{v}_q, q) \geq h^2 \|\nabla q\|_{0,h}^2 - C|q|_{0,h}^2$$

and

$$(7.2) \quad \|\mathbf{v}_q\| \leq Ch \|\nabla q\|_{0,h},$$

where  $C$  is a constant independent of  $h$ .

*Proof.* For a given  $q = \{q_0, q_b\} \in W_h$  and  $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_{h,0}$ , from the definition of the discrete weak gradient we obtain

$$\begin{aligned}
b(\mathbf{v}, q) &= \sum_{T \in \mathcal{T}_h} (\mathbf{v}_0, \nabla_w q)_T \\
&= \sum_{T \in \mathcal{T}_h} (\langle \mathbf{v}_0 \cdot \mathbf{n}, q_b \rangle_{\partial T} - (\nabla \cdot \mathbf{v}_0, q_0)_T) \\
&= \sum_{T \in \mathcal{T}_h} ((\mathbf{v}_0, \nabla q_0)_T + \langle \mathbf{v}_0 \cdot \mathbf{n}, q_b - q_0 \rangle_{\partial T}),
\end{aligned}$$

where we have used the usual integration by parts in the last equation. By choosing  $\mathbf{v}_0 = 2h^2 \nabla q_0$  and  $\mathbf{v}_b = 0$  we arrive at

$$b(\mathbf{v}, q) = 2h^2 \sum_{T \in \mathcal{T}_h} (\nabla q_0, \nabla q_0)_T + 2h^2 \sum_{T \in \mathcal{T}_h} \langle \nabla q_0 \cdot \mathbf{n}, q_b - q_0 \rangle_{\partial T}.$$

Now by the Cauchy-Schwarz inequality and the trace inequality (6.9) we obtain

$$\begin{aligned} b(\mathbf{v}, q) &\geq 2h^2 \sum_{T \in \mathcal{T}_h} (\nabla q_0, \nabla q_0)_T - 2h^2 \sum_{T \in \mathcal{T}_h} \|\nabla q_0 \cdot \mathbf{n}\|_{\partial T} \|q_b - q_0\|_{\partial T} \\ &\geq 2h^2 \sum_{T \in \mathcal{T}_h} (\nabla q_0, \nabla q_0)_T - Ch^{1.5} \sum_{T \in \mathcal{T}_h} \|\nabla q_0\|_T \|q_b - q_0\|_{\partial T} \\ &\geq h^2 \sum_{T \in \mathcal{T}_h} (\nabla q_0, \nabla q_0)_T - Ch \sum_{T \in \mathcal{T}_h} \|q_b - q_0\|_{\partial T}^2, \end{aligned}$$

which gives rise to the inequality (7.1). The boundedness estimate (7.2) can be obtained by computing the triple bar norm of  $\mathbf{v}_q$  directly. This completes the proof of the lemma.  $\square$

The following is an error estimate for the WG finite element solutions.

**THEOREM 7.2.** *Let  $(\mathbf{u}; p) \in [H^{t+1}(\Omega)]^3 \times [H_0^1(\Omega) \cap H^{\max\{1,t\}}(\Omega)]$  with  $\frac{1}{2} < t \leq k$  and  $(\mathbf{u}_h; p_h) \in V_h \times W_h$  be the solution of (1.1)-(1.4) and (4.7)-(4.8) respectively. Then*

$$(7.3) \quad \|\mathbf{e}_h\| + |\epsilon_h|_{0,h} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t),$$

$$(7.4) \quad h \|\nabla \epsilon_h\|_{0,h} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t).$$

*Proof.* By letting  $\mathbf{v} = \mathbf{e}_h$  in (5.5) and  $q = \epsilon_h$  in (5.6) and adding the two resulting equations, we have

$$(7.5) \quad \|\mathbf{e}_h\|^2 + |\epsilon_h|_{0,h}^2 = \varphi_{\mathbf{u}}(\mathbf{e}_h) + \phi_{\mathbf{u},p}(\epsilon_h).$$

The right-hand side of (7.5) can be handled by using Lemma 6.2 as follows. Using (6.10) with  $\mathbf{w}$  and  $\mathbf{v}$  replaced by  $\mathbf{u}$  and  $\mathbf{e}_h$  we obtain

$$(7.6) \quad |\varphi_{\mathbf{u}}(\mathbf{e}_h)| \leq Ch^t \|\mathbf{u}\|_{t+1} \|\mathbf{e}_h\|.$$

Similarly, using (6.11) with  $\mathbf{w}$  and  $q$  replaced by  $\mathbf{u}$  and  $\epsilon_h$  we obtain

$$(7.7) \quad |\phi_{\mathbf{u},p}(\epsilon_h)| \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t) |\epsilon_h|_{0,h}.$$

Substituting (7.6) and (7.7) into (7.5) yields

$$(7.8) \quad \|\mathbf{e}_h\|^2 + |\epsilon_h|_{0,h}^2 \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t) (\|\mathbf{e}_h\| + |\epsilon_h|_{0,h}),$$

which implies the error estimate (7.3).

Next we will bound  $\|\nabla \epsilon_h\|_{0,h}$ . It follows from (5.5) that

$$b(\mathbf{v}, \epsilon_h) = a(\mathbf{e}_h, \mathbf{v}) - \varphi_{\mathbf{u}}(\mathbf{v}) \quad \forall \mathbf{v} \in V_{h,0}.$$

From Lemma 7.1, by choosing  $\mathbf{v} = \mathbf{v}_{\epsilon_h} = h^2\{\nabla\epsilon_h, 0\}$  we come up with

$$\begin{aligned}
(7.9) \quad h^2 \|\nabla\epsilon_h\|_{0,h}^2 &\leq |b(\mathbf{v}_{\epsilon_h}, \epsilon_h)| + C|\epsilon_h|_{0,h}^2 \\
&\leq |a(\mathbf{e}_h, \mathbf{v}_{\epsilon_h})| + |\varphi_{\mathbf{u}}(\mathbf{v}_{\epsilon_h})| + C|\epsilon_h|_{0,h}^2 \\
&\leq \|\mathbf{e}_h\| \|\mathbf{v}_{\epsilon_h}\| + Ch^t \|\mathbf{u}\|_{t+1} \|\mathbf{v}_{\epsilon_h}\| + C|\epsilon_h|_{0,h}^2 \\
&\leq C(\|\mathbf{e}_h\| + Ch^t \|\mathbf{u}\|_{t+1}) h \|\nabla\epsilon_h\|_{0,h} + C|\epsilon_h|_{0,h}^2,
\end{aligned}$$

where we have used the estimate (7.2) in the last inequality. It follows from (7.9) and (7.3) that (7.4) holds true. This completes the proof of the theorem.  $\square$

Recall that  $\|\mathbf{v}\|$  is merely a semi-norm in the finite element space  $V_{h,0}$ . Thus, the error estimate (7.3) only provides a partial answer to the convergence of the WG finite element method, particularly for the vector component  $\mathbf{u}_h$ . The norm  $\|\cdot\|_1$ , as defined by (6.2), involves two additional terms. The following theorem shall provide some estimates with respect to those additional terms.

**THEOREM 7.3.** *Let  $(\mathbf{u}; p) \in [H^{t+1}(\Omega)]^3 \times (H_0^1(\Omega) \cap H^{\max\{1,t\}}(\Omega))$  with  $\frac{1}{2} < t \leq k$  and  $(\mathbf{u}_h; p_h) \in V_h \times W_h$  be the solution of (1.1)-(1.4) and (4.7)-(4.8) respectively. Then, we have*

$$(7.10) \quad \left( \sum_{e \in \mathcal{E}_h^0} h^{-1} \|\llbracket \mathbf{e}_0 \cdot \mathbf{n} \rrbracket_e\|_e^2 \right)^{\frac{1}{2}} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t),$$

$$(7.11) \quad \left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2 \right)^{\frac{1}{2}} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t).$$

*Proof.* Using the error equation (5.6) we have

$$(7.12) \quad b(\mathbf{e}_h, q) = \phi_{\mathbf{u},p}(q) - s_2(\epsilon_h, q).$$

The definition of the weak gradient implies that

$$(7.13) \quad b(\mathbf{e}_h, q) = (\mathbf{e}_0, \nabla_w q) = \sum_{T \in \mathcal{T}_h} ((\mathbf{e}_0 \cdot \mathbf{n}, q_b)_{\partial T} - (\nabla \cdot \mathbf{e}_0, q_0)_T).$$

By letting  $q = q_{\mathbf{e}_h} = \{0, h^{-1} \llbracket \mathbf{e}_0 \cdot \mathbf{n} \rrbracket\}$  on the interior edges, we obtain

$$b(\mathbf{e}_h, q_{\mathbf{e}_h}) = \sum_{e \in \mathcal{E}_h^0} h^{-1} \|\llbracket \mathbf{e}_0 \cdot \mathbf{n} \rrbracket_e\|_e^2.$$

Thus,

$$\sum_{e \in \mathcal{E}_h^0} h^{-1} \|\llbracket \mathbf{e}_0 \cdot \mathbf{n} \rrbracket_e\|_e^2 = \phi_{\mathbf{u},p}(q_{\mathbf{e}_h}) - s_2(\epsilon_h, q_{\mathbf{e}_h}).$$

It follows from (7.3) that

$$(7.14) \quad |s_2(\epsilon_h, q_{\mathbf{e}_h})| \leq |\epsilon_h|_{0,h} |q_{\mathbf{e}_h}|_{0,h} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t) |q_{\mathbf{e}_h}|_{0,h},$$

and from (6.11)

$$(7.15) \quad |\phi_{\mathbf{u},p}(q_{\mathbf{e}_h})| \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t) |q_{\mathbf{e}_h}|_{0,h}.$$

Also, it is easy to see that

$$\begin{aligned}
|q_{\mathbf{e}_h}|_{0,h}^2 &= \sum_{T \in \mathcal{T}_h} h \|q_0 - q_b\|_{\partial T \cap \Omega}^2 \\
&= \sum_{T \in \mathcal{T}_h} h^{-1} \|[\![\mathbf{e}_0 \cdot \mathbf{n}]\!] \|_{\partial T \cap \Omega}^2 \\
&\leq C \sum_{e \in \mathcal{E}_h^0} h^{-1} \|[\![\mathbf{e}_0 \cdot \mathbf{n}]\!] \|_e^2.
\end{aligned}$$

Combining the above four inequalities yields

$$(7.16) \quad \left( \sum_{e \in \mathcal{E}_h^0} h^{-1} \|[\![\mathbf{e}_0 \cdot \mathbf{n}]\!] \|_e^2 \right)^{\frac{1}{2}} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t),$$

which verifies the estimate (7.10).

To derive (7.11), we set  $q = q_{\mathbf{e}_h} = \{-\nabla \cdot \mathbf{e}_0, 0\} \in W_h$  in (7.13) so that

$$(7.17) \quad b(\mathbf{e}_h, q_{\mathbf{e}_h}) = \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2.$$

Thus, it follows from (7.12) that

$$(7.18) \quad \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2 = \phi_{\mathbf{u},p}(q_{\mathbf{e}_h}) - s_2(\epsilon_h, q_{\mathbf{e}_h}).$$

Substituting (7.14) and (7.15) into (7.18) implies

$$(7.19) \quad \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2 \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t) |q_{\mathbf{e}_h}|_{0,h}.$$

It follows from the definition of  $|q|_{0,h}$  and the trace inequality (6.9) that

$$|q_{\mathbf{e}_h}|_{0,h} \leq \left( \sum_{T \in \mathcal{T}_h} h \|\nabla \cdot \mathbf{e}_0\|_{\partial T}^2 \right)^{\frac{1}{2}} \leq C \left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2 \right)^{\frac{1}{2}},$$

which, together with (7.19), leads to the following estimate

$$\left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot \mathbf{e}_0\|_T^2 \right)^{\frac{1}{2}} \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t).$$

This completes the proof.  $\square$

To summarize, we have obtained the following error estimate for the WG finite element solution arising from (4.7)-(4.8).

**THEOREM 7.4.** *Under the assumptions of Theorem 7.2, we have the following error estimate for the WG finite element approximations:*

$$(7.20) \quad \|\mathbf{e}_h\|_1 + \|\epsilon_h\|_0 \leq Ch^t (\|\mathbf{u}\|_{t+1} + \|p\|_t).$$

**8. An Error Estimate in  $L^2$ .** To derive an  $L^2$ -error estimate for the WG approximation of the vector component, we consider an auxiliary problem that seeks  $(\psi; \xi)$  satisfying

$$(8.1) \quad \begin{aligned} \nabla \times (\nu \nabla \times \psi) - \nabla \xi &= \mathbf{e}_0 & \text{in } \Omega, \\ \nabla \cdot \psi &= 0 & \text{in } \Omega, \\ \psi \times \mathbf{n} &= 0 & \text{on } \partial\Omega, \\ \xi &= 0 & \text{on } \partial\Omega. \end{aligned}$$

Assume that the problem (8.1) has the  $[H^{1+s}(\Omega)]^3 \times H^s(\Omega)$ -regularity property in the sense that the solution  $(\psi; \xi) \in (H^{1+s}(\Omega))^3 \times H^s(\Omega)$  and the following a priori estimate holds true:

$$(8.2) \quad \|\psi\|_{1+s} + \|\xi\|_s \leq C \|\mathbf{e}_0\|,$$

where  $0 < s \leq 1$ .

**THEOREM 8.1.** *Let  $(\mathbf{u}; p) \in [H^{r+1}(\Omega)]^3 \times [H_0^1(\Omega) \cap H^{\max\{1,r\}}(\Omega)]$  and  $(\mathbf{u}_h; p_h) \in V_h \times W_h$  be the solutions of (1.1)-(1.4) and (4.7)-(4.8) respectively. Let  $\frac{1}{2} < r \leq k$  and  $0 < s \leq 1$ . Then,*

$$(8.3) \quad \|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\| \leq Ch^{r+s} (\|\mathbf{u}\|_{r+1} + \|p\|_r).$$

*Proof.* Testing the first equation of (8.1) by  $\mathbf{e}_0$  gives

$$\|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\|^2 = (\mathbf{e}_0, \mathbf{e}_0) = (\nabla \times (\nu \nabla \times \psi), \mathbf{e}_0) - (\nabla \xi, \mathbf{e}_0).$$

Using (5.12) and (5.15) (with  $\psi, \xi, \mathbf{e}_h$  in the place of  $\mathbf{u}, p, \mathbf{v}$  respectively), the above equation becomes

$$\|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\|^2 = (\nu \nabla_w \times \mathbf{Q}_h \psi, \nabla_w \times \mathbf{e}_h)_h - (\mathbf{e}_0, \nabla_w (Q_h \xi))_h + l_1(\psi, \mathbf{e}_h).$$

Adding and subtracting  $s_1(\mathbf{Q}_h \psi, \mathbf{e}_h)$  to the equation above yields

$$\|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\|^2 = a(\mathbf{Q}_h \psi, \mathbf{e}_h) - b(\mathbf{e}_h, Q_h \xi) - \varphi_\psi(\mathbf{e}_h).$$

The error equation (5.6) implies

$$b(\mathbf{e}_h, Q_h \xi) = -s_2(\epsilon_h, Q_h \xi) + \phi_{\mathbf{u},p}(Q_h \xi).$$

It now follows from the definition of  $\mathbf{Q}_0$ ,  $\nabla_w$  and the second equation of (8.1) that

$$\begin{aligned} b(\mathbf{Q}_h \psi, \epsilon_h) &= (\mathbf{Q}_0 \psi, \nabla_w \epsilon_h)_h = \sum_{T \in \mathcal{T}_h} (\langle \epsilon_b, \mathbf{Q}_0 \psi \cdot \mathbf{n} \rangle_{\partial T} - (\epsilon_0, \nabla \cdot (\mathbf{Q}_0 \psi))_T) \\ &= \sum_{T \in \mathcal{T}_h} (\langle \epsilon_b - \epsilon_0, \mathbf{Q}_0 \psi \cdot \mathbf{n} \rangle_{\partial T} + (\nabla \epsilon_0, \psi)_T) \\ &= \sum_{T \in \mathcal{T}_h} (\langle \epsilon_b - \epsilon_0, \mathbf{Q}_0 \psi \cdot \mathbf{n} \rangle_{\partial T} - \langle \epsilon_b - \epsilon_0, \psi \cdot \mathbf{n} \rangle_{\partial T}) \\ &= l_2(\psi, \epsilon_h), \end{aligned}$$

where we have used the fact that  $\sum_{T \in \mathcal{T}_h} \langle \epsilon_b, \psi \cdot \mathbf{n} \rangle_{\partial T} = 0$ . Using the equations above, we have

$$\|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\|^2 = a(\mathbf{Q}_h \psi, \mathbf{e}_h) - b(\mathbf{Q}_h \psi, \epsilon_h) - \phi_{\mathbf{u},p}(Q_h \xi) - \varphi_\psi(\mathbf{e}_h) + \phi_{\psi,\xi}(\epsilon_h).$$

Using (5.5) and the equation above, we have

$$(8.4) \quad \|\mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0\|^2 = \varphi_{\mathbf{u}}(\mathbf{Q}_h \psi) - \phi_{\mathbf{u},p}(Q_h \xi) - \varphi_\psi(\mathbf{e}_h) + \phi_{\psi,\xi}(\epsilon_h).$$

The four terms on the right-hand side of (8.4) can be handled by the estimates presented in Lemma 6.2. To this end, we use (6.10) and (6.11) with  $t = r$  to obtain

$$(8.5) \quad |\varphi_{\mathbf{u}}(\mathbf{Q}_h \psi) - \phi_{\mathbf{u},p}(Q_h \xi)| \leq Ch^r (\|\mathbf{u}\|_{r+1} + \|p\|_r) (|\mathbf{Q}_h \psi|_{1,h} + |Q_h \xi|_{0,h}).$$

Using the definition (6.3) we have

$$(8.6) \quad \begin{aligned} |\mathbf{Q}_h \psi|_{1,h}^2 &= \sum_{T \in \mathcal{T}_h} h^{-1} \|(\mathbf{Q}_0 \psi - Q_b \psi) \times \mathbf{n}\|_{\partial T}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} h^{-1} \|(\mathbf{Q}_0 \psi - \psi) \times \mathbf{n}\|_{\partial T}^2 \\ &\leq Ch^{2s} \|\psi\|_{s+1}^2. \end{aligned}$$

Similarly, we have from the definition of  $Q_b$ , (6.8) and (6.7)

$$(8.7) \quad \begin{aligned} |Q_h \xi|_{0,h}^2 &= \sum_{T \in \mathcal{T}_h} h \|Q_0 \xi - Q_b \xi\|_{\partial T}^2 \\ &\leq \sum_{T \in \mathcal{T}_h} h \|Q_0 \xi - \xi\|_{\partial T}^2 \\ &\leq Ch^{2s} \|\xi\|_s^2. \end{aligned}$$

Substituting (8.6) and (8.7) into (8.5) gives

$$(8.8) \quad \begin{aligned} |\varphi_{\mathbf{u}}(\mathbf{Q}_h \psi) - \phi_{\mathbf{u},p}(Q_h \xi)| &\leq Ch^{r+s} (\|\mathbf{u}\|_{r+1} + \|p\|_r) (\|\psi\|_{1+s} + \|\xi\|_s) \\ &\leq Ch^{r+s} (\|\mathbf{u}\|_{r+1} + \|p\|_r) \|\mathbf{e}_0\|, \end{aligned}$$

where the regularity estimate (8.2) was used in the second equation.

Analogously, we have from (6.10) and (6.11) with  $t = s$  that

$$(8.9) \quad \begin{aligned} |\varphi_\psi(\mathbf{e}_h) - \phi_{\psi,\xi}(\epsilon_h)| &\leq Ch^s (\|\psi\|_{s+1} + \|\xi\|_s) (|\mathbf{e}_h|_{1,h} + |\epsilon_h|_{0,h}) \\ &\leq Ch^s (\|\mathbf{e}_h\| + |\epsilon_h|_{0,h}) \|\mathbf{e}_0\| \\ &\leq Ch^{r+s} (\|\mathbf{u}\|_{r+1} + \|p\|_r) \|\mathbf{e}_0\|, \end{aligned}$$

where we have used the error estimate (7.3) and the regularity inequality (8.2). Finally, substituting (8.8) and (8.9) into (8.4) yields the desired error estimate (8.3). This completes the proof of the theorem.  $\square$

**9. An Effective Implementation through Variable Reduction.** The degree of freedoms for the WG formulation (4.7)-(4.8) is associated with  $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\}$  and  $p_h = \{p_0, p_b\}$ . In this section, we will demonstrate how  $\mathbf{u}_0$  and  $p_0$  can be eliminated from the system in order to obtain a global system that depends only on  $\mathbf{u}_b$

and  $p_b$ . With such a variable reduction, the number of unknowns of the WG method is reduced significantly for an efficient practical implementation.

Let  $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} \in V_h$  and  $p_h = \{p_0, p_b\} \in W_h$  be the solution of the WG method (4.7)-(4.8). Recall that  $(\mathbf{u}_h; p_h)$  satisfies  $\mathbf{u}_b \times \mathbf{n} = Q_b \phi$  on  $\partial\Omega$  and the following equations:

$$(9.1) \quad a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0), \quad \forall \mathbf{v} = \{\mathbf{v}_0, 0\} \in V_{h,0},$$

$$(9.2) \quad b(\mathbf{u}_h, q) + s_2(p_h, q) = -(g, q_0), \quad \forall q = \{q_0, 0\} \in W_h,$$

and

$$(9.3) \quad a(\mathbf{u}_h, \mathbf{v}) = 0, \quad \forall \mathbf{v} = \{0, \mathbf{v}_b\} \in V_{h,0},$$

$$(9.4) \quad b(\mathbf{u}_h, q) + s_2(p_h, q) = 0, \quad \forall q = \{0, q_b\} \in W_h.$$

Denote by  $V_k(T)$  and  $W_k(T)$  the restrictions of  $V_h$  and  $W_h$  on  $T$ :

$$V_k(T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b = v_1 \mathbf{t}_1 + v_2 \mathbf{t}_2\} : \mathbf{v}_0|_T \in [P_k(T)]^3, v_1, v_2 \in P_k(e), e \subset \partial T\}.$$

and

$$W_k(T) = \{q = \{q_0, q_b\}, q_0 \in P_{k-1}(T), q_b \in P_k(e), e \subset \partial T\}.$$

Since the testing functions  $\mathbf{v} = \{\mathbf{v}_0, 0\}$  and  $q = \{q_0, 0\}$  are locally supported on each element, then the system of equations (9.1)-(9.2) is equivalent to the following system of equations defined locally on each element  $T$ :

$$(9.5) \quad a(\mathbf{u}_h, \mathbf{v}) - b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}_0), \quad \forall \mathbf{v} = \{\mathbf{v}_0, 0\} \in V_k(T),$$

$$(9.6) \quad b(\mathbf{u}_h, q) + s_2(p_h, q) = -(g, q_0), \quad \forall q = \{q_0, 0\} \in W_k(T).$$

If the exact solution of  $\mathbf{u}_b$  and  $p_b$  were known on  $\partial T$ , then the corresponding  $\mathbf{u}_0$  and  $p_0$  can be obtained by solving (9.5) and (9.6) locally on each element. Therefore, the key is to derive a system of equations that shall determine  $\mathbf{u}_b$  and  $p_b$ .

For any given  $\mathbf{u}_b$  and  $p_b$  on  $\partial T$ , let us solve (9.5) and (9.6) to obtain  $\mathbf{u}_0$  and  $p_0$  on each element  $T$ . For simplicity, we introduce the following notation

$$(9.7) \quad \mathbf{u}_0 := D(\mathbf{u}_b, p_b, \mathbf{f}, g),$$

$$(9.8) \quad p_0 := E(\mathbf{u}_b, p_b, f, g).$$

Then the solution  $\mathbf{u}_h$  and  $p_h$  of (4.7)-(4.8) can be written as  $\mathbf{u}_h = \{\mathbf{u}_0, \mathbf{u}_b\} = \{D(\mathbf{u}_b, p_b, \mathbf{f}, g), \mathbf{u}_b\}$  and  $p_h = \{p_0, p_b\} = \{E(\mathbf{u}_b, p_b, \mathbf{f}, g), p_b\}$ .

Let  $D_1(\mathbf{u}_b, p_b) = D(\mathbf{u}_b, p_b, 0, 0)$  and  $D_2(\mathbf{f}, g) = D(0, 0, \mathbf{f}, g)$ . Similarly let  $E_1(\mathbf{u}_b, p_b) = E(\mathbf{u}_b, p_b, 0, 0)$  and  $E_2(\mathbf{f}, g) = E(0, 0, \mathbf{f}, g)$ . Since  $a(\cdot, \cdot)$ ,  $b(\cdot, \cdot)$  and  $s_2(\cdot, \cdot)$  are bilinear, then superposition implies

$$\begin{aligned} (\mathbf{u}_h; p_h) &= (\{\mathbf{u}_0, \mathbf{u}_b\}; \{p_0, p_b\}) \\ &= (\{D(\mathbf{u}_b, p_b, \mathbf{f}, g), \mathbf{u}_b\}; \{E(\mathbf{u}_b, p_b, \mathbf{f}, g), p_b\}) \\ &= (\{D(\mathbf{u}_b, p_b, 0, 0), \mathbf{u}_b\}; \{E(\mathbf{u}_b, p_b, 0, 0), p_b\}) \\ &\quad + (\{D(0, 0, \mathbf{f}, g), 0\}; \{E(0, 0, \mathbf{f}, g), 0\}) \\ &= (\{D_1(\mathbf{u}_b, p_b), \mathbf{u}_b\}; \{E_1(\mathbf{u}_b, p_b), p_b\}) + (\{D_2(\mathbf{f}, g), 0\}; \{E_2(\mathbf{f}, g), 0\}). \end{aligned}$$

Substituting  $\mathbf{u}_h = \{D(\mathbf{u}_b, p_b, \mathbf{f}, g), \mathbf{u}_b\}$  and  $p_h = \{E(\mathbf{u}_b, p_b, \mathbf{f}, g), p_b\}$  into the system (9.3)-(9.4) yields

$$(9.9) \quad a(\{D_1(\mathbf{u}_b, p_b), \mathbf{u}_b\}, \mathbf{v}) = \zeta_1(\mathbf{v}),$$

$$(9.10) \quad b(\{D_1(\mathbf{u}_b, p_b), \mathbf{u}_b\}, q) + s_2(\{E_1(\mathbf{u}_b, p_b), p_b\}, q) = \zeta_2(q),$$

for all  $\mathbf{v} = \{0, \mathbf{v}_b\} \in V_{h,0}$  and  $q = \{0, q_b\} \in W_h$ . Here

$$\begin{aligned} \zeta_1(\mathbf{v}) &= -a(\{D_2(\mathbf{f}, g), 0\}, \mathbf{v}) \\ \zeta_2(q) &= -b(\{D_2(\mathbf{f}, g), 0\}, q) - s_2(\{E_2(\mathbf{f}, g), 0\}, q). \end{aligned}$$

Note that the system (9.9)-(9.10) is a square matrix problem with  $\mathbf{u}_b$  and  $p_b$  as unknowns, and this is the system of equations that  $\mathbf{u}_b$  and  $p_b$  have to satisfy.

To summarize, our WG scheme (4.7)-(4.8) can be implemented as follows:

**Step 1.** Find  $\mathbf{u}_b$  and  $p_b$  with  $\mathbf{u}_b \times \mathbf{n} = Q_b \phi$  and  $p_b = 0$  on  $\partial\Omega$  satisfying (9.9)-(9.10).

**Step 2.** Recover  $\mathbf{u}_0$  and  $p_0$  by  $\mathbf{u}_0 = D(\mathbf{u}_b, p_b, \mathbf{f}, g)$  and  $p_0 = E(\mathbf{u}_b, p_b, \mathbf{f}, g)$  by solving (9.5) and (9.6) locally on each element.

The system of equations (9.9)-(9.10) is known as a Schur complement of the original WG finite element scheme (4.7)-(4.8).

**10. Numerical Results.** Our numerical tests are conducted for the Maxwell equations (1.1)–(1.4) on the unit cube  $\Omega = (0, 1)^3$ . The first level grid consists of one cube. Each grid is refined by subdividing a cube into eight half-sized cubes, to define the next level grid. We apply the first order weak Galerkin finite element method; i.e.,  $V_h$  and  $W_h$  are defined in (4.1) and (4.2) with  $k = 1$ , respectively. Thus, the vector component  $\mathbf{u}$  is approximated by using piecewise linear functions on each cube and its faces; the scalar component  $p$  is approximated by using constants on each cube and linear function on its faces.

We compute four sets of solutions of (1.1)–(1.4), which are

$$(10.1) \quad \mathbf{u} = \begin{pmatrix} y - z \\ z - x \\ 3z - 2y \end{pmatrix}, \quad p = 1.$$

$$(10.2) \quad \mathbf{u} = \begin{pmatrix} yz \\ zx \\ 3z - 2yx \end{pmatrix}, \quad p = xz.$$

$$(10.3) \quad \mathbf{u} = \begin{pmatrix} e^{yz} \\ z/(x+1) \\ e^{xy} \end{pmatrix}, \quad p = e^{-xyz}.$$

$$(10.4) \quad \mathbf{u} = \begin{pmatrix} \cos(\pi x) \sin(\pi y) \sin(\pi z) \\ \sin(\pi x) \cos(\pi y) \sin(\pi z) \\ \sin(\pi x) \sin(\pi y) \cos(\pi z) \end{pmatrix}, \quad p = \sin(2\pi x) \sin(2\pi y) \sin(2\pi z).$$

Observe that the solution  $p$  in the first three test cases does not satisfy the homogeneous boundary condition (1.4). The corresponding WG scheme (4.7)-(4.8) needs to be modified so that  $p_b$  assumes the given non-homogeneous boundary value; namely, the  $L^2$  projection of the value of  $p$  on the boundary.

The first numerical test is used to check the correctness of the code, where the exact solutions (10.1) are linear and constant, respectively. As expected, the weak

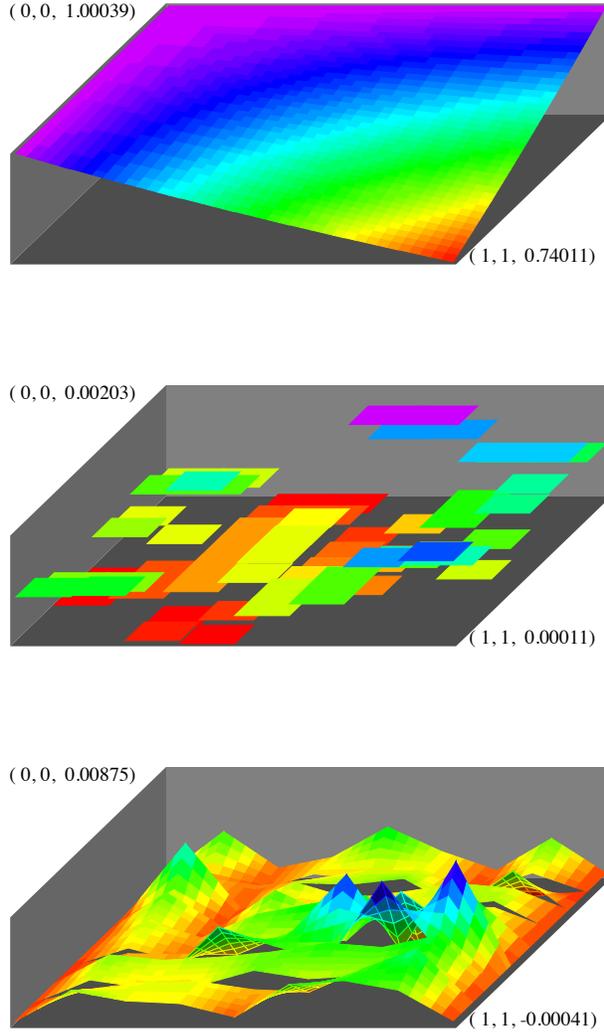


FIG. 10.1. The solution  $p$  in (10.3), and the errors  $(p - p_0)$  and  $(p - p_b)$  on level 4, at  $z = 0.3$ .

TABLE 10.1

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm  $\|\cdot\|_1$ ,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm  $\|\cdot\|_0$ ,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm, and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm, for (10.1) by  $k = 1$  finite elements (4.1)–(4.2).

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$
1	0.00000	0.0	0.00000	0.0	0.00000	0.0	0.00000	0.0
2	0.00000	0.0	0.00000	0.0	0.00000	0.0	0.00000	0.0
3	0.00000	0.0	0.00000	0.0	0.00000	0.0	0.00000	0.0
4	0.00000	0.0	0.00000	0.0	0.00000	0.0	0.00000	0.0

Galerkin finite element solutions are exact, up to computer accuracy. As shown in Table 10.1, all errors are zero.

In the second test (10.2), we choose some bilinear functions as the exact solution. The numerical results are reported in Table 10.2. It can be seen that the numerical solution for the unknown function  $p$  is numerically the same as the exact solution. Moreover, the order of convergence for  $\mathbf{u}$  is half-order higher than what was proved

TABLE 10.2

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm  $\|\mathbf{e}_h\|_1$ ,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm  $\|\mathbf{e}_0\|$ ,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm  $\|\epsilon_h\|_0$ , and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm  $\|\epsilon_0\|$ , for (10.2) by  $k = 1$  finite elements (4.1)–(4.2). And the order  $r$  as in  $O(h^r)$  of convergence.

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$
1	2.26e-08	-	7.76e-09	-	0.0000	-	0.0000	-
2	5.15e-02	-	9.46e-03	-	0.0000	-	0.0000	-
3	2.28e-02	1.1	2.14e-03	2.1	0.0000	-	0.0000	-
4	8.77e-03	1.4	4.15e-04	2.4	0.0000	-	0.0000	-
5	3.03e-03	1.5	7.66e-05	2.4	0.0000	-	0.0000	-

in the theory. This superconvergence is probably caused by the special format of the exact solution.

TABLE 10.3

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm  $\|\mathbf{e}_h\|_1$ ,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm  $\|\mathbf{e}_0\|$ ,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm  $\|\epsilon_h\|_0$ ,  $\|\epsilon_h\|_{0,h}$ , and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm  $\|\epsilon_0\|$ , for (10.3) by  $k = 1$  finite elements (4.1)–(4.2). And the order  $r$  as in  $O(h^r)$  of convergence.

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_h\ _{0,h}$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$
1	7.02e-1	-	3.32e-1	-	6.56e-3	-	6.56e-3	-	2.68e-3	-
2	3.69e-1	0.9	8.71e-2	1.9	7.34e-2	-	4.73e-3	0.5	2.33e-3	0.2
3	1.91e-1	0.9	2.10e-2	2.1	5.11e-2	0.5	1.09e-3	2.1	4.73e-4	2.3
4	1.02e-1	0.9	5.10e-3	2.0	2.91e-2	0.8	2.67e-4	2.0	1.18e-4	2.0
5	5.05e-2	1.0	1.26e-3	2.0	1.55e-2	0.9	6.59e-5	2.0	2.95e-5	2.0
6	2.52e-2	1.0	3.16e-4	2.0	7.73e-3	1.0	1.65e-5	2.0	7.39e-6	2.0

In the third test (10.3), the exact solution is chosen as a general function. The numerical results for this test case is presented in Table 10.3, confirming the theoretical convergence estimates as derived in Theorems 7.4 and 8.1.

Table 10.3 contains additional information for the scalar approximation  $p_h$ ; the fourth column is the error for the scalar approximation measured at the center of each face in a discrete  $L^2$  fashion. More precisely, for each  $q_h = \{q_0, q_b\} \in W_h$ , the semi-norm  $\|q_h\|_{0,h}$  is defined as follows:

$$\|q_h\|_{0,h}^2 = \sum_{T \in \mathcal{T}_h} h \|q_0 - \Pi q_b\|_{\partial T}^2,$$

where  $\Pi$  is the average operator on each face. It can be seen that the convergence in this discrete  $L^2$  norm is of order  $O(h^2)$ , which is higher than the theoretical prediction. For this purpose, we graph the solutions and errors in Figures 10.1 and 10.2. We believe that some superconvergence is playing a role in the weak Galerkin finite element method. This is left to interested readers for an investigation.

The fourth test (10.4) was conducted on another solution with general structure. The goal of this test is to re-confirm the convergence results developed in earlier Sections. The numerical results are presented in Table 10.4. The numerical performance of the weak Galerkin finite element method is similar to the test case three.

In the rest of our numerical experiments, we considered a version of the finite element scheme (4.7)–(4.8) for which no convergence theory was developed in the present paper. More precisely, the WG method makes use of piecewise linear functions

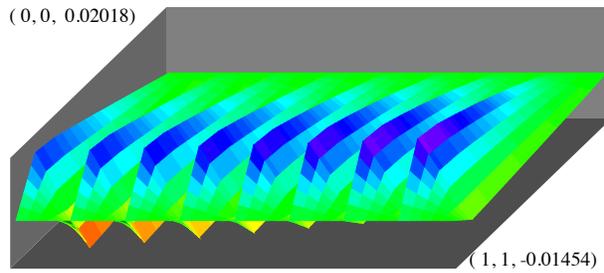
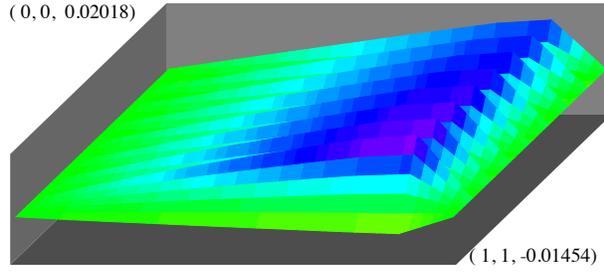
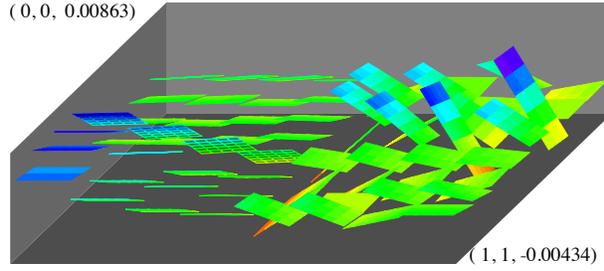
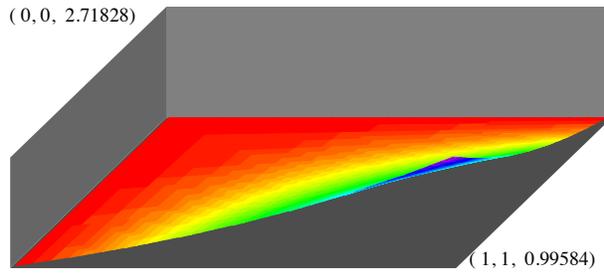


FIG. 10.2. The solution  $(\mathbf{u})_3$  (the third component) in (10.3), and the errors  $(\mathbf{u} - \mathbf{u}_0)_3$ ,  $(\mathbf{u} - \mathbf{u}_b)_{3,t_1}$  and  $(\mathbf{u} - \mathbf{u}_b)_{3,t_2}$  (the two tangential component of the third component) on level 4, at  $z = 0.3$ .

for the vector component  $V_h$ , but the scalar component is modified as follows:

$$W_h = \{w = \{w_0, w_b\} : \{w_0, w_b\}|_T \in P_0(T) \times P_0(e), e \subset \partial T, w_b = 0 \text{ on } \partial\Omega\}.$$

In other words, the scalar variable is approximated by using piecewise constants on both the interior and the boundary of each element. Again, it is not known if the

TABLE 10.4

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm  $\|\mathbf{e}_h\|_1$ ,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm  $\|\mathbf{e}_0\|$ ,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm  $\|\epsilon_h\|_0$ ,  $\|\epsilon_h\|_{0,h}$ , and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm  $\|\epsilon_0\|$ , for (10.4) by  $k = 1$  finite elements (4.1)–(4.2). And the order  $r$  as in  $O(h^r)$  of convergence.

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_h\ _{0,h}$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$
1	8.54e0	-	1.35e0	-	3.60e-1	-	3.60e-1	-	1.47e-1	-
2	2.27e0	1.8	4.77e-1	1.5	2.10e0	-	2.08e0	-	8.65e-1	-
3	9.86e-1	1.2	1.47e-1	1.7	5.17e-1	2.0	2.50e-1	3.1	1.78e-1	2.3
4	4.32e-1	1.1	3.86e-2	1.9	3.09e-1	0.8	4.53e-2	2.5	4.10e-2	2.1
5	1.97e-1	1.1	9.21e-3	2.0	1.71e-1	0.9	1.08e-2	2.0	1.06e-2	1.9
6	9.85e-2	1.0	2.26e-3	2.0	8.75e-2	1.0	2.69e-3	2.0	2.71e-3	2.0

current theoretical result can be extended to this simple WG element, though the numerical results show an excellent approximation to the exact solution. Table 10.5 contains the numerical results for the test case (10.3), and Table 10.6 is for the test case (10.4).

TABLE 10.5

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm, and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm, for (10.4) by lower order WG finite elements. And the order  $r$  as in  $O(h^r)$  of convergence.

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$
1	6.72e-1	-	3.24e-1	-	0.00e0	-	2.68e-3	-
2	3.66e-1	0.9	8.62e-2	1.9	5.73e-3	-	2.92e-3	-
3	1.94e-1	0.9	2.09e-2	2.0	1.03e-3	2.5	6.16e-4	2.2
4	1.02e-2	0.9	5.07e-3	2.0	1.89e-4	2.5	1.12e-4	2.4
5	5.05e-2	1.0	1.26e-3	2.0	9.44e-5	2.0	2.71e-5	2.1
6	2.52e-2	1.0	3.14e-4	2.0	4.72e-5	2.0	6.78e-6	2.0

TABLE 10.6

The errors,  $\mathbf{e}_h = \mathbf{Q}_h \mathbf{u} - \mathbf{u}_h$  in  $H^1$ -like norm,  $\mathbf{e}_0 = \mathbf{Q}_0 \mathbf{u} - \mathbf{u}_0$  in  $L^2$  norm,  $\epsilon_h = Q_h p - p_h$  in  $L^2$ -like norm, and  $\epsilon_0 = Q_0 p - p_0$  in  $L^2$  norm, for (10.4) by lower order WG finite elements. And the order  $r$  as in  $O(h^r)$  of convergence.

grid	$\ \mathbf{e}_h\ _1$	$h^r$	$\ \mathbf{e}_0\ _{L^2}$	$h^r$	$\ \epsilon_h\ _0$	$h^r$	$\ \epsilon_0\ _{L^2}$	$h^r$
1	8.54e0	-	1.35e0	-	3.60e-1	-	1.47e-1	-
2	2.21e0	1.9	4.27e-1	1.7	2.08e0	-	8.57e-1	-
3	1.07e0	1.1	1.63e-1	1.4	1.99e-1	3.4	1.23e-1	2.8
4	4.35e-1	1.2	3.88e-2	2.1	4.49e-2	2.1	2.71e-2	2.2
5	1.96e-2	1.1	9.19e-3	2.1	1.07e-2	2.1	7.23e-3	1.9
6	9.82e-2	1.0	2.26e-3	2.0	2.61e-3	2.0	1.85e-3	2.0

## REFERENCES

- [1] A. BOSSAVIT *Computational Electromagnetism*. Academic Press, San Diego, 1998.
- [2] S. BRENNER, F. LI AND L. SUNG, *A locally divergence-free interior penalty method for two-dimensional curl-curl problems*, SIAM J. Numer. Anal. 42, (2008), 1190–1211.
- [3] S. BRENNER, J. CUI, Z. NAN AND L. SUNG *Hodge Decomposition for Divergence-Free Vector Fields and Two-Dimensional Maxwell’s Equations*, *Mathematics of Computation*, 81 (2012), 643–659.
- [4] P. HOUSTON, I. PERUGIA, AND D. SCHOTZAU, *Mixed Discontinuous Galerkin Approximation of the Maxwell Operator*, Tech. report 02-16, University of Basel, Department of Mathematics, Basel, Switzerland, 2002.
- [5] P. HOUSTON, I. PERUGIA, AND D. SCHOTZAU, *hp-DGFEM for Maxwell’s equations*, in *Numerical Mathematics and Advanced Applications: ENUMATH 2001*, F. Brezzi, A. Buffa, S. Corsaro, and A. Murli, eds., Springer-Verlag, Berlin, 2003, 785–794.
- [6] P. HOUSTON, I. PERUGIA, AND D. SCHOTZAU, *Mixed Discontinuous Galerkin Approximation of the Maxwell Operator*, SIAM J. Numer. Anal. 42, (2004), 434–459.
- [7] J. JIN *The Finite Element Method in Electromagnetics* (Second Edition). John Wiley & Sons, Inc., New York, 2002.
- [8] P. MONK, *Finite Element Methods for Maxwell’s Equations*. Oxford University Press, New York, 2003.
- [9] L. MU, J. WANG, AND X. YE, *Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes*, *Numerical Methods for Partial Differential Equations* (to appear), (arXiv:1303.0927v1).
- [10] J. NEDELEC *Mixed finite elements in  $R^3$* . Numer. Math., 35 (1980), 315–341.
- [11] J. NEDELEC *A new family of mixed finite elements in  $R^3$* . Numer. Math., 50 (1986) 57–81.
- [12] I. PERUGIA AND D. SCHOTZAU, *The hp-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations*, *Math. Comp.*, 72 (2003), 1179–1214.
- [13] I. PERUGIA, D. SCHOTZAU, AND P. MONK, *Stabilized interior penalty methods for the time-harmonic Maxwell equations*, *Comput. Methods Appl. Mech. Engrg.*, 191 (2002), 4675–4697.
- [14] C. WANG AND J. WANG, *An efficient numerical scheme for the Biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes*, *Computers and Mathematics with Applications* (submitted), (arXiv:1309.5560v1).
- [15] J. WANG AND X. YE, *A weak Galerkin finite element method for second-order elliptic problems*, *J. Comp. and Appl. Math.*, 241 (2013), 103–115.
- [16] J. WANG AND X. YE, *A weak Galerkin mixed finite element method for second-order elliptic problems*, *Math. Comp.* (to appear), (arXiv:1202.3655v1).
- [17] J. WANG AND X. YE, *A weak Galerkin finite element method for the Stokes equations*, *Math. Comp.* (submitted), (arXiv:1302.2707v1).