

# ANALYSIS OF A REDUCED-ORDER HDG METHOD FOR THE STOKES EQUATIONS

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**ABSTRACT.** In this paper, we analyze a hybridized discontinuous Galerkin(HDG) method with reduced stabilization for the Stokes equations. The reduced stabilization enables us to reduce the number of facet unknowns and improve the computational efficiency of the method. We provide optimal error estimates in an energy and  $L^2$  norms. It is shown that the reduced method with the lowest-order approximation is closely related to the nonconforming Crouzeix-Raviart finite element method. We also prove that the solution of the reduced method converges to the nonconforming Gauss-Legendre finite element solution as a stabilization parameter  $\tau$  tends to infinity and that the convergence rate is  $O(\tau^{-1})$ .

discontinuous Galerkin method and hybridization and Gauss-Legendre element and Stokes equations

## 1. INTRODUCTION

The aim of this paper is to propose and analyze a reduced-order hybridized discontinuous Galerkin(HDG) method for the Stokes equations with no-slip boundary condition:

$$(1.1) \quad \begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= \mathbf{0} \text{ on } \partial\Omega, \end{aligned}$$

where  $\Omega \subset \mathbb{R}^d (d = 2, 3)$  is a convex polygonal or polyhedral domain and  $\mathbf{f} \in \mathbf{L}^2(\Omega) := [L^2(\Omega)]^d$  is a given function. For the Stokes problem, various HDG methods were already proposed and studied [8, 11, 13, 22, 12, 10, 9, 14, 21, 18]. We also refer to [15] for an overview. The method we investigate in this paper is the HDG-IP method proposed by Egger and Waluga in [21]. The HDG-IP method is based on the gradient-velocity-pressure formulation of the Stokes equations. We remark that the HDG method of the local discontinuous Galerkin type [22, 12] is close to the HDG-IP with a slight difference of a numerical flux.

A reduced stabilization was introduced to the DG methods and was analyzed for elliptic problems [6, 7]. In [4], Becker et al. studied the reduced-order DG method for the Stokes equations and analyzed the limit case as a stabilization parameter tends to infinity. In [21], Lehrenfeld proposed a reduced-order HDG method for the Poisson equation and the Stokes equations. Lehrenfeld also remarked that the convergence rate of the method is optimal, however, error analysis was not presented. In [23], for the Poisson problem, the author provided the optimal error estimates and showed the reduced-order HDG method with the lowest-order approximation is

closely related to the nonconforming Crouzeix-Raviart finite element method. Recently, Qiu and Shi analyzed the reduced methods for linear elasticity problems[25] and convection-diffusion equations[24].

The reduced-order HDG method uses polynomials of degree  $k + 1$  and  $k$  to approximate element and hybrid unknowns, respectively, whereas the standard HDG method uses polynomials of degree  $k$  for both unknowns. Although both the methods have the same number of globally coupled degrees of freedom, the reduced method can provide higher order convergence than the standard method. This is the main advantage of the reduced-order HDG method. For the Stokes problem, when we use polygonal or polyhedral elements, the reduced method is indeed better than the standard method in term of convergence orders. The standard method using polynomials of degree  $k$  for all unknowns obtains the suboptimal convergence; the orders are  $k + 1$  for velocity without postprocessing and  $k + 1/2$  for the gradient and pressure, according to [15]. In contrast, the reduced method uses polynomials of degree  $k + 1$  for velocity and polynomials of degree  $k$  for the hybrid part of velocity and pressure and can achieve the optimal order convergence.

In this paper, we provide optimal error estimates of the reduced method for the Stokes problem. Since we need to use a weaker energy norm in our analysis, it is necessary to modify the error analysis of the standard method. We note that the main difficulties can be overcome by the techniques used in the author's previous work [23] and the discrete inf-sup condition proved by Egger and Waluga [18]. We also show a relation between the reduced method and the Gauss-Legendre element (see [5, 26] for example). It is proved that the hybrid part of velocity and the pressure of the reduced-order HDG method with the lowest-order approximation coincides with those of the nonconforming Crouzeix-Raviart finite element solution. In the limit case as the stabilization parameter  $\tau$  tends to infinity, the solution of the reduced method converges to that of the nonconforming Gauss-Legendre method. The convergence rate is estimated to be  $O(\tau^{-1})$ . This result is inspired by [4, Theorem 3], however, our proof is completely different and novel.

The rest of this paper is organized as follows. Section 2 is devoted to the preliminaries. In Section 3, we introduce a reduced stabilization and present a reduced HDG method. In Section 4, we provide a priori error estimates in an energy and  $L^2$  norms. In Section 5, some relations between the nonconforming Gauss-Legendre finite element method and the reduced method are shown. In Section 6, numerical results are presented to confirm our theoretical results.

## 2. PRELIMINARIES

**2.1. Meshes and function spaces.** Let  $\{\mathcal{T}_h\}_h$  be a family of shape-regular triangulations of  $\Omega$  and define  $\Gamma_h = \bigcup_{K \in \mathcal{T}_h} \partial K$ . Let  $\mathcal{E}_h$  be the set of all edges in  $\mathcal{T}_h$ . The mesh size of  $\mathcal{T}_h$  is denoted by  $h$ , namely  $h := \max_{K \in \mathcal{T}_h} h_K$ , where  $h_K = \text{diam} K$ . The length of an edge  $e \in \mathcal{E}_h$  is denoted by  $h_e$ .

We use the usual Lebesgue and Sobolev spaces;  $L^2(\Omega)$ ,  $L^2(\Gamma_h)$  and  $H^m(\Omega)$ , and also  $L_0^2(\Omega) = \{q \in L^2(\Omega) : \int_{\Omega} q dx = 0\}$ . We introduce piecewise Sobolev spaces  $H^m(\mathcal{T}_h) = \{v \in L^2(\Omega) : v|_K \in H^m(K) \ \forall K \in \mathcal{T}_h\}$ . For vector-valued function spaces, we write them in bold, such as  $\mathbf{L}^2(\Omega) = [L^2(\Omega)]^d$  and  $\mathbf{H}^m(\Omega) = [H^m(\Omega)]^d$ . The usual  $L^2$  inner product is denoted by  $(\cdot, \cdot)_{\Omega}$ . Let us define the piecewise inner

products by

$$(u, v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_K u v dx, \quad \langle u, v \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \int_{\partial K} u v ds.$$

Let  $P_k(\mathcal{T}_h)$  and  $P_k(\mathcal{E}_h)$  denote the space of *element-wise* and *edge-wise* polynomials of degree  $k$ , respectively. We employ  $\mathbf{V}_h^{k+1} = \mathbf{P}_{k+1}(\mathcal{T}_h)$ ,  $\widehat{\mathbf{V}}_h^k = \mathbf{P}_k(\mathcal{E}_h) \cap \{\widehat{\mathbf{v}} \in \mathbf{L}^2(\Gamma_h) : \widehat{\mathbf{v}} = \mathbf{0} \text{ on } \partial\Omega\}$  and  $Q_h^k = P_k(\mathcal{T}_h) \cap L_0^2(\Omega)$  as finite element spaces, which we call  $P_{k+1}$ - $P_k$ / $P_k$  approximation. The  $L^2$ -projection from  $\prod_{K \in \mathcal{T}_h} \mathbf{L}^2(\partial K)$  onto  $\prod_{K \in \mathcal{T}_h} \mathbf{P}_k(\partial K)$  is denoted by  $\mathbf{P}_k$ , and  $\mathbf{I}$  stands for the identity operator.

**2.2. Norms and seminorms.** As usual, we use the Sobolev norms  $|\mathbf{v}|_m = |\mathbf{v}|_{\mathbf{H}^m(\Omega)}$  and  $\|\mathbf{v}\|_{m,D} = \|\mathbf{v}\|_{\mathbf{H}^m(\Omega)}$  for a domain  $D$ . The  $L^2$ -norm is denoted by  $\|\mathbf{v}\| = \|\mathbf{v}\|_{0,D} = \|\mathbf{v}\|_{\mathbf{L}^2(D)}$ . The energy norms are defined as follows: for  $(\mathbf{v}, \widehat{\mathbf{v}}) \in \mathbf{H}^2(\mathcal{T}_h) \times \mathbf{L}^2(\Gamma_h)$ ,

$$\begin{aligned} \|(\mathbf{v}, \widehat{\mathbf{v}})\|^2 &= |\mathbf{v}|_{1,h}^2 + |\mathbf{v}|_{2,h}^2 + |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j}}^2, \\ \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{\tau}^2 &= |\mathbf{v}|_{1,h}^2 + |\mathbf{v}|_{2,h}^2 + |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j},\tau}^2, \\ \|(\mathbf{v}, \widehat{\mathbf{v}})\|_h^2 &= |\mathbf{v}|_{1,h}^2 + |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j}}^2, \\ \|(\mathbf{v}, \widehat{\mathbf{v}})\|_{h,\tau}^2 &= |\mathbf{v}|_{1,h}^2 + |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j},\tau}^2, \end{aligned}$$

where

$$\begin{aligned} |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j}}^2 &= \sum_{K \in \mathcal{T}_h} \sum_{e \subset \partial K} \frac{1}{h_e} \|\mathbf{P}_k(\widehat{\mathbf{v}} - \mathbf{v})\|_{0,e}^2, \\ |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j},\tau}^2 &= \tau |(\mathbf{v}, \widehat{\mathbf{v}})|_{\mathbf{j}}^2 \\ |\mathbf{v}|_{1,h}^2 &= \sum_{K \in \mathcal{T}_h} |\mathbf{v}|_{1,K}^2, \\ |\mathbf{v}|_{2,h}^2 &= \sum_{K \in \mathcal{T}_h} h_K^2 |\mathbf{v}|_{2,K}^2. \end{aligned}$$

The symbol  $\tau$  is a stabilization parameter which will be defined in Section 3. The parameter-free energy norms  $\|\cdot\|$  and  $\|\cdot\|_h$  are used to analyze the convergence rate with respect to the mesh size  $h$ . We need the parameter-dependent energy norms in order to analyze the proposed method when  $\tau \rightarrow \infty$ . We also use the stronger  $L^2$  norm

$$\|q\|_h^2 = \|q\|^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |q|_{1,K}^2.$$

By the inverse inequality[1], we see that the two energy norms are equivalent to each other on  $\mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k$ , i.e.,

$$(2.1) \quad \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_h \leq \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\| \leq C \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_h,$$

$$(2.2) \quad \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{h,\tau} \leq \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{\tau} \leq C \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{h,\tau}$$

for some constant  $C > 0$  independent of  $h$ . Similarly, it holds that  $\|q_h\| \leq \|q_h\|_h \leq C \|q_h\|$  for all  $q_h \in Q_h^k$ . In the following, the symbol  $C$  will stand for a generic constant independent of the mesh size  $h$  and the stabilization parameter  $\tau$ .

**2.3. Approximation property.** The approximation property in the energy norm holds as well as in the standard energy norm.

**Theorem 1.** *If  $\psi \in \mathbf{H}^{k+2}(\Omega)$  and  $\pi \in H^{k+1}(\Omega)$ , then we have*

$$(2.3) \quad \inf_{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \|(\psi - \mathbf{v}_h, \psi|_{\Gamma_h} - \widehat{\mathbf{v}}_h)\| \leq Ch^{k+1} |\psi|_{\mathbf{H}^{k+2}(\Omega)},$$

$$(2.4) \quad \inf_{q_h \in Q_h^k} \|\pi - q_h\| \leq Ch^{k+1} |\pi|_{H^{k+1}(\Omega)}.$$

*Proof.* We refer to [23]. □

### 3. A REDUCED-ORDER HDG METHOD

In this section, we present a reduced-order HDG method based on the HDG method proposed by Egger and Waluga in [18]. By taking the  $L^2$ -projection onto the polynomial space of lower degree by one in the stabilization term of the standard method, we obtain the reduced-order HDG method: find  $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k \times Q_h^k$  such that

$$(3.1a) \quad a_h(\mathbf{u}_h, \widehat{\mathbf{u}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) + b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; p_h) = (\mathbf{f}, \mathbf{v}_h)_\Omega \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k,$$

$$(3.1b) \quad b_h(\mathbf{u}_h, \widehat{\mathbf{u}}_h; q_h) = 0 \quad \forall q_h \in Q_h^k,$$

where the bilinear forms are given by

$$\begin{aligned} a_h(\mathbf{u}_h, \widehat{\mathbf{u}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) &= (\nabla \mathbf{u}_h, \nabla \mathbf{v}_h)_{\mathcal{T}_h} + \langle \partial_{\mathbf{n}} \mathbf{u}_h, \widehat{\mathbf{v}}_h - \mathbf{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \partial_{\mathbf{n}} \mathbf{v}_h, \widehat{\mathbf{u}}_h - \mathbf{u}_h \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \tau h_e^{-1} \mathbf{P}_k(\widehat{\mathbf{u}}_h - \mathbf{u}_h), \mathbf{P}_k(\widehat{\mathbf{v}}_h - \mathbf{v}_h) \rangle_{\partial \mathcal{T}_h}, \\ b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; p_h) &= -(\operatorname{div} \mathbf{v}_h, p_h)_{\mathcal{T}_h} - \langle \widehat{\mathbf{v}}_h - \mathbf{v}_h, p_h \mathbf{n} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

Here  $\tau$  is a stabilization parameter assumed to be greater than or equal to one and sufficiently large. We recall that  $\mathbf{P}_k$ , which is defined in Section 2.1., is the  $L^2$ -projection onto the edge-wise polynomial space of degree  $k$ .

**Remark 2.** *In the two-dimensional case, we can easily implement the reduced method by using a reduced-order quadrature formula in the computations of the reduced stabilization term, see [23, Lemma 5] for details.*

### 4. ERROR ANALYSIS

In this section, we provide the optimal error estimates of the method in both the energy and  $L^2$  norms. To do that, we first show the consistency of the method, the boundedness of  $a_h$  and  $b_h$ , and the coercivity of  $a_h$ . In addition, the discrete inf-sup condition of  $b_h$  is proved based on the results of [18].

**4.1. Consistency.** We state the consistency and adjoint consistency of the method.

**Theorem 3.** *Let  $(\mathbf{u}, p)$  be the exact solution of the Stokes equations (1.1). Then we have*

$$(4.1) \quad \begin{aligned} a_h(\mathbf{u}, \mathbf{u}|_{\Gamma_h}; \mathbf{v}_h, \widehat{\mathbf{v}}_h) + b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; p) &= (\mathbf{f}, \mathbf{v}_h)_\Omega \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k, \\ b_h(\mathbf{u}, \mathbf{u}|_{\Gamma_h}; q_h) &= 0 \quad \forall q_h \in Q_h^k. \end{aligned}$$

*Proof.* Since  $\mathbf{u} - \mathbf{u}|_{\Gamma_h} = \mathbf{0}$  on  $\Gamma_h$ , we can easily see that the consistency (4.1) holds. □

Let  $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k \times Q_h^k$  be the solution of the method (3.1). From the consistency, the Galerkin orthogonality follows immediately:

$$(4.2) \quad \begin{aligned} a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) + b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; p - p_h) &= 0 \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k, \\ b_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; q_h) &= 0 \quad \forall q_h \in Q_h^k. \end{aligned}$$

Due to the symmetricity of  $a_h$ , we readily see that the adjoint consistency also holds:

$$(4.3) \quad \begin{aligned} a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{u}, \mathbf{u}|_{\Gamma_h}) + b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; p) &= (\mathbf{f}, \mathbf{v}_h)_\Omega \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k, \\ b_h(\mathbf{u}, \mathbf{u}|_{\Gamma_h}; q_h) &= 0 \quad \forall q_h \in Q_h^k. \end{aligned}$$

**4.2. Boundedness and coercivity.** We first prove the boundedness of  $a_h$  and  $b_h$ .

**Theorem 4.** *Let  $(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}) = (\mathbf{w} + \mathbf{w}_h, \mathbf{w}|_{\Gamma_h} + \widehat{\mathbf{w}}_h)$  and  $(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}}) = (\mathbf{v} + \mathbf{v}_h, \mathbf{v}|_{\Gamma_h} + \widehat{\mathbf{v}}_h)$ , where  $(\mathbf{w}_h, \widehat{\mathbf{w}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k$  and  $\mathbf{w}, \mathbf{v} \in \mathbf{H}_0^1(\Omega)$ . Then there exists a constant  $C > 0$  independent of  $h$  and  $\tau$  such that*

$$(4.4) \quad |a_h(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}; \boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})| \leq C\tau \|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})\| \|(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})\|.$$

With respect to the parameter-dependent energy norm, we have

$$(4.5) \quad |a_h(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}; \boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})| \leq C \|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})\|_\tau \|(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})\|_\tau.$$

*Proof.* By the Schwarz inequality, the first term of  $a_h$  is bounded as

$$(4.6) \quad |(\nabla \boldsymbol{\xi}, \nabla \boldsymbol{\eta})_{\mathcal{T}_h}| \leq |\boldsymbol{\xi}|_{1,h} |\boldsymbol{\eta}|_{1,h}.$$

We estimate the second term. Note that

$$(4.7) \quad \begin{aligned} \langle \partial_n \boldsymbol{\xi}, \widehat{\boldsymbol{\eta}} - \boldsymbol{\eta} \rangle_{\partial \mathcal{T}_h} &= \langle \partial_n \boldsymbol{\xi}, \mathbf{P}_k(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \rangle_{\partial \mathcal{T}_h} + \langle \partial_n \boldsymbol{\xi}, (\mathbf{I} - \mathbf{P}_k)(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \rangle_{\partial \mathcal{T}_h} \\ &= \langle \partial_n \boldsymbol{\xi}, \mathbf{P}_k(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \rangle_{\partial \mathcal{T}_h} - \langle \partial_n \boldsymbol{\xi}, (\mathbf{I} - \mathbf{P}_k)\mathbf{v}_h \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

Since  $\langle \partial_n \mathbf{w}, (\mathbf{I} - \mathbf{P}_k)\mathbf{v} \rangle_{\partial \mathcal{T}_h} = \langle \partial_n \mathbf{w}_h, (\mathbf{I} - \mathbf{P}_k)\mathbf{v} \rangle_{\partial \mathcal{T}_h} = 0$ , it follows that

$$\langle \partial_n \boldsymbol{\xi}, (\mathbf{I} - \mathbf{P}_k)\mathbf{v} \rangle_{\partial \mathcal{T}_h} = 0.$$

Using this, we deduce that  $\langle \partial_n \boldsymbol{\xi}, (\mathbf{I} - \mathbf{P}_k)\mathbf{v}_h \rangle_{\partial \mathcal{T}_h} = \langle \partial_n \boldsymbol{\xi}, (\mathbf{I} - \mathbf{P}_k)\boldsymbol{\eta} \rangle_{\partial \mathcal{T}_h}$ . Then we have

$$(4.8) \quad \begin{aligned} |\langle \partial_n \boldsymbol{\xi}, \widehat{\boldsymbol{\eta}} - \boldsymbol{\eta} \rangle_{\partial \mathcal{T}_h}| &= |\langle \partial_n \boldsymbol{\xi}, \mathbf{P}_k(\widehat{\mathbf{v}}_h - \mathbf{v}_h) - (\mathbf{I} - \mathbf{P}_k)\boldsymbol{\eta} \rangle_{\partial \mathcal{T}_h}| \\ &\leq C \max\{1, \tau^{-1/2}\} (|\boldsymbol{\xi}|_{1,h}^2 + h^2 |\boldsymbol{\xi}|_{2,h}^2)^{1/2} (|(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})|_{j,\tau}^2 + |\boldsymbol{\eta}|_{1,h}^2)^{1/2} \\ &\leq C \|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})\|_\tau \|(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})\|_\tau, \end{aligned}$$

where we have used the trace inequality and the following estimate (see [23] for the proof)

$$(4.9) \quad |(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})|_j \leq C |\boldsymbol{\eta}|_{1,h}^2.$$

The stabilization term is bounded as

$$(4.10) \quad |\langle \tau h_e^{-1} \mathbf{P}_k(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}), \mathbf{P}_k(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta}) \rangle_{\partial \mathcal{T}_h}| \leq |(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})|_{j,\tau} |(\boldsymbol{\eta}, \widehat{\boldsymbol{\eta}})|_{j,\tau}.$$

From (4.7), (4.8) and (4.10), we obtain the boundedness (4.5). With a slight modification, we can also show that (4.4).  $\square$

**Theorem 5** (Boundedness of  $b_h$ ). *Let  $(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})$  be the same as in Lemma 4 and  $r = q + q_h$  with  $q \in H^1(\mathcal{T}_h) \cap L_0^2(\Omega)$  and  $q_h \in Q_h^k$ . Then there exists a constant  $C > 0$  independent  $h$  and  $\tau$  such that*

$$(4.11) \quad |b_h(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}; r)| \leq C \|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})\| \|r\|_h.$$

*In particular, in the case of  $r = q_h \in Q_h^k$ , we have*

$$(4.12) \quad |b_h(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}}; q_h)| \leq C \|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})\| \|q_h\|.$$

*Proof.* By the Schwarz inequality, we have

$$(4.13) \quad |(\operatorname{div} \boldsymbol{\xi}, r)_{\mathcal{T}_h}| \leq \|\boldsymbol{\xi}\|_{1,h} \|r\|.$$

Next, we estimate the second term of the bilinear form  $b_h$ . In a similar manner of (4.7), we have

$$(4.14) \quad \begin{aligned} \langle \widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}, r\mathbf{n} \rangle_{\partial\mathcal{T}_h} &= \langle \mathbf{P}_k(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}), r\mathbf{n} \rangle_{\partial\mathcal{T}_h} + \langle (\mathbf{I} - \mathbf{P}_k)(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}), r\mathbf{n} \rangle_{\partial\mathcal{T}_h} \\ &= \langle \mathbf{P}_k(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}), r\mathbf{n} \rangle_{\partial\mathcal{T}_h} - \langle (\mathbf{I} - \mathbf{P}_k)\mathbf{w}_h, r\mathbf{n} \rangle_{\partial\mathcal{T}_h}. \end{aligned}$$

Since  $\langle (\mathbf{I} - \mathbf{P}_k)\mathbf{w}, r\mathbf{n} \rangle_{\partial\mathcal{T}_h} = 0$ , we get

$$\langle (\mathbf{I} - \mathbf{P}_k)\mathbf{w}_h, r\mathbf{n} \rangle_{\partial\mathcal{T}_h} = \langle (\mathbf{I} - \mathbf{P}_k)(\mathbf{w} + \mathbf{w}_h), r\mathbf{n} \rangle_{\partial\mathcal{T}_h} = \langle (\mathbf{I} - \mathbf{P}_k)\boldsymbol{\xi}, r\mathbf{n} \rangle_{\partial\mathcal{T}_h}.$$

Hence

$$\begin{aligned} |\langle \widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}, r\mathbf{n} \rangle_{\partial\mathcal{T}_h}| &= |\langle \mathbf{P}_k(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi}) - (\mathbf{I} - \mathbf{P}_k)\boldsymbol{\xi}, r\mathbf{n} \rangle_{\partial\mathcal{T}_h}| \\ &\leq C(|(\boldsymbol{\xi}, \widehat{\boldsymbol{\xi}})|_{\mathbf{j}}^2 + \|\boldsymbol{\xi}\|_{1,h}^2)^{1/2} \|r\|_h, \end{aligned}$$

where we have used the trace inequality and (4.9). Consequently, we obtain the inequality (4.11). From the inverse inequality, (4.12) follows immediately.  $\square$

**Theorem 6** (Coercivity). *Assume that  $\tau$  is sufficiently large. There exists a constant  $C > 0$  independent of  $h$  and  $\tau$  such that*

$$(4.15) \quad a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq C \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{\tau}^2 \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k.$$

*In particular, we have*

$$(4.16) \quad a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq C \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{\tau}^2 \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k.$$

*Proof.* We note that

$$(4.17) \quad \langle \boldsymbol{\partial}_n \mathbf{v}_h, \widehat{\mathbf{v}}_h - \mathbf{v}_h \rangle_{\partial\mathcal{T}_h} = \langle \boldsymbol{\partial}_n \mathbf{v}_h, \mathbf{P}_k(\widehat{\mathbf{v}}_h - \mathbf{v}_h) \rangle_{\partial\mathcal{T}_h}.$$

Then it follows that

$$(4.18) \quad a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq |\mathbf{v}_h|_{1,h}^2 - 2|\langle \boldsymbol{\partial}_n \mathbf{v}_h, \mathbf{P}_k(\widehat{\mathbf{v}}_h - \mathbf{v}_h) \rangle_{\partial\mathcal{T}_h}| + |(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|_{\mathbf{j},\tau}^2.$$

By the trace and inverse inequalities and Young's inequality, we have

$$(4.19) \quad 2|\langle \boldsymbol{\partial}_n \mathbf{v}_h, \mathbf{P}_k(\widehat{\mathbf{v}}_h - \mathbf{v}_h) \rangle_{\partial\mathcal{T}_h}| \leq C(\varepsilon |\mathbf{v}_h|_{1,h}^2 + \varepsilon^{-1} \tau^{-1} |(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|_{\mathbf{j},\tau}^2)$$

for any  $\varepsilon > 0$ . From (4.18) and (4.19), it follows that

$$(4.20) \quad a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq (1 - C\varepsilon) |\mathbf{v}_h|_{1,h}^2 + (1 - C\varepsilon^{-1} \tau^{-1}) |(\mathbf{v}_h, \widehat{\mathbf{v}}_h)|_{\mathbf{j},\tau}^2.$$

We can take  $\varepsilon = \tau^{-1/2}$  and deduce that, by assuming  $\tau \geq 4C^2$ ,

$$a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \mathbf{v}_h, \widehat{\mathbf{v}}_h) \geq \left(1 - C\tau^{-1/2}\right) \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{h,\tau}^2 \geq \frac{1}{2} \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{h,\tau}^2.$$

By the inverse inequality, we have  $\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{\tau} \leq C' \|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_{h,\tau}$  for some positive constant  $C'$ , which completes the proof.  $\square$

**4.3. The discrete inf-sup condition.** To prove the discrete inf-sup condition of the bilinear form  $b_h$ , we introduce a Fortin operator. The main idea and techniques for constructing the Fortin operator are entirely based on [18]. The global  $L^2$ -projection operators  $\Pi_h^k : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{V}_h^k$  and  $\widehat{\Pi}_h^k : \mathbf{H}^1(\Omega) \rightarrow \widehat{\mathbf{V}}_h^k$  are defined by

$$\begin{aligned} (\Pi_h^k \mathbf{v})|_K &= \Pi_K^k(\mathbf{v}|_K) \text{ for } K \in \mathcal{T}_h, \\ (\widehat{\Pi}_h^k \mathbf{v})|_e &= \widehat{\Pi}_e^k(\mathbf{v}|_e) \text{ for } e \in \mathcal{E}_h, \end{aligned}$$

where  $\Pi_K^k$  and  $\widehat{\Pi}_e^k$  are the  $L^2$ -projections onto  $\mathbf{P}_k(K)$  and  $\mathbf{P}_k(e)$ , respectively. We define the Fortin operator by

$$(\Pi_h^{k+1}, \widehat{\Pi}_h^k) : \mathbf{H}_0^1(\Omega) \rightarrow \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k.$$

In the following, we show this operator satisfies the Fortin properties.

**Theorem 7.** *For all  $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ , we have*

$$(4.21) \quad b_h(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v}; q_h) = -(\operatorname{div} \mathbf{v}, q_h)_\Omega \quad \forall q_h \in Q_h^k.$$

*Moreover, there exists a constant  $C > 0$  such that*

$$(4.22) \quad \|(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v})\|_h \leq C |\mathbf{v}|_{1,h} \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega).$$

*Proof.* First, we prove (4.21). Using the Green formula and the property of  $L^2$  projection, we have

$$\begin{aligned} b_h(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v}; q_h) &= -(\operatorname{div} \Pi_h^{k+1} \mathbf{v}, q_h)_{\mathcal{T}_h} - \langle \widehat{\Pi}_h^k \mathbf{v} - \Pi_h^{k+1} \mathbf{v}, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ (4.23) \quad &= (\Pi_h^{k+1} \mathbf{v}, \nabla q_h)_{\mathcal{T}_h} - \langle \widehat{\Pi}_h^k \mathbf{v}, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= (\mathbf{v}, \nabla q_h)_{\mathcal{T}_h} - \langle \mathbf{v}, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= -(\operatorname{div} \mathbf{v}, q_h)_\Omega. \end{aligned}$$

Next, we prove (4.22). Note that  $|\Pi_h^{k+1} \mathbf{v}|_{1,h} \leq |\mathbf{v}|_{1,h}$  and

$$(4.24) \quad \begin{aligned} h_e^{-1/2} \|\mathbf{P}_k(\widehat{\Pi}_h^k \mathbf{v} - \Pi_h^{k+1} \mathbf{v})\|_{0,e} &\leq h_e^{-1/2} \|\mathbf{v} - \Pi_h^{k+1} \mathbf{v}\|_{0,e} \\ &\leq C |\mathbf{v}|_{1,h}. \end{aligned}$$

From (4.23) and (4.24), it follows that

$$(4.25) \quad \|(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v})\|_h^2 = |\Pi_h^{k+1} \mathbf{v}|_{1,h}^2 + |(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v})|_j^2 \leq C |\mathbf{v}|_{1,h}^2,$$

which completes the proof.  $\square$

By using the above results, we can prove the discrete inf-sup condition for the bilinear form  $b_h$ .

**Theorem 8** (Discrete inf-sup condition). *There exists a constant  $\beta > 0$  independent of  $h$  such that*

$$(4.26) \quad \sup_{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \frac{b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; q_h)}{\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|} \geq \beta \|q_h\| \quad \forall q_h \in Q_h^k.$$

*Proof.* It is well-known that the continuous inf-sup condition holds: there exists  $\beta' > 0$  such that

$$(4.27) \quad \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\operatorname{div} \mathbf{v}, q)_\Omega}{|\mathbf{v}|_{1,h}} \geq \beta' \|q\| \quad \forall q \in L_0^2(\Omega).$$

For all  $q_h \in Q_h^k \subset L_0^2(\Omega)$ , we have

$$\begin{aligned}
\sup_{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \frac{b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; q_h)}{\|(\mathbf{v}_h, \widehat{\mathbf{v}}_h)\|_h} &\geq \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{b_h(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v}; q_h)}{\|(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v})\|_h} \\
&= \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\operatorname{div} \mathbf{v}, q_h)_\Omega}{\|(\Pi_h^{k+1} \mathbf{v}, \widehat{\Pi}_h^k \mathbf{v})\|_h} \\
&\geq C \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \frac{(\operatorname{div} \mathbf{v}, q_h)_\Omega}{\|\mathbf{v}\|_{1,h}} \\
&\geq \beta' \|q_h\|.
\end{aligned}$$

The proof is complete.  $\square$

**4.4. A priori error estimates.** We prove optimal error estimates by using the results in the previous section. In this section, the stabilization parameter  $\tau$  is fixed to be a sufficiently large value.

**Theorem 9** (Energy-norm error estimate). *Let  $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$  be the exact solution of the Stokes equations (1.1), and let  $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k \times Q_h^k$  be the solution of the method (3.1). Then we have*

$$\begin{aligned}
&\|(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h)\| + \|p - p_h\| \\
(4.28) \quad &\leq C \left( \inf_{(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \|(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{v}}_h)\| + \inf_{q_h \in Q_h^k} \|p - q_h\| \right).
\end{aligned}$$

If  $(\mathbf{u}, p) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ , we obtain

$$(4.29) \quad \|(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h)\| + \|p - p_h\| \leq Ch^{k+1} (\|\mathbf{u}\|_{\mathbf{H}^{k+2}(\Omega)} + \|p\|_{H^{k+1}(\Omega)}).$$

*Proof.* Let  $\mathbf{v}_h \in \mathbf{V}_h^{k+1}$ ,  $\widehat{\mathbf{v}}_h \in \widehat{\mathbf{V}}_h^k$  and  $r_h \in Q_h^k$  be arbitrary, and set  $\boldsymbol{\eta}_h = \mathbf{u}_h - \mathbf{v}_h$ ,  $\widehat{\boldsymbol{\eta}}_h = \widehat{\mathbf{u}}_h - \widehat{\mathbf{v}}_h$  and  $\delta_h = p_h - q_h$ . Then we have

$$\begin{aligned}
(4.30) \quad &a_h(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h; \mathbf{w}_h, \widehat{\mathbf{w}}_h) + b_h(\mathbf{w}_h, \widehat{\mathbf{w}}_h; \delta_h) = F(\mathbf{w}_h, \widehat{\mathbf{w}}_h) \quad \forall (\mathbf{w}_h, \widehat{\mathbf{w}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k, \\
&b_h(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h; r_h) = G(r_h) \quad \forall r_h \in Q_h^k,
\end{aligned}$$

where  $F : \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k \rightarrow \mathbb{R}$  and  $G : Q_h^k \rightarrow \mathbb{R}$  are defined by

$$\begin{aligned}
F(\mathbf{w}_h, \widehat{\mathbf{w}}_h) &= a_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{v}}_h; \mathbf{w}_h, \widehat{\mathbf{w}}_h) + b_h(\mathbf{w}_h, \widehat{\mathbf{w}}_h; p - q_h), \\
G(r_h) &= b_h(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h, r_h).
\end{aligned}$$

By the boundedness of  $a_h$  and  $b_h$ , we can estimate the dual norms of  $F$  and  $G$  as follows:

$$\begin{aligned}
(4.31) \quad \|F\| &= \sup_{(\mathbf{w}_h, \widehat{\mathbf{w}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \frac{|F(\mathbf{w}_h, \widehat{\mathbf{w}}_h)|}{\|(\mathbf{w}_h, \widehat{\mathbf{w}}_h)\|} \\
&\leq C (\|(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{v}}_h)\| + \|p - q_h\|), \\
\|G\| &= \sup_{r_h \in Q_h^k} \frac{|G(r_h)|}{\|r_h\|} \leq C \|(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{v}}_h)\|.
\end{aligned}$$



By the coercivity (4.15), we have

$$\begin{aligned}
 C\|(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h)\|^2 &\leq a_h(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h; \boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h) \\
 (4.32) \quad &= F(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h) - G(\delta_h) \\
 &\leq \|F\| \|(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h)\| + \|G\| \|\delta_h\|.
 \end{aligned}$$

By the discrete inf-sup condition, we deduce that

$$\begin{aligned}
 \beta \|\delta_h\| &\leq \sup_{(\mathbf{w}_h, \widehat{\mathbf{w}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \frac{b(\mathbf{w}_h, \widehat{\mathbf{w}}_h; \delta_h)}{\|(\mathbf{w}_h, \widehat{\mathbf{w}}_h)\|} \\
 (4.33) \quad &\leq \sup_{(\mathbf{w}_h, \widehat{\mathbf{w}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k} \frac{F(\mathbf{w}_h, \widehat{\mathbf{w}}_h) - a_h(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h; \mathbf{w}_h, \widehat{\mathbf{w}}_h)}{\|(\mathbf{w}_h, \widehat{\mathbf{w}}_h)\|} \\
 &\leq \|F\| + C\|(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h)\|.
 \end{aligned}$$

From (4.32) and (4.33), it follows that

$$\|(\boldsymbol{\eta}_h, \widehat{\boldsymbol{\eta}}_h)\| + \|\delta_h\| \leq C(\|(\mathbf{u} - \mathbf{v}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{v}}_h)\| + \|p - q_h\|).$$

By the triangle inequality, we obtain the estimate (4.28). In addition, using the approximation property, we see that (4.29) holds.  $\square$

We can prove the  $L^2$  error estimate of optimal order by the Aubin-Nitsche duality argument.

**Theorem 10** ( $L^2$ -error estimate). *Let the notation be the same as in Theorem 9. If  $(\mathbf{u}, p) \in \mathbf{H}^{k+2}(\Omega) \times H^{k+1}(\Omega)$ , then we have*

$$(4.34) \quad \|\mathbf{u} - \mathbf{u}_h\| \leq Ch^{k+2}(|\mathbf{u}|_{\mathbf{H}^{k+2}(\Omega)} + |p|_{H^{k+1}(\Omega)}).$$

*Proof.* We consider the following adjoint problem: find  $(\boldsymbol{\psi}, \pi) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times (H^1(\Omega) \cap L_0^2(\Omega))$  such that

$$\begin{aligned}
 -\Delta \boldsymbol{\psi} + \nabla \pi &= \mathbf{u} - \mathbf{u}_h, \\
 \operatorname{div} \boldsymbol{\psi} &= 0.
 \end{aligned}$$

Note that  $|\boldsymbol{\psi}|_{\mathbf{H}^2(\Omega)} + |\pi|_{H^1(\Omega)} \leq C\|\mathbf{u} - \mathbf{u}_h\|$ . From the adjoint consistency (4.3), the solution of the problem satisfies

$$\begin{aligned}
 (4.35) \quad a_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \boldsymbol{\psi}, \boldsymbol{\psi}|_{\Gamma_h}) + b_h(\mathbf{v}_h, \widehat{\mathbf{v}}_h; \pi) &= (\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)_\Omega \quad \forall (\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k, \\
 b_h(\boldsymbol{\psi}, \boldsymbol{\psi}|_{\Gamma_h}; q_h) &= 0 \quad \forall q_h \in Q_h^k.
 \end{aligned}$$

Let  $(\boldsymbol{\psi}_h, \widehat{\boldsymbol{\psi}}_h, \pi_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k \times Q_h^k$  be an approximation to  $(\boldsymbol{\psi}, \boldsymbol{\psi}|_{\Gamma_h}, \pi)$  satisfying

$$\|(\boldsymbol{\psi} - \boldsymbol{\psi}_h, \boldsymbol{\psi}|_{\Gamma_h} - \widehat{\boldsymbol{\psi}}_h)\| \leq Ch|\boldsymbol{\psi}|_{\mathbf{H}^2(\Omega)}, \quad \|\pi - \pi_h\| \leq Ch|\pi|_{H^1(\Omega)}.$$

Taking  $\mathbf{v}_h = \mathbf{u} - \mathbf{u}_h$  and  $\widehat{\mathbf{v}}_h = \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h$  in (4.35),

$$\begin{aligned}
 \|\mathbf{u} - \mathbf{u}_h\|^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; \boldsymbol{\psi}, \boldsymbol{\psi}|_{\Gamma_h}) + b_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; \pi) \\
 &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; \boldsymbol{\psi} - \boldsymbol{\psi}_h, \boldsymbol{\psi}|_{\Gamma_h} - \widehat{\boldsymbol{\psi}}_h) \\
 &\quad + b_h(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h; \pi - \pi_h) \\
 &\leq C\|(\mathbf{u} - \mathbf{u}_h, \mathbf{u}|_{\Gamma_h} - \widehat{\mathbf{u}}_h)\|(\|(\boldsymbol{\psi} - \boldsymbol{\psi}_h, \boldsymbol{\psi}|_{\Gamma_h} - \widehat{\boldsymbol{\psi}}_h)\| + \|\pi - \pi_h\|) \\
 &\leq Ch^{k+2}(|\mathbf{u}|_{\mathbf{H}^{k+2}(\Omega)} + |p|_{H^{k+1}(\Omega)})\|\mathbf{u} - \mathbf{u}_h\|.
 \end{aligned}$$

Thus we obtain the assertion.  $\square$

## 5. RELATIONS WITH THE NONCONFORMING GAUSS-LEGENDRE FINITE ELEMENT METHOD

**5.1. The Gauss-Legendre element.** The approximation space of the  $(k+1)$ -th Gauss-Legendre element for velocity is defined by

$$(5.1) \quad \tilde{\mathbf{V}}_h^{k+1} = \{\mathbf{v}_h \in \mathbf{V}_h^{k+1} : [\![\mathbf{P}_k \mathbf{v}_h]\!] = 0\},$$

where  $[\![\cdot]\!]$  is a jump operator (see [2] for example). This space is known as the Crouzeix-Raviart [17] ( $k=0$ ), Fortin-Soulie [19] ( $k=1$ ) or Crouzeix-Falk [16] ( $k=2$ ) element. Note that  $\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}$  is continuous at the  $(k+1)$ -th order Gauss-Legendre points, and thereby  $\mathbf{P}_k \tilde{\mathbf{v}}_h$  is single-valued on  $\Gamma_h$ .

The nonconforming Gauss-Legendre finite element method reads as: find  $(\mathbf{u}_h^*, p_h^*) \in \tilde{\mathbf{V}}_h^{k+1} \times Q_h^k$  such that

$$(5.2a) \quad (\nabla \mathbf{u}_h^*, \nabla \tilde{\mathbf{v}}_h)_{\mathcal{T}_h} - (\operatorname{div} \tilde{\mathbf{v}}_h, p_h^*)_{\mathcal{T}_h} = (\mathbf{f}, \tilde{\mathbf{v}}_h)_\Omega \quad \forall \mathbf{v}_h \in \tilde{\mathbf{V}}_h^{k+1},$$

$$(5.2b) \quad (\operatorname{div} \mathbf{u}_h^*, q_h)_{\mathcal{T}_h} = 0 \quad \forall q_h \in Q_h^k.$$

The nonconforming method is well-posed for  $k=0, 1, 2$ . For  $k \geq 3$ , it is the case under some assumption on a mesh, see [3, Lemma 3.1]. We here assume that the method (5.2) is well-posed for simplicity. In our analysis, we will use the following inf-sup condition for the Gauss-Legendre element, see also [3].

**Theorem 11** (Discrete inf-sup condition for the Gauss-Legendre element). *There exists a constant  $\tilde{\beta} > 0$  such that*

$$(5.3) \quad \sup_{\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \frac{(\operatorname{div} \tilde{\mathbf{v}}_h, p_h)_{\mathcal{T}_h}}{|\tilde{\mathbf{v}}_h|_{1,h}} \geq \tilde{\beta} \|q_h\| \quad \forall q_h \in Q_h^k.$$

**5.2. A relation between the reduced-order HDG method with the lowest-order approximation and the Crouzeix-Raviart element.** It is known that there are relations between HDG methods and the conforming or nonconforming finite element methods for Poisson's equation. The hybrid part the solution of the embedded discontinuous Galerkin (EDG) method [20] with the continuous  $P_1$  element is identical to the conforming finite element solution on  $\Gamma_h$ . In [23], it was proved that the hybrid part of the solution of the reduced method using the discontinuous  $P_1$ - $P_0$  approximation coincides with the nonconforming Crouzeix-Raviart finite element method at the mid-points of edges. In this section, we discover a relation between the reduced method with the  $P_1$ - $P_0$ / $P_0$  approximation and the nonconforming Crouzeix-Raviart finite element method.

Let  $\tilde{V}_h^1$  be the usual Crouzeix-Raviart finite element space. The Crouzeix-Raviart interpolation operator for scalar-valued functions,  $\Pi_h^* : L^2(\Gamma_h) \rightarrow \tilde{V}_h^1$ , is defined as

$$(5.4) \quad \int_e \Pi_h^* \hat{v} ds = \int_e \hat{v} ds \quad \forall e \in \mathcal{E}_h.$$

For a vector-valued function  $\hat{\mathbf{v}} = (\hat{v}_1, \dots, \hat{v}_d)^T \in \mathbf{L}^2(\Gamma_h)$ , we set

$$\mathbf{\Pi}_h^* \hat{\mathbf{v}} := (\Pi_h^* \hat{v}_1, \dots, \Pi_h^* \hat{v}_d)^T.$$

We are now in a position to prove the relation.

**Theorem 12.** *Let  $(\mathbf{u}_h, \hat{\mathbf{u}}_h, p_h) \in \mathbf{V}_h^1 \times \hat{\mathbf{V}}_h^0 \times Q_h^0$  be the solution of the reduced method (3.1) with  $k=0$  and  $(\mathbf{u}_h^*, p_h^*) \in \tilde{\mathbf{V}}_h^1 \times Q_h^0$  be the solution of the nonconforming*

*Crouzeix-Raviart finite element method. Then we have*

$$\mathbf{\Pi}_h^* \hat{\mathbf{u}}_h = \mathbf{u}_h^*, \quad p_h = p_h^*.$$

*Proof.* Since  $\mathbf{\Pi}_h^* \hat{\mathbf{v}}_h$  is a piecewise linear polynomial, it follows that, by the Green formula,

$$\begin{aligned} \langle \partial_{\mathbf{n}} \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h, \hat{\mathbf{u}}_h \rangle_{\partial \mathcal{T}_h} &= \langle \partial_{\mathbf{n}} \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h, \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h \rangle_{\partial \mathcal{T}_h} \\ &= (\nabla \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h, \nabla \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h)_{\mathcal{T}_h}. \end{aligned}$$

Choosing  $\mathbf{v}_h = \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h$  in (3.1a) and noting that  $\hat{\mathbf{v}}_h - \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h = \mathbf{0}$  at the mid-points of edges, we have

$$(5.5) \quad (\nabla \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h, \nabla \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h)_{\mathcal{T}_h} - (\operatorname{div} \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h, p_h)_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h)_{\Omega} \quad \forall \hat{\mathbf{v}}_h \in \hat{\mathbf{V}}_h^0.$$

The equation (3.1b) can be rewritten as

$$\begin{aligned} (5.6) \quad -(\operatorname{div} \mathbf{u}_h, q_h)_{\mathcal{T}_h} - \langle \hat{\mathbf{u}}_h - \mathbf{u}_h, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} &= -\langle \hat{\mathbf{u}}_h, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= -\langle \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &= -(\operatorname{div} \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h, q_h \mathbf{n})_{\mathcal{T}_h}. \end{aligned}$$

From (5.5) and (5.6), we obtain the following equations to determine  $\hat{\mathbf{u}}_h$ :

$$(5.7) \quad (\nabla \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h, \nabla \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h)_{\mathcal{T}_h} - (\operatorname{div} \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h, p_h)_{\mathcal{T}_h} = (\mathbf{f}, \mathbf{\Pi}_h^* \hat{\mathbf{v}}_h)_{\Omega} \quad \forall \hat{\mathbf{v}}_h \in \hat{\mathbf{V}}_h^0,$$

$$(5.8) \quad (\operatorname{div} \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h, q_h \mathbf{n})_{\mathcal{T}_h} = 0 \quad \forall q_h \in Q_h^0,$$

which is nothing but (5.2) in the case of  $k = 1$ . Therefore we have  $\mathbf{\Pi}_h^* \hat{\mathbf{u}}_h = \mathbf{u}_h^*$  and  $p_h = p_h^*$ .  $\square$

**Remark 13.** From the definition of the Crouzeix-Raviart interpolation, we see that

$$\int_e \hat{\mathbf{u}}_h ds = \int_e \mathbf{\Pi}_h^* \hat{\mathbf{u}}_h ds = \int_e \mathbf{u}_h^* ds,$$

which implies  $\hat{\mathbf{u}}_h$  and  $\mathbf{u}_h^*$  are equal at the mid-point of  $e \in \mathcal{E}_h$ .

**5.3. The limit case as  $\tau$  tends to infinity.** We will show that, as the stabilization parameter  $\tau$  tends to infinity, the solution of the reduced method converges to the nonconforming Gauss-Legendre finite element solution with rate  $O(\tau^{-1})$ . To do that, we first prove the following key lemma.

**Theorem 14.** *There exists a constant  $C > 0$  such that, for any  $(\mathbf{v}_h, \hat{\mathbf{v}}_h) \in \mathbf{V}_h^{k+1} \times \hat{\mathbf{V}}_h^k$ ,*

$$(5.9) \quad \inf_{\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \|(\mathbf{v}_h - \tilde{\mathbf{w}}_h, \hat{\mathbf{v}}_h - \mathbf{P}_k \tilde{\mathbf{w}}_h)\|_h \leq C |(\mathbf{v}_h, \hat{\mathbf{v}}_h)|_j.$$

*Proof.* Let us define  $\mathbf{G}(\tilde{\mathbf{V}}_h^{k+1}) = \{(\tilde{\mathbf{v}}_h, \mathbf{P}_k \tilde{\mathbf{v}}_h) : \tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}\}$  which is a closed subspace of  $\mathbf{V}_h^{k+1} \times \hat{\mathbf{V}}_h^k$ . We consider the quotient space  $\mathbf{V}_h^{k+1} \times \hat{\mathbf{V}}_h^k / \mathbf{G}(\tilde{\mathbf{V}}_h^{k+1})$ , where the standard quotient norm is given by

$$\|(\mathbf{v}_h, \hat{\mathbf{v}}_h)\|_{\mathbf{V}_h^{k+1} \times \hat{\mathbf{V}}_h^k / \mathbf{G}(\tilde{\mathbf{V}}_h^{k+1})} = \inf_{\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \|(\mathbf{v}_h - \tilde{\mathbf{w}}_h, \hat{\mathbf{v}}_h - \mathbf{P}_k \tilde{\mathbf{w}}_h)\|_h.$$

We shall prove that  $|(\mathbf{v}_h, \hat{\mathbf{v}}_h)|_j$  is also a norm on  $\mathbf{V}_h^{k+1} \times \hat{\mathbf{V}}_h^k / \mathbf{G}(\tilde{\mathbf{V}}_h^{k+1})$ . Suppose that  $|(\mathbf{v}_h, \hat{\mathbf{v}}_h)|_j = 0$ , then we readily have  $\mathbf{v}_h \in \tilde{\mathbf{V}}_h^{k+1}$  and  $\hat{\mathbf{v}}_h = \mathbf{P}_k \mathbf{v}_h$ . Hence it follows that  $(\mathbf{v}_h, \hat{\mathbf{v}}_h) \in \mathbf{G}(\tilde{\mathbf{V}}_h^{k+1})$ , which implies that  $(\mathbf{v}_h, \hat{\mathbf{v}}_h)$  is the zero element

of  $\mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k / \mathbf{G}(\widetilde{\mathbf{V}}_h^{k+1})$ . Since any two norms on a finite-dimensional space are equivalent to each other, we conclude the assertion.  $\square$

Let us denote  $(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau, p_h^\tau)$  the solution of the reduced method (3.1) with the stabilization parameter  $\tau$ . By using Lemma 14, it can be proved that  $|(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)|_j$  converges to zero as  $\tau \rightarrow \infty$  with rate  $O(\tau^{-1})$ .

**Theorem 15.** *If  $\tau$  is sufficiently large, we have*

$$(5.10) \quad |(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)|_j \leq C\tau^{-1} \|\mathbf{f}\|.$$

*Proof.* Let  $\widetilde{\mathbf{v}}_h \in \widetilde{\mathbf{V}}_h^{k+1}$  be arbitrary. Since  $\mathbf{P}_k(\widetilde{\mathbf{v}}_h - \mathbf{P}_k \widetilde{\mathbf{v}}_h)$  vanishes on  $\Gamma_h$ , we have

$$\begin{aligned} |(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)|_{j,\tau} &= |(\mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h, \widehat{\mathbf{u}}_h^\tau - \mathbf{P}_k \widetilde{\mathbf{v}}_h)|_{j,\tau} \\ &\leq \|(\mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h, \widehat{\mathbf{u}}_h^\tau - \mathbf{P}_k \widetilde{\mathbf{v}}_h)\|_\tau. \end{aligned}$$

Therefore,

$$(5.11) \quad |(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)|_{j,\tau} \leq \inf_{\widetilde{\mathbf{w}}_h \in \widetilde{\mathbf{V}}_h^{k+1}} \|(\mathbf{u}_h^\tau - \widetilde{\mathbf{w}}_h, \widehat{\mathbf{u}}_h^\tau - \mathbf{P}_k \widetilde{\mathbf{w}}_h)\|_\tau.$$

Taking  $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) = (\widetilde{\mathbf{v}}_h, \mathbf{P}_k \widetilde{\mathbf{v}}_h)$  in (3.1), we have

$$(5.12) \quad (\nabla \mathbf{u}_h^\tau, \nabla \widetilde{\mathbf{v}}_h)_{\mathcal{T}_h} + \langle \partial_{\mathbf{n}} \widetilde{\mathbf{v}}_h, \widehat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h} - (\operatorname{div} \widetilde{\mathbf{v}}_h, p_h^\tau)_{\mathcal{T}_h} = (\mathbf{f}, \widetilde{\mathbf{v}}_h)_\Omega.$$

Subtracting this from (3.1a) by taking  $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) = (\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)$  gives

$$\begin{aligned} |(\mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau)|_{j,\tau}^2 &= (\mathbf{f}, \mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h)_\Omega - (\nabla \mathbf{u}_h^\tau, \nabla(\mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h))_{\mathcal{T}_h} \\ (5.13) \quad &\quad - \langle \partial_{\mathbf{n}} \mathbf{u}_h^\tau, \widehat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h} - \langle \partial_{\mathbf{n}}(\mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h), \widehat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\operatorname{div}(\mathbf{u}_h^\tau - \widetilde{\mathbf{v}}_h), p_h^\tau)_{\mathcal{T}_h} + \langle \widehat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau, p_h^\tau \mathbf{n} \rangle_{\partial \mathcal{T}_h}. \end{aligned}$$

Here we note that the energy norm is stronger than the  $L^2$  norm (see [23] for the proof):

$$(5.14) \quad \|\mathbf{z}_h\| \leq C \|(\mathbf{z}_h, \widehat{\mathbf{z}}_h)\|_h \quad \forall (\mathbf{z}_h, \widehat{\mathbf{z}}_h) \in \mathbf{V}_h^{k+1} \times \widehat{\mathbf{V}}_h^k,$$

and that by the standard argument, we see that  $\mathbf{u}_h^\tau$  and  $p_h^\tau$  are uniformly bounded in the energy and  $L^2$  norms, respectively, i.e.

$$\|\mathbf{u}_h^\tau\|_h + \|p_h^\tau\| \leq C \|\mathbf{f}\|,$$

where  $C$  is independent of  $\tau$ . The terms on the right-hand side of (5.13) are bounded as follows:

$$\begin{aligned}
|(\mathbf{f}, \mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h)_\Omega| &\leq \|\mathbf{f}\| \|\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h\| \\
&\leq C \|\mathbf{f}\| \|(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{v}}_h)\|_h, \\
|(\nabla \mathbf{u}_h^\tau, \nabla(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h))_{\mathcal{T}_h}| &\leq \|\nabla \mathbf{u}_h^\tau\| \|\nabla(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h)\| \\
&\leq C \|\mathbf{f}\| \|(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{v}}_h)\|_h, \\
|\langle \partial_n \mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h}| &\leq C \|\nabla \mathbf{u}_h^\tau\| |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \\
&\leq C \|\mathbf{f}\| |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j, \\
|\langle \partial_n(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h), \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h}| &\leq C \|\nabla(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h)\| |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \\
&\leq C \|(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{v}}_h)\|_h |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \\
|(\operatorname{div}(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h), p_h^\tau)_{\mathcal{T}_h}| &\leq \|\nabla(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h)\| \|p_h^\tau\| \\
&\leq C \|(\mathbf{u}_h^\tau - \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{v}}_h)\|_h \|\mathbf{f}\| \\
|\langle \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau, p_h^\tau \mathbf{n} \rangle_{\partial \mathcal{T}_h}| &\leq C |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \|p_h^\tau\| \\
&\leq C |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \|\mathbf{f}\|.
\end{aligned}$$

From the above estimates, we obtain, by using Lemma 14,

$$\begin{aligned}
|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_{j,\tau}^2 &\leq C(\|\mathbf{f}\| + |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j) \cdot \inf_{\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \|(\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{w}}_h)\|_h \\
&\leq C_*(\|\mathbf{f}\| + |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j) |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j.
\end{aligned}$$

Since we can assume  $\tau > C_*$ , it finally follows that

$$|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \leq \frac{C}{\tau - C_*} \|\mathbf{f}\| \leq C\tau^{-1} \|\mathbf{f}\|.$$

The proof is complete.  $\square$

In the following theorem, we prove that  $(\mathbf{u}_h^\tau, p_h^\tau)$  converges to  $(\mathbf{u}_h^*, p_h^*)$  with the same rate as  $|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j$ , namely  $O(\tau^{-1})$ .

**Theorem 16.** *If  $\tau$  is sufficiently large, we have*

$$(5.15) \quad \|\mathbf{u}_h^* - \mathbf{u}_h^\tau\|_{1,h} + \|p_h^* - p_h^\tau\| \leq C\tau^{-1} \|\mathbf{f}\|.$$

*Proof.* Choosing  $\mathbf{v}_h = \tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^k$  and  $\hat{\mathbf{v}}_h = \mathbf{P}_k \tilde{\mathbf{v}}_h$  in (3.1), we have

$$\begin{aligned}
(5.16a) \quad &(\nabla \mathbf{u}_h^\tau, \nabla \tilde{\mathbf{v}}_h)_{\mathcal{T}_h} + \langle \partial_n \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h} - (\operatorname{div} \tilde{\mathbf{v}}_h, p_h^\tau)_{\mathcal{T}_h} = (\mathbf{f}, \tilde{\mathbf{v}}_h)_\Omega \quad \forall \tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}, \\
(5.16b) \quad &(\operatorname{div} \mathbf{u}_h^\tau, q_h)_{\mathcal{T}_h} + \langle \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall q_h \in Q_h^k.
\end{aligned}$$

Let us denote  $\boldsymbol{\xi}_h^\tau = \mathbf{u}_h^* - \mathbf{u}_h^\tau$ ,  $\hat{\boldsymbol{\xi}}_h^\tau = \mathbf{P}_k \mathbf{u}_h^* - \hat{\mathbf{u}}_h^\tau$  and  $\delta_h^\tau = p_h^* - p_h^\tau$ . Subtracting (5.16) from (5.2) leads to

$$(5.17a) \quad (\nabla \boldsymbol{\xi}_h^\tau, \nabla \tilde{\mathbf{v}}_h)_{\mathcal{T}_h} - \langle \partial_n \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial \mathcal{T}_h} - (\operatorname{div} \tilde{\mathbf{v}}_h, \delta_h^\tau)_{\mathcal{T}_h} = 0 \quad \forall \tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1},$$

$$(5.17b) \quad (\operatorname{div} \boldsymbol{\xi}_h^\tau, q_h)_{\mathcal{T}_h} + \langle \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau, q_h \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall q_h \in Q_h^k.$$

Let  $\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h^{k+1}$  be arbitrary, then we have

$$\begin{aligned}
(\nabla \boldsymbol{\xi}_h^\tau, \nabla \tilde{\mathbf{w}}_h)_{\mathcal{T}_h} &= (\nabla(\mathbf{u}_h^* - \tilde{\mathbf{w}}_h), \nabla \boldsymbol{\xi}_h^\tau)_{\mathcal{T}_h} + (\nabla(\tilde{\mathbf{w}}_h - \mathbf{u}_h^\tau), \nabla \boldsymbol{\xi}_h^\tau)_{\mathcal{T}_h} \\
&=: I + II.
\end{aligned}$$

By (5.17a) with  $\tilde{\mathbf{v}}_h = \mathbf{u}_h^* - \tilde{\mathbf{w}}_h$ , we get

$$\begin{aligned} I &= \langle \partial_{\mathbf{n}}(\mathbf{u}_h^* - \tilde{\mathbf{w}}_h), \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial\mathcal{T}_h} + (\operatorname{div}(\mathbf{u}_h^* - \tilde{\mathbf{w}}_h), \delta_h^\tau)_{\mathcal{T}_h} \\ &= \langle \partial_{\mathbf{n}}\boldsymbol{\xi}_h^\tau, \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial\mathcal{T}_h} + \langle \partial_{\mathbf{n}}(\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h), \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial\mathcal{T}_h} \\ &\quad + (\operatorname{div}\boldsymbol{\xi}_h^\tau, \delta_h^\tau)_{\mathcal{T}_h} + (\operatorname{div}(\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h), \delta_h^\tau)_{\mathcal{T}_h} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

These terms can be bounded as follows:

$$\begin{aligned} |I_1| &\leq C|\boldsymbol{\xi}_h^\tau|_{1,h} |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j, \\ |I_2| &\leq C|\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h|_{1,h} |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j, \\ |I_3| &= |-\langle \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau, \delta_h^\tau \mathbf{n} \rangle_{\partial\mathcal{T}_h}| \quad (\text{by (5.17b)}) \\ &\leq C|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j \|\delta_h^\tau\|, \\ |I_4| &\leq C|\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h|_{1,h} \|\delta_h^\tau\|. \end{aligned}$$

Therefore, we have

$$|I| \leq C(|\boldsymbol{\xi}_h^\tau|_{1,h} + \|\delta_h^\tau\|) |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j + C(|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j + \|\delta_h^\tau\|) |\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h|_{1,h}.$$

Taking the infimum with respect to  $\tilde{\mathbf{w}}_h$  and using Lemma 14, we have

$$(5.18) \quad |I| \leq C(|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j + |\boldsymbol{\xi}_h^\tau|_{1,h} + \|\delta_h^\tau\|) |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j.$$

On the other hand, by the inf-sup condition for the Gauss-Legendre element, we have

$$\begin{aligned} \tilde{\beta} \|\delta_h^\tau\| &\leq \sup_{\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \frac{(\operatorname{div} \tilde{\mathbf{v}}_h, \delta_h^\tau)_{\mathcal{T}_h}}{|\tilde{\mathbf{v}}_h|_{1,h}} \\ (5.19) \quad &= \sup_{\tilde{\mathbf{v}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \frac{(\nabla \boldsymbol{\xi}_h^\tau, \nabla \tilde{\mathbf{v}}_h)_{\mathcal{T}_h} - \langle \partial_{\mathbf{n}} \tilde{\mathbf{v}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{u}_h^\tau \rangle_{\partial\mathcal{T}_h}}{|\tilde{\mathbf{v}}_h|_{1,h}} \\ &\leq |\boldsymbol{\xi}_h^\tau|_{1,h} + C|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j. \end{aligned}$$

Combining (5.18) with (5.19) gives us

$$(5.20) \quad |I| \leq C(|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j + |\boldsymbol{\xi}_h^\tau|_{1,h}) |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j.$$

The term  $II$  is estimated as follows by using Lemma 14:

$$\begin{aligned} |II| &\leq \inf_{\tilde{\mathbf{w}}_h \in \tilde{\mathbf{V}}_h^{k+1}} \|(\mathbf{u}_h^\tau - \tilde{\mathbf{w}}_h, \hat{\mathbf{u}}_h^\tau - \mathbf{P}_k \tilde{\mathbf{w}}_h)\|_h \cdot |\boldsymbol{\xi}_h^\tau|_{1,h} \\ (5.21) \quad &\leq C|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j |\boldsymbol{\xi}_h^\tau|_{1,h}. \end{aligned}$$

From (5.20) and (5.21), it follows that

$$|\boldsymbol{\xi}_h^\tau|_{1,h}^2 \leq C(|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j + |\boldsymbol{\xi}_h^\tau|_{1,h}) |(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j.$$

By Young's inequality, we get  $|\boldsymbol{\xi}_h^\tau|_{1,h} \leq C|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j$ . Using (5.19) again, we have  $\|\delta_h^\tau\| \leq C|(\mathbf{u}_h^\tau, \hat{\mathbf{u}}_h^\tau)|_j$ . By Theorem 15, we obtain the assertion.  $\square$

## 6. NUMERICAL RESULTS

As a test problem, we consider the case of  $\Omega = (0, 1)^2$  and

$$\mathbf{f}(x, y) = (4\pi^2 \sin(2\pi y), 4\pi^2 \sin(2\pi x)(-1 + 4 \cos(2\pi y)))^T.$$

The source term is chosen so that the exact solution  $(\mathbf{u}, p)$  is

$$\mathbf{u}(x, y) = (2 \sin^2(\pi x) \sin(\pi y), -2 \sin(\pi x) \sin^2(\pi y))^T,$$

$$p(x, y) = 4\pi \sin(2\pi x) \sin(2\pi y).$$

We carry out numerical computations to examine the convergence rates of the reduced method. We employ unstructured triangular meshes and the  $P_{k+1}-P_k/P_k$  approximations for  $k = 0, 1, 2$ . The convergence history is shown at Table 1, and the convergence diagrams are displayed in Figure 1. From these results, we observe that the convergence orders are optimal in all the cases, which agrees with Theorems 9 and 10.

TABLE 1. Convergence history of the reduced method.

Degree	Mesh size	$\ \mathbf{u} - \mathbf{u}_h\ $		$\ \nabla(\mathbf{u} - \mathbf{u}_h)\ $		$\ p - p_h\ $	
$k$	$h$	Error	Order	Error	Order	Error	Order
0	0.2634	1.789E-01	–	3.672E+00	–	2.307E+00	–
	0.1414	4.938E-02	2.07	1.901E+00	1.06	1.249E+00	0.99
	0.0701	1.200E-02	2.01	9.340E-01	1.01	6.054E-01	1.03
	0.0373	2.907E-03	2.25	4.596E-01	1.13	2.925E-01	1.15
1	0.2634	1.038E-02	–	8.486E-01	–	4.520E-01	–
	0.1414	1.639E-03	2.97	2.290E-01	2.11	1.259E-01	2.06
	0.0701	1.815E-04	3.13	5.003E-02	2.17	2.768E-02	2.16
	0.0373	2.295E-05	3.28	1.254E-02	2.19	6.859E-03	2.20
2	0.2634	1.338E-03	–	1.439E-01	–	3.105E-02	–
	0.1415	8.656E-05	4.40	1.900E-02	3.26	3.920E-03	3.33
	0.0701	5.089E-06	4.03	2.298E-03	3.01	4.310E-04	3.14
	0.0373	3.168E-07	4.40	2.858E-04	3.31	4.956E-05	3.43

## REFERENCES

- [1] D. N. Arnold. An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.*, 19(4):742–760, 1982.
- [2] D. N. Arnold, F. Brezzi, B. Cockburn, and L. D. Marini. Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.*, 39:1749–1779, 2002.
- [3] Á. Baran and G. Stoyan. Gauss-Legendre elements: a stable, higher order non-conforming finite element family. *Computing*, 79(1):1–21, 2007.
- [4] R. Becker, D. Capatina, and J. Joie. Connections between discontinuous Galerkin and nonconforming finite element methods for the Stokes equations. *Numer. Methods Partial Differential Equations*, 28(3):1013–1041, 2012.
- [5] D. Boffi, F. Brezzi, and M. Fortin. *Mixed finite element methods and applications*, volume 44 of *Springer Series in Computational Mathematics*. Springer, Berlin, 2013.

- [6] E. Burman and B. Stamm. Low order discontinuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.*, 47(1):508–533, 2008.
- [7] E. Burman and B. Stamm. Local discontinuous Galerkin method with reduced stabilization for diffusion equations. *Commun. Comput. Phys.*, 5(2-4):498–514, 2009.
- [8] J. Carrero, B. Cockburn, and D. Schötzau. Hybridized globally divergence-free LDG methods. I. The Stokes problem. *Math. Comp.*, 75(254):533–563, 2006.
- [9] B. Cockburn and J. Cui. An analysis of HDG methods for the vorticity-velocity-pressure formulation of the Stokes problem in three dimensions. *Math. Comp.*, 81:1355–1368, 2012.
- [10] B. Cockburn and J. Cui. Divergence-free HDG methods for the vorticity-velocity formulation of the Stokes problem. *J. Sci. Comput.*, 52:256–270, 2012.
- [11] B. Cockburn and J. Gopalakrishnan. The derivation of hybridizable discontinuous Galerkin methods for Stokes flow. *SIAM J. Numer. Anal.*, 47:1092–1125, 2009.
- [12] B. Cockburn, J. Gopalakrishnan, N. C. Nguyen, J. Peraire, and F.-J. Sayas. Analysis of HDG methods for Stokes flow. *Math. Comp.*, 80(274):723–760, 2011.
- [13] B. Cockburn, N. C. Nguyen, and J. Peraire. A comparison of HDG methods for Stokes flow. *J. Sci. Comput.*, 45(1-3):215–237, 2010.
- [14] B. Cockburn and F.-J. Sayas. Divergence-conforming HDG methods for Stokes flows. *Math. Comp.*, 83(288):1571–1598, 2014.
- [15] B. Cockburn and K. Shi. Devising **HDG** methods for Stokes flow: an overview. *Comput. & Fluids*, 98:221–229, 2014.

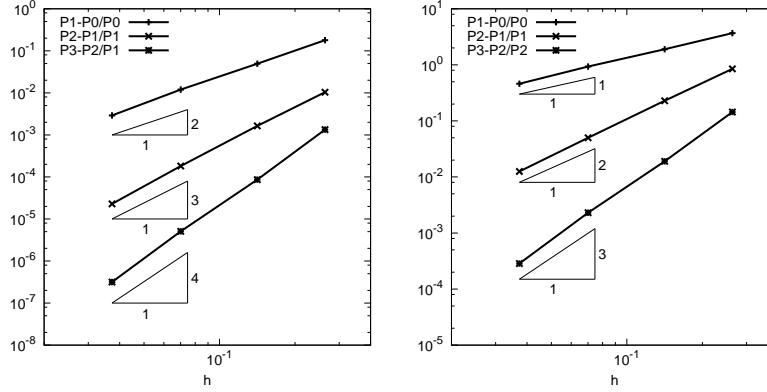
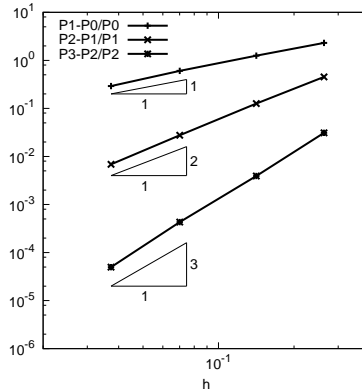
(a)  $L^2$ -error of velocity(b)  $H^1$ -error of velocity(c)  $L^2$ -error of pressure

FIGURE 1. Convergence diagrams of the reduced method.



- [16] M. Crouzeix and R. S. Falk. Nonconforming finite elements for the Stokes problem. *Math. Comp.*, 52(186):437–456, 1989.
- [17] M. Crouzeix and P.-A. Raviart. Conforming and nonconforming finite element methods for solving the stationary Stokes equations. *ESAIM: Math. Model. Numer. Anal.*, 7:33–75, 1973.
- [18] H. Egger and C. Waluga. *hp* analysis of a hybrid DG method for Stokes flow. *IMA J. Numer. Anal.*, 33(2):687–721, 2013.
- [19] M. Fortin and M. Soulie. A nonconforming piecewise quadratic finite element on triangles. *Internat. J. Numer. Methods Engrg.*, 19(4):505–520, 1983.
- [20] S. Güzey, B. Cockburn, and H. K. Stolarski. The embedded discontinuous Galerkin method: application to linear shell problems. *Internat. J. Numer. Methods Engrg.*, 70(7):757–790, 2007.
- [21] C. Lehrenfeld. Hybrid discontinuous Galerkin methods for solving incompressible flow problems. *PhD Thesis, RWTH Aachen University*, 2010.
- [22] N. C. Nguyen, J. Peraire, and B. Cockburn. A hybridizable discontinuous Galerkin method for Stokes flow. *Comput. Methods Appl. Mech. Engrg.*, 199(9-12):582–597, 2010.
- [23] I. Oikawa. A hybridized discontinuous Galerkin method with reduced stabilization. *J. Sci. Comput.*, to appear.
- [24] W. Qiu and K. Shi. An HDG method for linear elasticity with strong symmetric stresses. *submitted*, 2014.
- [25] W. Qiu and K. Shi. An HDG method for convection diffusion equation. *J. Sci. Comput.*, to appear.
- [26] G. Stoyan and Á. Baran. Crouzeix-Velte decompositions for higher-order finite elements. *Comput. Math. Appl.*, 51(6-7):967–986, 2006.