# INTERIOR PENALTY DISCONTINUOUS GALERKIN METHODS FOR SECOND ORDER LINEAR NON-DIVERGENCE FORM ELLIPTIC PDES* 

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#### Abstract

This paper develops interior penalty discontinuous Galerkin (IP-DG) methods to approximate $W^{2, p}$ strong solutions of second order linear elliptic partial differential equations (PDEs) in non-divergence form with continuous coefficients. The proposed IP-DG methods are closely related to the IP-DG methods for advection-diffusion equations, and they are easy to implement on existing standard IP-DG software platforms. It is proved that the proposed IP-DG methods have unique solutions and converge with optimal rate to the $W^{2, p}$ strong solution in a discrete $W^{2, p}$-norm. The crux of the analysis is to establish a DG discrete counterpart of the Calderon-Zygmund estimate and to adapt a freezing coefficient technique used for the PDE analysis at the discrete level. As a byproduct of our analysis, we also establish broken $W^{1, p}$-norm error estimates for IP-DG approximations of constant coefficient elliptic PDEs. Numerical experiments are provided to gauge the performance of the proposed IP-DG methods and to validate the theoretical convergence results.


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1. Introduction. In this paper we develop interior penalty discontinuous Galerkin methods for approximating the $W^{2, p}$ strong solution to the following second order linear elliptic PDE in non-divergence form:

$$
\begin{align*}
\mathcal{L} u(x):=-A(x): D^{2} u(x) & =f(x) & & x \in \Omega  \tag{1.1a}\\
u(x) & =0 & & x \in \partial \Omega \tag{1.1b}
\end{align*}
$$

where $\Omega \subset \mathbb{R}^{n}$ is an open, bounded domain with boundary $\partial \Omega, f \in L^{p}(\Omega)$ with $1<p<\infty$, and $A \in\left[C^{0}(\bar{\Omega})\right]^{n \times n}$ is positive definite in $\bar{\Omega}$.

Non-divergence form elliptic PDEs arrive naturally from many applications such as stochastic optimal control and game theory [12]; they are also encountered in the linearization of fully nonlinear PDEs such as Monge-Ampère-type equations [5]. If $A$ is differentiable, then it is easy to check that equation (1.1a) can be rewritten as a diffusion-convection equation with $A$ as the diffusion coefficient and $(\nabla \cdot A)$ as the convection coefficient. However, if $A \in\left[C^{0}(\bar{\Omega})\right]^{n \times n}$, then this formulation is not possible since $(\nabla \cdot A)$ does not exist as a function, but rather only as a measure. We recall that in the literature there are three well-established PDE theories for nondivergence form elliptic PDEs depending on the smoothness of $A$ and $f$ (as well as the smoothness of the boundary $\partial \Omega$ which we assume sufficiently smooth here to ease the presentation). The first theory is the classical solution (or Schauder's) theory [13, Chapter 6] which seeks solutions in the Hölder space $C^{2, \alpha}(\bar{\Omega})$ for $0<\alpha<1$ when $A \in\left[C^{\alpha}(\bar{\Omega})\right]^{n \times n}$ and $f \in C^{\alpha}(\bar{\Omega})$. The second one is the $W^{2, p}$ (strong) solution theory [13, Chapter 9] which seeks solutions in the Sobolev space $W^{2, p}(\Omega)$ for $1<p<\infty$ that satisfy the PDE almost everywhere in $\Omega$ under the assumptions that $A \in\left[C^{0}(\bar{\Omega})\right]^{n \times n}$

[^0]and $f \in L^{p}(\Omega)$. If the coefficient matrix satisfies the Cordes condition and if the domain is convex, then the notion of strong solutions may be extended to the case $A \in\left[L^{\infty}(\Omega)\right]^{n \times n}[26,16]$. The third one is the viscosity (weak) solution theory [8] which seeks solutions in $C^{0}(\Omega)$ (or even in $B(\Omega)$, the space of bounded functions in $\Omega$ ) that satisfy the PDE in the viscosity sense under the assumptions that $A \in$ $\left[L^{\infty}(\Omega)\right]^{n \times n}$ and $f \in L^{\infty}(\Omega)$. We note that all the three solution concepts and PDE theories are non-variational; this is a main difference between divergence form PDEs [13, Chapter 8] and non-divergence form PDEs. We also note that in the case of the viscosity solution theory, the uniqueness and regularity of solutions had been the main focus of the study for second order non-divergence form PDEs (see [8, 26, 17, 23] and the references therein).

In contrast to the advances of the PDE analysis, almost no progress on numerical methods and numerical analysis was achieved until very recently for second order elliptic PDEs in non-divergence form with non-differentiable coefficient matrix $A$ (cf. $[9,11,15,18,20,26,28])$. The main difficulty is caused by the non-divergence structure of the PDEs which prevents any straightforward application of Galerkintype numerical methodologies such as finite element methods, discontinuous Galerkin methods and spectral methods. Moreover, the non-variational nature of the strong and viscosity solution concepts make convergence analysis and/or error estimates of any convergent numerical methods very delicate and difficult. Nonstandard numerical techniques are often required to do the job (cf. [11, 18, 20, 26, 28]).

The primary goal of this paper is to develop convergent interior penalty discontinuous Galerkin (IP-DG) methods for approximating the $W^{2, p}$ strong solution of problem (1.1) under the assumption that $A \in\left[C^{0}(\bar{\Omega})\right]^{n \times n}$ and its solution satisfies the Calderon-Zygmund estimate [13, Chapter 9]. This paper can be viewed as a DG counterpart of [11] where non-standard convergent finite element methods were proposed and analyzed for approximating the $W^{2, p}$ strong solution of (1.1). The reason for undertaking such an extension is twofold. First, we intend to take advantage of some of the features of DG methods such as simplicity and ease of computation, and flexibility of mesh and ease for adaptivity, to design better numerical methods for problem (1.1a)-(1.1b); in the meantime, we develop new numerical analysis techniques and machineries for non-divergence form PDEs. Second, since the jumps of the normal derivatives across element edges and mesh-dependent bilinear forms are already used in the finite element methods of [11], it is natural to use totally discontinuous piecewise polynomial approximation spaces to explore the full potential of the interior penalty technique and idea. It should be noted that although the generalization of the finite element formulations of [11] to the DG case is not very hard, the DG convergence analysis is more involved because of the extra difficulty caused by the non-conformity of the DG finite element space. The crux of the analysis of this paper is to establish a discrete Calderon-Zygmund estimate for the proposed IP-DG methods and to adapt a freezing coefficient technique used in the PDE analysis at the discrete level. Moreover, in order to prove the desired discrete Calderon-Zygmund estimate, we need to establish the $W_{h}^{1, p}$ stability and error estimates for the IP-DG approximations of constant coefficient elliptic PDEs. Such estimates seem to be new and have independent interest.

The remainder of this paper is organized as follows. In Section 2 we provide the notation and a collection of preliminary estimates, in particular, some properties of DG functions are either cited or proved. Section 3 analyzes the IP-DG methods for constant coefficient elliptic PDEs. The $W_{h}^{1, p}$ stability and error estimates are derived,
which in turn lead to global $W_{h}^{2, p}$ stability estimates. The latter estimate can be regarded as a discrete Calderon-Zygmund estimate for the proposed IP-DG methods. To the best of our knowledge, such an estimate is new and of independent interest. Section 4 is devoted to the formulation, stability and convergence analysis and error estimate for the proposed IP-DG methods. Here, using the continuity of the coefficient matrix $A$, the freezing coefficient technique is adapted to establish local stability (or a left-side inf-sup condition) for the IP-DG discrete operators, which together with a covering argument leads to a global Gärding-type inequality for the formal adjoint operators of the IP-DG operators. Next, using a duality argument we obtain a global left-side inf-sup condition for the IP-DG discrete operators. Finally, by employing another duality argument, we derive the desired discrete Calderon-Zygmund estimate for the DG discrete operators. Once this stability estimate is shown, well-posedness and convergence of the IP-DG methods follow easily. In Section 5 we present a number of numerical experiments to verify our theoretical results and to gauge the performance of the proposed IP-DG methods, even for the case $A \in\left[L^{\infty}(\bar{\Omega})\right]^{n \times n}$ which is not covered by our convergence theory.

## 2. Preliminary Results.

2.1. Notation. Let $\Omega$ be an open and bounded domain in $\mathbb{R}^{n}$. For a subdomain $D$ of $\Omega$ with boundary $\partial D$, let $L^{p}(D)$ and $W^{s, p}(D)$ for $s \geq 0$ and $1 \leq p \leq \infty$ denote the standard Lebesgue and Sobolev spaces respectively, and $W_{0}^{1, p}(D)$ be the closure of $C_{c}^{\infty}(D)$ in $W^{1, p}(D)$. Let $(\cdot, \cdot)_{D}$ be the $L^{2}$ inner product on $D$ and $(\cdot, \cdot):=(\cdot, \cdot)_{\Omega}$. To improve the readability of the paper, we adopt the convention that $a \lesssim b$ stands for $a \leq C b$ for some $C>0$ which does not depend on any discretization parameters.

Let $\mathcal{T}_{h}$ be a shape-regular and conforming triangulation of $\Omega$. Let $\mathcal{E}_{h}^{I}$ and $\mathcal{E}_{h}^{B}$ denote respectively the sets of all interior and boundary edges/faces of $\mathcal{T}_{h}$, and set $\mathcal{E}_{h}:=\mathcal{E}_{h}^{I} \cup \mathcal{E}_{h}^{B}$. We introduce the broken Sobolev spaces

$$
\begin{array}{rlrl}
W^{s, p}\left(\mathcal{T}_{h}\right): & =\prod_{T \in \mathcal{T}_{h}} W^{s, p}(T), & L^{p}\left(\mathcal{T}_{h}\right):=W^{0, p}\left(\mathcal{T}_{h}\right) \\
W_{h}^{s, p}(D):=\left.W^{s, p}\left(\mathcal{T}_{h}\right)\right|_{D}, & L_{h}^{p}(D):=\left.L^{p}\left(\mathcal{T}_{h}\right)\right|_{D}
\end{array}
$$

For any interior edge/face $e=\partial T^{+} \cap \partial T^{-} \in \mathcal{E}_{h}^{I}$ we define the jump and average of a scalar or vector valued function $v$ as

$$
\left.[v]\right|_{e}:=v^{+}-v^{-},\left.\quad\{v\}\right|_{e}:=\frac{1}{2}\left(v^{+}+v^{-}\right)
$$

where $v^{ \pm}=\left.v\right|_{T^{ \pm}}$. On a boundary edge/face $e \in \mathcal{E}_{h}^{B}$ with $e=\partial T^{+} \cap \partial \Omega$, we set $\left.[v]\right|_{e}=\left.\{v\}\right|_{e}=v^{+}$. For any $e \in \mathcal{E}_{h}^{I}$ we use $\nu_{e}$ to denote the unit outward normal vector pointing in the direction of the element with the smaller global index. For $e \in \mathcal{E}_{h}^{B}$ we set $\nu_{e}$ to be the outward normal to $\partial \Omega$ restricted to $e$. The standard DG finite element space is defined as

$$
V_{h}:=\left\{v_{h} \in W^{2, p}\left(\mathcal{T}_{h}\right) ;\left.v_{h}\right|_{T} \in \mathbb{P}_{k}(T) \quad \forall T \in \mathcal{T}_{h}\right\}
$$

where $\mathbb{P}_{k}(T)$ denotes the set of all polynomials of degree less than or equal to $k$ on $T$. We also introduce for any $D \subset \Omega$

$$
V_{h}(D):=\left\{v \in V_{h} ;\left.v\right|_{\Omega \backslash \bar{D}} \equiv 0\right\} .
$$

Note that $V_{h}(D)$ is nontrivial provided that there exists an inscribed ball $B$ with radius $r \geq h$ such that $B \subset D$. We also adopt the convention $V_{h}(\Omega)=V_{h}$.

For each $e \in \mathcal{E}_{h}$, let $\gamma_{e}>0$ be constant on $e$. We define the following meshdependent norms on $W_{h}^{1, p}(D)$ and $W_{h}^{2, p}(D)$ :

$$
\begin{align*}
\|v\|_{W_{h}^{2, p}(D)}:= & \left\|D_{h}^{2} v\right\|_{L^{p}(D)}+\left(\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-p}\| \|[\nabla v]\| \|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}}  \tag{2.1}\\
& +\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{1-2 p}\|[v]\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}}, \\
\|v\|_{W_{h}^{1, p}(D)}:= & \left\|\nabla_{h} v\right\|_{L^{p}(D)}+\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{1-p}\|[v]\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}}  \tag{2.2}\\
& +\left(\sum_{e \in \mathcal{E}_{h}} h_{e}\|\{\nabla v\}\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}},
\end{align*}
$$

where $\nabla_{h} v$ and $D_{h}^{2} v$ denote the piecewise gradient and Hessian of $v$.
In addition, we define the discrete $W_{h}^{-2, p}$-norm and $W_{h}^{-1, p}$-norm as follows:

$$
\begin{align*}
& \|q\|_{W_{h}^{-2, p}(D)}:=\sup _{0 \neq v_{h} \in V_{h}} \frac{\left(q, v_{h}\right)_{D}}{\left\|v_{h}\right\|_{W_{h}^{2, p^{\prime}}(D)}}  \tag{2.3}\\
& \|q\|_{W_{h}^{-1, p}(D)}:=\sup _{0 \neq v \in W_{h}^{1, p^{\prime}}(D)} \frac{(q, v)_{D}}{\|v\|_{W_{h}^{1, p^{\prime}}(D)}} \tag{2.4}
\end{align*}
$$

where $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. Finally, for any domain $D \subseteq \Omega$ and any $w \in L_{h}^{p}(D)$, we introduce the following mesh-dependent semi-norm

$$
\begin{equation*}
\|w\|_{L_{h}^{p}(D)}:=\sup _{0 \neq v_{h} \in V_{h}(D)} \frac{\left(w, v_{h}\right)_{D}}{\left\|v_{h}\right\|_{L^{p^{\prime}}(D)}} . \tag{2.5}
\end{equation*}
$$

It can be proved that (cf. [11])

$$
\begin{equation*}
\left\|w_{h}\right\|_{L^{p}(\Omega)} \lesssim\left\|w_{h}\right\|_{L_{h}^{p}(\Omega)} \quad \forall w_{h} \in V_{h} \tag{2.6}
\end{equation*}
$$

2.2. Properties of the DG space $V_{h}$. In this subsection we collect some technical lemmas that cover the basic properties of functions in the DG space $V_{h}$. These facts will be used many times in the later sections. We first state the standard trace inequalities for broken Sobolev functions, a proof of this lemma can be found in [3].

Lemma 2.1. For any $T \in \mathcal{T}_{h}$ and $1<p<\infty$, there holds

$$
\begin{equation*}
\|v\|_{L^{p}(\partial T)}^{p} \lesssim\left(h_{T}^{p-1}\|\nabla v\|_{L^{p}(T)}^{p}+h_{T}^{-1}\|v\|_{L^{p}(T)}^{p}\right) \quad \forall v \in W^{1, p}(T) \tag{2.7}
\end{equation*}
$$

Therefore by scaling we have

$$
\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}\|v\|_{L^{p}(e \cap \bar{D})}^{p} \lesssim \begin{cases}\|v\|_{L^{p}(D)}^{p} & \forall v \in V_{h}(D)  \tag{2.8}\\ \|v\|_{L^{p}(D)}^{p}+h^{p}\|\nabla v\|_{L^{p}(D)}^{p} & \forall v \in W_{h}^{2, p}(D)\end{cases}
$$

Next, we prove an inverse inequality between the $W_{h}^{2, p}$-norm and the $W_{h}^{1, p}$-norm.

Lemma 2.2. For any $v_{h} \in V_{h}, D \subseteq \Omega$, there holds for $1<p<\infty$

$$
\begin{equation*}
\left\|v_{h}\right\|_{W_{h}^{2, p}(D)} \lesssim h^{-1}\left\|v_{h}\right\|_{W_{h}^{1, p}\left(D_{h}\right)} \tag{2.9}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{h}=\{x \in \Omega ; \operatorname{dist}(x, D) \leq h\} \tag{2.10}
\end{equation*}
$$

Proof. To show (2.9), we use (2.1), (2.7), and standard inverse estimates [3] to obtain

$$
\begin{aligned}
&\left\|v_{h}\right\|_{W_{h}^{2, p}(D)} \lesssim\left\|D_{h}^{2} v_{h}\right\|_{L^{p}(D)}+\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{1-2 p}\|[v]\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}} \\
&+\sum_{\substack{T \in \mathcal{T}_{h} \\
T \subset D_{h}}}\left(h_{T}^{1-p}\left(h_{T}^{p-1}\left\|D^{2} v_{h}\right\|_{L^{p}(T)}^{p}+h_{T}^{-1}\left\|\nabla v_{h}\right\|_{L^{p}(T)}^{p}\right)\right)^{\frac{1}{p}} \\
& \lesssim\left\|D_{h}^{2} v_{h}\right\|_{L^{p}(D)}+\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{-p} h_{e}^{1-p}\|[v]\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}} \\
&+\sum_{\substack{T \in \mathcal{T}_{h}}}\left(\left\|D^{2} v_{h}\right\|_{L^{p}(T)}^{p}+h_{T}^{-p}\left\|\nabla v_{h}\right\|_{L^{p}(T)}^{p}\right)^{\frac{1}{p}} \\
& \lesssim h^{-1}\left\|v_{h}\right\|_{W_{h}^{1, p}\left(D_{h}\right)}+h^{-1}\left(\sum_{e \in \mathcal{E}_{h}^{I}} \gamma_{e}^{p} h_{e}^{1-p}\|[v]\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{\frac{1}{p}} \\
& \lesssim h^{-1}\left\|v_{h}\right\|_{W_{h}^{1, p}\left(D_{h}\right)} .
\end{aligned}
$$

We also prove an inverse inequality between the $L^{p}$-norm and the $W_{h}^{-1, p}$-norm. Lemma 2.3. Let $v \in V_{h}(D)$. For any $1<p<\infty$ and subdomain $D \subset \Omega$ we have

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p}(D)} \lesssim h^{-1}\left\|v_{h}\right\|_{W_{h}^{-1, p}(D)} \tag{2.11}
\end{equation*}
$$

Proof. Using the relation (2.6) and the definition of $\|\cdot\|_{L_{h}^{p}(\Omega)}$, we find that

$$
\left\|v_{h}\right\|_{L^{p}(D)} \leq\left\|v_{h}\right\|_{L^{p}(\Omega)} \lesssim\left\|v_{h}\right\|_{L_{h}^{p}(\Omega)}=\sup _{0 \neq w_{h} \in V_{h}} \frac{\left(v_{h}, w_{h}\right)_{D}}{\left\|w_{h}\right\|_{L^{p^{\prime}}(\Omega)}} \quad \forall v_{h} \in V_{h}(D)
$$

Therefore, by the standard inverse estimate $h\left\|w_{h}\right\|_{W_{h}^{1, p^{\prime}}(D)} \leq h\left\|w_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)} \lesssim\left\|w_{h}\right\|_{L^{p^{\prime}}(\Omega)}$ and noting that $V_{h}(D) \subset W_{h}^{1, p^{\prime}}(D)$, we obtain
$\left\|v_{h}\right\|_{L^{p}(D)} \lesssim h^{-1} \sup _{0 \neq w_{h} \in V_{h}} \frac{\left(v_{h}, w_{h}\right)_{D}}{\left\|w_{h}\right\|_{W_{h}^{1, p^{\prime}}(D)}} \leq h^{-1} \sup _{0 \neq w \in W_{h}^{1, p^{\prime}(D)}} \frac{\left(v_{h}, w\right)_{D}}{\|w\|_{W_{h}^{1, p^{\prime}}(D)}}=\left\|v_{h}\right\|_{W_{h}^{-1, p}(D)}$.
The proof is complete.
The following lemma shows that the broken Sobolev norms are controlled by their corresponding Sobolev norms.

Lemma 2.4. For any $1<p<\infty$ there holds the following inequality:

$$
\|\varphi\|_{W_{h}^{2, p}(\Omega)} \leq\|\varphi\|_{W^{2, p}(\Omega)} \quad \forall \varphi \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega) .
$$

Proof. Since the inequality holds for all $\varphi \in C^{\infty}(\Omega) \cap W_{0}^{1, p}(\Omega)$ and it can be extended to all $\varphi \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)$ by a density argument.

The next lemma establishes a Poincaré-Friedrichs' inequality for DG functions.
Lemma 2.5. Let $D \subset \Omega$ such that $V_{h}(D) \neq\{0\}$ and $\operatorname{diam}(D) \geq h$. Then for any $v_{h} \in V_{h}(D)$ there holds the following inequalities:

$$
\begin{align*}
\left\|v_{h}\right\|_{L^{p}(D)} & \lesssim \operatorname{diam}(D)\left\|v_{h}\right\|_{W_{h}^{1, p}(D)},  \tag{2.12}\\
\left\|v_{h}\right\|_{W_{h}^{1, p}(D)} & \lesssim \operatorname{diam}(D)\left\|v_{h}\right\|_{W_{h}^{2, p}(D)} . \tag{2.13}
\end{align*}
$$

Proof. Let $\tilde{V}_{h}$ denote the generalized Hsiegh-Clough-Tochner space [10], and let $E_{h}: V_{h} \rightarrow \tilde{V}_{h}$ be the reconstruction operator constructed in [14]. The arguments given in [14] show that, for $v_{h} \in V_{h}(D)$,

$$
\begin{align*}
& E_{h} v_{h} \in H_{0}^{2}\left(D_{h}\right)  \tag{2.14}\\
& \left\|v_{h}-E_{h} v_{h}\right\|_{L^{p}(\Omega)} \lesssim h\left\|v_{h}\right\|_{W_{h}^{1, p}(D)} \\
& \left\|v_{h}-E_{h} v_{h}\right\|_{W_{h}^{m, p}(\Omega)} \lesssim h^{s-m}\left\|v_{h}\right\|_{W_{h}^{s, p}(D)}, \quad 1 \leq m \leq s \leq 2
\end{align*}
$$

where $D_{h}$ is the same as in Lemma 2.2. Therefore, by the triangle inequality, the Poincarè-Friedrichs inequality, and the assumption $\operatorname{diam}(D) \geq h$,

$$
\begin{aligned}
\left\|v_{h}\right\|_{L^{p}(D)} & \leq\left\|E_{h} v_{h}\right\|_{L^{p}(D)}+\left\|v_{h}-E_{h} v_{h}\right\|_{L^{p}(\Omega)} \\
& \lesssim \operatorname{diam}(D)\left\|E_{h} v_{h}\right\|_{W^{1, p}\left(D_{h}\right)}+h\left\|v_{h}\right\|_{W_{h}^{1, p}(D)} \lesssim \operatorname{diam}(D)\left\|v_{h}\right\|_{W_{h}^{1, p}(D)} .
\end{aligned}
$$

Likewise, we find

$$
\begin{aligned}
\left\|v_{h}\right\|_{W_{h}^{1, p}(D)} & \leq\left\|E_{h} v_{h}\right\|_{W^{1, p}\left(D_{h}\right)}+\left\|v_{h}-E_{h} v_{h}\right\|_{W_{h}^{1, p}(\Omega)} \\
& \lesssim \operatorname{diam}(D)\left\|E_{h} v_{h}\right\|_{W^{2, p}\left(D_{h}\right)}+h\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim \operatorname{diam}(D)\left\|v_{h}\right\|_{W_{h}^{2, p}(D)}
\end{aligned}
$$

The proof is complete.
Next we establish a discrete Sobolev interpolation estimate for DG functions.
Lemma 2.6. Let $1<p<\infty$. For all $v_{h} \in V_{h}$ we have

$$
\begin{equation*}
\left\|v_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{2} \lesssim\left\|v_{h}\right\|_{L^{p}(\Omega)}\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)} . \tag{2.15}
\end{equation*}
$$

Proof. Let $E_{h}: V_{h} \rightarrow \tilde{V}_{h}$ be the enriching operator in the proof of Lemma 2.5. By the triangle inequality and scaling we find

$$
\begin{equation*}
\left\|v_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{2} \lesssim\left\|v_{h}-E_{h} v_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{2}+\left\|E_{h} v_{h}\right\|_{W^{1, p}(\Omega)}^{2} \tag{2.16}
\end{equation*}
$$

Since $E_{h} v_{h} \in W^{2, p}(\Omega)$ we can apply the Gagliardo-Nirenberg estimate [4] to get

$$
\left\|E_{h} v_{h}\right\|_{W^{1, p}(\Omega)}^{2} \lesssim\left\|E_{h} v_{h}\right\|_{W^{2, p}(\Omega)}\left\|E v_{h}\right\|_{L^{p}(\Omega)}
$$

Applying estimates (2.14), we conclude that

$$
\begin{equation*}
\left\|E_{h} v_{h}\right\|_{W^{1, p}(\Omega)}^{2} \lesssim\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)}\left\|v_{h}\right\|_{L^{p}(\Omega)} \tag{2.17}
\end{equation*}
$$

Likewise, by (2.14) and an inverse estimate,

$$
\begin{equation*}
\left\|v_{h}-E_{h} v_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{2} \lesssim h^{2}\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)}^{2} \lesssim\left\|v_{h}\right\|_{L^{p}(\Omega)}\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)} \tag{2.18}
\end{equation*}
$$

Combining (2.16)-(2.18) completes the proof.
Next we prove some local super approximation estimates for the DG nodal interpolation in various discrete norms. The derivation of the lemma is standard (cf. [19]); for completeness we provide the proof in the Appendix.

LEMMA 2.7. Let $I_{h}: C^{0}\left(\mathcal{T}_{h}\right):=\Pi_{T \in \mathcal{T}_{h}} C^{0}(\bar{T}) \rightarrow V_{h}$ denote the nodal interpolation operator, and $\eta \in C^{\infty}(\Omega)$ with $|\eta|_{W^{j, \infty}(\Omega)} \lesssim d^{-j}$ for $0 \leq j \leq k$. Then for any $v_{h} \in V_{h}$ and $D \subseteq \Omega$ we have

$$
\begin{align*}
\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p}(D)} & \lesssim \frac{h}{d}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)}  \tag{2.19}\\
h\left\|\nabla_{h}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(D)} & \lesssim \frac{h}{d}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)},  \tag{2.20}\\
h^{2}\left\|D_{h}^{2}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(D)} & \lesssim \frac{h}{d}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)},  \tag{2.21}\\
\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{W_{h}^{2, p}(D)} & \lesssim \frac{1}{d^{2}}\left(\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)}+\left\|\nabla_{h} v_{h}\right\|_{L^{p}\left(D_{h}\right)}\right), \tag{2.22}
\end{align*}
$$

where $D_{h}$ is the same as in Lemma 2.2. Moreover, there holds

$$
\begin{equation*}
\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{W_{h}^{2, p}(D)} \lesssim \frac{h}{d^{3}}\left\|v_{h}\right\|_{W_{h}^{2, p}\left(D_{h}\right)} \tag{2.23}
\end{equation*}
$$

if the polynomial degree $k \geq 2$.
3. DG discrete $W^{1, p}$ and Calderon-Zygmund estimates for PDEs with constant coefficients. In this section we consider the constant coefficient case, that is, $A(x) \equiv A_{0} \in \mathbb{R}^{n \times n}$ on $\Omega$. We define three interior-penalty discontinuous Galerkin discretizations $\mathcal{L}_{0, h}^{\varepsilon}$ to the PDE operator $\mathcal{L}$ and extend their domains to the broken Sobolev space $W^{2, p}\left(\mathcal{T}_{h}\right)$. Our goal in this subsection is to prove global stability estimates for $\mathcal{L}_{0, h}^{\varepsilon}$ which will be crucially used in the next section. The final global stability estimate given in Theorem 3.6 can be regarded as a DG discrete CalderonZygmund estimate for $\mathcal{L}_{0, h}^{\varepsilon}$.

Let $A_{0}$ be a constant, positive-definite matrix in $\mathbb{R}^{n \times n}$ and define

$$
\begin{equation*}
\mathcal{L}_{0} w:=-A_{0}: D^{2} w=-\nabla \cdot\left(A_{0} \nabla w\right) \tag{3.1}
\end{equation*}
$$

From this we gather the standard PDE weak form:

$$
\begin{equation*}
a_{0}(w, v):=\int_{\Omega} A_{0} \nabla w \cdot \nabla v \mathrm{~d} x \quad \forall w, v \in H_{0}^{1}(\Omega) \tag{3.2}
\end{equation*}
$$

The Lax-Milgram theorem [3] yields the existence and boundedness of $\mathcal{L}_{0}^{-1}: H^{-1}(\Omega) \rightarrow$ $H_{0}^{1}(\Omega)$. Moreover if $\partial \Omega \in C^{1,1}$ we have from Calderon-Zygmund theory [13] that $\mathcal{L}_{0}^{-1}: L^{p}(\Omega) \rightarrow W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)$ exists and

$$
\left\|\mathcal{L}_{0}^{-1} \varphi\right\|_{W^{2, p}(\Omega)} \lesssim\|\varphi\|_{L^{p}(\Omega)} \quad \forall \varphi \in L^{p}(\Omega)
$$

and therefore

$$
\|w\|_{W^{2, p}(\Omega)} \lesssim\left\|\mathcal{L}_{0} w\right\|_{L^{p}(\Omega)} \quad \forall w \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)
$$

Define $\mathcal{L}_{0, h}^{\varepsilon}: V_{h} \rightarrow V_{h}$ by

$$
\begin{equation*}
\left(\mathcal{L}_{0, h}^{\varepsilon} w_{h}, v_{h}\right):=a_{0, h}^{\varepsilon}\left(w_{h}, v_{h}\right) \quad \forall v_{h}, w_{h} \in V_{h} \tag{3.3}
\end{equation*}
$$

where the IP-DG bilinear form is defined by

$$
\begin{align*}
a_{0, h}\left(w_{h}, v_{h}\right):= & \int_{\Omega} A_{0} \nabla_{h} w_{h} \cdot \nabla_{h} v_{h} \mathrm{~d} x-\sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A_{0} \nabla w_{h} \cdot \nu_{e}\right\}\left[v_{h}\right] \mathrm{d} S  \tag{3.4}\\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A_{0} \nabla v_{h} \cdot \nu_{e}\right\}\left[w_{h}\right] \mathrm{d} S+\sum_{e \in \mathcal{E}_{h}} \int_{e} \frac{\gamma_{e}}{h_{e}}\left[w_{h}\right]\left[v_{h}\right] \mathrm{d} S
\end{align*}
$$

and $\gamma_{e}>0$ is a penalization parameter. The parameter choices $\varepsilon \in\{1,0,-1\}$ give respectively the SIP-DG, IIP- DG, and NIP-DG formulations. For the sake of clarity and readability we shall assume for the rest of the paper that $\varepsilon$ may be either 1,0 , or -1 unless otherwise stated.

We recall the following well-known DG integration by parts formula:

$$
\begin{equation*}
\int_{\Omega} \tau \cdot \nabla_{h} v \mathrm{~d} x=-\int_{\Omega}\left(\nabla_{h} \cdot \tau\right) v \mathrm{~d} x+\sum_{e \in \mathcal{E}_{h}^{I}} \int_{e}\left[\tau \cdot \nu_{e}\right]\{v\} \mathrm{d} S+\sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{\tau \cdot \nu_{e}\right\}[v] \mathrm{d} S \tag{3.5}
\end{equation*}
$$

which holds for any piecewise scalar-valued function $v$ and vector-valued function $\tau$. Applying (3.5) to the first term on the right-hand side of (3.4) yields

$$
\begin{align*}
a_{0, h}^{\varepsilon}\left(w_{h}, v_{h}\right)= & -\int_{\Omega}\left(A_{0}: D_{h}^{2} w_{h}\right) v_{h} \mathrm{~d} x+\sum_{e \in \mathcal{E}_{h}^{I}} \int_{e}\left[A_{0} \nabla w_{h} \cdot \nu_{e}\right]\left\{v_{h}\right\} \mathrm{d} S  \tag{3.6}\\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A_{0} \nabla v_{h} \cdot \nu_{e}\right\}\left[w_{h}\right] \mathrm{d} S+\sum_{e \in \mathcal{E}_{h}} \int_{e} \frac{\gamma_{e}}{h_{e}}\left[w_{h}\right]\left[v_{h}\right] \mathrm{d} S
\end{align*}
$$

for any $w_{h}, v_{h} \in V_{h}$. By Hölder's inequality, it is easy to check that the above new form of $a_{0, h}^{\varepsilon}(\cdot, \cdot)$ is also well-defined on $W^{2, p}\left(\mathcal{T}_{h}\right) \times W^{2, p^{\prime}}\left(\mathcal{T}_{h}\right)$ with $\frac{1}{p}+\frac{1}{p^{\prime}}=1$. As a result, this new form enables us to extend the domain of $a_{0, h}^{\varepsilon}(\cdot, \cdot)$ to $W^{2, p}\left(\mathcal{T}_{h}\right) \times W^{2, p^{\prime}}\left(\mathcal{T}_{h}\right)$ and $\mathcal{L}_{0, h}^{\varepsilon}: W^{2, p}\left(\mathcal{T}_{h}\right) \rightarrow\left(W^{2, p}\left(\mathcal{T}_{h}\right)\right)^{*}$.
3.1. DG discrete $W^{1, p}$ error estimates. From the standard IP-DG theory [22], there exists $\gamma^{*}=\gamma^{*}\left(\left\|A_{0}\right\|_{L^{\infty}(\Omega)}, \mathcal{T}_{h}\right)>0$ depending only on the shape regularity of the mesh and on $\left\|A_{0}\right\|_{L^{\infty}(\Omega)}$ such that $\mathcal{L}_{0, h}^{\varepsilon}$ is invertible on $V_{h}$ provided $\gamma_{e} \geq \gamma^{*}$; in the non-symmetric case $\varepsilon=-1, \gamma^{*}$ can be any positive number. Moreover, if $w \in W^{2,2}\left(\mathcal{T}_{h}\right) \cap H_{0}^{1}(\Omega)$ and $w_{h} \in V_{h}$ satisfy

$$
\begin{equation*}
a_{0, h}^{\varepsilon}\left(w-w_{h}, v_{h}\right)=0 \quad \forall v_{h} \in V_{h} \tag{3.7}
\end{equation*}
$$

then the quasi-optimal error estimate

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{W_{h}^{1,2}(\Omega)} \lesssim \inf _{v_{h} \in V_{h}}\left\|w-v_{h}\right\|_{W_{h}^{1,2}(\Omega)} \tag{3.8}
\end{equation*}
$$

is satisfied. The goal of this subsection is to generalize this result to general exponent $p \in(1, \infty)$ for the SIP-DG method. In particular, we have

THEOREM 3.1. Suppose $w \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)(1<p<\infty)$ and $w_{h} \in V_{h}$ satisfy (3.7) with $\varepsilon=1$. Then there holds

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)} \lesssim h|\log h|^{t}\|w\|_{W^{2, p}(\Omega)} \tag{3.9}
\end{equation*}
$$

where $t=(p+1) / p$ if $k=1$ and $t=0$ if $k \geq 2$.
To prove Theorem 3.1 we introduce some notation given in [6] (also see [25]). For given $z \in \bar{\Omega}$, we define the weight function $\sigma_{z}$ as

$$
\begin{equation*}
\sigma_{z}(x)=\frac{h}{|x-z|+h} . \tag{3.10}
\end{equation*}
$$

For $p \in[1, \infty)$, and $s \in \mathbb{R}$, we define the following weighted norms

$$
\begin{align*}
\|v\|_{L^{p}(D), z, s}= & \left(\int_{D}\left|\sigma_{z}^{s}(x) v(x)\right|^{p} \mathrm{~d} x\right)^{1 / p} \\
\|v\|_{W^{1, p}(D), z, s}= & \|v\|_{L^{p}(D), z, s}+\left\|\nabla_{h} v\right\|_{L^{p}(D), z, s} \\
\|v\|_{W_{h}^{1, p}(D), z, s}= & \|v\|_{W^{1, p}(D), z, s}+\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\sigma_{z}^{s}[v]\right\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{1 / p}  \tag{3.11}\\
& +\left(\sum_{e \in \mathcal{E}_{h}} h_{e}\left\|\sigma_{z}^{s}\left\{\nabla_{h} v\right\}\right\|_{L^{p}(e \cap \bar{D})}^{p}\right)^{1 / p}
\end{align*}
$$

The weighted norms in the case $p=\infty$ are defined analogously.
The derivation of $W^{1, p}$ error estimates of DG approximations is based on the work [6], where localized pointwise estimates of DG approximations are obtained. There it was shown that if $w \in W^{2, \infty}(\Omega)$ and $w_{h} \in V_{h}$ satisfy (3.7) with $\varepsilon=1$, then

$$
\begin{equation*}
\left|\nabla\left(w-w_{h}\right)(z)\right| \lesssim \inf _{v_{h} \in V_{h}}\left\|w-v_{h}\right\|_{W_{h}^{1, \infty}(\Omega), z, s} \quad 0 \leq s<k \tag{3.12}
\end{equation*}
$$

for all $z \in \bar{\Omega}$. Similar to pointwise estimates of finite element approximations (e.g., [25, $3]$ ), the ingredients to prove (3.12) include duality arguments and DG approximation estimates of regularized Green functions in a weighted (discrete) $W^{1,1}$-norm. These results are rather technical and involve dyadic decompositions of $\Omega$, local DG error estimates, and Green function estimates.

Here, we follow a similar argument to derive $W^{1, p}$ estimates; the main difference being that we derive DG approximation estimates of regularized Green functions in a weighted (discrete) $W^{1, p^{\prime}}$-norm with $1 / p+1 / p^{\prime}=1$ (cf. Lemma 3.4). Using these estimates and applying similar arguments in $[25,6]$ then yield the estimate

$$
\left|\nabla\left(w-w_{h}\right)(z)\right|^{p} \lesssim h^{-n} \inf _{v_{h} \in V_{h}}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}^{p}
$$

for certain values of $s$. Integrating this expression with respect to $z$ and applying Fubini's theorem (cf. Lemma 3.2) then yields $L^{p}$ estimates of the piecewise gradient error.

Unfortunately, the strategy just described does not immediately give us estimates for the terms $h_{e}^{1-p}\left\|\left[w-w_{h}\right]\right\|_{L^{p}(e)}^{p}$ appearing in the $W_{h}^{1, p}$-norm. To bypass this difficulty, we first use the trace inequality

$$
\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\left[w-w_{h}\right]\right\|_{L^{p}(e)}^{p} \lesssim\left\|\nabla_{h}\left(w-w_{h}\right)\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p}
$$

and then derive estimates for $h^{-p}\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p}$. We note that the standard duality argument to derive $L^{p}$ estimates yields

$$
\left\|w-w_{h}\right\|_{L^{p}(\Omega)} \lesssim h\left\|w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)}
$$

which is of little benefit. Rather, our strategy is to modify the arguments given in [6, Theorem 5.1] and estimate $\left|\left(w-w_{h}\right)(z)\right|$ in terms of $\inf _{v_{h} \in V_{h}}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}$ (cf. Lemma 3.3) and then apply Fubini's theorem. We note that it is due to this term that the $|\log h|^{t}$ factor appears in Theorem 3.1.

Lemma 3.2. Let $p \in[2, \infty)$ and $v \in L^{p}(\Omega)$. Let $z \in \Omega$ and $T_{z} \in \mathcal{T}_{h}$ such that $z \in T_{z}$. Then there holds

$$
\begin{equation*}
\int_{\Omega} \int_{T_{z}}|v(x)|^{p} \mathrm{~d} x \mathrm{~d} z \lesssim h^{n}\|v\|_{L^{p}(\Omega)}^{p} \tag{3.13}
\end{equation*}
$$

Moreover for any $s>n / p$ and $w \in W^{2, p}\left(\mathcal{T}_{h}\right)$, there holds

$$
\begin{equation*}
\int_{\Omega}\|v\|_{W_{h}^{1, p}(\Omega), z, s}^{p} \mathrm{~d} z \lesssim \frac{h^{n}}{p s-n}\left(\left\|\nabla_{h} v\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\|v\|_{L^{p}(\Omega)}^{p}+h^{p}\left\|D_{h}^{2} v\right\|_{L^{p}(\Omega)}^{p}\right) \tag{3.14}
\end{equation*}
$$

If $s=n / p$, then we have

$$
\begin{equation*}
\int_{\Omega}\|v\|_{W_{h}^{1, p}(\Omega), z, n / p}^{p} \mathrm{~d} z \lesssim|\log h| h^{n}\left(\left\|\nabla_{h} v\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\|v\|_{L^{p}(\Omega)}^{p}+h^{p}\left\|D_{h}^{2} v\right\|_{L^{p}(\Omega)}^{p}\right) . \tag{3.15}
\end{equation*}
$$

Proof. (i) Let $v \in L^{p}(\Omega)$ and extend $v$ to $\mathbb{R}^{n}$ by zero. Denote by $B_{h}(z)$ the ball of radius $h$ and center $z$. Then by a change of variables and interchanging integrals, we find

$$
\begin{aligned}
\int_{\Omega} \int_{T_{z}}|v(x)|^{p} \mathrm{~d} x \mathrm{~d} z & \leq \int_{\Omega} \int_{B_{h}(z)}|v(x)|^{p} \mathrm{~d} x \mathrm{~d} z \\
& =h^{n} \int_{\Omega} \int_{B_{1}(0)}|v(z+h y)|^{p} \mathrm{~d} y \mathrm{~d} z \\
& =h^{n} \int_{B_{1}(0)} \int_{\Omega}|v(z+h y)|^{p} \mathrm{~d} z \mathrm{~d} y \\
& \lesssim h^{n} \int_{B_{1}(0)}\|v\|_{L^{p}(\Omega)}^{p} \mathrm{~d} y \lesssim h^{n}\|v\|_{L^{p}(\Omega)}^{p}
\end{aligned}
$$

This proves (3.13).
(ii) To prove (3.14) we again extend $v$ to $\mathbb{R}^{n}$ by zero and make a change of variables to obtain

$$
\begin{aligned}
\int_{\Omega}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p} \mathrm{~d} z & =\int_{\Omega} \int_{\Omega}\left(\frac{h}{|x-z|+h}\right)^{s p}|v(x)|^{p} \mathrm{~d} x \mathrm{~d} z \\
& \leq h^{n} \int_{\Omega} \int_{\hat{\Omega}} \frac{h^{s p}}{(|h y|+h)^{s p}}|v(z+h y)|^{p} \mathrm{~d} y \mathrm{~d} z \\
& =h^{n} \int_{\hat{\Omega}}\left(\int_{\Omega}|v(z+h y)|^{p} \mathrm{~d} z\right) \frac{1}{(|y|+1)^{s p}} \mathrm{~d} y
\end{aligned}
$$

where $\hat{\Omega}=\left\{2 h^{-1} x: x \in \Omega\right\}$ is a dilation of $\Omega$. Therefore,

$$
\begin{equation*}
\int_{\Omega}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p} \mathrm{~d} z \lesssim h^{n}\|v\|_{L^{p}(\Omega)}^{p} \int_{\hat{\Omega}} \frac{1}{(|y|+1)^{s p}} \mathrm{~d} y . \tag{3.16}
\end{equation*}
$$

For $s p>n$, there holds

$$
\int_{\hat{\Omega}} \frac{1}{(|y|+1)^{s p}} \mathrm{~d} y \lesssim \int_{0}^{\infty} \frac{r^{n-1}}{(r+1)^{s p}} \mathrm{~d} r=(n-1)!\prod_{j=1}^{n}(s p-j)^{-1} \leq \frac{(n-1)!}{s p-n}
$$

Combining this identity with (3.16) yields the inequality

$$
\begin{equation*}
\int_{\Omega}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p} \mathrm{~d} z \lesssim \frac{h^{n}}{s p-n}\|v\|_{L^{p}(\Omega)}^{p} \tag{3.17}
\end{equation*}
$$

If $s p=n$, then we find by a direct calculation that

$$
\int_{\hat{\Omega}} \frac{1}{(|y|+1)^{n}} \mathrm{~d} y \lesssim \int_{0}^{h^{-1}} \frac{r^{n-1}}{(r+1)^{n}} \mathrm{~d} r=-\sum_{j=1}^{n-1} \frac{1}{(h+1)^{n-j}}+\log \left(1+h^{-1}\right) \lesssim|\log h|
$$

and therefore by (3.16),

$$
\begin{equation*}
\int_{\Omega}\left\|\sigma_{z}^{p / n} v\right\|_{L^{p}(\Omega)}^{p} \mathrm{~d} z \lesssim|\log h| h^{n}\|v\|_{L^{p}(\Omega)} \tag{3.18}
\end{equation*}
$$

Next, by trace inequalities given in Lemma 2.1, we have

$$
\begin{aligned}
\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\left[\sigma_{z}^{s} v\right]\right\|_{L^{p}(e)}^{p} & \lesssim h^{-p}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p}+\left\|\nabla_{h}\left(\sigma_{z}^{s} v\right)\right\|_{L^{p}(\Omega)}^{p} \\
& \lesssim h^{-p}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p}+\left\|v \nabla\left(\sigma_{z}^{s}\right)\right\|_{L^{p}(\Omega)}^{p}+\left\|\sigma_{z}^{s} \nabla_{h} v\right\|_{L^{p}(\Omega)}^{p}
\end{aligned}
$$

Noting that

$$
\left|\nabla\left(\sigma_{z}^{s}\right)\right| \lesssim \frac{h^{s}}{(|x-z|+h)^{s+1}}=\frac{\sigma_{z}^{s}}{|x-z|+h} \lesssim h^{-1} \sigma_{z}^{s}
$$

we obtain

$$
\begin{equation*}
\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\left[\sigma_{z}^{s} v\right]\right\|_{L^{p}(e)}^{p} \lesssim h^{-p}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p}+\left\|\sigma_{z}^{s} \nabla_{h} v\right\|_{L^{p}(\Omega)}^{p} \tag{3.19}
\end{equation*}
$$

Likewise we have

$$
\begin{equation*}
\sum_{e \in \mathcal{E}_{h}} h_{e}\left\|\sigma_{z}^{s}\{\nabla v\}\right\|_{L^{p}(e)}^{p} \lesssim\left\|\sigma_{z}^{s} \nabla_{h} v\right\|_{L^{p}(\Omega)}^{p}+h^{p}\left\|\sigma_{z}^{s} D_{h}^{2} v\right\|_{L^{p}(\Omega)} \tag{3.20}
\end{equation*}
$$

Combining (3.19)-(3.20) yields

$$
\begin{equation*}
\|v\|_{W_{h}^{1, p}(\Omega), z, s} \lesssim\left\|\sigma_{z}^{s} \nabla_{h} v\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\left\|\sigma_{z}^{s} v\right\|_{L^{p}(\Omega)}^{p}+h^{p}\left\|\sigma_{z}^{s} D_{h}^{2} v\right\|_{L^{p}(\Omega)}^{p} . \tag{3.21}
\end{equation*}
$$

Finally applying the identities (3.17)-(3.18) to (3.21) yields the desired result (3.14)(3.15). The proof is complete.

Lemma 3.3. Let $w \in W^{2, p}\left(\mathcal{T}_{h}\right)(2 \leq p \leq \infty)$ and $w_{h} \in V_{h}$ satisfy (3.7) with $\varepsilon=1$. Then for any $0 \leq s \leq k-1+n / p$ and $z \in \bar{\Omega}$,

$$
\left|\left(w-w_{h}\right)(z)\right| \lesssim h^{1-n / p}|\log h|^{\bar{s}(p)} \inf _{v_{h} \in V_{h}}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}
$$

where $\bar{s}(p)=1$ if $k=s+1-n / p$ and $\bar{s}(p)=0$ for $k>s+1-n / p$.
Proof. Step 1: Set-up. By the triangle inequality, an inverse estimate, and Hölder's inequality we obtain

$$
\begin{aligned}
\left|\left(w-w_{h}\right)(z)\right| & \leq\left|\left(w-v_{h}\right)(z)\right|+\left\|v_{h}-w_{h}\right\|_{L^{\infty}\left(T_{z}\right)} \\
& \leq\left|\left(w-v_{h}\right)(z)\right|+h^{-n / 2}\left\|v_{h}-w_{h}\right\|_{L^{2}\left(T_{z}\right)} \\
& \leq\left|\left(w-v_{h}\right)(z)\right|+h^{-n / 2}\left(\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)}+\left\|w-v_{h}\right\|_{L^{2}\left(T_{z}\right)}\right) \\
& \leq\left\|w-v_{h}\right\|_{L^{\infty}\left(T_{z}\right)}+h^{-n / 2}\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)}
\end{aligned}
$$

Therefore by standard approximation theory, and since $\sigma_{z} \approx 1$ on $T_{z}$, we have

$$
\begin{aligned}
\left|\left(w-w_{h}\right)(z)\right| & \leq h^{1-n / p}\|w\|_{W_{h}^{1, p}\left(T_{z}\right)}+h^{-n / 2}\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)} \\
& \leq h^{1-n / p}\|w\|_{W_{h}^{1, p}(\Omega), z, s}+h^{-n / 2}\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)}
\end{aligned}
$$

Replacing $w$ and $w_{h}$ by $w-v_{h}$ and $w_{h}-v_{h}$, respectively, yields

$$
\begin{equation*}
\left|\left(w-w_{h}\right)(z)\right| \lesssim h^{1-n / p}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}+h^{-n / 2}\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)} \tag{3.22}
\end{equation*}
$$

Next, define $\rho \in L^{2}(\Omega)$ by

$$
\rho(x)= \begin{cases}\frac{h^{-n / 2}\left(w-w_{h}\right)(x)}{\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)}} & \text { if } x \in T_{z} \\ 0 & \text { otherwise }\end{cases}
$$

and let $g_{z} \in H_{0}^{1}(\Omega)$ be the regularized Green's function satisfying

$$
\begin{equation*}
\mathcal{L}_{0} g_{z}=\rho \tag{3.23}
\end{equation*}
$$

Setting $g_{z, h}$ to be the DG approximation of $g_{z}$, i.e., $a_{0, h}\left(v_{h}, g_{z}-g_{z, h}\right)=0, \forall v_{h} \in V_{h}$, and $e_{z}:=g_{z}-g_{z, h}$, we have by Galerkin orthogonality and the continuity of the bilinear form,

$$
\begin{aligned}
h^{-n / 2}\left\|w-w_{h}\right\|_{L^{2}\left(T_{z}\right)} & =\left(\rho, w-w_{h}\right)=a_{0, h}\left(w-w_{h}, g_{z}\right) \\
& =a_{0, h}\left(w-v_{h}, e_{z}\right) \lesssim\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), s, z}\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega),-s, z}
\end{aligned}
$$

Consequently, by (3.22), we have
(3.24) $\left|\left(w-w_{h}\right)(z)\right| \lesssim\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}\left(h^{1-n / p}+\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}(\Omega),-s, z}}\right) \quad \forall v_{h} \in V_{h}$.

Thus, the proof will be completed once it is shown that $\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega),-s, z} \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}$. This result is derived in the following steps.

Step 2: Dyadic decomposition of $\Omega$. To estimate $\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega),-s, z}$ we require some more notation. Without loss of generality, assume that $\operatorname{diam}(\Omega)=1$. Let $d_{j}=2^{-j}$ and set

$$
\Omega_{j}=\left\{x \in \Omega: d_{j+1}<|z-x|<d_{j}\right\}
$$

$$
\begin{aligned}
\Omega_{j}^{\prime} & =\left\{x \in \Omega: d_{j+2}<|z-x|<d_{j-1}\right\} \\
\Omega_{j}^{\prime \prime} & =\left\{x \in \Omega: d_{j+3}<|z-x|<d_{j-2}\right\}
\end{aligned}
$$

Let $M>1$ be a real number to be determined later, and let $J \approx|\log h|$ be an integer such that $M h=2^{-J}$. We then write

$$
\begin{equation*}
\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \lesssim\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(B_{M h}(z)\right), z,-s}+\sum_{j=0}^{J}\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(\Omega_{j}\right), z,-s} \tag{3.25}
\end{equation*}
$$

Note that, by the definition of $\Omega_{j}$, the weighted norms, and Hölder's inequality that

$$
\begin{aligned}
\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(\Omega_{j}\right), z,-s} & \lesssim d_{j}^{n / q+s} h^{-s}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \\
\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(B_{M h}\right), z,-s} & \lesssim h^{n / q}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(B_{M h}(z)\right)} \leq h^{n / q}\left\|e_{z}\right\|_{W_{h}^{1,2}(\Omega)},
\end{aligned}
$$

where

$$
q \in[2, \infty] \text { satisfies } 1 / q+1 / p=1 / 2
$$

Applying these estimates to (3.25) yields

$$
\begin{align*}
\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} & \lesssim h^{n / q}\left\|e_{z}\right\|_{W_{h}^{1,2}(\Omega)}+\sum_{j=0}^{J} d_{j}^{n / q+s} h^{-s}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)}  \tag{3.26}\\
& =h^{n / q}\left\|e_{z}\right\|_{W_{h}^{1,2}(\Omega)}+Q_{h}
\end{align*}
$$

where

$$
\begin{equation*}
Q_{h}:=h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} . \tag{3.27}
\end{equation*}
$$

To estimate the first term in the right-hand side of (3.26), we apply elliptic regularity and the identity $\|\rho\|_{L^{2}(\Omega)}=h^{-n / 2}$ to obtain

$$
\left\|e_{z}\right\|_{W_{h}^{1,2}(\Omega)} \lesssim h\left\|g_{z}\right\|_{W^{2,2}(\Omega)} \lesssim h\|\rho\|_{L^{2}(\Omega)}=h^{1-n / 2}
$$

Applying this estimate in (3.26) and using the identity $1-n / 2+n / q=1-n / p$ yields

$$
\begin{equation*}
\left\|e_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \lesssim h^{1-n / p}+Q_{h} \tag{3.28}
\end{equation*}
$$

It remains to find an appropriate upper bound of $Q_{h}$ to complete the proof.
Step 3: Estimate of $Q_{h}$-Local error estimates. Lemma 4.4 in [6] states that

$$
\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \lesssim h^{k} d_{j}^{1-k-n / 2}+d_{j}^{-1}\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)}
$$

Applying this estimate to the definition of $Q_{h}(3.27)$ yields

$$
Q_{h} \lesssim h^{k-s} \sum_{j=0}^{J} d_{j}^{n / q+s+1-k-n / 2}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)}
$$

$$
\begin{aligned}
& =h^{k-s} \sum_{j=0}^{J} d_{j}^{-(k-s+n / p-1)}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} \\
& =h^{1-n / p} \Theta(k-s+n / p-1)+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)}
\end{aligned}
$$

where

$$
\Theta(\gamma):=\sum_{j=0}^{J}\left(\frac{h}{d_{j}}\right)^{\gamma}
$$

Therefore, since (cf. [6, (5.19)])

$$
\Theta(\gamma) \lesssim \begin{cases}|\log h| & \text { if } \gamma=0 \\ \frac{1}{M^{\gamma}\left(1-2^{-\gamma}\right)} & \text { if } \gamma>0\end{cases}
$$

we find that

$$
\begin{equation*}
Q_{h} \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} \tag{3.29}
\end{equation*}
$$

Step 4: Estimate of $Q_{h}-$ Duality Arguments. Applying [6, (5.24)] yields

$$
\begin{equation*}
\left\|e_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} \lesssim h^{k} d_{j}^{1-k-n / 2}\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}+h\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}^{\prime \prime}\right)} \tag{3.30}
\end{equation*}
$$

Using estimates (3.30) and (3.29), and noting that $\max _{0 \leq j \leq J} d_{j}^{-1}=2^{J}=1 /(h M)$, we find

$$
\begin{aligned}
Q_{h} & \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{k-s} \sum_{j=0}^{J} d_{j}^{s-k-n / p}\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}+h^{1-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}^{\prime \prime}\right)} \\
& \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{k-s} \sum_{j=0}^{J} d_{j}^{s-k-n / p}\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}+\frac{h^{-s}}{M} \sum_{j=0}^{J} d_{j}^{n / q+s}\left\|e_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}^{\prime \prime}\right)} \\
& \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}+\frac{1}{M} Q_{h} .
\end{aligned}
$$

Taking $M$ sufficiently large yields

$$
Q_{h} \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}
$$

Applying this estimate to (3.28) then yields

$$
\begin{equation*}
\left\|e_{z}\right\|_{W_{h}^{1, p}(\Omega), z,-s} \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)} \tag{3.31}
\end{equation*}
$$

In particular, the case $s=0, p=\infty, p^{\prime}=1$ gives

$$
\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)} \lesssim|\log h|^{\bar{s}(\infty)} h+\Theta(k)\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)}
$$

Since

$$
\Theta(k) \lesssim \frac{1}{M^{k}\left(1-2^{-k}\right)},
$$

we can take $M$ sufficiently large to conclude that

$$
\left\|e_{z}\right\|_{W_{h}^{1,1}(\Omega)} \lesssim|\log h|^{\bar{s}(\infty)} h
$$

Finally, applying this last estimate to (3.31) yields

$$
\left\|e_{z}\right\|_{W_{h}^{1, p}(\Omega), z,-s} \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}(1+\Theta(k-s+n / p)) \lesssim|\log h|^{\bar{s}(p)} h^{1-n / p}
$$

Applying this last estimate to (3.24) completes the proof.
Lemma 3.4. Let $z$ and $T_{z}$ be as in Lemma 3.2. For arbitrary $\varphi \in C_{0}^{\infty}\left(T_{z}\right)$, with $\|\varphi\|_{W^{1,2}\left(T_{z}\right)}=1$, we extend $\varphi$ to $\Omega$ by zero, and let $\hat{g}_{z}$ be the solution to

$$
\mathcal{L}_{0}^{*} \hat{g}_{z}=h^{-n / 2-1} \partial \varphi / \partial x_{i} \quad \text { in } \Omega, \quad \hat{g}_{z}=0 \quad \text { on } \partial \Omega .
$$

Let $\hat{g}_{z, h} \in V_{h}$ satisfy the discrete adjoint problem

$$
a_{0, h}\left(v_{h}, \hat{g}_{z, h}\right)=h^{-n / 2-1} \int_{\Omega}\left(\partial \varphi / \partial x_{i}\right) v_{h} \mathrm{~d} x \quad \forall v_{h} \in V_{h}
$$

where we have dropped the superscript of the bilinear form for notational simplicity. Let $p \in[2, \infty], p^{\prime} \in[1,2]$ such that $1 / p+1 / p^{\prime}=1$. Then for any $0 \leq s \leq k+n / p$ there holds

$$
\left\|\hat{g}_{z}-\hat{g}_{z, h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}
$$

where $\overline{\bar{s}}(p)=1$ if $s=k+n / p$ and $\overline{\bar{s}}(p)=0$ otherwise.
Proof. Set $\hat{e}_{z}=\hat{g}_{z}-\hat{g}_{z, h}$, and for $M>0$, let $J$ satisfy $M h=2^{-J}$. Then by applying similar arguments as the proof of Lemma 3.3, we obtain

$$
\begin{aligned}
\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} & \leq\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(B_{M h}(z)\right), z,-s}+\sum_{j=0}^{J}\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}\left(\Omega_{j}\right)} \\
& \lesssim h^{n / q}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}(\Omega)}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \\
& \lesssim h^{n / q+1}\left\|\hat{g}_{z}\right\|_{W_{h}^{2,2}(\Omega)}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \\
& \lesssim h^{-n / p}+\hat{F}_{h},
\end{aligned}
$$

with

$$
\begin{equation*}
\hat{F}_{h}:=h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \tag{3.33}
\end{equation*}
$$

By the local error estimate given in [6, Lemma 4.2] we have

$$
\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}\right)} \lesssim h^{k}\left\|\hat{g}_{z}\right\|_{W^{k+1,2}\left(\Omega_{j}^{\prime}\right)}+d_{j}^{-1}\left\|\hat{e}_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)}
$$

and Green function estimates show that $\left\|\hat{g}_{z}\right\|_{W^{k+1,2}\left(\Omega_{j}^{\prime}\right)} \lesssim d_{j}^{-n / 2-k}$. Applying these estimates into (3.33) yield

$$
\hat{F}_{h} \lesssim h^{k-s} \sum_{j=0}^{J} d_{j}^{n(1 / q-1 / 2)+s-k}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|\hat{e}_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)}
$$

$$
\begin{aligned}
& =h^{-n / p} \Theta(k-s+n / p)+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|\hat{e}_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} \\
& \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}+h^{-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|\hat{e}_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} .
\end{aligned}
$$

Applying [6, (5.39)], we have

$$
\left\|\hat{e}_{z}\right\|_{L^{2}\left(\Omega_{j}^{\prime}\right)} \lesssim h^{k} d_{j}^{1-k-n / 2}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}+h\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}^{\prime \prime}\right)}
$$

and therefore

$$
\begin{aligned}
\hat{F}_{h} & \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}+h^{1-s} \sum_{j=0}^{J} d_{j}^{n / q+s-1}\left\|\hat{e}_{z}\right\|_{W_{h}^{1,2}\left(\Omega_{j}^{\prime \prime}\right)} \\
& \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}+\frac{\hat{F}_{h}}{M}
\end{aligned}
$$

By taking $M$ sufficiently large, we obtain

$$
\hat{F}_{h} \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}
$$

and therefore

$$
\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}+h^{-n / p} \Theta(k-s+n / p)\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}
$$

The case $s=0, p^{\prime}=1, p=\infty$ yields

$$
\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)} \lesssim 1+\Theta(k)\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)}
$$

and therefore, we conclude by taking $M>0$ sufficiently large that

$$
\left\|\hat{e}_{z}\right\|_{W_{h}^{1,1}(\Omega)} \lesssim 1
$$

We then conclude that

$$
\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \lesssim|\log h|^{\overline{\bar{s}}(p)} h^{-n / p}
$$

The proof is complete.
3.1.1. Proof of Theorem $\mathbf{3 . 1}$ for $\boldsymbol{p} \geq \mathbf{2}$. We now prove Theorem 3.1 in the case $p \in[2, \infty)$. To this end, let $z \in \Omega$ and $T_{z} \in \mathcal{T}_{h}$ such that $z \in T_{z}$. Using an inverse estimate, (2.11), and the triangle inequality we obtain

$$
\begin{align*}
\left|\partial w_{h}(z) / \partial x_{i}\right| & \lesssim h^{-n / 2}\left\|\partial w_{h} / \partial x_{i}\right\|_{L^{2}\left(T_{z}\right)}  \tag{3.34}\\
& \lesssim h^{-n / 2-1}\left\|\partial w_{h} / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)} \\
& \lesssim h^{-n / 2-1}\left(\left\|\partial\left(w-w_{h}\right) / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)}+\left\|\partial w / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)}\right) .
\end{align*}
$$

Note that, by the Poincaré-Friedrichs and Hölder inequalities,

$$
\left\|\partial w / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)}=\sup _{\substack{\varphi \in C_{0}^{\infty}\left(T_{z}\right) \\\|\varphi\|_{W^{1,2}\left(T_{z}\right)}=1}}\left(\partial w / \partial x_{i}, \varphi\right)_{T_{z}}
$$

$$
\begin{aligned}
& \lesssim \sup _{\varphi \in C_{0}^{\infty}\left(T_{z}\right)}\left|T_{z}\right|^{\frac{p-2}{2 p}}\left\|\partial w / \partial x_{i}\right\|_{L^{p}\left(T_{z}\right)}\|\varphi\|_{L^{2}\left(T_{z}\right)} \\
& \lesssim \mid \|_{W^{1,2}\left(T_{z}\right)}=1 \\
& \lesssim\left|T_{z}\right|^{\frac{p-2}{2 p}} \operatorname{diam}\left(T_{z}\right)\left\|\partial w / \partial x_{i}\right\|_{L^{p}\left(T_{z}\right)} \lesssim h^{1+n / 2-n / p}\left\|\partial w / \partial x_{i}\right\|_{L^{p}\left(T_{z}\right)}
\end{aligned}
$$

Inserting this estimate into (3.34) yields

$$
\left|\partial w_{h}(z) / \partial x_{i}\right| \lesssim h^{-n / p}\left\|\partial w / \partial x_{i}\right\|_{L^{p}\left(T_{z}\right)}+h^{-n / 2-1}\left\|\partial\left(w-w_{h}\right) / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)}
$$

Replacing $w$ by $w-v_{h}$ and $w_{h}$ by $w_{h}-v_{h}$ for some $v_{h} \in V_{h}$ in the argument above, we conclude

$$
\begin{align*}
&\left|\partial\left(w_{h}-v_{h}\right)(z) / \partial x_{i}\right| \lesssim h^{-n / p}\left\|\partial\left(w-v_{h}\right) / \partial x_{i}\right\|_{L^{p}\left(T_{z}\right)}  \tag{3.35}\\
& \quad+h^{-n / 2-1}\left\|\partial\left(w-w_{h}\right) / \partial x_{i}\right\|_{W^{-1,2}\left(T_{z}\right)}
\end{align*}
$$

Let $\varphi, \hat{g}_{z}$ and $\hat{g}_{z, h}$ be as in Lemma 3.4. Setting $\hat{e}_{z}=\hat{g}_{z}-\hat{g}_{z, h}$, we have for arbitrary $v_{h} \in V_{h}$

$$
\begin{aligned}
h^{-n / 2-1} \int_{T_{z}}\left(w-w_{h}\right) \partial \varphi / \partial x_{i} d x & =a_{0, h}\left(w-v_{h}, \hat{e}_{z}\right) \\
& \lesssim\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}\left\|\hat{e}_{z}\right\|_{W_{h}^{1, p^{\prime}}(\Omega), z,-s} \\
& \lesssim\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}|\log h|^{\bar{s}(p)} h^{-n / p}
\end{aligned}
$$

where $\overline{\bar{s}}(p)$ is defined in Lemma 3.4.
Applying this last estimate into (3.35) yields

$$
\begin{align*}
\left|\nabla\left(w_{h}-v_{h}\right)(z)\right| \lesssim h^{-n / p} & \left\|\nabla\left(w-v_{h}\right)\right\|_{L^{p}\left(T_{z}\right)}  \tag{3.36}\\
& +h^{-n / p}|\log h|^{\bar{s}(p)}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}
\end{align*}
$$

Raising (3.36) by the power $p$ and integrating over $\Omega$ with respect to $z$, we conclude

$$
\begin{aligned}
\left\|\nabla_{h}\left(w_{h}-v_{h}\right)\right\|_{L^{p}(\Omega)} \lesssim & \left(h^{-n} \int_{\Omega}\left\|\nabla_{h}\left(w-v_{h}\right)\right\|_{L^{p}\left(T_{z}\right)}^{p} \mathrm{~d} z\right)^{1 / p} \\
& +\left(h^{-n}|\log h|^{\bar{s}(p) p} \int_{\Omega}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}^{p} \mathrm{~d} z\right)^{1 / p}
\end{aligned}
$$

Next, we choose $s$ such that $n / p<s<k+n / p$. Then $\overline{\bar{s}}(p)=0$, and by (3.13)-(3.14)
$\left\|\nabla_{h}\left(w_{h}-v_{h}\right)\right\|_{L^{p}(\Omega)}^{p} \lesssim\left\|\nabla_{h}\left(w-v_{h}\right)\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\left\|w-v_{h}\right\|_{L^{p}(\Omega)}^{p}+h^{p}\left\|D_{h}^{2}\left(w-v_{h}\right)\right\|_{L^{p}(\Omega)}^{p}$,
and therefore by the triangle inequality, and by taking $v_{h}=I_{h} w$, the nodal interpolant of $w$,

$$
\begin{equation*}
\left\|\nabla_{h}\left(w-w_{h}\right)\right\|_{L^{p}(\Omega)} \lesssim h\|w\|_{W^{2, p}(\Omega)} \tag{3.37}
\end{equation*}
$$

Next we bound the jumps $\left\|\left[w-w_{h}\right]\right\|_{L^{p}(e)}$. First, by the trace inequalities stated in Lemma 2.1 we have

$$
\begin{equation*}
\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\left[w-w_{h}\right]\right\|_{L^{p}(e)}^{p} \lesssim C\left(\left\|\nabla_{h}\left(w-w_{h}\right)\right\|_{L^{p}(\Omega)}^{p}+h^{-p}\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p}\right) \tag{3.38}
\end{equation*}
$$

By Lemma 3.3 we have for any $z \in \bar{\Omega}$ and $v_{h} \in V_{h}$,

$$
\left|\left(w-w_{h}\right)(z)\right|^{p} \lesssim h^{p-n}|\log h|^{p \bar{s}(p)}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}^{p}
$$

where $\bar{s}(p)=1$ if $k=s+1-n / p$ and $\bar{s}(p)=0$ for $k>s+1-n / p$. Integrating this expression with respect to $z$ yields

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p} \lesssim h^{p-n}|\log h|^{p \bar{s}(p)} \int_{\Omega}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, s}^{p} \mathrm{~d} z . \tag{3.39}
\end{equation*}
$$

If $k=1$, then we set $s=n / p$, so that $\bar{s}(p)=1$, and by (3.15) with $v_{h}=I_{h} w$,

$$
\begin{align*}
\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p} & \lesssim h^{p-n}|\log h|^{p} \int_{\Omega}\left\|w-v_{h}\right\|_{W_{h}^{1, p}(\Omega), z, n / p}^{p} \mathrm{~d} z  \tag{3.40}\\
& \lesssim h^{2 p}|\log h|^{p+1}\|w\|_{W^{2, p}(\Omega)}^{p}
\end{align*}
$$

On the other hand, if $k \geq 2$, then we choose $s$ such that $n / p<s<k-1+n / p$. Then $\bar{s}(p)=0$, and by (3.39) and (3.14),

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{L^{p}(\Omega)}^{p} \lesssim h^{2 p}\|w\|_{W^{2, p}(\Omega)}^{p} \tag{3.41}
\end{equation*}
$$

Combining (3.38) with (3.37), (3.40) and (3.41) then yields,

$$
\begin{equation*}
\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p}\left\|\left[w-w_{h}\right]\right\|_{L^{p}(e)}^{p} \lesssim|\log h|^{p+1} h^{p}\|w\|_{W^{2, p}(\Omega)}^{p} \tag{3.42}
\end{equation*}
$$

Finally combining (3.37), (3.42) and applying standard scaling arguments yields (3.9). This completes the proof of Theorem 3.1 in the case $p \geq 2$.
3.1.2. Proof of Theorem $\mathbf{3 . 1}$ for $\mathbf{1}<\boldsymbol{p}<\mathbf{2}$. The proof of $W^{1, p}$ error estimates in the range $p \in(1,2)$ is based on the following result.

Lemma 3.5. There holds, for $p^{\prime} \in[2, \infty)$,

$$
\left\|v_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)} \lesssim|\log h|^{t^{\prime}} \sup _{0 \neq z_{h} \in V_{h}} \frac{a_{0, h}\left(v_{h}, z_{h}\right)}{\left\|z_{h}\right\|_{W_{h}^{1, p}(\Omega)}} \quad \forall v_{h} \in V_{h}
$$

where $p \in(1,2]$ satisfies $1 / p+1 / p^{\prime}=1$ and $t^{\prime}=\left(p^{\prime}+1\right) / p^{\prime}$ if $k=1$ and $t^{\prime}=0$ for $k \geq 2$.

Proof. For a fixed $v_{h} \in V_{h}$, let $v \in H_{0}^{1}(\Omega)$ satisfy $\mathcal{L}_{0} v=\mathcal{L}_{0, h} v_{h}$ in $\Omega$. Then $v \in W^{2, p^{\prime}}(\Omega)$ with

$$
\begin{equation*}
\|v\|_{W^{2, p^{\prime}}(\Omega)} \lesssim\left\|\mathcal{L}_{0, h} v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \tag{3.43}
\end{equation*}
$$

Moreover, due to the definition of $\mathcal{L}_{0, h}$ and the consistency of $a_{0, h}(\cdot, \cdot)$, we find that

$$
a_{0, h}\left(v, z_{h}\right)=a_{0, h}\left(v_{h}, z_{h}\right) \quad \forall z_{h} \in V_{h}
$$

Since $p^{\prime} \geq 2$, we can apply the results of the previous section to conclude that

$$
\begin{equation*}
\left\|v_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)} \lesssim|\log h|^{t^{\prime}}\left(\|v\|_{W^{1, p^{\prime}}(\Omega)}+h\|v\|_{W^{2, p^{\prime}}(\Omega)}\right) \tag{3.44}
\end{equation*}
$$

Denote by $\mathcal{P}_{h}: L^{2}(\Omega) \rightarrow V_{h}$ the $L^{2}$ projection onto $V_{h}$. We then write
$\|v\|_{W^{1, p^{\prime}}(\Omega)} \lesssim \sup _{z \in W_{0}^{1, p}(\Omega)} \frac{\left(A_{0} \nabla v, \nabla z\right)}{\|z\|_{W^{1, p}(\Omega)}}=\sup _{z \in W_{0}^{1, p}(\Omega)} \frac{\left(\mathcal{L}_{0} v, z\right)}{\|z\|_{W^{1, p}(\Omega)}}=\sup _{z \in W_{0}^{1, p}(\Omega)} \frac{\left(\mathcal{L}_{0, h} v_{h}, \mathcal{P}_{h} z\right)}{\|z\|_{W^{1, p}(\Omega)}}$.

Standard arguments show that $\left\|\mathcal{P}_{h} z\right\|_{W_{h}^{1, p}(\Omega)} \lesssim\|z\|_{W^{1, p}(\Omega)}$ for all $z \in W^{1, p}(\Omega)$; thus,

$$
\begin{equation*}
\|v\|_{W^{1, p^{\prime}}(\Omega)} \lesssim \sup _{0 \neq z_{h} \in V_{h}} \frac{\left(\mathcal{L}_{0, h} v_{h}, z_{h}\right)}{\left\|z_{h}\right\|_{W_{h}^{1, p}(\Omega)}}=\sup _{0 \neq z_{h} \in V_{h}} \frac{a_{0, h}\left(v_{h}, z_{h}\right)}{\left\|z_{h}\right\|_{W_{h}^{1, p}(\Omega)}} \tag{3.45}
\end{equation*}
$$

Likewise, using (3.43), (2.6) and an inverse estimate yields

$$
\begin{equation*}
\|v\|_{W^{2, p^{\prime}}(\Omega)} \lesssim\left\|\mathcal{L}_{0, h} v_{h}\right\|_{L_{h}^{p^{\prime}}(\Omega)}=\sup _{0 \neq z_{h} \in V_{h}} \frac{a_{0, h}\left(v_{h}, z_{h}\right)}{\left\|z_{h}\right\|_{L^{p}(\Omega)}} \lesssim h^{-1} \sup _{0 \neq z_{h} \in V_{h}} \frac{a_{0, h}\left(v_{h}, z_{h}\right)}{\left\|z_{h}\right\|_{W^{1, p}(\Omega)}} \tag{3.46}
\end{equation*}
$$

Applying the estimates (3.45)-(3.46) to (3.44) then gives the desired result.
We now prove Theorem 3.1 for $1<p<2$. To this end, for $w_{h} \in V_{h}$ and $w \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)$ satisfying (3.7), let $v_{h} \in V_{h}$ be the unique solution to

$$
a_{0, h}\left(v_{h}, z_{h}\right)=\int_{\Omega}\left|\nabla_{h} w_{h}\right|^{p-2} \nabla_{h} w_{h} \cdot \nabla_{h} z_{h} \mathrm{~d} x+\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-p} \int_{e}\left|\left[w_{h}\right]\right|^{p-2}\left[w_{h}\right]\left[z_{h}\right] \mathrm{d} S
$$

for all $z_{h} \in V_{h}$. Setting $z_{h}=w_{h}$ and using a scaling argument yields

$$
\left\|w_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{p} \lesssim a_{0, h}\left(v_{h}, w_{h}\right)
$$

Moreover, Lemma 3.5 and Hölder's inequality gets

$$
\left\|v_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)} \lesssim|\log h|^{t^{\prime}}\left\|w_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{p-1}
$$

Consequently,

$$
\begin{aligned}
\left\|w_{h}\right\|_{W_{h}^{1, p}(\Omega)} & =\frac{\left\|w_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{p}}{\left\|w_{h}\right\|_{W_{h}^{1, p}(\Omega)}^{p-1}} \lesssim|\log h|^{t^{\prime}} \frac{a_{0, h}\left(v_{h}, w_{h}\right)}{\left\|v_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)}} \\
& =|\log h|^{t^{\prime}} \frac{a_{0, h}\left(w, v_{h}\right)}{\left\|v_{h}\right\|_{W_{h}^{1, p^{\prime}}(\Omega)}} \lesssim|\log h|^{t^{\prime}}\|w\|_{W_{h}^{1, p}(\Omega)}
\end{aligned}
$$

Standard arguments then show that this estimate implies

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)} \lesssim|\log h|^{t^{\prime}} h\|w\|_{W^{2, p}(\Omega)} \quad 1<p<2 \tag{3.47}
\end{equation*}
$$

This completes the proof of Theorem 3.1 upon noting that $t^{\prime}=\left(p^{\prime}+1\right) / p^{\prime}=(2 p-$ 1) $/ p \leq(p+1) / p=t$ for $p \in(1,2]$.
3.2. DG discrete Calderon-Zygmund estimates for PDEs with constant coefficients. The goal of this subsection is to establish a stability result for the operator $\mathcal{L}_{0, h}^{\varepsilon}$ in the $W_{h}^{2, p}$-norm, which is a discrete counterpart of (3.3). Such an estimate can be regarded as a DG discrete Calderon-Zygmund estimate for $\mathcal{L}_{0, h}^{\varepsilon}$.

Theorem 3.6. (i) For $\varepsilon=1$ and $1<p<\infty$ we have

$$
\begin{equation*}
\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim|\log h|^{t}\left\|\mathcal{L}_{0, h}^{\varepsilon} w_{h}\right\|_{L^{p}(\Omega)} \quad \forall w_{h} \in V_{h} \tag{3.48}
\end{equation*}
$$

where $t=(p+1) / p$ if $k=1$ and $t=0$ if $k \geq 2$.
(ii) (3.48) also holds with $t=0$ for $\varepsilon \in\{1,0,-1\}$ and $p=2$.

Proof. (i) We observe that (3.48) is equivalent to showing

$$
\begin{equation*}
\left\|\left(\mathcal{L}_{0, h}^{\varepsilon}\right)^{-1} \varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim|\log h|^{t}\left\|\varphi_{h}\right\|_{L^{p}(\Omega)} \quad \forall \varphi_{h} \in V_{h} \tag{3.49}
\end{equation*}
$$

For any $\varphi_{h} \in V_{h}$, let $w:=\mathcal{L}_{0}^{-1} \varphi_{h} \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)$ and $w_{h}:=\left(\mathcal{L}_{0, h}^{\varepsilon}\right)^{-1} \varphi_{h} \in$ $V_{h}$. Since $w \in W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega)$ we have

$$
a_{0, h}^{\varepsilon}\left(w, v_{h}\right)=\left(\varphi_{h}, v_{h}\right)=a_{0, h}^{\varepsilon}\left(w_{h}, v_{h}\right) \quad \forall v_{h} \in V_{h} .
$$

Thus $w_{h}$ is the IP-DG approximate solution to $w$. Applying Theorem 3.1 and the elliptic regularity estimate, we obtain

$$
\begin{equation*}
\left\|w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)} \lesssim|\log h|^{t} h\|w\|_{W^{2, p}(\Omega)} \lesssim|\log h|^{t} h\left\|\varphi_{h}\right\|_{L^{p}(\Omega)} . \tag{3.50}
\end{equation*}
$$

Moreover, by Lemma 2.4 and the Calderon-Zygmund estimate for $\mathcal{L}_{0}$ we have

$$
\begin{equation*}
\|w\|_{W_{h}^{2, p}(\Omega)} \leq\|w\|_{W^{2, p}(\Omega)} \lesssim\left\|\varphi_{h}\right\|_{L^{p}(\Omega)} . \tag{3.51}
\end{equation*}
$$

Denote by $I_{h}: C^{0}(\Omega) \rightarrow V_{h}$ the nodal interpolation operator onto $V_{h}$. By finite element interpolation theory [7] we have

$$
\begin{equation*}
h^{-1}\left\|w-I_{h} w\right\|_{W_{h}^{1, p}(\Omega)}+\left\|w-I_{h} w\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\|w\|_{W^{2, p}(\Omega)} . \tag{3.52}
\end{equation*}
$$

Therefore by the triangle inequality, an inverse estimate, Lemma 2.4, (3.50), and (3.51), we obtain

$$
\begin{aligned}
\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)} & \leq\left\|w-I_{h} w\right\|_{W_{h}^{2, p}(\Omega)}+\left\|I_{h} w-w_{h}\right\|_{W_{h}^{2, p}(\Omega)}+\|w\|_{W_{h}^{2, p}(\Omega)} \\
& \lesssim h^{-1}\left\|I_{h} w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)}+\left\|\varphi_{h}\right\|_{L^{p}(\Omega)} \\
& \leq h^{-1}\left(\left\|w-w_{h}\right\|_{W_{h}^{1, p}(\Omega)}+\left\|w-I_{h} w\right\|_{W_{h}^{1, p}(\Omega)}\right)+\left\|\varphi_{h}\right\|_{L^{p}(\Omega)} \\
& \lesssim|\log h|^{t}\left\|\varphi_{h}\right\|_{L^{p}(\Omega)}=|\log h|^{t}\left\|\mathcal{L}_{0, h} w_{h}\right\|_{L^{p}(\Omega)} .
\end{aligned}
$$

(ii) The proof of this part is exactly same as that of Part (i), the only difference is that now (3.8), instead of (3.9), should be called in the proof.
4. IP-DG methods and their convergence analysis. As mentioned in Section 1, our primary goal in this paper to develop convergent IP-DG methods for approximating the $W^{2, p}$ strong solution to the boundary value problem (1.1). We assume that equation (1.1a) is uniformly elliptic, precisely, we assume $A \in\left[C^{0}(\bar{\Omega})\right]^{n \times n}$ is positive definite, that is, there exist constants $\Lambda>\lambda>0$ such that

$$
\begin{equation*}
\lambda|\xi|^{2} \leq A(x) \leq \Lambda|\xi|^{2} \quad \text { for all } \xi \in \mathbb{R}^{n} \text { and } x \in \bar{\Omega} . \tag{4.1}
\end{equation*}
$$

We also assume that the solution $u$ satisfies the following Calderon-Zygmund estimate:

$$
\begin{equation*}
\|u\|_{W^{2, p}(\Omega)} \lesssim\|f\|_{L^{p}(\Omega)} . \tag{4.2}
\end{equation*}
$$

It is well-known [13, Chapter 9] that the above estimate holds for any $f \in L^{p}(\Omega)$ with $1<p<\infty$ if $\partial \Omega \in C^{1,1}$. Moreover, when $n, p=2,(4.2)$ also holds if $\Omega$ is a convex domain [21, 2].
4.1. Formulation of IP-DG methods. We follow the same recipe as in the constant coefficient case to build our IP-DG methods. To this end, we momentarily assume $A \in\left[C^{1}(\Omega)\right]^{n \times n}$, so that we can rewrite the $\operatorname{PDE}$ (1.1a) in divergence form as follows:

$$
\begin{equation*}
-\nabla \cdot(A \nabla u)+(\nabla \cdot A) \cdot \nabla u=f \tag{4.3}
\end{equation*}
$$

where $\nabla \cdot A$ is defined row-wise. We then define the following (standard) IP-DG methods for problem (4.3) by seeking $u_{h} \in V_{h}$ such that

$$
\begin{align*}
& \int_{\Omega}\left(A \nabla_{h} u_{h}\right) \cdot \nabla_{h} v_{h} \mathrm{~d} x+\int_{\Omega}\left((\nabla \cdot A) \cdot \nabla_{h} u_{h}\right) v_{h} \mathrm{~d} x  \tag{4.4}\\
&-\sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A \nabla u_{h} \cdot \nu_{e}\right\}\left[v_{h}\right] \mathrm{d} S-\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A \nabla v_{h} \cdot \nu_{e}\right\}\left[u_{h}\right] \mathrm{d} S \\
&+\sum_{e \in \mathcal{E}_{h}} \int_{e} \frac{\gamma_{e}}{h_{e}}\left[u_{h}\right]\left[v_{h}\right] \mathrm{d} S=\int_{\Omega} f v_{h} \mathrm{~d} x,
\end{align*}
$$

where $\gamma_{e} \geq \gamma_{*}\left(\|A\|_{L^{\infty}(\Omega)}, \mathcal{T}_{h}\right)>0$. We emphasize that $\gamma^{*}$ is independent of the derivatives of $A$.

Now come back to our case in hand with $A \in\left[C^{0}(\Omega)\right]^{n \times n}$. Clearly, the term $\nabla \cdot A$ does not exist as a function (it is in fact a Radon measure), so the above formulation is not defined for the case we are considering. To overcome this difficulty, our idea is to apply the DG integration by parts formula (3.5) to the first term on the left-hand side of (4.4), yielding

$$
\begin{align*}
a_{h}\left(w_{h}, v_{h}\right):= & -\int_{\Omega}\left(A: D_{h}^{2} w_{h}\right) v_{h} \mathrm{~d} x+\sum_{e \in \mathcal{E}_{h}^{I}} \int_{e}\left[A \nabla w_{h} \cdot \nu_{e}\right]\left\{v_{h}\right\} \mathrm{d} S  \tag{4.5}\\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e}\left\{A \nabla v_{h} \cdot \nu_{e}\right\}\left[w_{h}\right] \mathrm{d} S+\sum_{e \in \mathcal{E}_{h}} \int_{e} \frac{\gamma_{e}}{h_{e}}\left[w_{h}\right]\left[v_{h}\right] \mathrm{d} S .
\end{align*}
$$

No derivative of $A$ appears in the above new form of $a_{h}^{\varepsilon}(\cdot, \cdot)$; thus, it is well-defined on $V_{h} \times V_{h}$. This leads to the following definition.

Definition 4.1. Our IP-DG methods are defined by seeking $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
a_{h}^{\varepsilon}\left(u_{h}, v_{h}\right)=\left(f, v_{h}\right) \quad \forall v_{h} \in V_{h}, \quad \varepsilon \in\{1,0,-1\} \tag{4.6}
\end{equation*}
$$

When $\varepsilon=1$ we will refer to the method as "symmetrically induced" even though the bilinear form is not symmetric. Likewise, $\varepsilon=0$ and $\varepsilon=-1$ yields an "incompletely induced" and "non-symmetrically induced" method, respectively.
4.2. Stability analysis. As in Section 3 we can define the IP-DG approximation $\mathcal{L}_{h}^{\varepsilon}$ of $\mathcal{L}$ on $V_{h}$ using the bilinear form $a_{h}^{\varepsilon}(\cdot, \cdot) ;$ precisely, we define $\mathcal{L}_{h}^{\varepsilon}: V_{h} \rightarrow V_{h}$ by

$$
\begin{equation*}
\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, v_{h}\right):=a_{h}^{\varepsilon}\left(w_{h}, v_{h}\right) \quad \forall w_{h}, v_{h} \in V_{h} \tag{4.7}
\end{equation*}
$$

Since we can extend the domain of $a_{h}^{\varepsilon}(\cdot, \cdot)$ to $W^{2, p}\left(\mathcal{T}_{h}\right) \times W^{2, p^{\prime}}\left(\mathcal{T}_{h}\right)$, then the domain and co-domain of $\mathcal{L}_{h}$ can be extended to the broken Sobolev spaces $W^{2, p}\left(\mathcal{T}_{h}\right)$ and $\left(W^{2, p^{\prime}}\left(\mathcal{T}_{h}\right)\right)^{*}$ respectively.

The goal of this subsection is to establish a DG discrete Calderon-Zygmund estimate similar to (3.48) for the operator $\mathcal{L}_{h}^{\varepsilon}$. To this end, our main idea is to mimic, at
the discrete level, the "freezing the coefficients" technique and the covering argument found in Schauder theory and $W^{2, p}$ strong solution theory [13, Chapters 6 and 9 ]. Since $A$ is continuous, we show that in a small ball $B_{\delta}(\subset \Omega) A$ behaves as if it were constant. This allows us to conclude that $\mathcal{L}_{h}^{\varepsilon}$ is locally very close to $\mathcal{L}_{0, h}^{\varepsilon}$ in the ball $B_{\delta}\left(x_{0}\right)$ for any $x_{0} \in \Omega$. By applying the above mentioned "freezing the coefficients" technique and covering argument to the formal adjoint of $\mathcal{L}_{h}^{\varepsilon}$, we are able to prove a global left-side inf-sup condition for $\mathcal{L}_{h}^{\varepsilon}$. Then by employing a duality argument, we derive the desired discrete Calderon-Zygmund estimate for $\mathcal{L}_{h}^{\varepsilon}$.

We now proceed to establish a few auxiliary lemmas which will be needed to show the desired estimate.

Lemma 4.1. For all $\delta>0$, there exists $R_{\delta}>0$ and $h_{\delta}>0$ such that for all $x_{0} \in \Omega$ and $A_{0} \equiv A\left(x_{0}\right)$

$$
\begin{equation*}
\left\|\left(\mathcal{L}_{h}^{\varepsilon}-\mathcal{L}_{0, h}^{\varepsilon}\right) w\right\|_{L_{h}^{p}\left(B_{R_{\delta}}\left(x_{0}\right)\right)} \lesssim \delta\|w\|_{W_{h}^{2, p}\left(B_{R_{\delta}}\left(x_{0}\right)\right)} \quad \forall w \in W^{2, p}\left(\mathcal{T}_{h}\right), \forall h \leq h_{\delta} \tag{4.8}
\end{equation*}
$$

Here, $B_{R_{\delta}}\left(x_{0}\right):=\left\{x \in \Omega:\left|x-x_{0}\right|<R_{\delta}\right\}$ denotes the ball with center $x_{0}$ and radius $R_{\delta}$.

Proof. Since $A$ is continuous on $\bar{\Omega}$, then it is uniformly continuous. Therefore, for every $\delta>0$ there exists $R_{\delta}>0$ such that if $x, y \in \Omega$ satisfies $|x-y|<R_{\delta}$, we have $|A(x)-A(y)|<\delta$. Consequently, for any $x_{0} \in \Omega$

$$
\begin{equation*}
\left\|A-A_{0}\right\|_{L^{\infty}\left(B_{R_{\delta}}\right)} \leq \delta \tag{4.9}
\end{equation*}
$$

where we have used the shorthand notation $B_{R_{\delta}}:=B_{R_{\delta}}\left(x_{0}\right)$.
Set $h_{\delta}=\min \left\{h_{0}, \frac{R_{\delta}}{4}\right\}$ and let $0<h<h_{\delta}, w \in W^{2, p}\left(\mathcal{T}_{h}\right)$, and $v_{h} \in V_{h}\left(B_{R_{\delta}}\right)$. Since $\left(\mathcal{L}_{0, h}^{\varepsilon}-\mathcal{L}_{h}^{\varepsilon}\right) w \in W^{2, p}\left(\mathcal{T}_{h}\right)$, it follows from (3.6) and (4.5) that for every $v_{h} \in V_{h}\left(B_{R_{\delta}}\right)$ we have

$$
\begin{aligned}
& \left(\left(\mathcal{L}_{0, h}^{\varepsilon}-\mathcal{L}_{h}^{\varepsilon}\right) w, v_{h}\right)=-\int_{\Omega \cap B_{R_{\delta}}}\left(\left(A_{0}-A\right): D_{h}^{2} w\right) v_{h} \mathrm{~d} x \\
& \quad+\sum_{e \in \mathcal{E}_{h}^{I}} \int_{e \cap \bar{B}_{R_{\delta}}}\left[\left(A_{0}-A\right) \nabla w \cdot \nu_{e}\right]\left\{v_{h}\right\} \mathrm{d} S-\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e \cap \bar{B}_{R_{\delta}}}\left\{\left(A-A_{0}\right) \nabla v_{h} \cdot \nu_{e}\right\}\left[w_{h}\right] \mathrm{d} S \\
& \leq\left\|A-A_{0}\right\|_{L^{\infty}\left(B_{R_{\delta}}\right)}\left(\left\|D_{h}^{2} w\right\|_{L^{p}\left(\Omega \cap B_{R_{\delta}}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(\Omega \cap B_{R_{\delta}}\right)}\right. \\
& \quad+\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-2 p}\|[w]\|_{\left.L^{p}\left(e \cap \bar{B}_{R_{\delta}}\right)\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e} h_{e}^{p^{\prime}}\left\|\left\{\nabla v_{h} \cdot \nu_{e}\right\}\right\|_{\left.L^{p^{\prime}}\left(e \cap \bar{B}_{R_{\delta}}\right)\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}} \\
& \left.\quad+\left(\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-p}\|[\nabla w]\|_{L^{p}\left(e \cap \bar{B}_{\left.R_{\delta}\right)}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}\left\|\left\{v_{h}\right\}\right\|_{L^{p^{\prime}}\left(e \cap \bar{B}_{\left.R_{\delta}\right)}\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}}\right) \\
& \lesssim\left\|A-A_{0}\right\|_{L^{\infty}\left(B_{R_{\delta}}\right)}\|w\|_{W_{h}^{2, p}\left(B_{R_{\delta}}\right)}\left(\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{R_{\delta}}\right)}+h\left\|\nabla_{h} v_{h}\right\|_{L^{p^{\prime}}\left(B_{\left.R_{\delta}\right)}\right)}\right) \\
& \lesssim \delta\|w\|_{W_{h}^{2, p}\left(B_{R_{\delta}}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{R_{\delta}}\right)}}=\delta\|w\|_{W_{h}^{2, p}\left(B_{R_{\delta}}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{R_{\delta}}\right)}} .
\end{aligned}
$$

Dividing both sides by $\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{R_{\delta}}\right)}$ yields the desired estimate. The proof is complete. ■

The next lemma shows that $\mathcal{L}_{h}^{\varepsilon}$ is locally a bounded operator on $W^{2, p}\left(\mathcal{T}_{h}\right)$.
Lemma 4.2. For any $x_{0} \in \Omega$ and $R \geq h$, there holds

$$
\begin{equation*}
\left\|\mathcal{L}_{h}^{\varepsilon} w\right\|_{L_{h}^{p}\left(B_{R}\left(x_{0}\right)\right)} \lesssim\|w\|_{W_{h}^{2, p}\left(B_{R}\left(x_{0}\right)\right)} \quad \forall w \in W^{2, p}\left(\mathcal{T}_{h}\right) \tag{4.10}
\end{equation*}
$$

Proof. Set $B_{R}:=B_{R}\left(x_{0}\right)$ and let $v_{h} \in V_{h}\left(B_{R}\right)$. For $e \in \mathcal{E}_{h}$, set $e_{R}:=e \cap \bar{B}_{R}$. By the trace estimate (2.8)), the definition of $\mathcal{L}_{h}^{\varepsilon}$, and an inverse inequality we have

$$
\begin{aligned}
\left(\mathcal{L}_{h}^{\varepsilon} w, v_{h}\right)= & -\int_{\Omega \cap B_{R}}\left(A: D_{h}^{2} w\right) v_{h} \mathrm{~d} x+\sum_{e \in \mathcal{E}_{h}^{I}} \int_{e_{R}}\left[A \nabla w \cdot \nu_{e}\right]\left\{v_{h}\right\} \mathrm{d} S \\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e_{R}}\left\{A \nabla v_{h} \cdot \nu_{e}\right\}[w] \mathrm{d} S+\sum_{e \in \mathcal{E}_{h}} \int_{e_{R}} \frac{\gamma_{e}}{h_{e}}[w]\left[v_{h}\right] \mathrm{d} S \\
\lesssim & \left\|D^{2} w\right\|_{L^{p}\left(\Omega \cap B_{R}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}\left(\Omega \cap B_{R}\right)}} \\
& \left.+\left(\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-p}\|[\nabla w]\|_{L^{p}\left(e_{R}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}\left\|\left\{v_{h}\right\}\right\|_{L^{p}\left(e_{R}\right)}^{p^{\prime}}\right)\right)^{\frac{1}{p^{\prime}}} \\
& +\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{1-2 p}\|[w]\|_{L^{p}\left(e_{R}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{p^{\prime}} h_{e}\left\|\left\{\nabla_{h} v_{h} \cdot \nu_{e}\right\}\right\|_{L^{p^{\prime}}\left(e_{R}\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}} \\
& +\left(\sum_{e \in \mathcal{E}_{h}} \gamma_{e}^{p} h_{e}^{1-2 p}\|[w]\|_{L^{p}\left(e_{R}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}\left\|\left[v_{h}\right]\right\|_{L^{p^{\prime}}\left(e_{R}\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}} \\
\lesssim & \|w\|_{W_{h}^{2, p}\left(B_{R}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{R}\right)} .
\end{aligned}
$$

Dividing both sides by $\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{R}\right)}$ yields the desired estimate. $\square$
Our last lemma establishes a left-side inf-sup condition for $\mathcal{L}_{h}^{\varepsilon}$. This estimate relies on the formal adjoint operator $\mathcal{L}_{h}^{*}:=\left(\mathcal{L}_{h}^{\varepsilon}\right)^{*}$ and some techniques from [24].

Lemma 4.3. There exists an $h_{0}>0$ such that for all $h \leq h_{0}$ and $k \geq 2$ we have

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim \sup _{0 \neq w_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, v_{h}\right)}{\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}} \quad \forall v_{h} \in V_{h} \tag{4.11}
\end{equation*}
$$

where $1<p<\infty$ if $\varepsilon=1$ and $p=2$ if $\varepsilon \in\{0,-1\}$.
Proof. Note that (4.11) is equivalent to

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim \sup _{0 \neq w_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, v_{h}\right)}{\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}}=\sup _{0 \neq w_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{*} v_{h}, w_{h}\right)}{\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}}=\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)} \tag{4.12}
\end{equation*}
$$

for all $v_{h} \in V_{h}$. We divide the remaining proof into three steps.
Step 1: Local estimates. Let $x_{0} \in \Omega, A_{0} \equiv A\left(x_{0}\right), \delta_{0}, h_{\delta_{0}}, R_{\delta_{0}}, R_{1}:=(1 / 3) R_{\delta_{0}}$, and $B_{1}:=B_{R_{1}}\left(x_{0}\right)$ be as in Lemma 4.1 with $\delta_{0}>0$ to be determined, and set $h \leq h_{\delta_{0}}$.

By the elliptic regularity of $\mathcal{L}$, for any $v_{h} \in V_{h}\left(B_{1}\right)$, there exists $\varphi \in W^{2, p}(\Omega) \cap$ $W_{0}^{1, p}(\Omega)$ such that $\mathcal{L} \varphi=v_{h}\left|v_{h}\right|^{p-2}$ in $\Omega$ and satisfies the estimate

$$
\begin{equation*}
\|\varphi\|_{W^{2, p}(\Omega)} \lesssim\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}^{p^{\prime}-1}=\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)}^{p^{\prime}-1} . \tag{4.13}
\end{equation*}
$$

Since $\mathcal{L}_{h}^{\varepsilon}$ is consistent with $\mathcal{L}$ for any $\varphi_{h} \in V_{h}$ we have

$$
\begin{align*}
\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} & =\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}=\left(\mathcal{L} \varphi, v_{h}\right)=\left(\mathcal{L}_{h}^{\varepsilon} \varphi, v_{h}\right)  \tag{4.14}\\
& =\left(\mathcal{L}_{h}^{\varepsilon} \varphi_{h}, v_{h}\right)+\left(\mathcal{L}_{h}^{\varepsilon}\left(\varphi-\varphi_{h}\right), v_{h}\right) \\
& =\left(\mathcal{L}_{h}^{*} v_{h}, \varphi_{h}\right)+\left(\mathcal{L}_{0, h}^{\varepsilon}\left(\varphi-\varphi_{h}\right), v_{h}\right)+\left(\left(\mathcal{L}_{h}^{\varepsilon}-\mathcal{L}_{0, h}^{\varepsilon}\right)\left(\varphi-\varphi_{h}\right), v_{h}\right) .
\end{align*}
$$

From the existence-uniqueness of the IP-DG scheme (3.4), there exists $\varphi_{h} \in V_{h}$ such that

$$
\left(\mathcal{L}_{0, h}^{\varepsilon}\left(\varphi-\varphi_{h}\right), w_{h}\right)=0 \quad \forall w_{h} \in V_{h}
$$

Combining Galerkin orthogonality, Theorem 3.6, and (4.13) gives us the solution estimate

$$
\begin{equation*}
\left\|\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\left\|\mathcal{L}_{0, h}^{\varepsilon} \varphi_{h}\right\|_{L_{h}^{p}(\Omega)}=\left\|\mathcal{L}_{0, h}^{\varepsilon} \varphi\right\|_{L_{h}^{p}(\Omega)} \lesssim\|\varphi\|_{W^{2, p}(\Omega)} \lesssim\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)}^{p^{\prime}-1} \tag{4.15}
\end{equation*}
$$

Using Lemma 4.1 and (4.13)-(4.15) we have

$$
\begin{aligned}
\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{1}\right)}}^{p^{\prime}} & =\left(\mathcal{L}_{h}^{*} v_{h}, \varphi_{h}\right)+\left(\left(\mathcal{L}_{h}^{\varepsilon}-\mathcal{L}_{0, h}^{\varepsilon}\right)\left(\varphi-\varphi_{h}\right), v_{h}\right) \\
& \leq\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}\left\|\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)}+\left\|\left(\mathcal{L}_{h}^{\varepsilon}-\mathcal{L}_{0, h}^{\varepsilon}\right)\left(\varphi-\varphi_{h}\right)\right\|_{L_{h}^{p}\left(B_{1}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{1}\right)}} \\
& \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)}^{p^{p^{\prime}}}+\delta_{0}\left\|\varphi-\varphi_{h}\right\|_{W_{h}^{2, p}\left(B_{1}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} \\
& \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}\left\|v_{h}\right\|_{L^{p^{p^{\prime}}\left(B_{1}\right)}}^{\left.p^{\prime}\right)}+\delta_{0}\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{1}\right)}}^{p^{\prime}}
\end{aligned}
$$

Taking $\delta_{0}$ sufficiently small to move the right hand term to the left side and dividing by $\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{1}\right)}}^{p^{\prime}-1}$ gives us the local estimate

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p}\left(B_{1}\right)} \quad \forall v_{h} \in V_{h}\left(B_{1}\right) \tag{4.16}
\end{equation*}
$$

Step 2: A Gärding type inequality by a covering argument. Given $R_{1}$ from Step 1, let $R_{2}=2 R_{1}$ and $R_{3}=3 R_{1}$. Let $\eta \in C^{3}(\Omega)$ be a cutoff function satisfying

$$
\begin{equation*}
0 \leq \eta \leq 1,\left.\quad \eta\right|_{B_{1}}=1,\left.\quad \eta\right|_{\Omega \backslash B_{2}}=0, \quad|\eta|_{W^{m, \infty}(\Omega)}=O\left(R_{1}^{-m}\right) \tag{4.17}
\end{equation*}
$$

For any $v_{h} \in V_{h}$, we have by (4.16),

$$
\begin{align*}
& \left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)}=\left\|\eta v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} \leq\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p^{\prime}}\left(B_{1}\right)}+\left\|I_{h}\left(\eta v_{h}\right)\right\|_{L^{p^{\prime}}\left(B_{1}\right)}  \tag{4.18}\\
& \lesssim\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p^{\prime}}\left(B_{1}\right)}+\left\|\mathcal{L}_{h}^{*}\left(I_{h}\left(\eta v_{h}\right)\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{1}\right)} \\
& \lesssim\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p^{\prime}}\left(B_{1}\right)}+\left\|\mathcal{L}_{h}^{*}\left(I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{1}\right)}+\left\|\mathcal{L}_{h}^{*}\left(\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{1}\right)} .
\end{align*}
$$

We now bound the second term on the right hand side of (4.18). By the definition of $\|\cdot\|_{W_{h}^{-2, p}}$, Lemma 4.2 and (2.6), for any $w_{h} \in V_{h}$ we have

$$
\begin{aligned}
& \left\|\mathcal{L}_{h}^{*}\left(I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{1}\right)}=\sup _{0 \neq w_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{*}\left(I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right), w_{h}\right)}{\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{1}\right)}} \\
& \quad \leq \sup _{w_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right)}{\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{1}\right)} \lesssim \sup _{w_{h} \in V_{h}} \frac{\left.\left\|\mathcal{L}_{h}^{\varepsilon} w_{h}\right\|_{L_{h}^{p}\left(B_{1}\right)} \| I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right) \|_{L^{p^{\prime}}\left(B_{1}\right)}}{\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{1}\right)}}} \begin{array}{l}
\quad \lesssim \sup _{w_{h} \in V_{h}} \frac{\left.\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{1}\right)} \| I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right)\left\|_{L^{p^{\prime}\left(B_{1}\right)}}\right\| w_{h} \|_{W_{h}^{2, p}\left(B_{1}\right)}}{}=\left\|I_{h}\left(\eta v_{h}\right)-\eta v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} .
\end{array} .
\end{aligned}
$$

Thus (4.18) becomes

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} \lesssim\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p^{\prime}}\left(B_{1}\right)}+\left\|\mathcal{L}_{h}^{*}\left(\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{1}\right)} \tag{4.19}
\end{equation*}
$$

Using Lemmas 2.3, 2.7, and 4.2 with (4.19) yields

$$
\begin{align*}
\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} & \lesssim \frac{h}{R_{1}}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}+\left\|\mathcal{L}_{h}^{*}\left(\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}  \tag{4.20}\\
& \lesssim \frac{1}{R_{1}}\left\|v_{h}\right\|_{W^{-1, p^{\prime}}\left(B_{3}\right)}+\left\|\mathcal{L}_{h}^{*}\left(\eta v_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}
\end{align*}
$$

We now want to remove the cutoff function $\eta$ from the adjoint operator appearing in the right-hand side of $(4.20)$. For $w_{h} \in V_{h}\left(B_{3}\right)$, we break up $\mathcal{L}_{h}^{*}\left(\eta v_{h}\right)$ as follows:

$$
\begin{align*}
\left(\mathcal{L}_{h}^{*}\right. & \left.\left(\eta v_{h}\right), w_{h}\right)=\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, \eta v_{h}\right)=\left(\mathcal{L}_{h}^{\varepsilon} w_{h} \eta, v_{h}\right)+\left[\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, \eta v_{h}\right)-\left(\mathcal{L}_{h}^{\varepsilon} w_{h} \eta, v_{h}\right)\right]  \tag{4.21}\\
= & \left(\mathcal{L}_{h}^{\varepsilon}\left(I_{h}\left(w_{h} \eta\right)\right), v_{h}\right)+\left(\mathcal{L}_{h}^{\varepsilon}\left(w_{h} \eta-I_{h}\left(w_{h} \eta\right)\right), v_{h}\right) \\
& +\left[\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, \eta v_{h}\right)-\left(\mathcal{L}_{h}^{\varepsilon} w_{h} \eta, v_{h}\right)\right] \\
= & I_{1}+I_{2}+I_{3}
\end{align*}
$$

We then seek to bound each $I$ in order. To bound $I_{1}$, we will use the definition of $\|\cdot\|_{W_{h}^{-2, p}}$, the stability of $I_{h}$, and Lemma 2.5 to obtain

$$
\begin{align*}
I_{1} & =\left(\mathcal{L}_{h}^{*} v_{h}, I_{h}\left(w_{h} \eta\right)\right) \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}\left\|I_{h}\left(\eta w_{h}\right)\right\|_{W_{h}^{2, p}\left(B_{3}\right)}  \tag{4.22}\\
& \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}{ }_{\left(B_{3}\right)}\left\|\eta w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)} \lesssim \frac{1}{R_{1}^{2}}\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}} .
\end{align*}
$$

For $I_{2}$ we use Lemmas 2.3, 2.7, 4.2 to get

$$
\begin{align*}
I_{2} & =\left(\mathcal{L}_{h}^{\varepsilon}\left(w_{h} \eta-I_{h}\left(w_{h} \eta\right)\right), v_{h}\right) \lesssim\left\|w_{h} \eta-I_{h}\left(w_{h} \eta\right)\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}  \tag{4.23}\\
& \lesssim \frac{h}{R_{1}^{3}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)} \lesssim \frac{1}{R_{1}^{3}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{W^{-1, p^{\prime}\left(B_{3}\right)}}
\end{align*}
$$

To bound $I_{3}$ we introduce the operator $\mathcal{L}_{0, h}^{\varepsilon}$. For $e \in \mathcal{E}_{h}$ let $e_{3}:=e \cap B_{3}$, and define $\tilde{A}:=A-A_{0}$. We then write

$$
\begin{align*}
I_{3}= & \left(\mathcal{L}_{h}^{\varepsilon} w_{h}, \eta v_{h}\right)-\left(\mathcal{L}_{h}^{\varepsilon} w_{h} \eta, v_{h}\right)  \tag{4.24}\\
= & \left(\mathcal{L}_{0, h}^{\varepsilon} w_{h}, \eta v_{h}\right)-\left(\mathcal{L}_{0, h}^{\varepsilon} w_{h} \eta, v_{h}\right)+ \\
& +\left[\left(\mathcal{L}_{h}^{\varepsilon} w_{h}, \eta v_{h}\right)-\left(\mathcal{L}_{h}^{\varepsilon} w_{h} \eta, v_{h}\right)-\left(\mathcal{L}_{0, h}^{\varepsilon} w_{h}, \eta v_{h}\right)+\left(\mathcal{L}_{0, h}^{\varepsilon} w_{h} \eta, v_{h}\right)\right] \\
= & -\int_{B_{3}}\left(w_{h} A_{0}: D^{2} \eta+\left(A_{0}+A_{0}^{T}\right) \nabla \eta \cdot \nabla_{h} w_{h}\right) v_{h} \mathrm{~d} x \\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e_{3}}\left(A_{0} \nabla \eta \cdot \nu_{e}\right)\left\{v_{h}\right\}\left[w_{h}\right] \mathrm{d} S \\
& -\int_{B_{3}}\left(w_{h}\left(\tilde{A}: D^{2} \eta+\left(\tilde{A}+\tilde{A}^{T}\right) \nabla \eta \cdot \nabla_{h} w_{h}\right) v_{h} \mathrm{~d} x\right. \\
& -\varepsilon \sum_{e \in \mathcal{E}_{h}} \int_{e_{3}}\left(\tilde{A} \nabla \eta \cdot \nu_{e}\right)\left\{v_{h}\right\}\left[w_{h}\right] \mathrm{d} S=: K_{1}+K_{2}+K_{3}+K_{4}
\end{align*}
$$

We now must bound each $K_{i}$. To bound $K_{1}$ we use the definition of $\|\cdot\|_{W_{h}^{-1, p}\left(B_{3}\right)}$ and Lemma 2.3 to get

$$
\begin{equation*}
K_{1} \lesssim\left(\left\|w_{h} A_{0}: D^{2} \eta\right\|_{W_{h}^{1, p}\left(B_{3}\right)}+\left\|\left(A_{0}+A_{0}^{T}\right) \nabla \eta \cdot \nabla_{h} w_{h}\right\|_{W_{h}^{1, p}\left(B_{3}\right)}\right)\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}\left(B_{3}\right)} \tag{4.25}
\end{equation*}
$$

$$
\lesssim \frac{1}{R_{1}^{3}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}\left(B_{3}\right)} .
$$

The bound of $K_{2}$ uses Lemmas 2.1, 2.5 to obtain

$$
\begin{align*}
K_{2} & \lesssim \frac{1}{R_{1}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-2 p}\left\|\left[w_{h}\right]\right\|_{L^{p}\left(e_{3}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e} h_{e}^{p^{\prime}}\left\|\left\{v_{h}\right\}\right\|_{L^{p^{\prime}}\left(e_{3}\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}}  \tag{4.26}\\
& \lesssim \frac{1}{R_{1}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left(h\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}\right) \\
& \lesssim \frac{1}{R_{1}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}\left(B_{3}\right)}} .
\end{align*}
$$

We use similar techniques as $(4.25),(4.26)$ and the fact that $\|\tilde{A}\|_{L^{\infty}\left(B_{3}\right)} \leq \delta_{0}$ to get

$$
\begin{align*}
K_{3} & \lesssim\left(\left\|w_{h} \tilde{A}: D^{2} \eta\right\|_{L^{p}\left(B_{3}\right)}+\left\|\left(\tilde{A}+\tilde{A}^{T}\right) \nabla \eta \cdot \nabla_{h} w_{h}\right\|_{L^{p}\left(B_{3}\right)}\right)\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}  \tag{4.27}\\
& \lesssim \delta_{0}\left(\frac{1}{R_{1}^{2}}\left\|w_{h}\right\|_{L^{p}\left(B_{3}\right)}+\frac{1}{R_{1}}\left\|w_{h}\right\|_{W_{h}^{1, p}\left(B_{3}\right)}\right)\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{3}\right)}} \\
& \lesssim \delta_{0}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)},
\end{align*}
$$

where we have used Lemma 2.5 to derive the last inequality. Likewise, we find

$$
\begin{align*}
K_{4} & \lesssim \frac{1}{R_{1}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-2 p}\left\|\left[w_{h}\right]\right\|_{L^{p}\left(e_{3}\right)}^{p}\right)^{\frac{1}{p}}\left(\sum_{e \in \mathcal{E}_{h}} h_{e} h_{e}^{p^{\prime}}\left\|\left\{v_{h}\right\}\right\|_{L^{p^{\prime}}\left(e_{3}\right)}^{p^{\prime}}\right)^{\frac{1}{p^{\prime}}}  \tag{4.28}\\
& \lesssim \frac{\delta_{0}}{R_{1}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left(h\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}\right) \\
& \lesssim \delta_{0}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}\left(B_{3}\right)}}
\end{align*}
$$

where we have used the inequality $h \leq R_{1}$. Combining (4.24)-(4.28) we get

$$
\begin{equation*}
I_{3} \lesssim \frac{1}{R_{1}^{3}}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}\left(B_{3}\right)}+\delta_{0}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)} \tag{4.29}
\end{equation*}
$$

and bringing together (4.21)-(4.23), and (4.29) gives us

$$
\begin{gather*}
\left(\mathcal{L}_{h}^{*}\left(\eta v_{h}\right), w_{h}\right) \lesssim  \tag{4.30}\\
\frac{1}{R_{1}^{3}}\left(\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}+\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}\left(B_{3}\right)}\right)\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}  \tag{4.31}\\
+\delta_{0}\left\|w_{h}\right\|_{W_{h}^{2, p}\left(B_{3}\right)}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)} .
\end{gather*}
$$

By the definition of $\|\cdot\|_{W_{h}^{-2, p^{\prime}}{ }_{\left(B_{3}\right)}}$ and (4.30) we get

$$
\begin{equation*}
\left\|\mathcal{L}_{h}^{*}\left(\eta w_{h}\right)\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)} \lesssim \frac{1}{R_{1}^{3}}\left(\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}{ }_{\left(B_{3}\right)}}+\left\|v_{h}\right\|_{\left.W_{h}^{-1, p^{\prime}}{ }_{\left(B_{3}\right)}\right)}\right)+\delta_{0}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)} . \tag{4.32}
\end{equation*}
$$

Using (4.20) and (4.32) gives us

$$
\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{1}\right)} \lesssim \frac{1}{R_{1}^{3}}\left(\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}\left(B_{3}\right)}+\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}\left(B_{3}\right)}\right)+\delta_{0}\left\|v_{h}\right\|_{L^{p^{\prime}}\left(B_{3}\right)}
$$

Since $\bar{\Omega}$ is compact, employing a covering argument (cf. [11, 13]) then yields

$$
\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}+\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}(\Omega)}+\delta_{0}\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}
$$

Because $\delta_{0}$ is small, we can absorb the last term on the right-hand side to the left-hand side to arrive at the global estimate

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}+\left\|v_{h}\right\|_{W_{h}^{-1, p^{\prime}}(\Omega)} \tag{4.33}
\end{equation*}
$$

which is a Gärding-type inequality.
Step 3: Duality argument on the adjoint operator. To control the last term in (4.33) we now use a duality argument for $\mathcal{L}_{h}^{*}$. This argument uses the regularity estimate of the original problem $\mathcal{L}$.

Define the set

$$
X=\left\{g \in W_{h}^{1, p}(\Omega) ;\|g\|_{W_{h}^{1, p}(\Omega)}=1\right\}
$$

By the discrete Poincaré inequality, with constant $C=C(p, \Omega)$, we have for all $g \in X$

$$
\|g\|_{L^{p}(\Omega)} \leq C\|g\|_{W_{h}^{1, p}(\Omega)}<\infty
$$

since $X$ is bounded in $W_{h}^{1, p}(\Omega)$. Thus, $X$ is precompact in $L^{p}(\Omega)$ by Sobolev embedding. Next we define the set

$$
W=\left\{\varphi:=\mathcal{L}^{-1} g ; g \in X\right\}
$$

Note that $\mathcal{L}^{-1}: L^{p}(\Omega) \rightarrow W^{2, p}(\Omega) \cap W_{0}^{1, p}(\Omega) \subset W^{2, p}\left(\mathcal{T}_{h}\right)$ is well defined by wellposedness of the PDE. Also since $\mathcal{L}^{-1}$ is linear and satisfies the estimate

$$
\left\|\mathcal{L}^{-1} g\right\|_{W_{h}^{2, p}(\Omega)}=\|\varphi\|_{W_{h}^{2, p}(\Omega)} \leq\|\varphi\|_{W^{2, p}(\Omega)} \lesssim\|g\|_{L^{p}(\Omega)}
$$

it is bounded in $W^{2, p}\left(\mathcal{T}_{h}\right)$. Thus $W$ is precompact in $W^{2, p}\left(\mathcal{T}_{h}\right)$. From [24, Lemma 5], for every $\tau>0$ there exists $h_{*}>0$ that only depends on $\tau$ and $\bar{W}$ such that for each $\varphi \in W$ and $0<h \leq h_{*}$ there is a $\varphi_{h} \in V_{h}$ such that if $k \geq 2$ we have

$$
\begin{equation*}
\left\|\varphi-\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)} \leq \tau \tag{4.34}
\end{equation*}
$$

Note by the reverse triangle inequality and (4.34) we have

$$
\left\|\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)} \leq\|\varphi\|_{W_{h}^{2, p}(\Omega)} \lesssim\|g\|_{L^{p}(\Omega)} \leq C
$$

and hence

$$
\left\{\varphi_{h} \in V_{h} ;\left|\varphi_{h}-\varphi\right| \leq \tau\right\}
$$

is uniformly bounded in $\varphi$ and $h$. Let $g \in X$ and choose $\varphi_{g}=\mathcal{L}^{-1} g \in W$ which tells us that $\mathcal{L} \varphi_{g}=g$. Let $v_{h} \in V_{h}$ and $\varphi_{h} \in V_{h}$. By Lemma 4.2 and the definition of $\|\cdot\|_{W_{h}^{-2, p^{\prime}}(\Omega)}$ we have

$$
\int_{\Omega} v_{h} g \mathrm{~d} x=\left(\mathcal{L}_{h}^{\varepsilon} \varphi_{g}, v_{h}\right)=\left(\mathcal{L}_{h}^{\varepsilon} \varphi_{h}, v_{h}\right)+\left(\mathcal{L}_{h}^{\varepsilon}\left(\varphi_{g}-\varphi_{h}\right), v_{h}\right)
$$

$$
\begin{aligned}
& =\left(\mathcal{L}_{h}^{*} v_{h}, \varphi_{h}\right)+\left(\mathcal{L}_{h}^{\varepsilon}\left(\varphi_{g}-\varphi_{h}\right), v_{h}\right) \\
& \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}\left\|\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)}+\left\|\varphi_{g}-\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)}\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}
\end{aligned}
$$

Selecting $\varphi_{h}$ to satisfy (4.34) and taking the supremum on $g$ gives us

$$
\begin{equation*}
\left\|v_{h}\right\|_{W^{-1, p}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}\left\|\varphi_{h}\right\|_{W_{h}^{2, p}(\Omega)}+\tau\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \tag{4.35}
\end{equation*}
$$

Combining (4.33) and (4.35) yields

$$
\begin{equation*}
\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{*} v_{h}\right\|_{W_{h}^{-2, p^{\prime}}(\Omega)}+\tau\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)} \tag{4.36}
\end{equation*}
$$

By choosing $\tau$ sufficiently small to kick back the right-most term we have (4.12). This completes the proof upon taking $h_{0}=\min \left\{h_{\delta_{0}}, h_{*}\right\}$.

We are now ready to prove the global stability of the operator $\mathcal{L}_{h}^{\varepsilon}$.
THEOREM 4.4. Suppose that $h \leq h_{0}$ and $k \geq 2$. Then there holds the following stability estimate:

$$
\begin{equation*}
\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{\varepsilon} w_{h}\right\|_{L_{h}^{p}(\Omega)} \quad \forall w_{h} \in V_{h} \tag{4.37}
\end{equation*}
$$

where $1<p<\infty$ if $\varepsilon=1$, and $p=2$ if $\varepsilon \in\{0,-1\}$.
Proof. Let $w_{h} \in V_{h}$ be fixed, and consider the auxiliary problem of finding $q_{h} \in V_{h}$ such that

$$
\begin{align*}
& \left(v_{h}, \mathcal{L}_{h}^{*} q_{h}\right)=\left(\mathcal{L}_{h}^{\varepsilon} v_{h}, q_{h}\right)=\int_{\Omega}\left|D_{h}^{2} w_{h}\right|^{p-2} D_{h}^{2} w_{h}: D^{2} v_{h} \mathrm{~d} x  \tag{4.38}\\
& \quad+\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-p} \int_{e}\left|\left[\nabla w_{h}\right]\right|^{p-2}\left[\nabla w_{h}\right] \cdot\left[\nabla v_{h}\right] \mathrm{d} S \\
& \quad+\sum_{e \in \mathcal{E}_{h}} h_{e}^{1-2 p} \int_{e}\left|\left[w_{h}\right]\right|^{p-2}\left[w_{h}\right]\left[v_{h}\right] \mathrm{d} S \quad \forall v_{h} \in V_{h}
\end{align*}
$$

Since $V_{h}$ is finite dimensional and the operator is linear, the existence is equivalent to the uniqueness. To show the uniqueness, let $q_{h}^{(1)}$ and $q_{h}^{(2)}$ both solve (4.38). Then by Lemma 4.3 we get

$$
\left\|q_{h}^{(1)}-q_{h}^{(2)}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim \sup _{0 \neq v_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} v_{h}, q_{h}^{(1)}-q_{h}^{(2)}\right)}{\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)}}=0
$$

Hence, (4.38) has a unique solution $q_{h} \in V_{h}$. Also by Lemma 4.3 and Hölder's inequality,

$$
\left\|q_{h}\right\|_{L^{p^{\prime}}(\Omega)} \lesssim \sup _{0 \neq v_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} v_{h}, q_{h}\right)}{\left\|v_{h}\right\|_{W_{h}^{2, p}(\Omega)}} \lesssim\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}^{p-1}
$$

Consequently, we find

$$
\begin{aligned}
\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}^{p} & \lesssim\left(w_{h}, \mathcal{L}_{h}^{*} q_{h}\right)=\left(\mathcal{L}_{h} w_{h}, q_{h}\right) \lesssim\left\|\mathcal{L}_{h} w_{h}\right\|_{L_{h}^{p}(\Omega)}\left\|q_{h}\right\|_{L^{p^{\prime}}(\Omega)} \\
& \lesssim\left\|\mathcal{L}_{h} w_{h}\right\|_{L_{h}^{p}(\Omega)}\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}^{p-1}
\end{aligned}
$$

Dividing by $\left\|w_{h}\right\|_{W_{h}^{2, p}(\Omega)}^{p-1}$ now yields the desired result.
4.3. Well-posedness and error estimates. The goals of this subsection are to establish the well-posedness for the IP-DG scheme (4.6) and to derive the optimal order error estimates in $W_{h}^{2, p}$-norm for the IP-DG solutions.

THEOREM 4.5. Under the assumptions of Lemma 4.3, the IP-DG scheme (4.6) has a unique solution $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
\left\|u_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\|f\|_{L^{p}(\Omega)} \tag{4.39}
\end{equation*}
$$

Proof. Since (4.6) is equivalent to a linear system, hence it suffices to prove the uniqueness. To show the uniqueness, we first prove (4.39).

Let $u_{h} \in V_{h}$ be a solution of (4.6), then from (4.37) and the definition of $\|\cdot\|_{L_{h}^{p}(\Omega)}$ we have

$$
\left\|u_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\left\|\mathcal{L}_{h}^{\varepsilon} u_{h}\right\|_{L_{h}^{p}(\Omega)}=\sup _{v_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon} u_{h}, v_{h}\right)}{\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}}=\sup _{v_{h} \in V_{h}} \frac{\left(f, v_{h}\right)}{\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}} \leq\|f\|_{L^{p}(\Omega)}
$$

Hence, (4.39) holds.
Suppose that $u_{h}^{1}, u_{h}^{2} \in V_{h}$ solve (4.6). Let $\tilde{u}_{h}=u_{h}^{1}-u_{h}^{2}$. Then by (4.39) we have

$$
\left\|\tilde{u}_{h}\right\|_{W_{h}^{2, p}(\Omega)} \leq\|0\|_{L^{p}(\Omega)}=0
$$

Since $\tilde{u}_{h} \in V_{h}$ with $\left\|\tilde{u}_{h}\right\|_{W_{h}^{2, p}(\Omega)}=0$ we conclude that $\tilde{u}_{h} \in C^{1}(\Omega),\left.\tilde{u}_{h}\right|_{\partial \Omega}=0$, and $D_{h}^{2} \tilde{u}_{h}=0$ in $\Omega$. The only way this can happen is if $\tilde{u}_{h}=0$. Thus, the IP-DG solution must be unique. The proof is complete.

Next we show a Céa-type lemma for the IP-DG scheme, which immediately deduces the optimal order error estimates in the $W_{h}^{2, p}$-norm.

THEOREM 4.6. Suppose that $h \leq h_{0}$ and $k \geq 2$. Let $u \in W^{2, p} \cap W_{0}^{1, p}(\Omega)$ be the solution of problem (1.1) and $u_{h} \in V_{h}$ solve (4.6). Then

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim \inf _{w_{h} \in V_{h}}\left\|u-w_{h}\right\|_{W_{h}^{2, p}(\Omega)} \tag{4.40}
\end{equation*}
$$

Moreover, if $u \in W^{s, p}(\Omega)$ for some $s \geq 2$, we have

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim h^{r-2}\|u\|_{W^{r, p}(\Omega)}, \quad r=\min \{s, k+1\} \tag{4.41}
\end{equation*}
$$

Proof. By the consistency of $\mathcal{L}_{h}^{\varepsilon}$ we have the following Galerkin orthogonality:

$$
\begin{equation*}
\left(\mathcal{L}_{h}^{\varepsilon}\left(u-u_{h}\right), v_{h}\right)=0 \quad \forall v_{h} \in V_{h} \tag{4.42}
\end{equation*}
$$

Let $w_{h} \in V_{h}$, by Theorem 4.4, Lemma 4.2, (4.42), and the definition of $\|\cdot\|_{L_{h}^{p}(\Omega)}$ we have

$$
\begin{align*}
\left\|u_{h}-w_{h}\right\|_{W_{h}^{2, p}(\Omega)} & \lesssim\left\|\mathcal{L}_{h}^{\varepsilon}\left(u_{h}-w_{h}\right)\right\|_{L_{h}^{p}(\Omega)}=\sup _{0 \neq v_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon}\left(u_{h}-w_{h}\right), v_{h}\right)}{\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}}  \tag{4.43}\\
& =\sup _{0 \neq v_{h} \in V_{h}} \frac{\left(\mathcal{L}_{h}^{\varepsilon}\left(u-w_{h}\right), v_{h}\right)}{\left\|v_{h}\right\|_{L^{p^{\prime}}(\Omega)}}=\left\|\mathcal{L}_{h}^{\varepsilon}\left(u-w_{h}\right)\right\|_{L_{h}^{p}(\Omega)} \\
& \lesssim\left\|u-w_{h}\right\|_{W_{h}^{2, p}(\Omega)} .
\end{align*}
$$

Thus by (4.43) and the triangle inequality we get

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{W_{h}^{2, p}(\Omega)} \leq\left\|u-w_{h}\right\|_{W_{h}^{2, p}(\Omega)}+\left\|u_{h}-w_{h}\right\|_{W_{h}^{2, p}(\Omega)} \lesssim\left\|u-w_{h}\right\|_{W_{h}^{2, p}(\Omega)} \tag{4.44}
\end{equation*}
$$

Taking the infimum on both sides over all $w_{h} \in V_{h}$ yields (4.40). Finally, (4.41) follows from taking $w_{h}=I_{h} u$ and using the finite element interpolation theory [3]. The proof is complete.
5. Numerical Experiments. In this section we present a number of 2-D numerical tests to verify our error estimate and to gauge the performance of our IP-DG methods. In particular, we shall compare our IP-DG methods to the related conforming finite element counterpart developed in [11]. Moreover, we shall also perform numerical tests which are not covered by our convergence theory, this includes the cases when the coefficient matrix is either discontinuous or degenerate.
5.1. Hölder continuous coefficient. For this test we take $A$ as the following Hölder continuous matrix-valued function:

$$
A(x)=\left[\begin{array}{cc}
|x|^{1 / 2}+1 & -|x|^{1 / 2} \\
-|x|^{1 / 2} & 5|x|^{1 / 2}+1
\end{array}\right], \quad x \in \mathbb{R}^{2}
$$

Let $\Omega=(-1 / 2,1 / 2)^{2}$ and choose $f$ such that the exact solution is given by

$$
u\left(x_{1}, x_{2}\right)=\sin \left(2 \pi x_{1}\right) \sin \left(2 \pi x_{2}\right) \exp \left(x_{1} \cos \left(x_{2}\right)\right)
$$

which has zero trace on the boundary.
Figure 5.1 shows the errors in the $L^{2}(\Omega), W_{h}^{1,2}(\Omega)$, and $W_{h}^{2,2}(\Omega)$ norms of both the symmetrically and incompletely induced methods. The convergence rates observed for the symmetrically induced method are

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{k+1}\right) \text { for all } k, \\
\left\|\nabla_{h}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{k}\right) \text { for all } k \\
\left\|D_{h}^{2}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{k-1}\right) \text { for } k=2,3
\end{aligned}
$$

As expected, these convergence rates are optimal. However, for the incompletely induced method we find that the rate of convergence in the $L^{2}$-norm is sub-optimal for even degree polynomials and optimal with all other norms and degrees. This should be expected since the incomplete scheme is sub-optimal even for smooth $A$ [22].
5.2. Uniformly continuous coefficients. In this test we take $\Omega=(0,1 / 2)^{2}$ and let

$$
A(x)=\left[\begin{array}{cc}
-\frac{5}{\log (|x|)}+15 & 1 \\
1 & -\frac{1}{\log (|x|)}+3
\end{array}\right]
$$

$f$ is chosen such that $u(x)=|x|^{7 / 4}$ is the exact solution. From [11] we see that the expected convergence rates are

$$
\begin{aligned}
\left\|\nabla_{h}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{\min \{k, 7 / 4-\delta\}}\right) \text { for all } k \\
\left\|D_{h}^{2}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{\min \{k, 7 / 4-\delta\}-1}\right) \text { for } k=2,3
\end{aligned}
$$

for any $\delta>0$.
Figure 5.2 gives the computed results for both the symmetrically and incompletely induced schemes which match exactly the expected rates of convergence.


Fig. 5.1. The $L^{2}$ (top), piecewise $H^{1}$ (middle), and piecewise $H^{2}$ (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k=1,2,3$. $\gamma_{e} \equiv 100$ is used as the penalty parameter.
5.3. Degenerate coefficients. In this test we take $\Omega=(0,1)^{2}$ and the matrix

$$
A(x)=\frac{16}{9}\left[\begin{array}{cc}
x_{1}^{2 / 3} & -x_{1}^{1 / 3} x_{2}^{1 / 3} \\
-x_{1}^{1 / 3} x_{2}^{1 / 3} & x_{2}^{2 / 3}
\end{array}\right] .
$$

$f=0$ and the exact solution $u(x)=x_{1}^{4 / 3}-x_{2}^{4 / 3}$. For an explanation for this example we refer to [11]. Note that $\operatorname{det}(A)=0$ for every $x \in \Omega$ so this PDE is degenerate everywhere and is outside the case considered in this paper. We also observe that $u \in W^{m, p}(\Omega)$ provided $(4-3 m) p>-1$.

Figure 5.3 shows the $L^{2}$ and piecewise $H^{1}$ errors for both the symmetrically and incompletely induced methods. The numerical results suggest the following rates of


Fig. 5.2. The piecewise $H^{1}$ (top) and piecewise $H^{2}$ (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k=1,2,3 . \gamma_{e} \equiv 1000$ is used as the penalty parameter.
convergence:

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{4 / 3}\right) \\
\left\|\nabla_{h}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)} & =\mathcal{O}\left(h^{5 / 6}\right)
\end{aligned}
$$

for $k=1,2,3$. These rates are consistent with the results of the related conforming finite element method given in [11].
5.4. $L^{\infty}$ Cordès coefficients. Our next test is taken from [26, 28] where a different DG method and a weak Galerkin method were used to solve this problem. Let $\Omega=[-1,1]^{2}$ and

$$
A(x)=\frac{16}{9}\left[\begin{array}{cc}
2 & x_{1} x_{2} /\left|x_{1} x_{2}\right| \\
x_{1} x_{2} /\left|x_{1} x_{2}\right| & 2
\end{array}\right]
$$

$f$ is chosen so that the exact solution is $u(x)=x_{1} x_{2}\left(1-e^{1-\left|x_{1}\right|}\right)\left(1-e^{1-\left|x_{2}\right|}\right)$. Notice that the matrix $A$ is discontinuous across the $x_{1}$-axis and $x_{2}$-axis, and it satisfies the Cordès condition. While our convergence theory does not apply to this example, we still compute the numerical solution on a uniform triangulation that has edges on all discontinuities of $A$. Due to its inconsistent behavior we list the $L^{2}$ error and convergence rates in Table 5.1. The following $H^{1}$ semi-norm rates are observed:

$$
\left\|\nabla_{h}\left(u-u_{h}\right)\right\|_{L^{2}(\Omega)}=\mathcal{O}\left(h^{k}\right)
$$



FIG. 5.3. The $L^{2}$ (top) and piecewise $H^{1}$ (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k=1,2,3 . \gamma_{e} \equiv 100$ is used as the penalty parameter.

TABLE 5.1
The $L^{2}$ errors and rates for the symmetrically induced method. The rates for the incompletely induced method are similar. $\gamma_{e} \equiv 10000$ is used as the penalty parameter.

|  | $k=1$ |  | $k=2$ |  | $k=3$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $h$ | $\left\\|u-u_{h}\right\\|_{L^{2}(\Omega)}$ | rate | $\left\\|u-u_{h}\right\\|_{L^{2}(\Omega)}$ | rate | $\left\\|u-u_{h}\right\\|_{L^{2}(\Omega)}$ | rate |
| 1 | $1.3 \mathrm{e}-1$ | - | $7.7 \mathrm{e}-2$ | - | $2.6 \mathrm{e}-2$ | - |
| $1 / 2$ | $8.9 \mathrm{e}-2$ | 0.58 | $1.8 \mathrm{e}-2$ | 2.09 | $1.5 \mathrm{e}-3$ | 4.12 |
| $1 / 4$ | $4.6 \mathrm{e}-2$ | 0.95 | $2.9 \mathrm{e}-3$ | 2.62 | $7.6 \mathrm{e}-4$ | 4.27 |
| $1 / 8$ | $1.9 \mathrm{e}-2$ | 1.22 | $4.8 \mathrm{e}-2$ | 2.62 | $4.2 \mathrm{e}-6$ | 4.19 |
| $1 / 16$ | $7.6 \mathrm{e}-3$ | 1.35 | $8.0 \mathrm{e}-5$ | 2.57 | $3.3 \mathrm{e}-7$ | 3.65 |
| $1 / 32$ | $2.9 \mathrm{e}-3$ | 1.41 | $1.4 \mathrm{e}-5$ | 2.54 | $3.2 \mathrm{e}-8$ | 3.36 |

for $k=1,2,3$ as shown in Figure 5.4.

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Fig. 5.4. The $L^{2}$ (top) and piecewise $H^{1}$ (bottom) errors for both the symmetrically (left) and incompletely (right) induced schemes with polynomial degree $k=1,2,3 . \gamma_{e} \equiv 10000$ is used as the penalty parameter.

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## Appendix A. Proof of Lemma 2.7.

Proof. From [1, Lemma 3] we have the following estimates for $I_{h}$

$$
\begin{equation*}
h^{m p}\left|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right|_{W^{m, p}(T)}^{p} \lesssim h^{p(k+1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p}, \quad 0 \leq m \leq k+1 \tag{A.1}
\end{equation*}
$$

By the assumptions on $\eta$, the fact that $\left|v_{h}\right|_{W^{k+1, p}(T)}=0$, and a standard inverse inequality we get

$$
\begin{align*}
\left|\eta v_{h}\right|_{W^{k+1, p}(T)} & \lesssim \sum_{|\alpha|+|\beta|=k+1} \int_{T}\left|D^{\alpha} \eta\right|^{p}\left|D^{\beta} v_{h}\right|^{p} \mathrm{~d} x  \tag{A.2}\\
& \lesssim \sum_{j=0}^{k} \frac{1}{d^{p(k+1-j}}\left|v_{h}\right|_{W^{j, p}(T)}^{p} \lesssim \sum_{j=0}^{k} \frac{h^{-j p}}{d^{p(k+1-j)}}\left\|v_{h}\right\|_{L^{p}(T)}^{p}
\end{align*}
$$

It follows from (A.1) and (A.2) with $h \leq d$ that

$$
h^{m p}\left|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right|_{W^{m, p}(T)}^{p} \lesssim \sum_{j=0}^{k} \frac{h^{p(k+1-j)}}{d^{p(k+1-j)}}\left\|v_{h}\right\|_{L^{p}(T)}^{p} \lesssim \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}(T)}
$$

Thus we have

$$
\begin{aligned}
\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p}(D)} & \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}}\left|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right|_{L^{p}(T)}^{p} \\
& \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}(T)} \lesssim \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)},
\end{aligned}
$$

$$
\begin{aligned}
h^{p}\left\|\nabla_{h}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(D)} & \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} h^{p}\left|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right|_{W^{1, p}(T)}^{p} \\
& \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}(T)} \lesssim \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)}, \\
h^{2 p}\left\|\nabla_{h}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(D)} & \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} h^{2 p}\left|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right|_{W^{2, p}(T)}^{p} \\
& \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}(T)} \lesssim \frac{h^{p}}{d^{p}}\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)}
\end{aligned}
$$

Hence (2.19), (2.20), and (2.21) hold.
To show (2.22), using (A.2) and an inverse estimate we have

$$
\begin{align*}
h^{p(k-1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p} & \lesssim \sum_{j=0}^{k} \frac{h^{p(k-1)}}{d^{p(k+1-j)}}\left|v_{h}\right|_{W^{j, p}(T)}^{p}  \tag{A.3}\\
& \lesssim \frac{1}{d^{2 p}}\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\sum_{j=1}^{k} \frac{h^{p(k-j)}}{d^{p(k+1-j)}}\left|v_{h}\right|_{W^{1, p}(T)}^{p} \\
& \lesssim \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left|v_{h}\right|_{W^{1, p}(T)}^{p}\right) .
\end{align*}
$$

It follows from (A.1) and (A.3) that

$$
\begin{aligned}
&\left\|D^{2}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(T)}^{p} \lesssim h^{p(k-1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p} \\
& \lesssim \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left|v_{h}\right|_{W^{1, p}(T)}^{p}\right), \\
& h^{-p}\left\|\nabla\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(T)}^{p} \lesssim h^{p(k-1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p} \lesssim \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left|v_{h}\right|_{W^{1, p}(T)}^{p}\right) \\
& h^{-2 p}\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p}(T)}^{p} \lesssim h^{p(k-1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p} \lesssim \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left|v_{h}\right|_{W^{1, p}(T)}^{p}\right) .
\end{aligned}
$$

Using the previous three estimates and Lemma 2.1 we get

$$
\begin{aligned}
& \| \eta v_{h}-I_{h}\left(\eta v_{h}\right)\left\|_{W_{h}^{2, p}(D)}^{p} \lesssim \sum_{T \in \mathcal{T}_{h}}\right\| D^{2}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right) \|_{L^{p}(T)}^{p} \\
&+\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-p}\left\|\left[\nabla\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right]\right\|_{L^{p}(e)}^{p}+\sum_{e \in \mathcal{E}_{h}^{I}} h_{e}^{1-2 p}\left\|\left[\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right]\right\|_{L^{p}(e)}^{p} \\
&+\sum_{e \in \mathcal{E}_{h}^{B}} h_{e}^{1-2 p}\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p}(e)}^{p} \\
& \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}}\left\|D^{2}\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(T)}^{p}+\sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} h^{-p}\left\|\nabla\left(\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right)\right\|_{L^{p}(T)}^{p} \\
& \quad+\sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} h^{-2 p}\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{L^{p}(T)}^{p} \\
& \lesssim \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}} \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left|v_{h}\right|_{W^{1, p}(T)}^{p}\right) \lesssim \frac{1}{d^{2 p}}\left(\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\left\|\nabla_{h} v_{h}\right\|_{L^{p}(T)}^{p}\right) .
\end{aligned}
$$

Thus, (2.22) holds.
Finally, the proof of (2.23) is similar to that of (2.22) except one minor detail. Since $k \geq 2$, by (A.2) and an inverse inequality we get

$$
\begin{align*}
& h^{p(k-1)}\left|\eta v_{h}\right|_{W^{k+1, p}(T)}^{p} \lesssim \sum_{j=0}^{k} \frac{h^{p(k-1)}}{d^{p(k+1-j)}}\left|v_{h}\right|_{W^{j, p}(T)}^{p}  \tag{A.4}\\
& \quad=h^{p}\left(\sum_{j=0}^{k} \frac{h^{p(k-2)}}{d^{p(k+1-j)}}\left|v_{h}\right|_{W^{j, p}(T)}^{p}\right) \\
& \quad=h^{p}\left(\frac{1}{d^{3 p}}\left\|v_{h}\right\|_{L^{p}(T)}^{p}+\frac{1}{d^{2 p}}\left|v_{h}\right|_{W^{1, p}(T)}^{p}+\sum_{j=2}^{k} \frac{h^{p(k-j)}}{d^{p(k+1-j)}}|v|_{W^{2, p}(T)}^{p}\right) \\
& \quad \lesssim \frac{h^{p}}{d^{3 p}}\left\|v_{h}\right\|_{W^{2, p}(T)}^{p} .
\end{align*}
$$

Thus, we can obtain (as in the derivation on (2.22)) using Lemma 2.5 that

$$
\begin{aligned}
\left\|\eta v_{h}-I_{h}\left(\eta v_{h}\right)\right\|_{W_{h}^{2, p}(D)} & \lesssim \frac{h}{d^{3}} \sum_{\substack{T \in \mathcal{T}_{h} \\
T \cap D \neq \emptyset}}\left\|v_{h}\right\|_{W^{2, p}(T)} \\
& =\frac{h}{d^{3}}\left(\left\|v_{h}\right\|_{L^{p}\left(D_{h}\right)}+\left\|\nabla_{h} v_{h}\right\|_{L^{p}\left(D_{h}\right)}+\left\|D_{h}^{2} v_{h}\right\|_{L^{p}\left(D_{h}\right)}\right) \\
& \lesssim \frac{h}{d^{3}}\left\|v_{h}\right\|_{W_{h}^{2, p}\left(D_{h}\right)} .
\end{aligned}
$$

The proof is complete. $\quad \square$


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