# An HDG method with orthogonal projections in facet integrals 

Issei Oikawa

Received: date / Accepted: date


#### Abstract

We propose and analyze a new hybridizable discontinuous Galerkin (HDG) method for second-order elliptic problems. Our method is obtained by inserting the $L^{2}$-orthogonal projection onto the approximate space for a numerical trace into all facet integrals in the usual HDG formulation. The orders of convergence for all variables are optimal if we use polynomials of degree $k+l, k+1$ and $k$, where $k$ and $l$ are any non-negative integers, to approximate the vector, scalar and trace variables, which implies that our method can achieve superconvergence for the scalar variable without postprocessing. Numerical results are presented to verify the theoretical results.


Keywords Discontinuous Galerkin • Hybridization • Superconvergence
Mathematics Subject Classification (2000) 65N12 • 65N30

## 1 Introduction

In this paper, we propose a new hybridizable discontinuous Galerkin (HDG) method for second-order elliptic problems. For simplicity, the following diffusion problem is considered:

$$
\begin{align*}
& \boldsymbol{q}+\nabla u=0 \text { in } \Omega,  \tag{1a}\\
& \nabla \cdot \boldsymbol{q}=f \text { in } \Omega,  \tag{1b}\\
& u=0  \tag{1c}\\
& \text { on } \partial \Omega,
\end{align*}
$$

This work was supported by JSPS KAKENHI Numbers 15H03635, 15K13454, 17K14243 and 17 K 18738.

Issei Oikawa
Waseda Research Institute for Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan
E-mail: oikawa@aoni.waseda.jp
where $\Omega \subset \mathbb{R}^{d}(d=2,3)$ is a bounded and convex polygonal or polyhedral domain and $f$ is a given $L^{2}$-function.

To begin with, let us define notations for the description of the standard HDG method. Let $\mathcal{T}_{h}$ be a mesh of $\Omega$, which consists of polygons or polyhedrons, where $h$ stands for the mesh size. Let $\mathcal{E}_{h}$ denote the set of faces of all elements in $\mathcal{T}_{h}$. A family of meshes $\left\{\mathcal{T}_{h}\right\}_{h}$ is assumed to satisfy the chunkiness condition [2], under which the trace and inverse inequalities hold. We use the usual notation of the Sobolev spaces [1], such as $H^{m}(D),\|w\|_{m, D}:=\|w\|_{H^{m}(D)}$, $|w|_{m, D}:=|w|_{H^{m}(D)}$ for an integer $m$ and a domain $D \subset \mathbb{R}^{d}$. When $D=\Omega$ or $m=0$, we omit the domain or the index and simply write $\|w\|_{m}=\|w\|_{m, \Omega}$, $|w|_{m}=|w|_{m, \Omega}$ and $\|w\|=\|w\|_{0, \Omega}$. The piecewise or broken Sobolev space of order $m$ is defined by $H^{m}\left(\mathcal{T}_{h}\right):=\left\{v \in L^{2}(\Omega):\left.v\right|_{K} \in H^{m}(K) \forall K \in \mathcal{T}_{h}\right\}$. We denote by $L^{2}\left(\mathcal{E}_{h}\right)$ the $L^{2}$ space on the union of all faces of $\mathcal{E}_{h}$ and by $P_{k}\left(\mathcal{T}_{h}\right)$ the space of piecewise polynomials of degree $k$. The piecewise inner products are written as

$$
(u, v)_{\mathcal{T}_{h}}=\sum_{K \in \mathcal{T}_{h}} \int_{K} u v d x, \quad\langle u, v\rangle_{\partial \mathcal{T}_{h}}=\sum_{K \in \mathcal{T}_{h}} \int_{\partial K} u v d s
$$

The induced piecewise norm are denoted as $\|v\|_{\mathcal{T}_{h}}=(v, v)_{\mathcal{T}_{h}}^{1 / 2}$ and $\|v\|_{\partial \mathcal{T}_{h}}=$ $\langle v, v\rangle_{\partial \mathcal{T}_{h}}^{1 / 2}$, and the piecewise Sobolev seminorm is defined by $|w|_{1, \mathcal{T}_{h}}=\left(\sum_{K \in \mathcal{T}_{h}}|w|_{1, K}^{2}\right)^{1 / 2}$.

Throughout the paper, we will use the symbol $C$ to denote generic constants independent of $h$. Vector variables and function spaces are displayed in boldface, such as $\boldsymbol{P}_{k}\left(\mathcal{T}_{h}\right)=P_{k}\left(\mathcal{T}_{h}\right)^{d}$.

We define finite element spaces for $\boldsymbol{q}, u$ and the trace of $u$ by

$$
\begin{aligned}
& \boldsymbol{V}_{h}=\left\{\boldsymbol{v} \in \boldsymbol{L}^{2}(\Omega):\left.\boldsymbol{v}\right|_{K} \in \boldsymbol{V}(K) \forall K \in \mathcal{T}_{h}\right\}, \\
& W_{h}=\left\{w \in L^{2}(\Omega):\left.w\right|_{K} \in P_{k+1}(K) \forall K \in \mathcal{T}_{h}\right\}, \\
& M_{h}=\left\{\mu \in L^{2}\left(\mathcal{E}_{h}\right):\left.\mu\right|_{F} \in P_{k}(F) \forall F \in \mathcal{E}_{h}\right\},
\end{aligned}
$$

respectively, where $\boldsymbol{V}(K)$ is a finite-dimensional spaces satisfying $P_{k}(K) \subset$ $\boldsymbol{V}(K)$. The $L^{2}$-orthogonal projections onto $\boldsymbol{V}_{h}, W_{h}$ and $M_{h}$ are denoted by $\boldsymbol{P}_{\boldsymbol{V}}, P_{W}$ and $P_{M}$, respectively. We simply write $P_{M} w=P_{M}\left(\left.w\right|_{\mathcal{E}_{h}}\right)$ for $w \in$ $H^{2}\left(\mathcal{T}_{h}\right)$. Note that $P_{M} w$ may not belong to $M_{h}$ since it is double-valued in general.

The standard HDG method reads as follows: Find $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times$ $M_{h}$ such that

$$
\begin{array}{rlrl}
\left(\boldsymbol{q}_{h}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left(u_{h}, \nabla \cdot \boldsymbol{v}\right)_{\mathcal{T}_{h}}+ & \left\langle\widehat{u}_{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} & =0 & \\
-\left(\boldsymbol{q}_{h}, \nabla w\right)_{\mathcal{T}_{h}}+ & \left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, w\right\rangle_{\partial \mathcal{T}_{h}} & =(f, w) & \\
& \left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}} & =0 &  \tag{2c}\\
, & \forall \mu \in W_{h}, \\
, ~
\end{array}
$$

where $\widehat{\boldsymbol{q}}_{h}$ is the numerical flux defined by

$$
\begin{equation*}
\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}=\boldsymbol{q}_{h} \cdot \boldsymbol{n}+\tau\left(u_{h}-\widehat{u}_{h}\right) . \tag{3}
\end{equation*}
$$

Here, $\tau$ is a positive parameter and is set to be of order $O\left(h^{-1}\right)$ in the paper.

The so-called Lehrenfeld-Schöberl (LS) numerical flux [8] is obtained by inserting $P_{M}$ into the stabilization part of the numerical flux:

$$
\widehat{\boldsymbol{q}}_{h}^{L S} \cdot \boldsymbol{n}=\boldsymbol{q}_{h} \cdot \boldsymbol{n}+\tau\left(P_{M} u_{h}-\widehat{u}_{h}\right) .
$$

In [10, it was proved that the HDG method using the LS numerical flux (the HDG-LS method) achieves optimal-order convergence for all variables if we use polynomials of degree $k, k+1$ and $k$ for $\boldsymbol{V}_{h}, W_{h}$ and $M_{h}$, respectively. It can be said that the HDG-LS method is superconvergent for the scalar variable $u$ without postprocessing. Another more elaborate flux was introduced in the hybrid higher-order ( HHO ) method [6, 5], which was recently linked to the HDG method in [4]. The LS numerical flux approach has been applied to various problems; linear elasticity [12], convection-diffusion problems [13], Stokes equations [11, Navier-Stokes equations [14, 9] and Maxwell's equations [3].

Let us here point out that the superconvergence of the HDG-LS method is sensitive to the choice of $\boldsymbol{V}_{h}$. For example, the superconvergence property is no longer maintained if $\boldsymbol{V}_{h}$ is taken to be $\boldsymbol{P}_{k+1}\left(\mathcal{T}_{h}\right)$ instead of $\boldsymbol{P}_{k}\left(\mathcal{T}_{h}\right)$. We now demonstrate that by numerical experiments for the test problem (19) which will be provided in Section 4. The numerical results are shown in Table 1 In the case of $\boldsymbol{V}_{h}=\boldsymbol{P}_{1}\left(\mathcal{T}_{h}\right)$, the orders of convergence are optimal for both variables $\boldsymbol{q}$ and $u$. On the other hand, all the orders become sub-optimal for $\boldsymbol{V}_{h}=\boldsymbol{P}_{2}\left(\mathcal{T}_{h}\right)$.

Table 1 Convergence history of the HDG-LS method for $\boldsymbol{V}_{h} \times W_{h} \times M_{h}=\boldsymbol{P}_{l}\left(\mathcal{T}_{h}\right) \times$ $P_{2}\left(\mathcal{T}_{h}\right) \times P_{1}\left(\mathcal{E}_{h}\right)$

|  |  | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|$ |  | $\left\\|u-u_{h}\right\\|$ |  | $\left\\|h^{-1 / 2}\left(P_{M} u_{h}-\widehat{u}_{h}\right)\right\\|_{\partial \mathcal{T}_{h}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $l$ | $1 / h$ | Error | Order | Error | Order | Error | Order |
| 1 | 10 | $1.236 \mathrm{E}-02$ | - | $7.400 \mathrm{E}-04$ | - | $2.719 \mathrm{E}-02$ | - |
|  | 20 | $3.083 \mathrm{E}-03$ | 2.00 | $9.085 \mathrm{E}-05$ | 3.03 | $6.676 \mathrm{E}-03$ | 2.03 |
|  | 40 | $7.655 \mathrm{E}-04$ | 2.01 | $1.140 \mathrm{E}-05$ | 2.99 | $1.662 \mathrm{E}-03$ | 2.01 |
|  | 80 | $1.915 \mathrm{E}-04$ | 2.00 | $1.414 \mathrm{E}-06$ | 3.01 | $4.113 \mathrm{E}-04$ | 2.01 |
| 2 | 10 | $1.464 \mathrm{E}-01$ | - | $2.738 \mathrm{E}-03$ | - | $1.064 \mathrm{E}-02$ | - |
|  | 20 | $7.177 \mathrm{E}-02$ | 1.03 | $6.517 \mathrm{E}-04$ | 2.07 | $4.999 \mathrm{E}-03$ | 1.09 |
|  | 40 | $3.543 \mathrm{E}-02$ | 1.02 | $1.568 \mathrm{E}-04$ | 2.06 | $2.363 \mathrm{E}-03$ | 1.08 |
|  | 80 | $1.744 \mathrm{E}-02$ | 1.02 | $3.815 \mathrm{E}-05$ | 2.04 | $1.180 \mathrm{E}-03$ | 1.00 |

The aim of the paper is to recover the superconvergence property for such cases. The key idea is to insert the orthogonal projection $P_{M}$ into the facet integrals in the usual HDG formulation. The resulting method can achieve optimal convergence in $\boldsymbol{q}$ and superconvergence in $u$ without postprocessing if $\boldsymbol{V}_{h}$ contains $\boldsymbol{P}_{k}\left(\mathcal{T}_{h}\right)$, see Theorems 1 and 2,

The rest of the paper is organized as follows. In Section 2, we introduce a new HDG method. In Section 3, error estimates for both variables $u$ and $\boldsymbol{q}$ are provided. Numerical results are presented to verify our theoretical results in Section 4.

## 2 An HDG method with orthogonal projections

We begin by introducing our method: Find $\left(\boldsymbol{q}_{h}, u_{h}, \widehat{u}_{h}\right) \in \boldsymbol{V}_{h} \times W_{h} \times M_{h}$ such that

$$
\begin{align*}
\left(\boldsymbol{q}_{h}+\nabla u_{h}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle P_{M} u_{h}-\widehat{u}_{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} & =0, & & \forall \boldsymbol{v} \in \boldsymbol{V}_{h},  \tag{4a}\\
-\left(\boldsymbol{q}_{h}, \nabla w\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}} & =(f, w), & & \forall w \in W_{h},  \tag{4b}\\
\left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}} & =0, & & \forall \mu \in M_{h}, \tag{4c}
\end{align*}
$$

where $\widehat{\boldsymbol{q}}_{h}$ is the standard numerical flux defined by (3). The derivation of our method is simple. Integrating by parts in (2a) and replacing $u_{h}$ by $P_{M} u_{h}$, we get (4a). The second equation (4a) is obtained by replacing $w$ by $P_{M} w$ in (2b). The third equation (4C) is just the same as (2C). Since $\mu=P_{M} \mu$ in (4c) (and (2c)), we can also consider that our method is obtained by inserting the orthogonal projection $P_{M}$ in all facet integrals in the standard HDG method.

Remark 1 If $\left.\boldsymbol{v} \cdot \boldsymbol{n}\right|_{F} \in P_{k}(F)$ for any $F \in \mathcal{E}_{h}$, then our method is identical to the HDG-LS method since

$$
\begin{array}{ll}
\left\langle P_{M} u_{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle u_{h}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} & \text { in (4a), } \\
\left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle P_{M}\left(\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}\right), w\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle\widehat{\boldsymbol{q}}_{h}^{L S} \cdot \boldsymbol{n}, w\right\rangle_{\partial \mathcal{T}_{h}} & \text { in (4b), } \\
\left\langle\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle P_{M}\left(\widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}\right), \mu\right\rangle_{\partial \mathcal{T}_{h}}=\left\langle\widehat{\boldsymbol{q}}_{h}^{L S} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}} & \text { in (4c). }
\end{array}
$$

## 3 Error analysis

In this section, we provide the optimal-order error estimates of our method. We are going to use the following approximation properties:

$$
\begin{aligned}
\left\|\boldsymbol{v}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{v}\right\| & \leq C h^{j}|\boldsymbol{v}|_{j}, & & 1 \leq j \leq k+1, \\
\left\|\boldsymbol{v}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{v}\right\|_{\partial \mathcal{T}_{h}} & \leq C h^{j-1 / 2}|\boldsymbol{v}|_{j}, & & 1 \leq j \leq k+1, \\
\left\|\nabla\left(w-P_{W} w\right)\right\|_{\mathcal{T}_{h}} & \leq C h^{j-1}|w|_{j}, & & 1 \leq j \leq k+2, \\
\left\|w-P_{W} w\right\| & \leq C h^{j}|w|_{j}, & & 1 \leq j \leq k+2, \\
\left\|w-P_{W} w\right\|_{\partial \mathcal{T}_{h}} & \leq C h^{j-1 / 2}|w|_{j}, & & 1 \leq j \leq k+2, \\
\left\|w-P_{M} w\right\|_{\partial \mathcal{T}_{h}} & \leq C h^{j-1 / 2}|w|_{j}, & & 1 \leq j \leq k+1,
\end{aligned}
$$

for any $\boldsymbol{v} \in \boldsymbol{H}^{j}(\Omega)$ and $w \in H^{j}(\Omega)$. For the piecewise Sobolev spaces, the following hold:

$$
\begin{array}{rlrl}
\left\|w-P_{M} w\right\|_{\partial \mathcal{T}_{h}} & \leq C h^{1 / 2}|w|_{1, \mathcal{T}_{h}} & \forall w \in H^{1}\left(\mathcal{T}_{h}\right) \\
\left\|\boldsymbol{v} \cdot \boldsymbol{n}-P_{M}(\boldsymbol{v} \cdot \boldsymbol{n})\right\|_{\partial \mathcal{T}_{h}} \leq C h^{1 / 2}|\boldsymbol{v}|_{1, \mathcal{T}_{h}} & \forall \boldsymbol{v} \in \boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right) \tag{6}
\end{array}
$$

Let $\boldsymbol{\Pi}_{k}$ be the orthogonal projection from $\boldsymbol{H}^{1}\left(\mathcal{T}_{h}\right)$ onto $\boldsymbol{P}_{k}\left(\mathcal{T}_{h}\right)$, which satisfies

$$
\begin{align*}
& \left.\boldsymbol{\Pi}_{k} \boldsymbol{v} \cdot \boldsymbol{n}\right|_{\partial K} \subset P_{k}(\partial K) \quad \forall K \in \mathcal{T}_{h}  \tag{7}\\
& \left\|\boldsymbol{v} \cdot \boldsymbol{n}-\boldsymbol{\Pi}_{k} \boldsymbol{v} \cdot \boldsymbol{n}\right\|_{\partial \mathcal{T}_{h}} \leq C h^{j-1 / 2}|\boldsymbol{v}|_{j} \quad \text { for } \boldsymbol{v} \in \boldsymbol{H}^{j}(\Omega), 1 \leq j \leq k+1 \tag{8}
\end{align*}
$$

The insertion of $P_{M}$ in (4b) gives rise to some terms in the form

$$
R(\boldsymbol{v}, w):=\left\langle\left(I-P_{M}\right) \boldsymbol{v} \cdot \boldsymbol{n}, w\right\rangle_{\partial \tau_{h}}
$$

in error analysis. We show the bound of $R(\cdot, \cdot)$ by the properties (7) and (8).
Lemma 1 For all $\boldsymbol{v} \in \boldsymbol{H}^{k+1}(\Omega)$ and $w \in H^{1}\left(\mathcal{T}_{h}\right)$, we have

$$
|R(\boldsymbol{v}, w)| \leq C h^{k+1}|\boldsymbol{v}|_{k+1}|w|_{1, \mathcal{T}_{h}} .
$$

Proof By (7), (8) and (5), we have

$$
\begin{aligned}
|R(\boldsymbol{v}, w)| & =\left|\left\langle\left(I-P_{M}\right)\left(\boldsymbol{v}-\boldsymbol{\Pi}_{k} \boldsymbol{v}\right) \cdot \boldsymbol{n},\left(I-P_{M}\right) w\right\rangle_{\partial \mathcal{T}_{h}}\right| \\
& \leq\left\|\boldsymbol{v} \cdot \boldsymbol{n}-\boldsymbol{\Pi}_{k} \boldsymbol{v} \cdot \boldsymbol{n}\right\|_{\partial \mathcal{T}_{h}}\left\|\left(I-P_{M}\right) w\right\|_{\partial \mathcal{T}_{h}} \\
& \leq C h^{k+1}|\boldsymbol{q}|_{k+1}|w|_{1, \mathcal{T}_{h}} .
\end{aligned}
$$

This completes the proof.

### 3.1 Error equations

As a lemma, we show the error equations in terms of the projections of the errors:

$$
\boldsymbol{e}_{\boldsymbol{q}}=\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}-\boldsymbol{q}_{h}, \quad e_{u}=P_{W} u-u_{h}, \quad e_{\widehat{u}}=P_{M} u-\widehat{u}_{h} .
$$

The approximation errors are denoted as

$$
\delta_{\boldsymbol{V}} \boldsymbol{q}=\boldsymbol{q}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}, \quad \delta_{W} u=u-P_{W} u, \quad \delta_{M} u=u-P_{M} u .
$$

Lemma 2 The following equations hold:

$$
\begin{array}{rrr}
\left(\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}+\left(\nabla e_{u}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle P_{M} e_{u}-e_{\widehat{u}}, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=F_{1}(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}_{h}, \\
-\left(\boldsymbol{e}_{\boldsymbol{q}}, \nabla w\right)+\left\langle\widehat{\boldsymbol{e}_{\boldsymbol{q}}} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}=F_{2}(w) & \forall w \in W_{h}, \\
\left\langle\widehat{\boldsymbol{e}_{\boldsymbol{q}}} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}}=F_{3}(\mu) & \forall \mu \in M_{h}, \tag{9c}
\end{array}
$$

where $\widehat{\boldsymbol{e}_{\boldsymbol{q}}} \cdot \boldsymbol{n}=\boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n}+\tau\left(P_{M} e_{u}-e_{\widehat{u}}\right)$ and

$$
\begin{aligned}
& F_{1}(\boldsymbol{v})=-\left(\nabla \delta_{W} u, \boldsymbol{v}\right)_{\mathcal{T}_{h}}+\left\langle P_{M} \delta_{W} u, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}, \\
& F_{2}(w)=-R(\boldsymbol{q}, w)-\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau \delta_{W} u, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}, \\
& F_{3}(\mu)=-\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau \delta_{W} u, \mu\right\rangle_{\partial \mathcal{T}_{h}}
\end{aligned}
$$

Proof We easily see that the exact solution satisfies

$$
\begin{align*}
(\boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_{h}}+(\nabla u, \boldsymbol{v})_{\mathcal{T}_{h}} & =0 & & \forall \boldsymbol{v} \in \boldsymbol{V}_{h},  \tag{10a}\\
-(\boldsymbol{q}, \nabla w)_{\mathcal{T}_{h}}+\langle\boldsymbol{q} \cdot \boldsymbol{n}, w\rangle_{\partial \mathcal{T}_{h}} & =(f, w) & & \forall w \in W_{h},  \tag{10b}\\
\langle\boldsymbol{q} \cdot \boldsymbol{n}, \mu\rangle_{\partial \mathcal{T}_{h}} & =0 & & \forall \mu \in M_{h} . \tag{10c}
\end{align*}
$$

Each term in (10) is rewritten in terms of $\boldsymbol{P}_{\boldsymbol{V}} q, P_{W} u$ and $P_{M} u$ as follows:

$$
\begin{aligned}
(\boldsymbol{q}, \boldsymbol{v})_{\mathcal{T}_{h}} & =\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}, \boldsymbol{v}\right)_{\mathcal{T}_{h}}, \\
(\nabla u, \boldsymbol{v})_{\mathcal{T}_{h}} & =\left(\nabla P_{W} u, \boldsymbol{v}\right)_{\mathcal{T}_{h}}+\left(\nabla \delta_{W} u, \boldsymbol{v}\right)_{\mathcal{T}_{h}}, \\
(\boldsymbol{q}, \nabla w)_{\mathcal{T}_{h}} & =\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}, \nabla w\right)_{\mathcal{T}_{h}}, \\
\langle\boldsymbol{q} \cdot \boldsymbol{n}, w\rangle_{\partial \mathcal{T}_{h}} & =\left\langle\boldsymbol{q} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}+R(\boldsymbol{q}, w) \\
& =\left\langle\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}+R(\boldsymbol{q}, w), \\
\langle\boldsymbol{q} \cdot \boldsymbol{n}, \mu\rangle_{\partial \mathcal{T}_{h}} & =\left\langle\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}}+\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}} .
\end{aligned}
$$

Taking the stabilization terms into account, we have

$$
\begin{array}{rlrl}
\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}+\nabla P_{W} u, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle P_{M}\left(P_{W} u\right)-P_{M} u, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}} & =F_{1}(\boldsymbol{v}) & & \forall \boldsymbol{v} \in \boldsymbol{V}_{h}, \\
-\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}, \nabla w\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}} & =(f, w)+F_{2}(w) & & \forall w \in W_{h}, \\
& & (11 \mathrm{a}) \\
\left\langle\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}} & =F_{3}(\mu) & & \forall \mu \in M_{h}, \tag{11c}
\end{array}
$$

where $\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q}} \cdot \boldsymbol{n}:=\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}+\tau\left(P_{M} u-P_{W} u\right)$. Subtracting (41) from (11), we obtain the required equations.

From Lemma 2 the below inequalities follow.
Lemma 3 If $u \in H^{k+2}(\Omega)$, then we have

$$
\begin{equation*}
\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}} \leq C\left(\left\|e_{\boldsymbol{q}}\right\|+h^{-1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}}+h^{k+1}|u|_{k+2}\right) \tag{12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\| \leq C\left(\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}}+h^{-1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}}+h^{k+1}|u|_{k+2}\right) \tag{13}
\end{equation*}
$$

Proof Taking $\boldsymbol{v}=\nabla e_{u}$ in (9a), we have

$$
\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}}^{2}=-\left(\boldsymbol{e}_{\boldsymbol{q}}, \nabla e_{u}\right)_{\mathcal{T}_{h}}+\left\langle P_{M} e_{u}-e_{\widehat{u}}, \nabla e_{u} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}+F_{1}\left(\nabla e_{u}\right) .
$$

The first two terms on the right-hand side are bounded as

$$
\begin{aligned}
& \left|\left(\boldsymbol{e}_{\boldsymbol{q}}, \nabla e_{u}\right)_{\mathcal{T}_{h}}\right| \leq\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}}, \\
& \left|\left\langle P_{M} e_{u}-e_{\widehat{u}}, \nabla e_{u} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}\right| \leq C h^{-1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}}\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}}
\end{aligned}
$$

By the inverse inequality, we can estimate the remaining term as

$$
\begin{align*}
\left|F_{1}\left(\nabla e_{u}\right)\right| & \leq\left|\left(\nabla \delta_{W} u, \nabla e_{u}\right)_{\mathcal{T}_{h}}\right|+\left|\left\langle P_{M} \delta_{W} u, \nabla e_{u} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}\right| \\
& \leq C h^{k+1}|u|_{k+2}\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}} . \tag{14}
\end{align*}
$$

Combining these results yields (12). Similarly, we can prove (13).
3.2 The estimate of $\boldsymbol{e}_{\boldsymbol{q}}$

We are now ready to show the main results of the paper.
Theorem 1 If $u \in H^{k+2}(\Omega)$, then we have

$$
\left\|e_{\boldsymbol{q}}\right\|+\left\|\tau^{1 / 2}\left(P_{M} e_{u}-e_{\widehat{u}}\right)\right\|_{\partial \tau_{h}} \leq C h^{k+1}|u|_{k+2}
$$

Proof Substituting $w=e_{u}$ in (9b) and $\mu=e_{\widehat{u}}$ in (9c), we have

$$
\begin{equation*}
-\left(\boldsymbol{e}_{\boldsymbol{q}}, \nabla e_{u}\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\boldsymbol{e}_{\boldsymbol{q}}} \cdot \boldsymbol{n}, P_{M} e_{u}-e_{\widehat{u}}\right\rangle_{\partial \mathcal{T}_{h}}=F_{2}\left(e_{u}\right)-F_{3}\left(e_{\widehat{u}}\right) \tag{15}
\end{equation*}
$$

Taking $\boldsymbol{v}=\boldsymbol{e}_{\boldsymbol{q}}$ in (9a) and adding it to (15), we get

$$
\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|^{2}+\left\|\tau^{1 / 2}\left(P_{M} e_{u}-e_{\widehat{u}}\right)\right\|_{\partial \mathcal{T}_{h}}^{2}=F_{1}\left(\boldsymbol{e}_{\boldsymbol{q}}\right)+F_{2}\left(e_{u}\right)-F_{3}\left(e_{\widehat{u}}\right) .
$$

In a similar way to (14), we have

$$
\left|F_{1}\left(\boldsymbol{e}_{\boldsymbol{q}}\right)\right| \leq C h^{k+1}|u|_{k+2}\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\| .
$$

The rest terms are written as

$$
\begin{aligned}
F_{2}\left(e_{u}\right)-F_{3}\left(e_{\widehat{u}}\right) & =-R\left(\boldsymbol{q}, e_{u}\right)+\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau P_{M} \delta_{W} u, P_{M} e_{u}-e_{\widehat{u}}\right\rangle_{\partial \mathcal{T}_{h}} \\
& =I_{1}+I_{2}
\end{aligned}
$$

By Lemmas 1 and 3, we have

$$
\begin{aligned}
\left|I_{1}\right| & \leq C h^{k+1}|u|_{k+2}\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}} \\
& \leq C h^{k+1}|u|_{k+2}\left(\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|+h^{-1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}}+h^{k+1}|u|_{k+2}\right)
\end{aligned}
$$

The term $I_{2}$ is bounded as, in view of $\tau=O\left(h^{-1}\right)$,

$$
\begin{aligned}
\left|I_{2}\right| & \leq\left(\left\|\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\partial \mathcal{T}_{h}}+\tau\left\|\delta_{W} u\right\|_{\partial \mathcal{T}_{h}}\right)\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}} \\
& =C h^{k+1}|u|_{k+2} \cdot \tau^{1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}} .
\end{aligned}
$$

Using Young's inequality and arranging the terms, we obtain

$$
\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|^{2}+\left\|\tau^{1 / 2}\left(P_{M} e_{u}-e_{\widehat{u}}\right)\right\|_{\partial \mathcal{T}_{h}}^{2} \leq C h^{2(k+1)}|u|_{k+2}^{2}
$$

which completes the proof.
3.3 The estimate of $e_{u}$

We show that the order of convergence in the variable $u$ is optimal by the duality argument. To this end, we consider the adjoint problem of (1): Find $\psi \in H^{2}(\Omega) \cap H_{0}^{1}(\Omega)$ and $\boldsymbol{\theta} \in \boldsymbol{H}^{1}(\Omega)$ such that

$$
\begin{array}{rlrl}
\nabla \psi+\boldsymbol{\theta} & =0 & \text { in } \Omega, \\
\nabla \cdot \boldsymbol{\theta} & =e_{u} & \text { in } \Omega \\
\psi & =0 & & \text { on } \partial \Omega .
\end{array}
$$

As is well known, the elliptic regularity holds:

$$
\|\boldsymbol{\theta}\|_{1}+\|\psi\|_{2} \leq C\left\|e_{u}\right\| .
$$

Let us denote the approximation errors of $\psi$ and $\boldsymbol{\theta}$ as follows:

$$
\delta_{\boldsymbol{V}} \boldsymbol{\theta}=\boldsymbol{\theta}-\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}, \quad \delta_{W} \psi=\psi-P_{W} \psi, \quad \delta_{M} \psi=\psi-P_{M} \psi .
$$

Theorem 2 If $u \in H^{k+2}(\Omega)$, then we have

$$
\left\|e_{u}\right\| \leq C h^{k+2}|u|_{k+2} .
$$

Proof Similarly to (11), we deduce

$$
\begin{array}{cc}
\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}+\nabla P_{W} \psi, \boldsymbol{v}\right)_{\mathcal{T}_{h}}-\left\langle P_{M}\left(\psi-P_{W} \psi\right), \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=G_{1}(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{V}_{h}, \\
-\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}, \nabla w\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}} \cdot \boldsymbol{n}, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}=\left(e_{u}, w\right)+G_{2}(w) & \forall w \in W_{h}, \\
& (16 \mathrm{a}) \\
\left\langle\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}} \cdot \boldsymbol{n}, \mu\right\rangle_{\partial \mathcal{T}_{h}}=G_{3}(\mu) & \forall \mu \in M_{h}, \tag{16c}
\end{array}
$$

where $\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}} \cdot \boldsymbol{n}=\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}+\tau\left(P_{W} \psi-P_{M} \psi\right)$ and

$$
\begin{aligned}
& G_{1}(\boldsymbol{v})=-\left(\nabla \delta_{W} \psi, \boldsymbol{v}\right)_{\mathcal{T}_{h}}+\left\langle P_{M} \delta_{W} \psi, \boldsymbol{v} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}, \\
& G_{2}(w)=-R(\boldsymbol{\theta}, w)-\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}-\tau P_{M} \delta_{W} \psi, P_{M} w\right\rangle_{\partial \mathcal{T}_{h}}, \\
& G_{3}(\mu)=-\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}-\tau P_{M} \delta_{W} \psi, \mu\right\rangle_{\partial \mathcal{T}_{h}} .
\end{aligned}
$$

Substituting $\boldsymbol{v}=\boldsymbol{e}_{\boldsymbol{q}}$ in (16a), $w=e_{u}$ in (16b) and $\mu=e_{\widehat{u}}$ in (16c) yields

$$
\begin{align*}
& \left(\boldsymbol{\theta}+\nabla P_{W} \psi, \boldsymbol{e}_{\boldsymbol{q}}\right)_{\mathcal{T}_{h}}-\left\langle P_{M}\left(\psi-P_{W} \psi\right), \boldsymbol{e}_{\boldsymbol{q}} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=G_{1}\left(\boldsymbol{e}_{\boldsymbol{q}}\right),  \tag{17a}\\
& -\left(\boldsymbol{\theta}, \nabla e_{u}\right)_{\mathcal{T}_{h}}+\left\langle\widehat{\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}} \cdot \boldsymbol{n}, P_{M} e_{u}-e_{\widehat{u}}\right\rangle_{\partial \mathcal{T}_{h}}=\left\|e_{u}\right\|^{2}+G_{2}\left(e_{u}\right)-G_{3}\left(e_{\widehat{u}}\right) . \tag{17~b}
\end{align*}
$$

Taking $\boldsymbol{v}=\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}, w=P_{W} \psi$ and $\mu=P_{M} \psi$ in the error equations (9), we have

$$
\begin{align*}
& \left(\boldsymbol{e}_{\boldsymbol{q}}+\nabla e_{u}, \boldsymbol{\theta}\right)_{\mathcal{T}_{h}}-\left\langle P_{M} e_{u}-e_{\widehat{u}}, \boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}=F_{1}\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}\right),  \tag{18a}\\
& -\left(\boldsymbol{e}_{\boldsymbol{q}}, \nabla P_{W} \psi\right)+\left\langle P_{M}\left(\widehat{\boldsymbol{e}_{\boldsymbol{q}}} \cdot \boldsymbol{n}\right), P_{W} \psi-\psi\right\rangle_{\partial \mathcal{T}_{h}}=F_{2}\left(P_{W} \psi\right)-F_{3}\left(P_{M} \psi\right) . \tag{18b}
\end{align*}
$$

Note that $P_{M} \psi \in M_{h}$ since $\psi$ is single-valued on $\mathcal{E}_{h}$. Adding (18b) to 17a) and (18a) to (17b), we have

$$
\begin{aligned}
& \left(\boldsymbol{\theta}, \boldsymbol{e}_{\boldsymbol{q}}\right)_{\mathcal{T}_{h}}-\left\langle\tau\left(P_{M} e_{u}-e_{\widehat{u}}\right), \delta_{W} \psi\right\rangle_{\partial \mathcal{T}_{h}}=G_{1}\left(\boldsymbol{e}_{\boldsymbol{q}}\right)+F_{2}\left(P_{W} \psi\right)-F_{3}\left(P_{M} \psi\right), \\
& \left(\boldsymbol{e}_{\boldsymbol{q}}, \boldsymbol{\theta}\right)_{\mathcal{T}_{h}}-\left\langle\tau\left(P_{M} e_{u}-e_{\widehat{u}}\right), \delta_{W} \psi\right\rangle_{\partial \mathcal{T}_{h}}=\left\|e_{u}\right\|^{2}+F_{1}\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}\right)+G_{2}\left(e_{u}\right)-G_{3}\left(e_{\widehat{u}}\right),
\end{aligned}
$$

respectively. Since the left-hand sides are equal to each other, we obtain

$$
\left\|e_{u}\right\|^{2}=G_{1}\left(\boldsymbol{e}_{\boldsymbol{q}}\right)-G_{2}\left(e_{u}\right)+G_{3}\left(e_{\widehat{u}}\right)-\left(F_{1}\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}\right)-F_{2}\left(P_{W} \psi\right)+F_{3}\left(P_{M} \psi\right)\right) .
$$

By the inverse and trace inequalities, we have

$$
\left|G_{1}\left(e_{\boldsymbol{q}}\right)\right| \leq C h|\psi|_{2}\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|
$$

By Lemma 3 and Theorem 11 we get

$$
\begin{aligned}
\left|G_{2}\left(e_{u}\right)-G_{3}\left(e_{\widehat{u}}\right)\right| & \leq\left|R\left(\boldsymbol{\theta}, e_{u}\right)\right|+\left|\left\langle-\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}+\tau \delta_{W} \psi, P_{M} e_{u}-e_{\widehat{u}}\right\rangle_{\partial \mathcal{T}_{h}}\right| \\
& \leq C h|\boldsymbol{\theta}|_{1}\left\|\nabla e_{u}\right\|_{\mathcal{T}_{h}}+C h\left(|\boldsymbol{\theta}|_{1}+|\psi|_{2}\right) \cdot \tau^{1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|_{\partial \mathcal{T}_{h}} \\
& =C h\left\|e_{u}\right\|\left(\left\|\boldsymbol{e}_{\boldsymbol{q}}\right\|+\tau^{1 / 2}\left\|P_{M} e_{u}-e_{\widehat{u}}\right\|+h^{k+1}|u|_{k+2}\right) \\
& \leq C h^{k+2}|u|_{k+2}\left\|e_{u}\right\|,
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
F_{1}\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}\right) & =-\left(\nabla \delta_{W} u, \boldsymbol{\theta}\right)+\left\langle\delta_{W} u, P_{M}\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}\right)\right\rangle_{\partial \tau_{h}} \\
& =\left(\delta_{W} u, \nabla \cdot \boldsymbol{\theta}\right)-\left\langle\left(I-P_{M}\right) \delta_{W} u, \boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta} \cdot \boldsymbol{n}\right\rangle_{\partial \tau_{h}} \\
& =: T_{1}+T_{2},
\end{aligned}
$$

and the terms are bounded as

$$
\begin{aligned}
\left|T_{1}\right| & \leq\left\|\delta_{W} u\right\|\left\|e_{u}\right\| \leq C h^{k+2}|u|_{k+2}\left\|e_{u}\right\|, \\
\left|T_{2}\right| & =\left|\left\langle\left(I-P_{M}\right) \delta_{W} u,\left(\boldsymbol{P}_{\boldsymbol{V}} \boldsymbol{\theta}-\boldsymbol{\theta}+\boldsymbol{\theta}-\boldsymbol{\Pi}_{k} \boldsymbol{\theta}\right) \cdot \boldsymbol{n}\right\rangle_{\partial \mathcal{T}_{h}}\right| \\
& \leq\left\|\delta_{W} u\right\|_{\partial \mathcal{T}_{h}}\left(\left\|\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{\theta}\right\|_{\partial \mathcal{T}_{h}}+\left\|\boldsymbol{\theta}-\boldsymbol{\Pi}_{k} \boldsymbol{\theta}\right\|_{\partial \mathcal{T}_{h}}\right) \\
& \leq C h^{k+3 / 2}|u|_{k+2} \cdot C h^{1 / 2}|\boldsymbol{\theta}|_{1} \\
& \leq C h^{k+2}|u|_{k+2}|\boldsymbol{\theta}|_{1} .
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
F_{2}\left(P_{W} \psi\right)-F_{3}\left(P_{M} \psi\right) & =-R\left(\boldsymbol{q}, P_{W} \psi\right)-\left\langle\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q} \cdot \boldsymbol{n}-\tau P_{M} \delta_{W} u, P_{M} \delta_{W} \psi\right\rangle_{\partial \mathcal{T}_{h}} \\
& =: T_{3}+T_{4}
\end{aligned}
$$

Since both $\boldsymbol{q}$ and $\psi$ are single-valued on $\mathcal{E}_{h}$, it follows that

$$
T_{3}=-\left\langle\left(I-P_{M}\right) \boldsymbol{q} \cdot \boldsymbol{n}, \delta_{W} \psi\right\rangle_{\partial \mathcal{T}_{h}}=R\left(\boldsymbol{q}, P_{W} \psi-\psi\right)
$$

By Lemma 1, we get

$$
\left|T_{3}\right| \leq C h^{k+1}|u|_{k+2}\left|\delta_{W} \psi\right|_{1, \mathcal{T}_{h}} \leq C h^{k+2}|u|_{k+2}|\psi|_{2}
$$

The other term is bounded as follows:

$$
\begin{aligned}
\left|T_{4}\right| & \leq C\left(\left\|\boldsymbol{\delta}_{\boldsymbol{V}} \boldsymbol{q}\right\|_{\partial \mathcal{T}_{h}}+\tau\left\|\delta_{W} u\right\|_{\partial \mathcal{T}_{h}}\right)\left\|\delta_{W} \psi\right\|_{\partial \mathcal{T}_{h}} \\
& \leq C\left(h^{k+1 / 2}|\boldsymbol{q}|_{k+1}+h^{-1} h^{k+3 / 2}|u|_{k+2}\right) \cdot C h^{3 / 2}|\psi|_{2} \\
& \leq C h^{k+2}|u|_{k+2}|\psi|_{2} .
\end{aligned}
$$

Combining these results and applying Young's inequality, we have

$$
\left\|e_{u}\right\|^{2} \leq C h^{k+2}\left(\left\|e_{u}\right\|+|\boldsymbol{\theta}|_{1}+|\psi|_{2}\right)
$$

Thanks to the elliptic regularity, we obtain the required inequality.

## 4 Numerical results

In this section, we carry out numerical experiments to verify our theoretical results. The following test problem is considered:

$$
\begin{align*}
-\Delta u & =2 \pi^{2} \sin (\pi x) \sin (\pi y) & & \text { in } \Omega  \tag{19a}\\
u & =0 & & \text { on } \partial \Omega \tag{19b}
\end{align*}
$$

where $\Omega=(0,1)^{2}$ and the exact solution is $\sin (\pi x) \sin (\pi y)$. All computations were done with FreeFem++ 7 . The meshes we used are unstructured triangular meshes. We set $\boldsymbol{V}_{h}=\boldsymbol{P}_{k+l}\left(\mathcal{T}_{h}\right), W_{h}=P_{k+1}\left(\mathcal{T}_{h}\right)$ and $M_{h}=P_{k}\left(\mathcal{E}_{h}\right)$ for $0 \leq k \leq$ 2 , varying $l$ from 0 to 2 . The stabilization parameter $\tau$ is set to be $1 / h$ in all cases.

The history of convergence of our method is displayed in Tables 24. From the results, we observe that the orders or convergence in $\boldsymbol{q}, u$ and the projected jump quantity are $k+1, k+2$ and $k+1$, respectively, which is in full agreement with Theorems 1 and 2 Note that, as mentioned in Remark 1, the errors of our method in Table 3 for $l=0$ coincide with those of the HDG-LS method in Table 1 .

## References

1. Adams, R.A., Fournier, J.J.F.: Sobolev spaces, Pure and Applied Mathematics (Amsterdam), vol. 140, second edn. Academic Press, Amsterdam (2003)
2. Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods, Texts in Applied Mathematics, vol. 15, third edn. Springer, New York (2008)
3. Chen, H., Qiu, W., Shi, K., Solano, M.: A superconvergent HDG method for the Maxwell equations. J. Sci. Comput. 70(3), 1010-1029 (2017)
4. Cockburn, B., Di Pietro, D.A., Ern, A.: Bridging the hybrid high-order and hybridizable discontinuous Galerkin methods. ESAIM Math. Model. Numer. Anal. 50(3), 635-650 (2016)
5. Di Pietro, D.A., Ern, A.: Hybrid high-order methods for variable-diffusion problems on general meshes. C. R. Math. Acad. Sci. Paris 353(1), 31-34 (2015)
6. Di Pietro, D.A., Ern, A., Lemaire, S.: An arbitrary-order and compact-stencil discretization of diffusion on general meshes based on local reconstruction operators. Comput. Methods Appl. Math. 14(4), 461-472 (2014)

Table 2 Convergence history for $k=0$

| $l$ |  | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|$ |  | $\left\\|u-u_{h}\right\\|$ |  | $\left\\|h^{-1 / 2}\left(P_{M} u_{h}-\widehat{u}_{h}\right)\right\\|_{\partial \mathcal{T}_{h}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $1 / h$ | Error | Order | Error | Order | Error | Order |
|  | 10 | $2.643 \mathrm{E}-01$ | - | $1.842 \mathrm{E}-02$ | - | $3.873 \mathrm{E}-01$ | - |
|  | 20 | $1.306 \mathrm{E}-01$ | 1.02 | $4.543 \mathrm{E}-03$ | 2.02 | $1.920 \mathrm{E}-01$ | 1.01 |
|  | 40 | $6.612 \mathrm{E}-02$ | 0.98 | $1.176 \mathrm{E}-03$ | 1.95 | $9.775 \mathrm{E}-02$ | 0.97 |
|  | 80 | $3.313 \mathrm{E}-02$ | 1.00 | $2.928 \mathrm{E}-04$ | 2.01 | $4.853 \mathrm{E}-02$ | 1.01 |
| 1 | 10 | $2.152 \mathrm{E}-01$ | - | $6.793 \mathrm{E}-03$ | - | $5.545 \mathrm{E}-02$ | - |
|  | 20 | $1.069 \mathrm{E}-01$ | 1.01 | $1.686 \mathrm{E}-03$ | 2.01 | $2.673 \mathrm{E}-02$ | 1.05 |
|  | 40 | $5.407 \mathrm{E}-02$ | 0.98 | $4.339 \mathrm{E}-04$ | 1.96 | $1.366 \mathrm{E}-02$ | 0.97 |
|  | 80 | $2.726 \mathrm{E}-02$ | 0.99 | $1.113 \mathrm{E}-04$ | 1.96 | $6.748 \mathrm{E}-03$ | 1.02 |
| 2 | 10 | $2.533 \mathrm{E}-01$ | - | $6.078 \mathrm{E}-03$ | - | $1.502 \mathrm{E}-02$ | - |
|  | 20 | $1.254 \mathrm{E}-01$ | 1.01 | $1.498 \mathrm{E}-03$ | 2.02 | $7.143 \mathrm{E}-03$ | 1.07 |
|  | 40 | $6.348 \mathrm{E}-02$ | 0.98 | $3.847 \mathrm{E}-04$ | 1.96 | $3.654 \mathrm{E}-03$ | 0.97 |
|  | 80 | $3.184 \mathrm{E}-02$ | 1.00 | $9.910 \mathrm{E}-05$ | 1.96 | $1.802 \mathrm{E}-03$ | 1.02 |

Table 3 Convergence history for $k=1$

| $l$ |  | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|$ |  | $\left\\|u-u_{h}\right\\|$ |  | $\left\\|h^{-1 / 2}\left(P_{M} u_{h}-\widehat{u}_{h}\right)\right\\|_{\partial \mathcal{T}_{h}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $1 / h$ | Error | Order | Error | Order | Error | Order |
|  | 10 | $1.236 \mathrm{E}-02$ | - | $7.400 \mathrm{E}-04$ | - | $2.719 \mathrm{E}-02$ | - |
|  | 20 | $3.083 \mathrm{E}-03$ | 2.00 | $9.085 \mathrm{E}-05$ | 3.03 | $6.676 \mathrm{E}-03$ | 2.03 |
|  | 40 | $7.655 \mathrm{E}-04$ | 2.01 | $1.140 \mathrm{E}-05$ | 2.99 | $1.662 \mathrm{E}-03$ | 2.01 |
|  | 80 | $1.915 \mathrm{E}-04$ | 2.00 | $1.414 \mathrm{E}-06$ | 3.01 | $4.113 \mathrm{E}-04$ | 2.01 |
| 1 | 10 | $7.212 \mathrm{E}-03$ | - | $1.023 \mathrm{E}-04$ | - | $3.157 \mathrm{E}-03$ | - |
|  | 20 | $1.744 \mathrm{E}-03$ | 2.05 | $1.194 \mathrm{E}-05$ | 3.10 | $7.704 \mathrm{E}-04$ | 2.03 |
|  | 40 | $4.468 \mathrm{E}-04$ | 1.96 | $1.544 \mathrm{E}-06$ | 2.95 | $1.863 \mathrm{E}-04$ | 2.05 |
|  | 80 | $1.127 \mathrm{E}-04$ | 1.99 | $1.922 \mathrm{E}-07$ | 3.01 | $4.525 \mathrm{E}-05$ | 2.04 |
| 2 | 10 | $8.936 \mathrm{E}-03$ | - | $1.059 \mathrm{E}-04$ | - | $1.763 \mathrm{E}-03$ | - |
|  | 20 | $2.187 \mathrm{E}-03$ | 2.03 | $1.254 \mathrm{E}-05$ | 3.08 | $4.422 \mathrm{E}-04$ | 2.00 |
|  | 40 | $5.529 \mathrm{E}-04$ | 1.98 | $1.616 \mathrm{E}-06$ | 2.96 | $1.062 \mathrm{E}-04$ | 2.06 |
|  | 80 | $1.387 \mathrm{E}-04$ | 1.99 | $2.019 \mathrm{E}-07$ | 3.00 | $2.594 \mathrm{E}-05$ | 2.03 |

7. Hecht, F.: New development in freefem++. J. Numer. Math. 20(3-4), 251-265 (2012)
8. Lehrenfeld, C.: Hybrid discontinuous Galerkin methods for solving incompressible flow problems. Master's Thesis, RWTH Aachen University (2010)
9. Lehrenfeld, C., Schöberl, J.: High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows. Comput. Methods Appl. Mech. Engrg. 307, 339-361 (2016)
10. Oikawa, I.: A hybridized discontinuous Galerkin method with reduced stabilization. J. Sci. Comput. 65(1), 327-340 (2015)
11. Oikawa, I.: Analysis of a reduced-order HDG method for the Stokes equations. J. Sci. Comput. 67(2), 475-492 (2016)
12. Qiu, W., Shen, J., Shi, K.: An HDG method for linear elasticity with strong symmetric stresses. Math. Comp. 87(309), 69-93 (2018)
13. Qiu, W., Shi, K.: An HDG method for convection diffusion equation. J. Sci. Comput. 66(1), 346-357 (2016)
14. Qiu, W., Shi, K.: A superconvergent HDG method for the incompressible Navier-Stokes equations on general polyhedral meshes. IMA J. Numer. Anal. 36(4), 1943-1967 (2016)

Table 4 Convergence history for $k=2$

| $l$ |  | $\left\\|\boldsymbol{q}-\boldsymbol{q}_{h}\right\\|$ |  | $\left\\|u-u_{h}\right\\|$ |  | $\left\\|h^{-1 / 2}\left(P_{M} u_{h}-\widehat{u}_{h}\right)\right\\|_{\partial \mathcal{T}_{h}}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $1 / h$ | Error | Order | Error | Order | Error | Order |
|  | 10 | $2.365 \mathrm{E}-04$ | - | $2.705 \mathrm{E}-05$ | - | $1.270 \mathrm{E}-03$ | - |
|  | 20 | $2.718 \mathrm{E}-05$ | 3.12 | $1.573 \mathrm{E}-06$ | 4.10 | $1.480 \mathrm{E}-04$ | 3.10 |
|  | 40 | $3.509 \mathrm{E}-06$ | 2.95 | $1.049 \mathrm{E}-07$ | 3.91 | $1.919 \mathrm{E}-05$ | 2.95 |
|  | 80 | $4.350 \mathrm{E}-07$ | 3.01 | $6.381 \mathrm{E}-09$ | 4.04 | $2.346 \mathrm{E}-06$ | 3.03 |
| 1 | 10 | $1.172 \mathrm{E}-04$ | - | $3.673 \mathrm{E}-06$ | - | $1.291 \mathrm{E}-04$ | - |
|  | 20 | $1.338 \mathrm{E}-05$ | 3.13 | $1.973 \mathrm{E}-07$ | 4.22 | $1.449 \mathrm{E}-05$ | 3.15 |
|  | 40 | $1.791 \mathrm{E}-06$ | 2.90 | $1.278 \mathrm{E}-08$ | 3.95 | $1.832 \mathrm{E}-06$ | 2.98 |
|  | 80 | $2.199 \mathrm{E}-07$ | 3.03 | $7.694 \mathrm{E}-10$ | 4.05 | $2.194 \mathrm{E}-07$ | 3.06 |
| 2 | 10 | $1.775 \mathrm{E}-04$ | - | $4.812 \mathrm{E}-06$ | - | $8.311 \mathrm{E}-05$ | - |
|  | 20 | $2.018 \mathrm{E}-05$ | 3.14 | $2.584 \mathrm{E}-07$ | 4.22 | $9.561 \mathrm{E}-06$ | 3.12 |
|  | 40 | $2.638 \mathrm{E}-06$ | 2.94 | $1.671 \mathrm{E}-08$ | 3.95 | $1.204 \mathrm{E}-06$ | 2.99 |
|  | 80 | $3.146 \mathrm{E}-07$ | 3.07 | $1.002 \mathrm{E}-09$ | 4.06 | $1.445 \mathrm{E}-07$ | 3.06 |

