A Superconvergent HDG Method for Distributed Control of Convection Diffusion PDEs

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November 6, 2018

Abstract

We consider a distributed optimal control problem governed by an elliptic convection diffusion PDE, and propose a hybridizable discontinuous Galerkin (HDG) method to approximate the solution. We use polynomials of degree k+1 and $k\geq 0$ to approximate the state, dual state, and their fluxes, respectively. Moreover, we use polynomials of degree k to approximate the numerical traces of the state and dual state on the faces, which are the only globally coupled unknowns. We prove optimal a priori error estimates for all variables when k>0. Furthermore, from the point of view of the number of degrees of freedom of the globally coupled unknowns, this method achieves superconvergence for the state, dual state, and control when $k\geq 1$. We illustrate our convergence results with numerical experiments.

1 Introduction

We consider the following distributed control problem. Let $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ be a Lipschitz polyhedral domain with boundary $\Gamma = \partial \Omega$. The goal is to minimize

$$J(u) = \frac{1}{2} \|y - y_d\|_{L^2(\Omega)}^2 + \frac{\gamma}{2} \|u\|_{L^2(\Omega)}^2, \quad \gamma > 0,$$
(1)

subject to

$$-\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega,$$

$$y = g \qquad \text{on } \partial \Omega,$$
 (2)

where $f \in L^2(\Omega)$ and the vector field $\boldsymbol{\beta}$ satisfies

$$\nabla \cdot \boldsymbol{\beta} \le 0. \tag{3}$$

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It is well known that this optimal control problem is equivalent to the optimality system

$$-\Delta y + \beta \cdot \nabla y = f + u \quad \text{in } \Omega, \tag{4a}$$

$$y = g$$
 on $\partial\Omega$, (4b)

$$y = g$$
 on $\partial \Omega$, (4b)
 $-\Delta z - \nabla \cdot (\beta z) = y_d - y$ in Ω , (4c)

$$z = 0$$
 on $\partial \Omega$, (4d)

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 on $\partial \Omega$, (4d)
 $z - \gamma u = 0$ in Ω . (4e)

Many different numerical methods have been investigated for this type of problem including approaches based on the finite element method [1-3, 10-14, 17], mixed finite elements [13, 26, 28], and discontinuous Galerkin (DG) methods [14,18,24,25,27,29,30]. Also, hybridizable discontinuous Galerkin (HDG) methods have recently been explored for various optimal control problems for the Poisson equation [16,31] and the above convection diffusion equation [15].

In this earlier work [15], we used a hybridizable discontinuous Galerkin (HDG) method to approximate the solution of the optimality system (4). We used polynomials of degree k to approximate all variables and obtained optimal convergence rates when β is divergence free.

In this work, we investigate a different HDG method for the above problem and prove that it is superconvergent. Specifically, we use polynomials of degree k+1 to approximate the state y and dual state z and polynomials of degree $k \geq 0$ for the fluxes $q = -\nabla y$ and $p = -\nabla z$. Moreover, we only use polynomials of degree k to approximate the numerical traces of the state and dual state on the faces, which are the only globally coupled unknowns. We describe the method in Section 2, and then in Section 3 we obtain the a priori error bounds

$$||y - y_h||_{0,\Omega} = O(h^{k+1+\min\{k,1\}}), ||z - z_h||_{0,\Omega} = O(h^{k+1+\min\{k,1\}}),$$

$$||\mathbf{q} - \mathbf{q}_h||_{0,\Omega} = O(h^{k+1}), ||\mathbf{p} - \mathbf{p}_h||_{0,\Omega} = O(h^{k+1}),$$

and

$$||u - u_h||_{0,\Omega} = O(h^{k+1+\min\{k,1\}}).$$

From the point of view of the global degrees of freedom, we obtain superconvergent approximations to y, z, and u without postprocessing if $k \geq 1$. We demonstrate the performance of the HDG method with numerical experiments in Section 4.

2 HDG scheme for the optimal control problem

We begin with notation and a complete description of the HDG method.

2.1 Notation

Throughout this work we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ with norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$. We also set $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \partial\Omega\}$ and $H(\operatorname{div},\Omega) = \{v \in [L^2(\Omega)]^d, \nabla \cdot v \in \mathbb{C}\}$ $L^2(\Omega)$. We denote the L^2 -inner products on $L^2(\Omega)$ and $L^2(\Gamma)$ by

$$(v, w) = \int_{\Omega} vw \quad \forall v, w \in L^2(\Omega),$$

 $\langle v, w \rangle = \int_{\Gamma} vw \quad \forall v, w \in L^2(\Gamma).$

Let \mathcal{T}_h be a collection of disjoint elements that partition Ω , and let $\partial \mathcal{T}_h$ be the set $\{\partial K : K \in \mathcal{T}_h\}$. For an element $K \in \mathcal{T}_h$, let $e = \partial K \cap \Gamma$ denote the boundary face of K if the d-1 Lebesgue measure of e is non-zero. For two elements K^+ and K^- in \mathcal{T}_h , let $e = \partial K^+ \cap \partial K^-$ denote the interior face between K^+ and K^- if the d-1 Lebesgue measure of e is non-zero. Let ε_h^o and ε_h^∂ denote the set of interior and boundary faces, respectively, and let ε_h be the union of ε_h^o and ε_h^∂ . Furthermore, we introduce

$$(w,v)_{\mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} (w,v)_K, \qquad \langle \zeta, \rho \rangle_{\partial \mathcal{T}_h} = \sum_{K \in \mathcal{T}_h} \langle \zeta, \rho \rangle_{\partial K}.$$

Let $\mathcal{P}^k(D)$ denote the set of polynomials of degree at most k on a domain D. We use the discontinuous finite element spaces

$$\mathbf{V}_h := \{ \mathbf{v} \in [L^2(\Omega)]^d : \mathbf{v}|_K \in [\mathcal{P}^k(K)]^d, \forall K \in \mathcal{T}_h \}, \tag{5}$$

$$W_h := \{ w \in L^2(\Omega) : w | K \in \mathcal{P}^{k+1}(K), \forall K \in \mathcal{T}_h \},$$

$$\tag{6}$$

$$M_h := \{ \mu \in L^2(\varepsilon_h) : \mu|_e \in \mathcal{P}^k(e), \forall e \in \varepsilon_h \}.$$
 (7)

Let $M_h(o)$ and $M_h(\partial)$ denote the spaces of discontinuous finite element functions of polynomial degree at most k defined on the set of interior faces ε_h^o and boundary faces ε_h^∂ , respectively. For any functions $w \in W_h$ and $r \in V_h$, let ∇w and $\nabla \cdot r$ denote the piecewise gradient and divergence on each element $K \in \mathcal{T}_h$.

2.2 The HDG Formulation

For the HDG method, we consider a mixed formulation of the optimality system (4) and approximate the state y, the dual state z, the fluxes $\mathbf{q} = -\nabla y$ and $\mathbf{p} = -\nabla z$, and the numerical traces of y and z on the faces. The approximate optimal distributed control is found directly using a discrete version of the optimality condition (4e). One important feature of HDG methods is the local solver: The unknowns corresponding to all variables except the numerical traces can be eliminated locally on each element, which leads to a globally coupled system involving only the coefficients of the numerical traces. This leads to a reduction in the computational cost. For more information on HDG methods, see, e.g., [4–9, 19–21, 23].

The mixed weak form of the optimality system (4a)-(4e) is given by

$$(\boldsymbol{q}, \boldsymbol{r}_1) - (y, \nabla \cdot \boldsymbol{r}_1) + \langle y, \boldsymbol{r}_1 \cdot \boldsymbol{n} \rangle = 0, \tag{8a}$$

$$(\nabla \cdot (\boldsymbol{q} + \boldsymbol{\beta} y), w_1) - (\nabla \cdot \boldsymbol{\beta} y, w_1) = (f + u, w_1), \tag{8b}$$

$$(\mathbf{p}, \mathbf{r}_2) - (z, \nabla \cdot \mathbf{r}_2) + \langle z, \mathbf{r}_2 \cdot \mathbf{n} \rangle = 0, \tag{8c}$$

$$(\nabla \cdot (\boldsymbol{p} - \boldsymbol{\beta}z), \boldsymbol{w}_2) = (y_d - y, w_2), \tag{8d}$$

$$(z - \gamma u, v) = 0, (8e)$$

for all $(\mathbf{r}_1, w_1, \mathbf{r}_2, w_2, v) \in H(\operatorname{div}, \Omega) \times L^2(\Omega) \times H(\operatorname{div}, \Omega) \times L^2(\Omega) \times L^2(\Omega)$. To approximate the solution of this problem, the HDG method seeks approximate fluxes $\mathbf{q}_h, \mathbf{p}_h \in \mathbf{V}_h$, states $y_h, z_h \in W_h$, interior element boundary traces $\hat{y}_h^o, \hat{z}_h^o \in M_h(o)$, and control $u_h \in W_h$ satisfying

$$(\boldsymbol{q}_{h}, \boldsymbol{r}_{1})_{\mathcal{T}_{h}} - (y_{h}, \nabla \cdot \boldsymbol{r}_{1})_{\mathcal{T}_{h}} + \langle \widehat{y}_{h}^{o}, \boldsymbol{r}_{1} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} = -\langle g, \boldsymbol{r}_{1} \cdot \boldsymbol{n} \rangle_{\varepsilon_{h}^{\partial}},$$
(9a)

$$-(\boldsymbol{q}_{h} + \boldsymbol{\beta}y_{h}, \nabla w_{1})_{\mathcal{T}_{h}} - (\nabla \cdot \boldsymbol{\beta}y_{h}, w_{1})_{\mathcal{T}_{h}} - (u_{h}, w_{1})_{\mathcal{T}_{h}}$$

$$+\langle \widehat{\boldsymbol{q}}_{h} \cdot \boldsymbol{n}, w_{1} \rangle_{\partial \mathcal{T}_{h}} + \langle \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{y}_{h}^{o}, w_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} = (f, w_{1})_{\mathcal{T}_{h}}$$

$$-\langle \boldsymbol{\beta} \cdot \boldsymbol{n}g, w_{1} \rangle_{\varepsilon_{h}^{\partial}},$$
(9b)

for all $(\mathbf{r}_1, w_1) \in \mathbf{V}_h \times W_h$,

$$(\boldsymbol{p}_h, \boldsymbol{r}_2)_{\mathcal{T}_h} - (z_h, \nabla \cdot \boldsymbol{r}_2)_{\mathcal{T}_h} + \langle \hat{z}_h, \boldsymbol{r}_2 \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_h^{\partial}} = 0, \tag{9c}$$

$$-(\boldsymbol{p}_{h} - \boldsymbol{\beta} z_{h}, \nabla w_{2})_{\mathcal{T}_{h}} + \langle \widehat{\boldsymbol{p}}_{h} \cdot \boldsymbol{n}, w_{2} \rangle_{\partial \mathcal{T}_{h}} -\langle \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{z}_{h}^{o}, w_{2} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{l}^{o}} + (y_{h}, w_{2})_{\mathcal{T}_{h}} = (y_{d}, w_{2})_{\mathcal{T}_{h}},$$
(9d)

for all $(\mathbf{r}_2, w_2) \in \mathbf{V}_h \times W_h$,

$$\langle \widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{y}_h^0, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_i^0} = 0,$$
 (9e)

for all $\mu_1 \in M_h(o)$,

$$\langle \widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n} - \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{\boldsymbol{z}}_h^o, \mu_2 \rangle_{\partial \mathcal{T}_b \setminus \varepsilon_c^o} = 0,$$
 (9f)

for all $\mu_2 \in M_h(o)$, and the optimality condition

$$(z_h - \gamma u_h, w_3)_{\mathcal{T}_h} = 0, \tag{9g}$$

for all $w_3 \in W_h$. The numerical traces on $\partial \mathcal{T}_h$ are defined by

$$\widehat{\mathbf{q}}_h \cdot \mathbf{n} = \mathbf{q}_h \cdot \mathbf{n} + h^{-1} (P_M y_h - \widehat{y}_h^o) + \tau_1 (y_h - \widehat{y}_h^o) \quad \text{on } \partial \mathcal{T}_h \backslash \varepsilon_h^{\partial},$$
 (9h)

$$\widehat{\boldsymbol{q}}_h \cdot \boldsymbol{n} = \boldsymbol{q}_h \cdot \boldsymbol{n} + h^{-1} (P_M y_h - P_M g) + \tau_1 (y_h - P_M g) \quad \text{on } \varepsilon_h^{\partial}, \tag{9i}$$

$$\widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n} = \boldsymbol{p}_h \cdot \boldsymbol{n} + h^{-1} (P_M z_h - \widehat{z}_h^o) + \tau_2 (z_h - \widehat{y}_h^o) \quad \text{on } \partial \mathcal{T}_h \backslash \varepsilon_h^{\partial},$$
(9j)

$$\widehat{\boldsymbol{p}}_h \cdot \boldsymbol{n} = \boldsymbol{p}_h \cdot \boldsymbol{n} + h^{-1} P_M z_h + \tau_2 z_h \qquad \text{on } \varepsilon_h^{\partial}, \tag{9k}$$

where τ_1 and τ_2 are stabilization functions defined on $\partial \mathcal{T}_h$. In the next section, we give conditions that the stabilization functions must satisfy in order to guarantee the convergence results.

The implementation of the above HDG method and the local solver is similar to the implementation of another HDG method described in our recent work [15]; therefore, we omit the details.

3 Error Analysis

Next, we perform an error analysis of the above HDG method. Throughout this section, we assume Ω is a bounded convex polyhedral domain, $\boldsymbol{\beta}$ is continuous on $\bar{\Omega}$, $\boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^d$, and the solution of the optimality system (4) is sufficiently smooth.

We choose the stabilization functions τ_1 and τ_2 so that the following conditions are satisfied:

(A1)
$$\tau_1 = \tau_2 + \boldsymbol{\beta} \cdot \boldsymbol{n}$$
.

(A2) For any
$$K \in \mathcal{T}_h$$
, $\min (\tau_1 - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})|_{\partial K} > 0$.

Note that (A1) and (A2) imply

$$\min \left(\tau_2 + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n}\right)|_{\partial K} > 0 \quad \text{for any } K \in \mathcal{T}_h.$$
 (10)

Below, we prove the main result:

Theorem 1. We have

$$\begin{aligned} &\|\boldsymbol{q} - \boldsymbol{q}_{h}\|_{\mathcal{T}_{h}} \lesssim h^{k+1}(|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+2} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+2}), \\ &\|\boldsymbol{p} - \boldsymbol{p}_{h}\|_{\mathcal{T}_{h}} \lesssim h^{k+1}(|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+2} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+2}), \\ &\|\boldsymbol{y} - \boldsymbol{y}_{h}\|_{\mathcal{T}_{h}} \lesssim h^{k+1+\min\{k,1\}}(|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+2} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+2}), \\ &\|\boldsymbol{z} - \boldsymbol{z}_{h}\|_{\mathcal{T}_{h}} \lesssim h^{k+1+\min\{k,1\}}(|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+2} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+2}), \\ &\|\boldsymbol{u} - \boldsymbol{u}_{h}\|_{\mathcal{T}_{h}} \lesssim h^{k+1+\min\{k,1\}}(|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+2} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+2}). \end{aligned}$$

3.1 Preliminary material

Let $\Pi: [L^2(\Omega)]^d \to V_h$, $\Pi: L^2(\Omega) \to W_h$, and $P_M: L^2(\varepsilon_h) \to M_h$ denote the standard L^2 projections, which satisfy

$$(\mathbf{\Pi} \boldsymbol{q}, \boldsymbol{r})_{K} = (\boldsymbol{q}, \boldsymbol{r})_{K}, \qquad \forall \boldsymbol{r} \in [\mathcal{P}_{k}(K)]^{d},$$

$$(\mathbf{\Pi} \boldsymbol{y}, \boldsymbol{w})_{K} = (\boldsymbol{y}, \boldsymbol{w})_{K}, \qquad \forall \boldsymbol{w} \in \mathcal{P}_{k+1}(K),$$

$$\langle P_{M} \boldsymbol{m}, \boldsymbol{\mu} \rangle_{e} = \langle \boldsymbol{m}, \boldsymbol{\mu} \rangle_{e}, \qquad \forall \boldsymbol{\mu} \in \mathcal{P}_{k}(e).$$

$$(11)$$

We use the following well-known bounds:

$$\|\boldsymbol{q} - \boldsymbol{\Pi} \boldsymbol{q}\|_{\mathcal{T}_h} \lesssim h^{k+1} \|\boldsymbol{q}\|_{k+1,\Omega}, \quad \|y - \boldsymbol{\Pi} y\|_{\mathcal{T}_h} \lesssim h^{k+2} \|y\|_{k+2,\Omega},$$
 (12a)

$$\|y - \Pi y\|_{\partial \mathcal{T}_h} \lesssim h^{k + \frac{3}{2}} \|y\|_{k + 2, \Omega}, \quad \|\boldsymbol{q} \cdot \boldsymbol{n} - \boldsymbol{\Pi} \boldsymbol{q} \cdot \boldsymbol{n}\|_{\partial \mathcal{T}_h} \lesssim h^{k + \frac{1}{2}} \|\boldsymbol{q}\|_{k + 1, \Omega}, \tag{12b}$$

$$||w||_{\partial \mathcal{T}_h} \lesssim h^{-\frac{1}{2}} ||w||_{\mathcal{T}_h}, \qquad \forall w \in W_h. \tag{12c}$$

We have the same projection error bounds for p and z.

Next, define HDG operators \mathcal{B}_1 and \mathcal{B}_2 by

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}; \boldsymbol{r}_{1}, w_{1}, \mu_{1})
= (\boldsymbol{q}_{h}, \boldsymbol{r}_{1})_{\mathcal{T}_{h}} - (y_{h}, \nabla \cdot \boldsymbol{r}_{1})_{\mathcal{T}_{h}} + \langle \widehat{y}_{h}^{o}, \boldsymbol{r}_{1} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}} - (\boldsymbol{q}_{h} + \boldsymbol{\beta} y_{h}, \nabla w_{1})_{\mathcal{T}_{h}}
- (\nabla \cdot \boldsymbol{\beta} y_{h}, w_{1})_{\mathcal{T}_{h}} + \langle \boldsymbol{q}_{h} \cdot \boldsymbol{n} + h^{-1} P_{M} y_{h} + \tau_{1} y_{h}, w_{1} \rangle_{\partial \mathcal{T}_{h}}
+ \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - h^{-1} - \tau_{1}) \widehat{y}_{h}^{o}, w_{1} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}}
- \langle \boldsymbol{q}_{h} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{y}_{h}^{o} + h^{-1} (P_{M} y_{h} - \widehat{y}_{h}^{o}) + \tau_{1} (y_{h} - \widehat{y}_{h}^{o}), \mu_{1} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}},$$

$$\mathcal{B}_{2}(\boldsymbol{p}_{h}, \boldsymbol{z}_{h}, \widehat{\boldsymbol{z}}_{h}^{o}; \boldsymbol{r}_{2}, w_{2}, \mu_{2})
= (\boldsymbol{p}_{h}, \boldsymbol{r}_{2})_{\mathcal{T}_{h}} - (\boldsymbol{z}_{h}, \nabla \cdot \boldsymbol{r}_{2})_{\mathcal{T}_{h}} + \langle \widehat{\boldsymbol{z}}_{h}^{o}, \boldsymbol{r}_{2} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}} - (\boldsymbol{p}_{h} - \boldsymbol{\beta} z_{h}, \nabla w_{2})_{\mathcal{T}_{h}}
+ \langle \boldsymbol{p}_{h} \cdot \boldsymbol{n} + h^{-1} P_{M} z_{h} + \tau_{2} z_{h}, w_{2} \rangle_{\partial \mathcal{T}_{h}} - \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} + h^{-1} + \tau_{2}) \widehat{\boldsymbol{z}}_{h}^{o}, w_{2} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}}
- \langle \boldsymbol{p}_{h} \cdot \boldsymbol{n} - \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{\boldsymbol{z}}_{h}^{o} + h^{-1} (P_{M} z_{h} - \widehat{\boldsymbol{z}}_{h}^{o}) + \tau_{2} (z_{h} - \widehat{\boldsymbol{z}}_{h}^{o}), \mu_{2} \rangle_{\partial \mathcal{T}_{h} \backslash \varepsilon_{h}^{\partial}}.$$
(14)

We use \mathcal{B}_1 and \mathcal{B}_2 to rewrite the HDG discretization of the optimality system (9): find

$$(\boldsymbol{q}_h, \boldsymbol{p}_h, y_h, z_h, u_h, \widehat{y}_h^o, \widehat{z}_h^o) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times W_h \times M_h(o) \times M_h(o)$$

satisfying

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}; \boldsymbol{r}_{1}, w_{1}, \mu_{1}) = (f + u_{h}, w_{1})_{\mathcal{T}_{h}} - \langle P_{M}g, \boldsymbol{r}_{1} \cdot \boldsymbol{n} \rangle - \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - h^{-1} - \tau_{1}) P_{M}g, w_{1} \rangle_{\varepsilon_{h}^{\partial}},$$

$$(15a)$$

$$\mathscr{B}_{2}(\mathbf{p}_{h}, z_{h}, \widehat{z}_{h}^{o}; \mathbf{r}_{2}, w_{2}, \mu_{2}) = (y_{d} - y_{h}, w_{2})_{\mathcal{T}_{h}}, \tag{15b}$$

$$(z_h - \gamma u_h, w_3)_{\mathcal{T}_h} = 0, \tag{15c}$$

for all $(\boldsymbol{r}_1, \boldsymbol{r}_2, w_1, w_2, w_3, \mu_1, \mu_2) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times W_h \times M_h(o) \times M_h(o)$.

Next, we prove an energy identity for the HDG operators and prove the discrete optimality system (15) is well-posed. The proofs of the next three results are similar to the proofs of the corresponding results in our earlier work [15]; we include them for completeness.

Lemma 1. For any $(\mathbf{v}_h, w_h, \mu_h) \in \mathbf{V}_h \times W_h \times M_h(o)$, we have

$$\mathcal{B}_{1}(\boldsymbol{v}_{h}, w_{h}, \mu_{h}; \boldsymbol{v}_{h}, w_{h}, \mu_{h})
= (\boldsymbol{v}_{h}, \boldsymbol{v}_{h})_{\mathcal{T}_{h}} + \langle (\tau_{1} - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})(w_{h} - \mu_{h}), w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}w_{h}, w_{h})_{\mathcal{T}_{h}}
+ \langle h^{-1}(P_{M}w_{h} - \mu_{h}), P_{M}w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + \langle (\tau_{1} - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})w_{h}, w_{h} \rangle_{\varepsilon_{h}^{\partial}}
+ \langle h^{-1}P_{M}w_{h}, P_{M}w_{h} \rangle_{\varepsilon_{h}^{\partial}},
\mathcal{B}_{2}(\boldsymbol{v}_{h}, w_{h}, \mu_{h}; \boldsymbol{v}_{h}, w_{h}, \mu_{h})
= (\boldsymbol{v}_{h}, \boldsymbol{v}_{h})_{\mathcal{T}_{h}} + \langle (\tau_{2} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})(w_{h} - \mu_{h}), w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}w_{h}, w_{h})_{\mathcal{T}_{h}}
+ \langle h^{-1}(P_{M}w_{h} - \mu_{h}), P_{M}w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + \langle (\tau_{2} + \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})w_{h}, w_{h} \rangle_{\varepsilon_{h}^{\partial}}
+ \langle h^{-1}P_{M}w_{h}, P_{M}w_{h} \rangle_{\varepsilon_{h}^{\partial}}.$$

Proof. We prove the first identity; the proof of the second identity is similar.

$$\mathcal{B}_{1}(\boldsymbol{v}_{h}, w_{h}, \mu_{h}; \boldsymbol{v}_{h}, w_{h}, \mu_{h})
= (\boldsymbol{v}_{h}, \boldsymbol{v}_{h})_{\mathcal{T}_{h}} - (w_{h}, \nabla \cdot \boldsymbol{v}_{h})_{\mathcal{T}_{h}} + \langle \mu_{h}, \boldsymbol{v}_{h} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - (\boldsymbol{v}_{h} + \boldsymbol{\beta} w_{h}, \nabla w_{h})_{\mathcal{T}_{h}}
- (\nabla \cdot \boldsymbol{\beta} w_{h}, w_{h})_{\mathcal{T}_{h}} + \langle \boldsymbol{v}_{h} \cdot \boldsymbol{n} + h^{-1} P_{M} w_{h} + \tau_{1} w_{h}, w_{h} \rangle_{\partial \mathcal{T}_{h}}
+ \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - h^{-1} - \tau_{1}) \mu_{h}, w_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
- \langle \boldsymbol{v}_{h} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \mu_{h} + h^{-1} (P_{M} w_{h} - \mu_{h}) + \tau_{1} (w_{h} - \mu_{h}), \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}},
= (\boldsymbol{v}_{h}, \boldsymbol{v}_{h})_{\mathcal{T}_{h}} - (\boldsymbol{\beta} w_{h}, \nabla w_{h})_{\mathcal{T}_{h}} - (\nabla \cdot \boldsymbol{\beta} w_{h}, w_{h})_{\mathcal{T}_{h}}
+ \langle h^{-1} P_{M} w_{h} + \tau_{1} w_{h}, w_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - h^{-1} - \tau_{1}) \mu_{h}, w_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
- \langle \boldsymbol{\beta} \cdot \boldsymbol{n} \mu_{h} + h^{-1} (P_{M} w_{h} - \mu_{h}) + \tau_{1} (w_{h} - \mu_{h}), \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}.$$

For the second term, we have

$$(\boldsymbol{\beta}w_h, \nabla w_h)_{\mathcal{T}_h} = (\boldsymbol{\beta} \cdot \nabla w_h, w_h)_{\mathcal{T}_h} = (\nabla \cdot (\boldsymbol{\beta}w_h), w_h)_{\mathcal{T}_h} - (\nabla \cdot \boldsymbol{\beta}w_h, w_h)_{\mathcal{T}_h} = (\boldsymbol{\beta} \cdot \boldsymbol{n}w_h, w_h)_{\partial \mathcal{T}_h} - (\boldsymbol{\beta}w_h, \nabla w_h)_{\mathcal{T}_h} - (\nabla \cdot \boldsymbol{\beta}w_h, w_h)_{\mathcal{T}_h},$$

which implies

$$(\boldsymbol{\beta}w_h, \nabla w_h)_{\mathcal{T}_h} = \frac{1}{2} \langle \boldsymbol{\beta} \cdot \boldsymbol{n}w_h, w_h \rangle_{\partial \mathcal{T}_h} - \frac{1}{2} (\nabla \cdot \boldsymbol{\beta}w_h, w_h)_{\mathcal{T}_h}. \tag{16}$$

This gives

$$\mathcal{B}_{1}(\boldsymbol{v}_{h}, w_{h}, \mu_{h}; \boldsymbol{v}_{h}, w_{h}, \mu_{h})
= (\boldsymbol{v}_{h}, \boldsymbol{v}_{h})_{\mathcal{T}_{h}} + \langle (\tau_{1} - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})(w_{h} - \mu_{h}), w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - \frac{1}{2}(\nabla \cdot \boldsymbol{\beta}w_{h}, w_{h})_{\mathcal{T}_{h}}
+ \langle h^{-1}(P_{M}w_{h} - \mu_{h}), P_{M}w_{h} - \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + \langle (\tau_{1} - \frac{1}{2}\boldsymbol{\beta} \cdot \boldsymbol{n})w_{h}, w_{h} \rangle_{\varepsilon_{h}^{\partial}}
+ \langle h^{-1}P_{M}w_{h}, P_{M}w_{h} \rangle_{\varepsilon_{h}^{\partial}} - \frac{1}{2}\langle \boldsymbol{\beta} \cdot \boldsymbol{n}\mu_{h}, \mu_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}.$$

Since μ_h is single-valued across the interfaces, we have

$$-\frac{1}{2}\langle \boldsymbol{\beta} \cdot \boldsymbol{n} \mu_h, \mu_h \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_h^{\partial}} = 0.$$

This completes the proof.

The following property of the HDG operators is crucial to our analysis.

Lemma 2. We have $\mathscr{B}_1(\boldsymbol{q}_h,y_h,\widehat{y}_h^o;\boldsymbol{p}_h,-z_h,-\widehat{z}_h^o)+\mathscr{B}_2(\boldsymbol{p}_h,z_h,\widehat{z}_h^o;-\boldsymbol{q}_h,y_h,\widehat{y}_h^o)=0.$

Proof. By definition:

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}; \boldsymbol{p}_{h}, -z_{h}, -\widehat{z}_{h}^{o}) + \mathcal{B}_{2}(\boldsymbol{p}_{h}, z_{h}, \widehat{z}_{h}^{o}; -\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}) \\
= (\boldsymbol{q}_{h}, \boldsymbol{p}_{h})_{\mathcal{T}_{h}} - (y_{h}, \nabla \cdot \boldsymbol{p}_{h})_{\mathcal{T}_{h}} + \langle \widehat{y}_{h}^{o}, \boldsymbol{p}_{h} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + (\boldsymbol{q}_{h} + \boldsymbol{\beta} y_{h}, \nabla z_{h})_{\mathcal{T}_{h}} \\
+ (\nabla \cdot \boldsymbol{\beta} y_{h}, z_{h})_{\mathcal{T}_{h}} - \langle \boldsymbol{q}_{h} \cdot \boldsymbol{n} + h^{-1} P_{M} y_{h} + \tau_{1} y_{h}, z_{h} \rangle_{\partial \mathcal{T}_{h}} \\
- \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - \tau_{1} - h^{-1}) \widehat{y}_{h}^{o}, z_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} \\
+ \langle \boldsymbol{q}_{h} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{y}_{h}^{o} + h^{-1} (P_{M} y_{h} - \widehat{y}_{h}^{o}) + \tau_{1} (y_{h} - \widehat{y}_{h}^{o}), \widehat{z}_{h}^{o} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} \\
- (\boldsymbol{p}_{h}, \boldsymbol{q}_{h})_{\mathcal{T}_{h}} + (z_{h}, \nabla \cdot \boldsymbol{q}_{h})_{\mathcal{T}_{h}} - \langle \widehat{z}_{h}^{o}, \boldsymbol{q}_{h} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - (\boldsymbol{p}_{h} - \boldsymbol{\beta} z_{h}, \nabla y_{h})_{\mathcal{T}_{h}} \\
+ \langle \boldsymbol{p}_{h} \cdot \boldsymbol{n} + h^{-1} P_{M} z_{h} + \tau_{2} z_{h}, y_{h} \rangle_{\partial \mathcal{T}_{h}} - \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} + \tau_{2} + h^{-1}) \widehat{z}_{h}^{o}, y_{h} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} \\
- \langle \boldsymbol{p}_{h} \cdot \boldsymbol{n} - \boldsymbol{\beta} \cdot \boldsymbol{n} \widehat{z}_{h}^{o} + h^{-1} (P_{M} z_{h} - \widehat{z}_{h}^{o}) + \tau_{2} (z_{h} - \widehat{z}_{h}^{o}), \widehat{y}_{h}^{o} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}.$$

Integration by parts gives

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}; \boldsymbol{p}_{h}, -z_{h}, -\widehat{z}_{h}^{o}) + \mathcal{B}_{2}(\boldsymbol{p}_{h}, z_{h}, \widehat{z}_{h}^{o}; -\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o})$$

$$= \langle (\tau_{2} + \boldsymbol{\beta} \cdot \boldsymbol{n} - \tau_{1})y_{h}, z_{h} \rangle_{\partial \mathcal{T}_{h}} + \langle (\tau_{2} + \boldsymbol{\beta} \cdot \boldsymbol{n} - \tau_{1})\widehat{y}_{h}^{o}, \widehat{z}_{h}^{o} \rangle_{\partial \mathcal{T}_{h} \setminus \boldsymbol{\epsilon}^{\partial}}.$$

Condition (A1) completes the proof.

Proposition 1. There exists a unique solution of the HDG equations (15).

Proof. Since the system (15) is finite dimensional, we only need to prove the uniqueness. Therefore, we assume $y_d = f = g = 0$ and show the system (15) only has the zero solution.

First, take $(\boldsymbol{r}_1, w_1, \mu_1) = (\boldsymbol{p}_h, -z_h, -\widehat{z}_h^o)$, $(\boldsymbol{r}_2, w_2, \mu_2) = (-\boldsymbol{q}_h, y_h, \widehat{y}_h^o)$, and $w_3 = z_h - \gamma u_h$ in the HDG equations (15a), (15b), and (15c), respectively, and sum to obtain

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o}; \boldsymbol{p}_{h}, -z_{h}, -\widehat{z}_{h}^{o}) + \mathcal{B}_{2}(\boldsymbol{p}_{h}, z_{h}, \widehat{z}_{h}^{o}; -\boldsymbol{q}_{h}, y_{h}, \widehat{y}_{h}^{o})$$

$$= \gamma(y_{h}, y_{h})_{\mathcal{T}_{h}} + (z_{h}, z_{h})_{\mathcal{T}_{h}}.$$

Since $\gamma > 0$, Lemma 2 gives $y_h = u_h = z_h = 0$.

Next, take $(\boldsymbol{r}_1, w_1, \mu_1) = (\boldsymbol{q}_h, y_h, \widehat{y}_h^o)$ and $(\boldsymbol{r}_2, w_2, \mu_2) = (\boldsymbol{p}_h, z_h, \widehat{z}_h^o)$ in Lemma 1, and then use $(\mathbf{A2})$ and (10) to get $\boldsymbol{q}_h = \boldsymbol{p}_h = \mathbf{0}$, $\widehat{y}_h^o = \widehat{z}_h^o = 0$.

3.2 Proof of the main result

We follow the proof strategy used in our earlier works [15,16], and split the proof of the main result into eight steps. We consider the following auxiliary problem: find

$$(\boldsymbol{q}_h(u), \boldsymbol{p}_h(u), y_h(u), z_h(u), \widehat{y}_h^o(u), \widehat{z}_h^o(u)) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times \boldsymbol{W}_h \times \boldsymbol{W}_h \times \boldsymbol{M}_h(o) \times \boldsymbol{M}_h(o)$$

such that

$$\mathcal{B}_{1}(\boldsymbol{q}_{h}(u), y_{h}(u), \widehat{y}_{h}(u); \boldsymbol{r}_{1}, w_{1}, \mu_{1}) = (f + u, w_{1})_{\mathcal{T}_{h}} - \langle P_{M}g, \boldsymbol{r}_{1} \cdot \boldsymbol{n} \rangle - \langle (\boldsymbol{\beta} \cdot \boldsymbol{n} - h^{-1} - \tau_{1}) P_{M}g, w_{1} \rangle_{\varepsilon^{\partial}},$$
(17a)

$$\mathscr{B}_{2}(\mathbf{p}_{h}(u), z_{h}(u), \widehat{z}_{h}(u); \mathbf{r}_{2}, w_{2}, \mu_{2}) = (y_{d} - y_{h}(u), w_{2})_{\mathcal{T}_{h}},$$
(17b)

for all $(\boldsymbol{r}_1, \boldsymbol{r}_2, w_1, w_2, \mu_1, \mu_2) \in \boldsymbol{V}_h \times \boldsymbol{V}_h \times W_h \times W_h \times M_h(o) \times M_h(o)$.

In the first three steps of the proof, we bound the error between the solution components $(y_h(u), \mathbf{q}_h(u))$ of part 1 of the auxiliary problem and (y, \mathbf{q}) of the mixed form of the optimality system. Since u is the exact optimal control in both problems and is fixed, the source terms in both problems are the same. We would use the results from [22] to obtain the error bounds; however, the authors of [22] pointed us to an error in their work in the k = 0 case. To be complete, we present most of the proofs in Steps 1–3, and we use many proof strategies from [22] in those steps.

3.2.1 Step 1: The error equation for part 1 of the auxiliary problem (17a).

Define

$$\delta^{\mathbf{q}} = \mathbf{q} - \mathbf{\Pi} \mathbf{q}, \qquad \qquad \varepsilon_{h}^{\mathbf{q}} = \mathbf{\Pi} \mathbf{q} - \mathbf{q}_{h}(u),$$

$$\delta^{y} = y - \mathbf{\Pi} y, \qquad \qquad \varepsilon_{h}^{y} = \mathbf{\Pi} y - y_{h}(u),$$

$$\delta^{\widehat{y}} = y - P_{M} y, \qquad \qquad \varepsilon_{h}^{\widehat{y}} = P_{M} y - \widehat{y}_{h}(u),$$

$$\widehat{\delta}_{1} = \delta^{\mathbf{q}} \cdot \mathbf{n} + h^{-1} P_{M} \delta^{y} + \beta \cdot \mathbf{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}).$$

$$(18)$$

where $\widehat{y}_h(u) = \widehat{y}_h^o(u)$ on ε_h^o and $\widehat{y}_h(u) = P_M g$ on ε_h^∂ . This gives $\varepsilon_h^{\widehat{y}} = 0$ on ε_h^∂ .

Lemma 3. We have

$$\mathcal{B}_{1}(\varepsilon_{h}^{q}, \varepsilon_{h}^{y}, \varepsilon_{h}^{\widehat{y}}, \boldsymbol{r}_{1}, w_{1}, \mu_{1}) = (\boldsymbol{\beta}\delta^{y}, \nabla w_{1})_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta}\delta^{y}, w_{1})_{\mathcal{T}_{h}} - \langle \widehat{\boldsymbol{\delta}}_{1}, w_{1} \rangle_{\partial \mathcal{T}_{h}} + \langle \widehat{\boldsymbol{\delta}}_{1}, \mu_{1} \rangle_{\partial \mathcal{T}_{h}} \rangle_{\varepsilon_{h}^{\partial}}.$$

$$(19)$$

Proof. By definition:

$$\mathcal{B}_{1}(\mathbf{\Pi}\boldsymbol{q},\Pi\boldsymbol{y},P_{M}\boldsymbol{y},\boldsymbol{r}_{1},w_{1},\mu_{1})
= (\mathbf{\Pi}\boldsymbol{q},\boldsymbol{r}_{1})_{\mathcal{T}_{h}} - (\Pi\boldsymbol{y},\nabla\cdot\boldsymbol{r}_{1})_{\mathcal{T}_{h}} + \langle P_{M}\boldsymbol{y},\boldsymbol{r}_{1}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} - (\mathbf{\Pi}\boldsymbol{q}+\boldsymbol{\beta}\Pi\boldsymbol{y},\nabla\boldsymbol{w}_{1})_{\mathcal{T}_{h}}
- (\nabla\cdot\boldsymbol{\beta}\Pi\boldsymbol{y},w_{1})_{\mathcal{T}_{h}} + \langle \mathbf{\Pi}\boldsymbol{q}\cdot\boldsymbol{n} + h^{-1}P_{M}\Pi\boldsymbol{y} + \tau_{1}\Pi\boldsymbol{y},w_{1}\rangle_{\partial\mathcal{T}_{h}}
+ \langle (\boldsymbol{\beta}\cdot\boldsymbol{n} - h^{-1} - \tau_{1})P_{M}\boldsymbol{y},w_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}}
- \langle \mathbf{\Pi}\boldsymbol{q}\cdot\boldsymbol{n} + \boldsymbol{\beta}\cdot\boldsymbol{n}P_{M}\boldsymbol{y} + h^{-1}(P_{M}\Pi\boldsymbol{y} - P_{M}\boldsymbol{y}) + \tau_{1}(\Pi\boldsymbol{y} - P_{M}\boldsymbol{y}),\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}},$$

Properties of the L^2 projections (11) give

$$\mathcal{B}_{1}(\mathbf{\Pi}\boldsymbol{q},\Pi\boldsymbol{y},P_{M}\boldsymbol{y},\boldsymbol{r}_{1},w_{1},\mu_{1}) \\
= (\boldsymbol{q},\boldsymbol{r}_{1})_{\mathcal{T}_{h}} - (\boldsymbol{y},\nabla\cdot\boldsymbol{r}_{1})_{\mathcal{T}_{h}} + \langle \boldsymbol{y},\boldsymbol{r}_{1}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} \\
- (\boldsymbol{q}+\boldsymbol{\beta}\boldsymbol{y},\nabla\boldsymbol{w}_{1})_{\mathcal{T}_{h}} + (\boldsymbol{\beta}\delta^{\boldsymbol{y}},\nabla\boldsymbol{w}_{1})_{\mathcal{T}_{h}} - (\nabla\cdot\boldsymbol{\beta}\boldsymbol{y},w_{1})_{\mathcal{T}_{h}} + (\nabla\cdot\boldsymbol{\beta}\delta^{\boldsymbol{y}},w_{1})_{\mathcal{T}_{h}} \\
+ \langle \boldsymbol{q}\cdot\boldsymbol{n},w_{1}\rangle_{\partial\mathcal{T}_{h}} - \langle \delta^{\boldsymbol{q}}\cdot\boldsymbol{n},w_{1}\rangle_{\partial\mathcal{T}_{h}} + \langle h^{-1}P_{M}\Pi\boldsymbol{y} + \tau_{1}\Pi\boldsymbol{y},w_{1}\rangle_{\partial\mathcal{T}_{h}} \\
+ \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\boldsymbol{y},w_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{\boldsymbol{y}}},w_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} - \langle (h^{-1}+\tau_{1})P_{M}\boldsymbol{y},w_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} \\
- \langle \boldsymbol{q}\cdot\boldsymbol{n},\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle \delta^{\boldsymbol{q}}\cdot\boldsymbol{n},\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\boldsymbol{y},\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} \\
+ \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{\boldsymbol{y}}},\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle h^{-1}P_{M}\delta^{\boldsymbol{y}},\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle \tau_{1}(\delta^{\boldsymbol{y}}-\delta^{\widehat{\boldsymbol{y}}}),\mu_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}}.$$

The exact state y and flux q satisfy

$$\begin{split} (\boldsymbol{q}, \boldsymbol{r}_1)_{\mathcal{T}_h} - (y, \nabla \cdot \boldsymbol{r}_1)_{\mathcal{T}_h} + \langle y, \boldsymbol{r}_1 \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_h^{\partial}} &= -\langle g, \boldsymbol{r}_1 \cdot \boldsymbol{n} \rangle_{\varepsilon_h^{\partial}}, \\ - (\boldsymbol{q} + \boldsymbol{\beta} y, \nabla w_1)_{\mathcal{T}_h} - (\nabla \cdot \boldsymbol{\beta} y, w_1)_{\mathcal{T}_h} \\ + \langle \boldsymbol{q} \cdot \boldsymbol{n}, w_1 \rangle_{\partial \mathcal{T}_h} + \langle \boldsymbol{\beta} \cdot \boldsymbol{n} y, w_1 \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_h^{\partial}} &= -\langle \boldsymbol{\beta} \cdot \boldsymbol{n} g, w_1 \rangle_{\varepsilon_h^{\partial}} + (f + u, w_1)_{\mathcal{T}_h}, \\ \langle (\boldsymbol{q} + \boldsymbol{\beta} y) \cdot \boldsymbol{n}, \mu_1 \rangle_{\partial \mathcal{T}_h \setminus \varepsilon_h^{\partial}} &= 0, \end{split}$$

for all $(\mathbf{r}_1, w_1, \mu_1) \in \mathbf{V}_h \times W_h \times M_h(o)$. Therefore,

$$\mathcal{B}_{1}(\mathbf{\Pi}\boldsymbol{q},\Pi\boldsymbol{y},P_{M}\boldsymbol{y},\boldsymbol{r}_{1},\boldsymbol{w}_{1},\boldsymbol{\mu}_{1}) \\
= -\langle g,\boldsymbol{r}_{1}\cdot\boldsymbol{n}\rangle_{\varepsilon_{h}^{\partial}} - \langle \boldsymbol{\beta}\cdot\boldsymbol{n}g,\boldsymbol{w}_{1}\rangle_{\varepsilon_{h}^{\partial}} + (f+u,\boldsymbol{w}_{1})_{\mathcal{T}_{h}} + (\boldsymbol{\beta}\delta^{y},\nabla\boldsymbol{w}_{1})_{\mathcal{T}_{h}} \\
+ (\nabla\cdot\boldsymbol{\beta}\delta^{y},\boldsymbol{w}_{1})_{\mathcal{T}_{h}} - \langle \delta^{\boldsymbol{q}}\cdot\boldsymbol{n},\boldsymbol{w}_{1}\rangle_{\partial\mathcal{T}_{h}} + \langle h^{-1}P_{M}\Pi\boldsymbol{y} + \tau_{1}\Pi\boldsymbol{y},\boldsymbol{w}_{1}\rangle_{\partial\mathcal{T}_{h}} \\
- \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}},\boldsymbol{w}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} - \langle (h^{-1}+\tau_{1})P_{M}\boldsymbol{y},\boldsymbol{w}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle \delta^{\boldsymbol{q}}\cdot\boldsymbol{n},\boldsymbol{\mu}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} \\
+ \langle \boldsymbol{\beta}\cdot\boldsymbol{n}\delta^{\widehat{y}},\boldsymbol{\mu}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle h^{-1}P_{M}\delta^{y},\boldsymbol{\mu}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}} + \langle \tau_{1}(\delta^{y}-\delta^{\widehat{y}}),\boldsymbol{\mu}_{1}\rangle_{\partial\mathcal{T}_{h}\backslash\varepsilon_{h}^{\partial}}.$$

Subtracting part 1 of the auxiliary problem (17a) from the above equality gives the result:

$$\mathcal{B}_{1}(\varepsilon_{h}^{\mathbf{q}}, \varepsilon_{h}^{y}, \varepsilon_{h}^{\widehat{y}}, \mathbf{r}_{1}, w_{1}, \mu_{1})
= (\boldsymbol{\beta}\delta^{y}, \nabla w_{1})_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta}\delta^{y}, w_{1})_{\mathcal{T}_{h}} - \langle \delta^{\mathbf{q}} \cdot \boldsymbol{n}, w_{1} \rangle_{\partial \mathcal{T}_{h}} + \langle h^{-1}P_{M}\Pi y, w_{1} \rangle_{\partial \mathcal{T}_{h}}
+ \langle \tau_{1}\Pi y, w_{1} \rangle_{\partial \mathcal{T}_{h}} - \langle \boldsymbol{\beta} \cdot \boldsymbol{n}\delta^{\widehat{y}}, w_{1} \rangle_{\partial \mathcal{T}_{h}} - \langle (h^{-1} + \tau_{1})P_{M}y, w_{1} \rangle_{\partial \mathcal{T}_{h}}
+ \langle \delta^{\mathbf{q}} \cdot \boldsymbol{n}, \mu_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + \langle \boldsymbol{\beta} \cdot \boldsymbol{n}\delta^{\widehat{y}}, \mu_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + \langle h^{-1}P_{M}\delta^{y}, \mu_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
+ \langle \tau_{1}(\delta^{y} - \delta^{\widehat{y}}), \mu_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
= (\boldsymbol{\beta}\delta^{y}, \nabla w_{1})_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta}\delta^{y}, w_{1})_{\mathcal{T}_{h}} - \langle \widehat{\boldsymbol{\delta}}_{1}, w_{1} \rangle_{\partial \mathcal{T}_{h}} + \langle \widehat{\boldsymbol{\delta}}_{1}, \mu_{1} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}.$$

3.2.2 Step 2: Estimate for ε_h^q .

The following key inequality is found in [22].

Lemma 4. We have

$$\|\nabla \varepsilon_h^y\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\varepsilon_h^y - \varepsilon_h^{\widehat{y}}\|_{\partial \mathcal{T}_h} \lesssim \|\varepsilon_h^q\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_h^{\widehat{y}}\|_{\partial \mathcal{T}_h}.$$

Lemma 5. We have

$$\|\varepsilon_h^{\mathbf{q}}\|_{\mathcal{T}_t} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^y - \varepsilon_h^{\widehat{y}}\|_{\partial \mathcal{T}_h} \lesssim h^{k+1} (\|\mathbf{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega}). \tag{20}$$

Proof. First, since $\varepsilon_h^{\widehat{y}} = 0$ on ε_h^{∂} , the energy identity for \mathscr{B}_1 in Lemma 1 gives

$$\mathcal{B}(\varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{y}}, \varepsilon_{h}^{\widehat{\boldsymbol{y}}}, \varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{y}}, \varepsilon_{h}^{\widehat{\boldsymbol{y}}})
= (\varepsilon_{h}^{\boldsymbol{q}}, \varepsilon_{h}^{\boldsymbol{q}})_{\mathcal{T}_{h}} + h^{-1} \| P_{M} \varepsilon_{h}^{\boldsymbol{y}} - \varepsilon_{h}^{\widehat{\boldsymbol{y}}} \|_{\partial \mathcal{T}_{h}}^{2} + \frac{1}{2} \| (-\nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}} \varepsilon_{h}^{\boldsymbol{y}} \|_{\mathcal{T}_{h}}^{2}
+ \| (\tau_{1} - \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n})^{\frac{1}{2}} (\varepsilon_{h}^{\boldsymbol{y}} - \varepsilon_{h}^{\widehat{\boldsymbol{y}}}) \|_{\partial \mathcal{T}_{h}}^{2}.$$

Taking $(r_1, w_1, \mu_1) = (\varepsilon_h^q, \varepsilon_h^y, \varepsilon_h^{\widehat{y}})$ in (19) in Lemma 3 gives

$$(\varepsilon_{h}^{q}, \varepsilon_{h}^{q})_{\mathcal{T}_{h}} + h^{-1} \| P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \|_{\partial \mathcal{T}_{h}}^{2} + \frac{1}{2} \| (-\nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}} \varepsilon_{h}^{y} \|_{\mathcal{T}_{h}}^{2}$$

$$\leq (\boldsymbol{\beta} \delta^{y}, \nabla \varepsilon_{h}^{y})_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta} \delta^{y}, \varepsilon_{h}^{y})_{\mathcal{T}_{h}} - \langle \widehat{\boldsymbol{\delta}}_{1}, \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}}$$

$$=: T_{1} + T_{2} + T_{3}.$$

$$(21)$$

For the terms T_1 and T_2 , apply Lemma 4 and Young's inequality to give

$$T_{1} = (\boldsymbol{\beta}\delta^{y}, \nabla \varepsilon_{h}^{y})_{\mathcal{T}_{h}} \leq C \|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{2} \|\delta^{y}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4} \|\varepsilon_{h}^{q}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4h} \|P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\|_{\partial \mathcal{T}_{h}}^{2},$$

$$T_{2} = (\nabla \cdot \boldsymbol{\beta}\delta^{y}, \varepsilon_{h}^{y})_{\mathcal{T}_{h}} \leq C \|\delta^{y}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{2} \|(-\nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}} \varepsilon_{h}^{y}\|_{\mathcal{T}_{h}}^{2}.$$

For the term T_3 ,

$$T_{3} = -\langle \widehat{\boldsymbol{\delta}}_{1}, \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}}$$

$$= -\langle \delta^{\boldsymbol{q}} \cdot \boldsymbol{n} + h^{-1} P_{M} \delta^{y} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}), \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}}$$

$$= -\langle \delta^{\boldsymbol{q}} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}), \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}} - \langle h^{-1} P_{M} \delta^{y}, \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}}$$

$$=: T_{4} + T_{5}.$$

Applying Lemma 4 and Young's inequality again gives

$$T_{4} = -\langle \delta^{\mathbf{q}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}), \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \rangle_{\partial \mathcal{T}_{h}}$$

$$\leq C \|h^{1/2} (\delta^{\mathbf{q}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}))\|_{\partial \mathcal{T}_{h}}^{2} + \frac{1}{C} \|h^{-1/2} (\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}})\|_{\partial \mathcal{T}_{h}}^{2}$$

$$\leq C \|h^{1/2} (\delta^{\mathbf{q}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\widehat{y}} + \tau_{1} (\delta^{y} - \delta^{\widehat{y}}))\|_{\partial \mathcal{T}_{h}}^{2} + \frac{1}{4} \|\varepsilon_{h}^{\mathbf{q}}\|_{\mathcal{T}_{h}}^{2}$$

$$+ \frac{1}{4h} \|P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\|_{\partial \mathcal{T}_{h}}^{2}.$$

Finally, for the term T_5 , we have

$$T_{5} = -\langle h^{-1}P_{M}\delta^{y}, \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\rangle_{\partial\mathcal{T}_{h}} = \langle h^{-1}\delta^{y}, P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\rangle_{\partial\mathcal{T}_{h}}$$

$$\leq 4\|h^{-1/2}\delta^{y}\|_{\partial\mathcal{T}_{h}}^{2} + \frac{1}{4h}\|P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\|_{\partial\mathcal{T}_{h}}^{2}.$$

Sum all the estimates for $\{T_i\}_{i=1}^5$ to obtain

$$\begin{split} \|\varepsilon_{h}^{\boldsymbol{q}}\|_{\mathcal{T}_{h}}^{2} + h^{-1} \|P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\|_{\partial \mathcal{T}_{h}}^{2} \\ &\lesssim h \|\delta^{\boldsymbol{q}}\|_{\partial \mathcal{T}_{h}}^{2} + h^{-1} \|\delta^{y}\|_{\partial \mathcal{T}_{h}}^{2} + h \|\delta^{\widehat{y}}\|_{\partial \mathcal{T}_{h}}^{2}, \\ &\lesssim h^{2k+2} (\|\boldsymbol{q}\|_{k+1,\Omega}^{2} + \|y\|_{k+2,\Omega}^{2}). \end{split}$$

3.2.3 Step 3: Estimate for ε_h^y by a duality argument.

Next, for any given Θ in $L^2(\Omega)$ the dual problem is given by

$$\Phi - \nabla \Psi = 0 \quad \text{in } \Omega,$$

$$\nabla \cdot \Phi + \nabla \cdot (\beta \Psi) = \Theta \quad \text{in } \Omega,$$

$$\Psi = 0 \quad \text{on } \partial \Omega.$$
(22)

Since the domain Ω is convex, we have the following regularity estimate

$$\|\Phi\|_{1,\Omega} + \|\Psi\|_{2,\Omega} \le C_{\text{reg}} \|\Theta\|_{\Omega},$$
 (23)

We use the following quantities in the proof below to estimate ε_h^y :

$$\delta^{\mathbf{\Phi}} = \mathbf{\Phi} - \mathbf{\Pi}\mathbf{\Phi}, \quad \delta^{\Psi} = \Psi - \Pi\Psi, \quad \delta^{\widehat{\Psi}} = \Psi - P_M\Psi. \tag{24}$$

Lemma 6. We have

$$\|\varepsilon_h^y\|_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}}(\|\boldsymbol{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega}).$$

Proof. Consider the dual problem (22) and let $\Theta = -\varepsilon_h^y$. Take $(\mathbf{r}_1, w_1, \mu_1) = (\mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_M \Psi)$ in (19) in Lemma 3, and since $\Psi = 0$ on ε_h^{∂} , we have

$$\begin{split} \mathscr{B}_{1}(\varepsilon_{h}^{\mathbf{q}}, \varepsilon_{h}^{y}, \varepsilon_{h}^{\widehat{y}}; \mathbf{\Pi} \mathbf{\Phi}, \mathbf{\Pi} \Psi, P_{M} \Psi) \\ &= (\varepsilon_{h}^{\mathbf{q}}, \mathbf{\Pi} \mathbf{\Phi})_{\mathcal{T}_{h}} - (\varepsilon_{h}^{y}, \nabla \cdot \mathbf{\Pi} \mathbf{\Phi})_{\mathcal{T}_{h}} + \langle \varepsilon_{h}^{\widehat{y}}, \mathbf{\Pi} \mathbf{\Phi} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} - (\varepsilon_{h}^{\mathbf{q}} + \boldsymbol{\beta} \varepsilon_{h}^{y}, \nabla \mathbf{\Pi} \Psi)_{\mathcal{T}_{h}} \\ &- (\nabla \cdot \boldsymbol{\beta} \varepsilon_{h}^{y}, \mathbf{\Pi} \Psi)_{\mathcal{T}_{h}} + \langle \varepsilon_{h}^{\mathbf{q}} \cdot \mathbf{n} + h^{-1} P_{M} \varepsilon_{h}^{y} + \tau_{1} \varepsilon_{h}^{y}, \mathbf{\Pi} \Psi \rangle_{\partial \mathcal{T}_{h}} \\ &+ \langle (\boldsymbol{\beta} \cdot \mathbf{n} - h^{-1} - \tau_{1}) \varepsilon_{h}^{\widehat{y}}, \mathbf{\Pi} \Psi \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} \\ &- \langle \varepsilon_{h}^{\mathbf{q}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \varepsilon_{h}^{\widehat{y}} + h^{-1} (P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1} (\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}), P_{M} \Psi \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} \\ &= (\varepsilon_{h}^{\mathbf{q}}, \mathbf{\Phi})_{\mathcal{T}_{h}} - (\varepsilon_{h}^{y}, \nabla \cdot \mathbf{\Phi})_{\mathcal{T}_{h}} + (\varepsilon_{h}^{y}, \nabla \cdot \delta^{\mathbf{\Phi}})_{\mathcal{T}_{h}} - \langle \varepsilon_{h}^{\widehat{y}}, \delta^{\mathbf{\Phi}} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_{h}} \\ &- (\varepsilon_{h}^{\mathbf{q}} + \boldsymbol{\beta} \varepsilon_{h}^{y}, \nabla \Psi)_{\mathcal{T}_{h}} + (\varepsilon_{h}^{\mathbf{q}} + \boldsymbol{\beta} \varepsilon_{h}^{y}, \nabla \delta^{\Psi})_{\mathcal{T}_{h}} - (\nabla \cdot \boldsymbol{\beta} \varepsilon_{h}^{y}, \Psi)_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta} \varepsilon_{h}^{y}, \delta^{\Psi})_{\mathcal{T}_{h}} \\ &- \langle \varepsilon_{h}^{\mathbf{q}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \varepsilon_{h}^{\widehat{y}} + h^{-1} (P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1} (\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}), \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}}. \end{split}$$

Here we used $\langle \varepsilon_h^{\widehat{y}}, \mathbf{\Phi} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_h} = 0$, which holds since $\varepsilon_h^{\widehat{y}}$ is a single-valued function on interior edges and $\varepsilon_h^{\widehat{y}} = 0$ on ε_h^{∂} .

Next, integration by parts gives

$$(\varepsilon_{h}^{y}, \nabla \cdot \delta^{\Phi})_{\mathcal{T}_{h}} = \langle \varepsilon_{h}^{y}, \delta^{\Phi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h}} - (\nabla \varepsilon_{h}^{y}, \delta^{\Phi})_{\mathcal{T}_{h}} = \langle \varepsilon_{h}^{y}, \delta^{\Phi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_{h}},$$

$$(\varepsilon_{h}^{q}, \nabla \delta^{\Psi})_{\mathcal{T}_{h}} = \langle \varepsilon_{h}^{q} \cdot \boldsymbol{n}, \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} - (\nabla \cdot \varepsilon_{h}^{q}, \delta^{\Psi})_{\mathcal{T}_{h}} = \langle \varepsilon_{h}^{q} \cdot \boldsymbol{n}, \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}},$$

$$(\beta \varepsilon_{h}^{y}, \nabla \delta^{\Psi})_{\mathcal{T}_{h}} = \langle \beta \cdot \boldsymbol{n} \varepsilon_{h}^{y}, \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} - (\nabla \cdot \beta \varepsilon_{h}^{y}, \delta^{\Psi})_{\mathcal{T}_{h}} - (\beta \nabla \varepsilon_{h}^{y}, \delta^{\Psi})_{\mathcal{T}_{h}}.$$

$$(25)$$

We have

$$\begin{split} \mathscr{B}_{1}(\varepsilon_{h}^{\mathbf{q}}, \varepsilon_{h}^{y}, \varepsilon_{h}^{\widehat{y}}; \mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_{M} \Psi) \\ &= \|\varepsilon_{h}^{y}\|_{\mathcal{T}_{h}}^{2} + \langle \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}, \delta^{\mathbf{\Phi}} \cdot \mathbf{n} + \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} - (\nabla \varepsilon_{h}^{y}, \boldsymbol{\beta} \delta^{\Psi})_{\mathcal{T}_{h}} \\ &- \langle h^{-1}(P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1}(\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}), \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}}. \end{split}$$

On the other hand, since $\Psi = 0$ on ε_h^{∂} the error equation (19) in Lemma 3 gives

$$\mathcal{B}_{1}(\varepsilon_{h}^{q}, \varepsilon_{h}^{y}, \varepsilon_{h}^{\widehat{y}}; \mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_{M} \Psi)$$

$$= (\boldsymbol{\beta} \delta^{y}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} + (\nabla \cdot \boldsymbol{\beta} \delta^{y}, \Pi \Psi)_{\mathcal{T}_{h}} + \langle \widehat{\boldsymbol{\delta}}_{1}, \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}}.$$

Comparing the above two equalities, we get

$$\begin{split} &|\varepsilon_{h}^{y}||_{\mathcal{T}_{h}}^{2} \\ &= -\langle \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}, \delta^{\Phi} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} + (\nabla \varepsilon_{h}^{y}, \boldsymbol{\beta} \delta^{\Psi})_{\mathcal{T}_{h}} + (\boldsymbol{\beta} \delta^{y}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} \\ &+ (\nabla \cdot \boldsymbol{\beta} \delta^{y}, \Pi \Psi)_{\mathcal{T}_{h}} + \langle h^{-1} (P_{M} \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1} (\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \widehat{\boldsymbol{\delta}}_{1}, \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}} \\ &=: R_{1} + R_{2} + R_{3} + R_{4} + R_{5}. \end{split}$$

For the terms R_1 and R_2 , Lemma 4 and Lemma 5 give

$$\begin{split} R_{1} &= -\langle \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}, \delta^{\Phi} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} \\ &\leq h^{-\frac{1}{2}} \| \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \|_{\partial \mathcal{T}_{h}} \ h^{\frac{1}{2}} \| \delta^{\Phi} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\Psi} \|_{\partial \mathcal{T}_{h}} \\ &\leq h^{-\frac{1}{2}} \| \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \|_{\partial \mathcal{T}_{h}} \| \delta^{\Phi} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\Psi} \|_{\mathcal{T}_{h}} \\ &\leq C h^{-\frac{1}{2}} \| \varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}} \|_{\partial \mathcal{T}_{h}} (\| \delta^{\Phi} \|_{\mathcal{T}_{h}} + \| \delta^{\Psi} \|_{\mathcal{T}_{h}}) \\ &\leq C h^{k+2} (\| \boldsymbol{q} \|_{k+1,\Omega} + \| \boldsymbol{y} \|_{k+2,\Omega}) \| \varepsilon_{h}^{y} \|_{\mathcal{T}_{h}}, \\ R_{2} &= (\nabla \varepsilon_{h}^{y}, \boldsymbol{\beta} \delta^{\Psi})_{\mathcal{T}_{h}} \leq C \| \nabla \varepsilon_{h}^{y} \|_{\mathcal{T}_{h}} \| \delta^{\Psi} \|_{\mathcal{T}_{h}} \\ &\leq C h^{k+2} (\| \boldsymbol{q} \|_{k+1,\Omega} + \| \boldsymbol{y} \|_{k+2,\Omega}) \| \varepsilon_{h}^{y} \|_{\mathcal{T}_{h}}. \end{split}$$

By a simple application of the triangle inequality for the terms R_3 and R_4 , we have

$$R_{3} = (\boldsymbol{\beta}\delta^{y}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} \|\nabla \Pi \Psi\|_{\mathcal{T}_{h}} \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} (\|\nabla \delta^{\Psi}\|_{\mathcal{T}_{h}} + \|\nabla \Psi\|_{\mathcal{T}_{h}})$$

$$\leq C \|\delta^{y}\|_{\mathcal{T}_{h}} (h\|\Psi\|_{2,\Omega} + \|\Psi\|_{1,\Omega}) \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} \|\Psi\|_{2,\Omega}$$

$$\leq C h^{k+2} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega}) \|\varepsilon_{h}^{y}\|_{\mathcal{T}_{h}},$$

$$R_{4} = (\nabla \cdot \boldsymbol{\beta}\delta^{y}, \Pi \Psi)_{\mathcal{T}_{h}} \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} \|\Pi \Psi\|_{\mathcal{T}_{h}} \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} (\|\delta^{\Psi}\|_{\mathcal{T}_{h}} + \|\Psi\|_{\mathcal{T}_{h}})$$

$$\leq C \|\delta^{y}\|_{\mathcal{T}_{h}} (h^{2}\|\Psi\|_{2,\Omega} + \|\Psi\|_{\Omega}) \leq C \|\delta^{y}\|_{\mathcal{T}_{h}} \|\Psi\|_{2,\Omega}$$

$$\leq C h^{k+2} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega}) \|\varepsilon_{h}^{y}\|_{\mathcal{T}_{h}}.$$

For the terms R_1 to R_4 , we obtain the optimal convergence rate for $k \geq 0$. However, we only get the optimal convergence rate for R_5 when $k \geq 1$.

$$R_{5} = \langle h^{-1}(P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1}(\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \widehat{\boldsymbol{\delta}}_{1}, \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial\mathcal{T}_{h}}$$

$$\leq \|h^{-1}(P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \tau_{1}(\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) + \widehat{\boldsymbol{\delta}}_{1}\|_{\partial\mathcal{T}_{h}} \|\delta^{\Psi} - \delta^{\widehat{\Psi}}\|_{\partial\mathcal{T}_{h}}$$

$$\leq C(h^{-1}\|(P_{M}\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}})\|_{\partial\mathcal{T}_{h}} + \|\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}\|_{\partial\mathcal{T}_{h}} + \|\widehat{\boldsymbol{\delta}}_{1}\|_{\partial\mathcal{T}_{h}})\|\delta^{\Psi} - \delta^{\widehat{\Psi}}\|_{\partial\mathcal{T}_{h}}.$$

It is straightforward to get

$$h^{-1} \| (P_M \varepsilon_h^y - \varepsilon_h^{\widehat{y}}) \|_{\partial \mathcal{T}_h} + \| \varepsilon_h^y - \varepsilon_h^{\widehat{y}} \|_{\partial \mathcal{T}_h} + \| \widehat{\delta}_1 \|_{\partial \mathcal{T}_h}$$

$$\leq C h^{k + \frac{1}{2}} (\| \boldsymbol{q} \|_{k+1,\Omega} + \| \boldsymbol{y} \|_{k+2,\Omega}),$$

and

$$\|\delta^{\Psi} - \delta^{\widehat{\Psi}}\|_{\partial \mathcal{T}_h} \le Ch^{\min\{k,1\} + \frac{1}{2}} \|\varepsilon_h^y\|_{\mathcal{T}_h}.$$

This gives

$$R_5 \le Ch^{k+1+\min\{k,1\}} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega}) \|\varepsilon_h^y\|_{\mathcal{T}_h}.$$

Finally, we complete the proof by summing the estimates for R_1 to R_5 .

The triangle inequality gives convergence rates for $\|\mathbf{q} - \mathbf{q}_h(u)\|_{\mathcal{T}_h}$ and $\|y - y_h(u)\|_{\mathcal{T}_h}$:

Lemma 7.

$$\|\boldsymbol{q} - \boldsymbol{q}_h(u)\|_{\mathcal{T}_h} \le \|\delta^{\boldsymbol{q}}\|_{\mathcal{T}_h} + \|\varepsilon_h^{\boldsymbol{q}}\|_{\mathcal{T}_h} \lesssim h^{k+1}(\|\boldsymbol{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega}),$$
(26a)

$$||y - y_h(u)||_{\mathcal{T}_h} \le ||\delta^y||_{\mathcal{T}_h} + ||\varepsilon_h^y||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (||q||_{k+1,\Omega} + ||y||_{k+2,\Omega}).$$
 (26b)

3.2.4 Step 4: The error equation for part 2 of the auxiliary problem (17b).

Next, we consider the dual variables, i.e., the state z and the flux p, and bound the error between the solutions of part 2 of the auxiliary problem and the mixed form (8a)-(8d) of the optimality system. Define

$$\delta^{\mathbf{p}} = \mathbf{p} - \mathbf{\Pi} \mathbf{p}, \qquad \qquad \varepsilon_{h}^{\mathbf{p}} = \mathbf{\Pi} \mathbf{p} - \mathbf{p}_{h}(u),
\delta^{z} = z - \Pi z, \qquad \qquad \varepsilon_{h}^{z} = \Pi z - z_{h}(u),
\delta^{\hat{z}} = z - P_{M} z, \qquad \qquad \varepsilon_{h}^{\hat{z}} = P_{M} z - \hat{z}_{h}(u),
\hat{\delta}_{2} = \delta^{\mathbf{p}} \cdot \mathbf{n} + h^{-1} P_{M} \delta^{z} + \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\hat{z}} + \tau_{2} (\delta^{z} - \delta^{\hat{z}}).$$

$$(27)$$

Lemma 8. We have

$$\mathcal{B}_{2}(\varepsilon_{h}^{\mathbf{p}}, \varepsilon_{h}^{z}, \varepsilon_{h}^{\widehat{z}}, \mathbf{r}_{2}, w_{2}, \mu_{2}) = (\boldsymbol{\beta}\delta^{z}, \nabla w_{2})_{\mathcal{T}_{h}} - \langle \widehat{\boldsymbol{\delta}}_{2}, w_{2} \rangle_{\partial \mathcal{T}_{h}} + \langle \widehat{\boldsymbol{\delta}}_{2}, \mu_{2} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}} + (y - y_{h}(u), w_{2})_{\mathcal{T}_{h}}.$$

$$(28)$$

The proof is similar to the proof of Lemma 3 and is omitted.

3.2.5 Step 5: Estimate for ε_h^p .

The following discrete Poincaré inequality can be found in [22].

Lemma 9. We have

$$\|\varepsilon_h^z\|_{\mathcal{T}_h} \le C(\|\nabla \varepsilon_h^z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}}\|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial \mathcal{T}_h}). \tag{29}$$

Lemma 10. We have

$$\|\varepsilon_{h}^{\mathbf{p}}\|_{\mathcal{T}_{h}} + h^{-\frac{1}{2}} \|P_{M}\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}\|_{\partial \mathcal{T}_{h}}$$

$$\lesssim h^{k+1} (\|\mathbf{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega} + \|\mathbf{p}\|_{k+1,\Omega} + \|z\|_{k+2,\Omega}),$$
(30a)

$$\|\varepsilon_h^z\|_{\mathcal{T}_h} \lesssim h^{k+1}(\|q\|_{k+1,\Omega} + \|y\|_{k+2,\Omega} + \|p\|_{k+1,\Omega} + \|z\|_{k+2,\Omega}).$$
 (30b)

Proof. First, we note the key inequality in Lemma 4 can be applied with (z, \mathbf{p}, \hat{z}) replaced by (y, \mathbf{q}, \hat{y}) . This gives

$$\|\nabla \varepsilon_h^z\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|\varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial \mathcal{T}_h} \lesssim \|\varepsilon_h^p\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^z - \varepsilon_h^{\widehat{z}}\|_{\partial \mathcal{T}_h}. \tag{31}$$

Next, since $\varepsilon_h^{\hat{z}} = 0$ on ε_h^{∂} , the energy identity for \mathscr{B}_2 in Lemma 1 gives

$$\begin{split} \mathscr{B}_{2}(\varepsilon_{h}^{\boldsymbol{p}}, \varepsilon_{h}^{z}, \varepsilon_{h}^{\widehat{z}}, \varepsilon_{h}^{\boldsymbol{p}}, \varepsilon_{h}^{z}, \varepsilon_{h}^{\widehat{z}}) \\ &= (\varepsilon_{h}^{\boldsymbol{p}}, \varepsilon_{h}^{\boldsymbol{p}})_{\mathcal{T}_{h}} + h^{-1} \| P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}} \|_{\partial \mathcal{T}_{h}}^{2} + \frac{1}{2} \| (-\nabla \cdot \boldsymbol{\beta})^{\frac{1}{2}} \varepsilon_{h}^{z} \|_{\mathcal{T}_{h}}^{2} \\ &+ \| (\tau_{2} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n})^{\frac{1}{2}} (\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}) \|_{\partial \mathcal{T}_{h}}^{2}. \end{split}$$

Then taking $(\mathbf{r}_2, w_2, \mu_2) = (\boldsymbol{\varepsilon}_h^{\mathbf{p}}, \boldsymbol{\varepsilon}_h^{\mathbf{z}}, \boldsymbol{\varepsilon}_h^{\widehat{\mathbf{z}}})$ in (28) in Lemma 8 gives

$$(\varepsilon_{h}^{\mathbf{p}}, \varepsilon_{h}^{\mathbf{p}})_{\mathcal{T}_{h}} + h^{-1} \| P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}} \|_{\partial \mathcal{T}_{h}}^{2} + \| (\tau_{2} + \frac{1}{2} \boldsymbol{\beta} \cdot \boldsymbol{n})^{\frac{1}{2}} (\varepsilon_{h}^{y} - \varepsilon_{h}^{\widehat{y}}) \|_{\partial \mathcal{T}_{h}}^{2}$$

$$\leq (\boldsymbol{\beta} \delta^{z}, \nabla \varepsilon_{h}^{z})_{\mathcal{T}_{h}} - \langle \widehat{\boldsymbol{\delta}}_{2}, \varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}} \rangle_{\partial \mathcal{T}_{h}} + (y - y_{h}(u), \varepsilon_{h}^{z})_{\mathcal{T}_{h}}$$

$$=: T_{1} + T_{2} + T_{3}.$$

As in the proof of Lemma 5, apply (31) and Young's inequality to obtain

$$T_{1} = (\boldsymbol{\beta}\delta^{z}, \nabla \varepsilon_{h}^{z})_{\mathcal{T}_{h}}$$

$$\leq C \|\delta^{z}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4} \|\varepsilon_{h}^{\boldsymbol{p}}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4h} \|P_{M}\varepsilon_{h}^{z} - \varepsilon_{h}^{z}\|_{\partial \mathcal{T}_{h}}^{2},$$

$$T_{2} = -\langle \widehat{\boldsymbol{\delta}}_{2}, \varepsilon_{h}^{z} - \varepsilon_{h}^{z} \rangle_{\partial \mathcal{T}_{h}}$$

$$\leq C (\|\delta^{\boldsymbol{p}}\|_{\mathcal{T}_{h}}^{2} + h^{-2} \|\delta^{z}\|_{\mathcal{T}_{h}}^{2} + h \|\delta^{z}\|_{\partial \mathcal{T}_{h}}^{2})$$

$$+ \frac{1}{4} \|\varepsilon_{h}^{\boldsymbol{p}}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4h} \|P_{M}\varepsilon_{h}^{z} - \varepsilon_{h}^{z}\|_{\partial \mathcal{T}_{h}}^{2}.$$

For the term T_3 , we have

$$T_{3} = (y - y_{h}(u), \varepsilon_{h}^{z})_{\mathcal{T}_{h}} \leq \|y - y_{h}(u)\|_{\mathcal{T}_{h}} \|\varepsilon_{h}^{z}\|_{\mathcal{T}_{h}}$$

$$\leq C\|y - y_{h}(u)\|_{\mathcal{T}_{h}} (\|\nabla \varepsilon_{h}^{z}\|_{\mathcal{T}_{h}} + h^{-\frac{1}{2}}\|\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}\|_{\partial \mathcal{T}_{h}})$$

$$\leq C\|y - y_{h}(u)\|_{\mathcal{T}_{h}} (\|\varepsilon_{h}^{p}\|_{\mathcal{T}_{h}} + h^{-\frac{1}{2}}\|P_{M}\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}\|_{\partial \mathcal{T}_{h}})$$

$$\leq C\|y - y_{h}(u)\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4}\|\varepsilon_{h}^{p}\|_{\mathcal{T}_{h}}^{2} + \frac{1}{4h}\|P_{M}\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}\|_{\partial \mathcal{T}_{h}}^{2}.$$

Summing T_1 to T_3 gives

$$\begin{aligned} \|\varepsilon_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} + h^{-\frac{1}{2}} \|P_M \varepsilon_h^{\tilde{z}} - \varepsilon_h^{\hat{z}}\|_{\partial \mathcal{T}_h} \\ &\leq C h^{k+1} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|\boldsymbol{z}\|_{k+2,\Omega}). \end{aligned}$$

Finally, (29), (30a), and (31) together imply (30b).

3.2.6 Step 6: Estimate for ε_h^z by a duality argument.

For Θ given in $L^2(\Omega)$, we consider the dual problem for z:

$$\Phi - \nabla \Psi = 0 \quad \text{in } \Omega,$$

$$\nabla \cdot \Phi - \beta \cdot \nabla \Psi = \Theta \quad \text{in } \Omega,$$

$$\Psi = 0 \quad \text{on } \partial \Omega.$$
(32)

Again since the domain Ω is convex, we have the regularity estimate

$$\|\Phi\|_{1,\Omega} + \|\Psi\|_{2,\Omega} \le C_{\text{reg}} \|\Theta\|_{\Omega},$$
 (33)

Before we estimate ε_h^z , we repeat the notation in (6):

$$\delta^{\mathbf{\Phi}} = \mathbf{\Phi} - \mathbf{\Pi}\mathbf{\Phi}, \quad \delta^{\Psi} = \Psi - \Pi\Psi, \quad \delta^{\widehat{\Psi}} = \Psi - P_M\Psi.$$

Lemma 11. We have

$$\|\varepsilon_h^z\|_{\mathcal{T}_h} \le Ch^{k+1+\min\{k,1\}}(\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|\boldsymbol{z}\|_{k+2,\Omega}).$$

Proof. Consider the dual problem (32) and let $\Theta = \varepsilon_h^z$. We take $(\mathbf{r}_2, w_2, \mu_2) = (\mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_M \Psi)$ in (28) in Lemma 8, and since $\Psi = 0$ on ε_h^{∂} , we have

$$\mathcal{B}_{2}(\varepsilon_{h}^{\mathbf{p}}, \varepsilon_{h}^{z}, \varepsilon_{h}^{z}; \mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_{M} \Psi)
= (\varepsilon_{h}^{\mathbf{p}}, \mathbf{\Pi} \mathbf{\Phi})_{\mathcal{T}_{h}} - (\varepsilon_{h}^{z}, \nabla \cdot \mathbf{\Pi} \mathbf{\Phi})_{\mathcal{T}_{h}} + \langle \varepsilon_{h}^{z}, \mathbf{\Pi} \mathbf{\Phi} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
- (\varepsilon_{h}^{\mathbf{p}} - \beta \varepsilon_{h}^{z}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} + \langle \varepsilon_{h}^{\mathbf{p}} \cdot \mathbf{n} + h^{-1} P_{M} \varepsilon_{h}^{z} + \tau_{2} \varepsilon_{h}^{z}, \Pi \Psi \rangle_{\partial \mathcal{T}_{h}}
- \langle (\beta \cdot \mathbf{n} + h^{-1} + \tau_{1}) \varepsilon_{h}^{\widehat{y}}, \Pi \Psi \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
- \langle \varepsilon_{h}^{\mathbf{p}} \cdot \mathbf{n} + \beta \cdot \mathbf{n} \varepsilon_{h}^{z} + h^{-1} (P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{z}) + \tau_{2} (\varepsilon_{h}^{z} - \varepsilon_{h}^{z}), P_{M} \Psi \rangle_{\partial \mathcal{T}_{h} \setminus \varepsilon_{h}^{\partial}}
= (\varepsilon_{h}^{\mathbf{p}}, \mathbf{\Phi})_{\mathcal{T}_{h}} - (\varepsilon_{h}^{z}, \nabla \cdot \mathbf{\Phi})_{\mathcal{T}_{h}} + (\varepsilon_{h}^{z}, \nabla \cdot \delta^{\mathbf{\Phi}})_{\mathcal{T}_{h}} - \langle \varepsilon_{h}^{z}, \delta^{\mathbf{\Phi}} \cdot \mathbf{n} \rangle_{\partial \mathcal{T}_{h}}
- (\varepsilon_{h}^{\mathbf{p}} - \beta \varepsilon_{h}^{z}, \nabla \Psi)_{\mathcal{T}_{h}} + (\varepsilon_{h}^{\mathbf{p}} - \beta \varepsilon_{h}^{z}, \nabla \delta^{\Psi})_{\mathcal{T}_{h}}
- \langle \varepsilon_{h}^{\mathbf{p}} \cdot \mathbf{n} - \beta \cdot \mathbf{n} \varepsilon_{h}^{z} + h^{-1} (P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{z}) + \tau_{2} (\varepsilon_{h}^{z} - \varepsilon_{h}^{z}), \delta^{\Psi} - \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}}.$$

Here, we have $\langle \varepsilon_h^{\widehat{z}}, \boldsymbol{\Phi} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} = 0$, which holds since $\varepsilon_h^{\widehat{z}}$ is single-valued function on interior edges and $\varepsilon_h^{\widehat{z}} = 0$ on ε_h^{∂} .

The same argument in (25) gives

$$(\varepsilon_h^z, \nabla \cdot \delta^{\mathbf{\Phi}})_{\mathcal{T}_h} = \langle \varepsilon_h^z, \delta^{\mathbf{\Phi}} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} - (\nabla \varepsilon_h^z, \delta^{\mathbf{\Phi}})_{\mathcal{T}_h} = \langle \varepsilon_h^z, \delta^{\mathbf{\Phi}} \cdot \boldsymbol{n} \rangle_{\partial \mathcal{T}_h},$$

$$(\varepsilon_h^{\boldsymbol{p}}, \nabla \delta^{\boldsymbol{\Psi}})_{\mathcal{T}_h} = \langle \varepsilon_h^{\boldsymbol{p}} \cdot \boldsymbol{n}, \delta^{\boldsymbol{\Psi}} \rangle_{\partial \mathcal{T}_h} - (\nabla \cdot \varepsilon_h^{\boldsymbol{p}}, \delta^{\boldsymbol{\Psi}})_{\mathcal{T}_h} = \langle \varepsilon_h^{\boldsymbol{p}} \cdot \boldsymbol{n}, \delta^{\boldsymbol{\Psi}} \rangle_{\partial \mathcal{T}_h},$$

$$(\beta \varepsilon_h^z, \nabla \delta^{\boldsymbol{\Psi}})_{\mathcal{T}_h} = \langle \beta \cdot \boldsymbol{n} \varepsilon_h^z, \delta^{\boldsymbol{\Psi}} \rangle_{\partial \mathcal{T}_h} - (\nabla \cdot \beta \varepsilon_h^z, \delta^{\boldsymbol{\Psi}})_{\mathcal{T}_h} - (\beta \nabla \varepsilon_h^z, \delta^{\boldsymbol{\Psi}})_{\mathcal{T}_h}.$$

Then,

$$\begin{split} \mathscr{B}_{2}(\varepsilon_{h}^{\mathbf{p}}, \varepsilon_{h}^{\tilde{z}}, \varepsilon_{h}^{\tilde{z}}; \mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_{M} \Psi) \\ &= \|\varepsilon_{h}^{z}\|_{\mathcal{T}_{h}}^{2} + \langle \varepsilon_{h}^{z} - \varepsilon_{h}^{\hat{z}}, \delta^{\mathbf{\Phi}} \cdot \mathbf{n} - \boldsymbol{\beta} \cdot \mathbf{n} \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} + (\nabla \varepsilon_{h}^{z}, \boldsymbol{\beta} \delta^{\Psi}) \\ &+ (\nabla \cdot \boldsymbol{\beta} \varepsilon_{h}^{z}, \delta^{\Psi})_{\mathcal{T}_{h}} - \langle h^{-1}(P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{\hat{z}}) + \tau_{1}(\varepsilon_{h}^{z} - \varepsilon_{h}^{\hat{z}}), \delta^{\Psi} - \delta^{\hat{\Psi}} \rangle_{\partial \mathcal{T}_{h}}, \end{split}$$

where we have used $\varepsilon_h^{\hat{z}}$ is single-valued function on interior edges and $\varepsilon_h^{\hat{z}} = 0$ on ε_h^{∂} . On the other hand,

$$\mathcal{B}_{2}(\varepsilon_{h}^{\mathbf{p}}, \varepsilon_{h}^{z}, \varepsilon_{h}^{\widehat{z}}; \mathbf{\Pi} \mathbf{\Phi}, \Pi \Psi, P_{M} \Psi)$$

$$= (\boldsymbol{\beta} \delta^{z}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} + \langle \widehat{\boldsymbol{\delta}}_{1}, \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}} + (y - y_{h}(u), \Pi \Psi)_{\mathcal{T}_{h}}.$$

Comparing the above two equalities gives

$$\begin{split} \|\varepsilon_{h}^{z}\|_{\mathcal{T}_{h}}^{2} &= -\langle \varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}, \delta^{\Phi} \cdot \boldsymbol{n} + \boldsymbol{\beta} \cdot \boldsymbol{n} \delta^{\Psi} \rangle_{\partial \mathcal{T}_{h}} - (\nabla \varepsilon_{h}^{z}, \boldsymbol{\beta} \delta^{\Psi})_{\mathcal{T}_{h}} + (\boldsymbol{\beta} \delta^{z}, \nabla \Pi \Psi)_{\mathcal{T}_{h}} \\ &+ \langle h^{-1}(P_{M} \varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}) + \tau_{2}(\varepsilon_{h}^{z} - \varepsilon_{h}^{\widehat{z}}) + \widehat{\boldsymbol{\delta}}_{2}, \delta^{\Psi} - \delta^{\widehat{\Psi}} \rangle_{\partial \mathcal{T}_{h}} \\ &- (\nabla \cdot \boldsymbol{\beta} \varepsilon_{h}^{z}, \delta^{\Psi})_{\mathcal{T}_{h}} + (y - y_{h}(u), \Pi \Psi)_{\mathcal{T}_{h}} \\ &=: S_{1} + S_{2} + S_{3} + S_{4} + S_{5} + S_{6}. \end{split}$$

We can estimate S_1 to S_4 as in the proof of Lemma 6 to get

$$\sum_{i=1}^{4} S_i \le Ch^{k+1+\min\{k,1\}} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|\boldsymbol{z}\|_{k+2,\Omega}).$$

By the estimate for ε_h^z in (30b) in Lemma 10, we have

$$S_5 = -(\nabla \cdot \boldsymbol{\beta} \varepsilon_h^z, \delta^{\Psi})_{\mathcal{T}_h} \le C \|\varepsilon_h^z\|_{\mathcal{T}_h} \|\delta^{\Psi}\|_{\mathcal{T}_h}$$

$$\le C h^{k+2} (\|\boldsymbol{q}\|_{k+1,\Omega} + \|\boldsymbol{y}\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|\boldsymbol{z}\|_{k+2,\Omega}) \|\varepsilon_h^z\|_{\mathcal{T}_h}.$$

The estimate of the last term S_6 can be easily obtained from (7):

$$S_6 = (y - y_h(u), \Pi \Psi)_{\mathcal{T}_h} \le ||y - y_h(u)||_{\mathcal{T}_h} (||\delta^{\Psi}||_{\mathcal{T}_h} + ||\Psi||_{\mathcal{T}_h})$$

$$\le Ch^{k+1+\min\{k,1\}} ||\varepsilon_h^z||_{\mathcal{T}_h}.$$

Finally, we complete the proof by combining the estimates for S_1 to S_6 .

The triangle inequality gives convergence rates for $\|\boldsymbol{p} - \boldsymbol{p}_h(u)\|_{\mathcal{T}_h}$ and $\|z - z_h(u)\|_{\mathcal{T}_h}$:

Lemma 12.

$$\|\boldsymbol{p} - \boldsymbol{p}_{h}(u)\|_{\mathcal{T}_{h}} \leq \|\delta^{\boldsymbol{p}}\|_{\mathcal{T}_{h}} + \|\varepsilon_{h}^{\boldsymbol{p}}\|_{\mathcal{T}_{h}} \lesssim h^{k+1}(\|\boldsymbol{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|z\|_{k+2,\Omega})$$

$$\|z - z_{h}(u)\|_{\mathcal{T}_{h}} \leq \|\delta^{z}\|_{\mathcal{T}_{h}} + \|\varepsilon_{h}^{z}\|_{\mathcal{T}_{h}} \lesssim h^{k+1+\min\{k,1\}}(\|\boldsymbol{q}\|_{k+1,\Omega} + \|y\|_{k+2,\Omega} + \|\boldsymbol{p}\|_{k+1,\Omega} + \|z\|_{k+2,\Omega}).$$

$$(34a)$$

3.2.7 Step 7: Estimate for $||u-u_h||_{\mathcal{T}_h}$, $||y-y_h||_{\mathcal{T}_h}$ and $||z-z_h||_{\mathcal{T}_h}$.

To obtain the main result, we bound the error between the solutions of the auxiliary problem and the HDG problem (15). The proofs of the results in Steps 7 and 8 are similar to the proofs of the corresponding results in our earlier work [15]; we include them for completeness.

For the final steps, let

$$\zeta_{\mathbf{q}} = \mathbf{q}_h(u) - \mathbf{q}_h, \quad \zeta_y = y_h(u) - y_h, \quad \zeta_{\widehat{y}} = \widehat{y}_h(u) - \widehat{y}_h,$$

 $\zeta_{\mathbf{p}} = \mathbf{p}_h(u) - \mathbf{p}_h, \quad \zeta_z = z_h(u) - z_h, \quad \zeta_{\widehat{z}} = \widehat{z}_h(u) - \widehat{z}_h.$

Subtracting the auxiliary problem and the HDG problem gives the error equations

$$\mathscr{B}_1(\zeta_{\boldsymbol{q}}, \zeta_{\boldsymbol{y}}, \zeta_{\widehat{\boldsymbol{y}}}; \boldsymbol{r}_1, w_1, \mu_1) = (u - u_h, w_1)_{\mathcal{T}_h}$$
(35a)

$$\mathscr{B}_2(\zeta_{\mathbf{p}}, \zeta_z, \zeta_{\widehat{z}}; \mathbf{r}_2, w_2, \mu_2) = -(\zeta_y, w_2)_{\mathcal{T}_h}. \tag{35b}$$

Lemma 13. We have

$$\gamma \| u - u_h \|_{\mathcal{T}_h}^2 + \| y_h(u) - y_h \|_{\mathcal{T}_h}^2
= (z_h - \gamma u_h, u - u_h)_{\mathcal{T}_h} - (z_h(u) - \gamma u, u - u_h)_{\mathcal{T}_h}.$$
(36)

Proof. First, we have

$$(z_h - \gamma u_h, u - u_h)_{\mathcal{T}_h} - (z_h(u) - \gamma u, u - u_h)_{\mathcal{T}_h}$$

= $-(\zeta_z, u - u_h)_{\mathcal{T}_h} + \gamma ||u - u_h||_{\mathcal{T}_h}^2$.

Next, Lemma 2 gives

$$\mathscr{B}_1(\zeta_{\boldsymbol{q}},\zeta_{\boldsymbol{y}},\zeta_{\widehat{\boldsymbol{v}}};\zeta_{\boldsymbol{p}},-\zeta_{\boldsymbol{z}},-\zeta_{\widehat{\boldsymbol{z}}})+\mathscr{B}_2(\zeta_{\boldsymbol{p}},\zeta_{\boldsymbol{z}},\zeta_{\widehat{\boldsymbol{z}}};-\zeta_{\boldsymbol{q}},\zeta_{\boldsymbol{y}},\zeta_{\widehat{\boldsymbol{v}}})=0.$$

On the other hand, working from the definitions yields

$$\mathcal{B}_{1}(\zeta_{\boldsymbol{q}}, \zeta_{y}, \zeta_{\widehat{y}}; \zeta_{\boldsymbol{p}}, -\zeta_{z}, -\zeta_{\widehat{z}}) + \mathcal{B}_{2}(\zeta_{\boldsymbol{p}}, \zeta_{z}, \zeta_{\widehat{z}}; -\zeta_{\boldsymbol{q}}, \zeta_{y}, \zeta_{\widehat{y}})$$

$$= -(u - u_{h}, \zeta_{z})_{\mathcal{T}_{h}} - \|\zeta_{y}\|_{\mathcal{T}_{h}}^{2}.$$

Comparing the above two equalities gives

$$-(u-u_h,\zeta_z)_{\mathcal{T}_h} = \|\zeta_y\|_{\mathcal{T}_h}^2,$$

which completes the proof.

Theorem 2. We have

$$||u - u_h||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|q|_{k+1} + |y|_{k+2} + |p|_{k+1} + |z|_{k+2}),$$
 (37a)

$$||y - y_h||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\mathbf{q}|_{k+1} + |y|_{k+2} + |\mathbf{p}|_{k+1} + |z|_{k+2}),$$
 (37b)

$$||z - z_h||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}).$$
 (37c)

Proof. The continuous and discretized optimality conditions (4e) and (15c) give $\gamma u = z$ and $\gamma u_h = z_h$. Use these equations and the previous lemma to obtain

$$\gamma \| u - u_h \|_{\mathcal{T}_h}^2 + \| \zeta_y \|_{\mathcal{T}_h}^2
= (z_h - \gamma u_h, u - u_h)_{\mathcal{T}_h} - (z_h(u) - \gamma u, u - u_h)_{\mathcal{T}_h}
= -(z_h(u) - z, u - u_h)_{\mathcal{T}_h}
\leq \| z_h(u) - z \|_{\mathcal{T}_h} \| u - u_h \|_{\mathcal{T}_h}
\leq \frac{1}{2\gamma} \| z_h(u) - z \|_{\mathcal{T}_h}^2 + \frac{\gamma}{2} \| u - u_h \|_{\mathcal{T}_h}^2.$$

By Lemma 12, we have

$$||u - u_h||_{\mathcal{T}_h} + ||\zeta_y||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}).$$

By the triangle inequality and Lemma 7 we obtain

$$||y - y_h||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}).$$

Finally, $z = \gamma u$ and $z_h = \gamma u_h$ give

$$||z - z_h||_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}).$$

3.2.8 Step 8: Estimate for $||q-q_h||_{\mathcal{T}_h}$ and $||p-p_h||_{\mathcal{T}_h}$.

Lemma 14. We have

$$\|\zeta_{\mathbf{q}}\|_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}}(|\mathbf{q}|_{k+1}+|y|_{k+2}+|\mathbf{p}|_{k+1}+|z|_{k+2}),$$
 (38a)

$$\|\zeta_{\mathbf{p}}\|_{\mathcal{T}_h} \lesssim h^{k+1+\min\{k,1\}} (|\mathbf{q}|_{k+1} + |y|_{k+2} + |\mathbf{p}|_{k+1} + |z|_{k+2}).$$
 (38b)

Proof. By Lemma 1 and the error equation (35a), we have

$$\begin{aligned} \|\zeta_{\boldsymbol{q}}\|_{\mathcal{T}_{h}}^{2} &\lesssim \mathscr{B}_{1}(\zeta_{\boldsymbol{q}}, \zeta_{y}, \zeta_{\widehat{y}}; \zeta_{\boldsymbol{q}}, \zeta_{y}, \zeta_{\widehat{y}}) \\ &= (u - u_{h}, \zeta_{y})_{\mathcal{T}_{h}} \\ &\leq \|u - u_{h}\|_{\mathcal{T}_{h}} \|\zeta_{y}\|_{\mathcal{T}_{h}} \\ &\lesssim h^{2k+2+2\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+1} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+1})^{2}. \end{aligned}$$

Similarly, by Lemma 1 and the error equation (35b), we have

$$\begin{aligned} \|\zeta_{\boldsymbol{p}}\|_{\mathcal{T}_{h}}^{2} &\lesssim \mathscr{B}_{2}(\zeta_{\boldsymbol{p}}, \zeta_{z}, \zeta_{\widehat{z}}; \zeta_{\boldsymbol{p}}, \zeta_{z}, \zeta_{\widehat{z}}) \\ &= -(\zeta_{y}, \zeta_{z})_{\mathcal{T}_{h}} \\ &\leq \|\zeta_{y}\|_{\mathcal{T}_{h}} \|\zeta_{z}\|_{\mathcal{T}_{h}} \\ &\lesssim h^{2k+2+2\min\{k,1\}} (|\boldsymbol{q}|_{k+1} + |\boldsymbol{y}|_{k+1} + |\boldsymbol{p}|_{k+1} + |\boldsymbol{z}|_{k+1})^{2}. \end{aligned}$$

The above lemma along with the triangle inequality, Lemma 7, and Lemma 12 complete the proof of the main result:

Theorem 3. We have

$$\|\boldsymbol{q} - \boldsymbol{q}_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}),$$
 (39a)

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_{\mathcal{T}_h} \lesssim h^{k+1}(|\boldsymbol{q}|_{k+1} + |y|_{k+2} + |\boldsymbol{p}|_{k+1} + |z|_{k+2}).$$
 (39b)

4 Numerical Experiments

To illustrate our convergence results, we consider two examples on a square domain $\Omega = [0,1] \times [0,1] \subset \mathbb{R}^2$ from our previous work [15]. We first take $\gamma = 1$ and choose the exact state, dual state, and function β . Then we generate the data f, g, and g using the optimality system (4).

Table 1–Table 4 show the computed errors and convergence rates for k = 0 and k = 1 for the two examples. The computational results match the theory.

Example 1.
$$\beta = [1, 1]$$
, state $y(x_1, x_2) = \sin(\pi x_1)$, dual state $z(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$

Example 2.
$$\beta = [x_2, x_1]$$
, state $y(x_1, x_2) = \sin(\pi x_1)$, dual state $z(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$

5 Conclusions

In our earlier work [15], we considered an HDG method with degree k polynomials for all variables to approximate the solution of an optimal distributed control problems for an elliptic convection diffusion equation. We proved optimal convergence rates for all variables in [15] when β is divergence

$h/\sqrt{2}$	1/16	1/32	1/64	1/128	1/256
$\ oldsymbol{q}-oldsymbol{q}_h\ _{0,\Omega}$	1.7274e-01	9.7054 e-02	5.2507e-02	2.7509e-02	1.4111e-02
order	-	0.83	0.89	0.93	0.96
$\left\lVert oldsymbol{p} - oldsymbol{p}_h ight Vert_{0,\Omega}$	2.5783e-01	1.4468e-01	7.7818e-02	4.0586e-02	2.0763e-02
order	-	0.833	0.89	0.94	0.97
$\ y-y_h\ _{0,\Omega}$	2.4430e-02	1.4046e-02	7.8371e-03	4.1908e-03	2.1744e-03
order	-	0.80	0.84	0.90	0.95
$ z-z_h _{0,\Omega}$	2.8132e-02	1.8225e-02	1.0659e-02	5.8061e-03	3.0363e-03
order	-	0.63	0.77	0.88	0.94

Table 1: Example 1: Errors for the state y, adjoint state z, and the fluxes \boldsymbol{q} and \boldsymbol{p} when k=0.

$h/\sqrt{2}$	1/8	1/16	1/32	1/64	1/128
$\left\ oldsymbol{q}-oldsymbol{q}_h ight\ _{0,\Omega}$	1.1365e-02	3.0743e-03	8.0051e-04	2.0438e-04	5.1648e-05
order	-	1.89	1.94	1.97	1.98
$\left\lVert oldsymbol{p} - oldsymbol{p}_h ight Vert_{0,\Omega}$	2.6923e-02	6.9736e-03	1.7764e-03	4.4849e-04	1.1269e-04
order	-	1.95	1.97	1.99	2.00
$\ y-y_h\ _{0,\Omega}$	1.9986e-03	2.8351e-04	3.7918e-05	4.9101e-06	6.2497e-07
order	-	2.82	2.90	2.95	2.97
$ z-z_h _{0,\Omega}$	3.8753e-03	5.3846e-04	7.1154e-05	9.1544e-06	1.1613e-06
order	-	2.85	2.92	2.96	2.98

Table 2: Example 1: Errors for the state y, adjoint state z, and the fluxes \boldsymbol{q} and \boldsymbol{p} when k=1.

$h/\sqrt{2}$	1/16	1/32	1/64	1/128	1/256
$\ oldsymbol{q}-oldsymbol{q}_h\ _{0,\Omega}$	1.7074e-01	9.5848e-02	5.1838e-02	2.7156e-02	1.3929e-02
order	-	0.83	0.89	0.93	0.96
$\left\ oldsymbol{p}-oldsymbol{p}_h ight\ _{0,\Omega}$	2.5679e-01	1.4404e-01	7.7454e-02	4.0391e-02	2.0661e-02
order	-	083	0.90	0.94	0.97
$ y-y_h _{0,\Omega}$	2.4537e-02	1.4150e-02	7.9032e-03	4.2273e- 03	2.1935e-03
order	-	0.79	0.84	0.90	0.95
$ z-z_h _{0,\Omega}$	2.8293e-02	1.8369e-02	1.0747e-02	5.8549e-03	3.0618e-03
order	_	0.62	0.77	0.88	0.94

Table 3: Example 2: Errors for the state y, adjoint state z, and the fluxes \boldsymbol{q} and \boldsymbol{p} when k=0.

$h/\sqrt{2}$	1/8	1/16	1/32	1/64	1/128
$\ oldsymbol{q}-oldsymbol{q}_h\ _{0,\Omega}$	1.0144e-02	2.7469e-03	7.1555e-04	1.8271e-04	4.6174e-05
order	-	1.88	1.94	1.97	1.98
$\ oldsymbol{p}-oldsymbol{p}_h\ _{0,\Omega}$	2.6378e-02	6.8203e- 03	1.7358e-03	4.3805e-04	1.1004e-04
order	-	1.95	1.97	1.99	1.99
$\ y-y_h\ _{0,\Omega}$	1.8869e-03	2.6762e-04	3.5771e-05	4.6297e-06	5.8909e-07
order	-	2.82	2.90	2.95	2.97
$\ z-z_h\ _{0,\Omega}$	3.8001e-03	5.2896e-04	6.9919e-05	8.9948e-06	1.1409e-06
order	-	2.84	2.92	2.96	2.98

Table 4: Example 2: Errors for the state y, adjoint state z, and the fluxes q and p when k = 1.

free; however, we did not obtain superconvergence. In this work, we considered the same control problem and approximated the solution using a different HDG method with degree k+1 polynomials for the flux variables and degree k polynomials for the other variables. When k>0 and $\nabla \cdot \boldsymbol{\beta} \leq 0$, we obtained superconvergence for the control, state, and dual state, and optimal convergence rates for the fluxes. We plan to consider HDG methods for more complicated optimal control problems for PDEs in the future.

Acknowledgements

W. Hu was supported in part by a postdoctoral fellowship for the annual program on Control Theory and its Applications at the Institute for Mathematics and its Applications (IMA) at the University of Minnesota. J. Singler and Y. Zhang were supported in part by National Science Foundation grant DMS-1217122. J. Singler and Y. Zhang thank the IMA for funding research visits, during which some of this work was completed. X. Zheng thanks Missouri University of Science and Technology for hosting him as a visiting scholar; some of this work was completed during his research visit. The authors thank Weifeng Qiu for many helpful discussions.

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