# Preconditioning of a hybridized discontinuous Galerkin finite element method for the Stokes equations 

Sander Rhebergen • Garth N. Wells


#### Abstract

We present optimal preconditioners for a recently introduced hybridized discontinuous Galerkin finite element discretization of the Stokes equations. Typical of hybridized discontinuous Galerkin methods, the method has degrees-offreedom that can be eliminated locally (cell-wise), thereby significantly reducing the size of the global problem. Although the linear system becomes more complex to analyze after static condensation of these element degrees-of-freedom, the pressure Schur complement of the original and reduced problem are the same. Using this fact, we prove spectral equivalence of this Schur complement to two simple matrices, which is then used to formulate optimal preconditioners for the statically condensed problem. Numerical simulations in two and three spatial dimensions demonstrate the good performance of the proposed preconditioners.


Keywords Stokes equations • preconditioning • hybridized methods • discontinuous Galerkin • finite element methods

## 1 Introduction

Recently, many hybridized discontinuous Galerkin (HDG) methods have been introduced for incompressible flows. For the Stokes problem these include 7, 8, 9, 18], and for the Oseen and Navier-Stokes problems we refer to [5, 6, 15, 17, 19, 21, 22]. We consider the method developed in [15] for the Navier-Stokes equations, but

[^0]with tighter restrictions on the 'facet' function spaces. The method is appealing in its simplicity and the fact that it can be formulated such that the approximate velocity field is automatically pointwise divergence-free. However, the implementation in [15] does not yield a $H$ (div)-conforming velocity field and, as consequence, cannot be simultaneously locally mass conserving, locally momentum conserving and energy stable. This issue was resolved in [23, 24], in which the method was modified for the Stokes and Navier-Stokes equations such that the approximate velocity fields are both pointwise divergence-free and $H$ (div)-conforming. This leads to a method that is locally mass conserving, momentum conserving, energy stable, and pressure-robust [14], as shown numerically in 24]. For the method in [23, 24] to be useful in practice, it is helpful if the discrete system arising from the method can be solved efficiently by iterative methods. In this work we introduce and analyze new preconditioners for the method applied to the Stokes problem, and show that optimal preconditioners can be constructed.

A feature of the HDG approach is static condensation; element degrees of freedom can be eliminated locally from the linear system, thereby significantly reducing the size of the global problem. There are different ways to apply static condensation to the HDG method of [23]. One may choose, for example, to eliminate both the element velocity and element pressure degrees-of-freedom. In this paper, however, we choose to eliminate only the element velocity degrees-of-freedom. In terms of reducing the global problem size, the effect of eliminating the element pressure degrees-of-freedom is minimal, whereas the reduction in global system size when eliminating the element velocity degrees-of-freedom is substantial. We do not consider elimination of the pressure degrees-of-freedom on cells as retaining the cell pressure field will lead to a formulation on which standard multigrid methods may be applied in the construction of optimal preconditioners. This would not be possible if cell pressure degrees-of-freedom are also eliminated from the original system.

The linear system obtained after static condensation of the cell-wise velocity degrees-of-freedom is more complex to analyze than the original linear system. However, the element/facet pressure Schur complement remains unchanged. Using boundedness and stability results of [23], and a suitable inf-sup condition, we prove spectral equivalence between the element/facet pressure Schur complement and an element/facet pressure mass matrix. This allows the general theory of Pestana and Wathen [20] for preconditioners for saddle point problems to be applied, which we use to develop two new preconditioners for the condensed HDG discretization of the Stokes equations. Optimality of the preconditioners for the HDG problem is proved, and numerical examples demonstrate very good performance.

The remainder of this paper is structured as follows. In section 2 we describe the HDG method for the Stokes equations and discuss and prove boundedness and stability results. These results are then used to develop and analyze preconditioners for the condensed form of the HDG discretization in section 3. We verify our analysis by two- and three-dimensional numerical simulations in section 4 and provide conclusions in section 5 .

2 Hybridizable discontinuous Galerkin method: formulation and analysis
We consider the Stokes system:

$$
\begin{array}{rlrl}
-\nabla^{2} u+\nabla p & =f & & \text { in } \Omega, \\
\nabla \cdot u & =0 & & \text { in } \Omega, \\
u & =0 & & \text { on } \partial \Omega, \\
\int_{\Omega} p \mathrm{~d} x & =0, & \tag{1d}
\end{array}
$$

where $\Omega \subset \mathbb{R}^{d}$ is a polygonal $(d=2)$ or polyhedral $(d=3)$ domain, $u: \Omega \rightarrow \mathbb{R}^{d}$ is the velocity, $p: \Omega \rightarrow \mathbb{R}$ is the pressure, and $f: \Omega \rightarrow \mathbb{R}^{d}$ is a prescribed body force.

### 2.1 Notation

To define the hybridizable discontinuous Galerkin method for the Stokes equations, we introduce first a triangulation $\mathcal{T}:=\{K\}$ of $\Omega$ consisting of non-overlapping cells. Each cell $K$ of the triangulation has a length measure $h_{K}$, and on the boundary of an element, $\partial K$, the outward unit normal vector is denoted by $n$. Two adjacent cells $K^{+}$and $K^{-}$share an interior facet $F$, while a boundary facet is a facet of $\partial K$ that lies on $\partial \Omega$. The set and union of all facets are denoted by $\mathcal{F}=\{F\}$ and $\Gamma^{0}$, respectively.

We will use the following finite element function spaces on $\Omega$ :

$$
\begin{align*}
V_{h} & :=\left\{v_{h} \in\left[L^{2}(\Omega)\right]^{d}: v_{h} \in\left[P_{k}(K)\right]^{d}, \forall K \in \mathcal{T}\right\},  \tag{2}\\
Q_{h} & :=\left\{q_{h} \in L^{2}(\Omega): q_{h} \in P_{k-1}(K), \forall K \in \mathcal{T}\right\},
\end{align*}
$$

and the following finite element spaces on $\Gamma^{0}$,

$$
\begin{align*}
& \bar{V}_{h}:=\left\{\bar{v}_{h} \in\left[L^{2}\left(\Gamma^{0}\right)\right]^{d}: \bar{v}_{h} \in\left[P_{k}(F)\right]^{d} \forall F \in \mathcal{F}, \bar{v}_{h}=0 \text { on } \partial \Omega\right\},  \tag{3}\\
& \bar{Q}_{h}:=\left\{\bar{q}_{h} \in L^{2}\left(\Gamma^{0}\right): \bar{q}_{h} \in P_{k}(F) \forall F \in \mathcal{F}\right\},
\end{align*}
$$

where $P_{k}(D)$ denotes the set of polynomials of degree at most $k$ on a domain $D$. For convenience, we introduce the spaces $V_{h}^{\star}:=V_{h} \times \bar{V}_{h}, Q_{h}^{\star}:=Q_{h} \times \bar{Q}_{h}$, and $X_{h}^{\star}:=V_{h}^{\star} \times Q_{h}^{\star}$. Function pairs in $V_{h}^{\star}$ and $Q_{h}^{\star}$ will be denoted by boldface, e.g., $\mathbf{v}_{h}:=\left(v_{h}, \bar{v}_{h}\right) \in V_{h}^{\star}$ and $\mathbf{q}_{h}:=\left(q_{h}, \bar{q}_{h}\right) \in Q_{h}^{\star}$.

On an element $K \subset \mathbb{R}^{d}$, for scalar functions $p, q \in L^{2}(K)$, we denote the standard inner-product by $(p, q)_{K}:=\int_{K} p q \mathrm{~d} x$, and we define $(p, q)_{\mathcal{T}}:=\sum_{K \in \mathcal{T}}(p, q)_{K}$. For scalar functions $p, q \in L^{2}(E)$, where $E \subset \mathbb{R}^{d-1}$, we define the inner-product $\langle p, q\rangle_{E}:=\int_{E} p q \mathrm{~d} s$ and $\langle p, q\rangle_{\partial \mathcal{T}}:=\sum_{K}\langle p, q\rangle_{\partial K}$. Similar inner-products hold for vector-valued functions.

We use various norms throughout, and which are defined now. On $V_{h}$ and $V_{h}^{\star}$ we define, respectively, the following 'discrete' $H^{1}$-norms:

$$
\begin{align*}
\left\|v_{h}\right\|_{D G}^{2} & :=\sum_{K \in \mathcal{T}}\left\|\nabla v_{h}\right\|_{K}^{2}+\sum_{K \in \mathcal{T}} \alpha h_{K}^{-1}\left\|v_{h}\right\|_{\partial K}^{2}  \tag{4}\\
\left\|\mathbf{v}_{h}\right\|_{v}^{2} & :=\sum_{K \in \mathcal{T}}\left\|\nabla v_{h}\right\|_{K}^{2}+\sum_{K \in \mathcal{T}} \alpha h_{K}^{-1}\left\|\bar{v}_{h}-v_{h}\right\|_{\partial K}^{2} \tag{5}
\end{align*}
$$

where $\alpha>0$ is a constant. For $\bar{v}_{h} \in \bar{V}_{h}$, we introduce the norm

$$
\begin{equation*}
\left\|\bar{v}_{h}\right\|_{h}^{2}:=\sum_{K \in \mathcal{T}_{h}} h_{K}^{-1}\left\|\bar{v}_{h}-m_{K}\left(\bar{v}_{h}\right)\right\|_{\partial K}^{2}, \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
m_{K}\left(\bar{v}_{h}\right):=\frac{1}{|\partial K|} \int_{\partial K} \bar{v}_{h} \mathrm{~d} s \tag{7}
\end{equation*}
$$

On $\bar{Q}_{h}$ and $Q_{h}^{\star}$ we define, respectively, 'discrete' $L^{2}$-norms,

$$
\begin{equation*}
\left\|\bar{q}_{h}\right\|_{p}^{2}:=\sum_{K \in \mathcal{T}} h_{K}\left\|\bar{q}_{h}\right\|_{\partial K}^{2} \quad \text { and } \quad\left\|\mathbf{q}_{h}\right\|_{p}^{2}:=\left\|q_{h}\right\|_{\Omega}^{2}+\left\|\bar{q}_{h}\right\|_{p}^{2} \tag{8}
\end{equation*}
$$

### 2.2 Weak formulation

The weak formulation for the Stokes problem in eq. (1) is given in [15, 23], and reads: given $f \in\left[L^{2}(\Omega)\right]^{d}$, find $\left(\mathbf{u}_{h}, \mathbf{p}_{h}\right) \in X_{h}^{\star}$ such that

$$
\begin{align*}
a_{h}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)+b_{h}\left(\mathbf{p}_{h}, \mathbf{v}_{h}\right) & =\left(v_{h}, f\right)_{\mathcal{T}} & & \forall \mathbf{v}_{h} \in V_{h}^{\star}  \tag{9a}\\
b_{h}\left(\mathbf{q}_{h}, \mathbf{u}_{h}\right) & =0 & & \forall \mathbf{q}_{h} \in Q_{h}^{\star} \tag{9b}
\end{align*}
$$

where

$$
\begin{align*}
a_{h}\left(\mathbf{w}_{h}, \mathbf{v}_{h}\right):= & \left(\nabla w_{h}, \nabla v_{h}\right)_{\mathcal{T}}+\left\langle\alpha h^{-1}\left(w_{h}-\bar{w}_{h}\right), v_{h}-\bar{v}_{h}\right\rangle_{\partial \mathcal{T}}  \tag{10a}\\
& -\left\langle w_{h}-\bar{w}_{h}, \partial_{n} v_{h}\right\rangle_{\partial \mathcal{T}}-\left\langle\partial_{n} w_{h}, v_{h}-\bar{v}_{h}\right\rangle_{\partial \mathcal{T}} \\
b_{h}\left(\mathbf{q}_{h}, \mathbf{v}_{h}\right):= & -\left(q_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}}+\left\langle v_{h} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} \tag{10b}
\end{align*}
$$

It is proven in [23, 25] that $\alpha$ can be chosen sufficiently large to ensure stability.
The formulation is a hybridized method in the sense that the facet function $\bar{p}_{h}$ acts as a Lagrange multiplier enforcing that the velocity field $u_{h}$ is $H$ (div)conforming, and specifically lies in a Brezzi-Douglas-Marini (BDM) finite element space [3].

The following results are from [23] and will be used in the analysis. For sufficiently large $\alpha$, the bilinear form $a_{h}(\cdot, \cdot)$ is coercive and bounded, i.e., there exist constants $c_{a}^{s}>0$ and $c_{a}^{b}>0$, independent of $h$, such that for all $\mathbf{u}_{h}, \mathbf{v}_{h} \in V_{h}^{\star}$,

$$
\begin{equation*}
a_{h}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right) \geq c_{a}^{s}\left\|\mathbf{v}_{h}\right\|_{v}^{2} \quad \text { and } \quad\left|a_{h}\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)\right| \leq c_{a}^{b}\left\|\mathbf{u}_{h}\right\|_{v}\left\|\mathbf{v}_{h}\right\|_{v} \tag{11}
\end{equation*}
$$

(see [23, Lemmas 4.2 and 4.3]) An immediate consequence of eq. (11) is:

$$
\begin{equation*}
c_{a}^{s}\left\|\mathbf{v}_{h}\right\|_{v}^{2} \leq a_{h}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right) \leq c_{a}^{b}\left\|\mathbf{v}_{h}\right\|_{v}^{2} \tag{12}
\end{equation*}
$$

From [23, Lemma 4.8 and Eq. 102], there exists a constant $c_{b}^{b}>0$, independent of $h$, such that for all $\mathbf{v}_{h} \in V_{h}^{\star}$ and for all $\mathbf{q}_{h} \in Q_{h}^{\star}$

$$
\begin{equation*}
\left|b_{h}\left(\mathbf{q}_{h}, \mathbf{v}_{h}\right)\right| \leq c_{b}^{b}\left\|\mathbf{v}_{h}\right\|_{v}\left\|\mathbf{q}_{h}\right\|_{p} \tag{13}
\end{equation*}
$$

2.3 The inf-sup condition

We present in this section a proof of inf-sup stability that is simpler than that in [23], and which better lends itself to the analysis of preconditioners.

The velocity-pressure coupling term in eq. (9) is

$$
\begin{equation*}
b_{h}\left(\mathbf{p}_{h}, \mathbf{v}_{h}\right):=b_{1}\left(p_{h}, \mathbf{v}_{h}\right)+b_{2}\left(\bar{p}_{h}, \mathbf{v}_{h}\right), \tag{14}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{1}\left(p_{h}, \mathbf{v}_{h}\right):=-\sum_{K \in \mathcal{T}} \int_{K} p_{h} \nabla \cdot v_{h} \mathrm{~d} x \quad \text { and } \quad b_{2}\left(\bar{p}_{h}, \mathbf{v}_{h}\right):=\sum_{K \in \mathcal{T}} \int_{\partial K} v_{h} \cdot n \bar{p}_{h} \mathrm{~d} s \tag{15}
\end{equation*}
$$

The main result of this section is stability of $b_{h}(\cdot, \cdot): Q_{h}^{\star} \times V_{h}^{\star} \rightarrow \mathbb{R}$, which we first state and then prove after some intermediate results.

Lemma 1 (Stability of $b_{h}$ ) There exists a constant $\beta_{p}>0$, independent of $h$, such that for all $\mathbf{q}_{h} \in Q_{h}^{\star}$

$$
\begin{equation*}
\beta_{p}\left\|\mathbf{q}_{h}\right\|_{p} \leq \sup _{\mathbf{v}_{h} \in V_{h}^{\star}} \frac{b_{h}\left(\mathbf{q}_{h}, \mathbf{v}_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{v}} \tag{16}
\end{equation*}
$$

Satisfaction of the stability condition does rely on a suitable combination of function spaces, as chosen in eqs. (2) and (3).

The following is a reduced version of 13, Theorem 3.1].
Theorem 1 Let $U, P_{1}$, and $P_{2}$ be reflexive Banach spaces, and let $b_{1}: P_{1} \times U \rightarrow \mathbb{R}$, and $b_{2}: P_{2} \times U \rightarrow \mathbb{R}$ be bilinear and continuous. Let

$$
\begin{equation*}
Z_{b_{i}}=\left\{v \in U: b_{i}\left(p_{i}, v\right)=0 \quad \forall p_{i} \in P_{i}\right\} \subset U, \quad i=1,2, \tag{17}
\end{equation*}
$$

then the following are equivalent:

1. There exists $c>0$ such that

$$
\sup _{v \in U} \frac{b_{1}\left(p_{1}, v\right)+b_{2}\left(p_{2}, v\right)}{\|v\|_{U}} \geq c\left(\left\|p_{1}\right\|_{P_{1}}+\left\|p_{2}\right\|_{P_{2}}\right) \quad\left(p_{1}, p_{2}\right) \in P_{1} \times P_{2} .
$$

2. There exists $c>0$ such that

$$
\sup _{v \in Z_{b_{2}}} \frac{b_{1}\left(p_{1}, v\right)}{\|v\|_{U}} \geq c\left\|p_{1}\right\|_{P_{1}}, p_{1} \in P_{1} \text { and } \sup _{v \in U} \frac{b_{2}\left(p_{2}, v\right)}{\|v\|_{U}} \geq c\left\|p_{2}\right\|_{P_{2}}, p_{2} \in P_{2} .
$$

Theorem 1 allows $b_{1}$ and $b_{2}$ in eq. (14) to be analyzed separately.
Lemma 2 (Stability of $b_{1}$ ) Let $V_{h}^{\mathrm{BDM}}$ be a Brezzi-Douglas-Marini (BDM) finite element space [3] :

$$
\begin{align*}
V_{h}^{\mathrm{BDM}}(K) & :=\left\{v_{h} \in\left[P_{k}(K)\right]^{d}: v_{h} \cdot n \in L^{2}(\partial K),\left.v_{h} \cdot n\right|_{F} \in P_{k}(F)\right\}, \\
V_{h}^{\mathrm{BDM}} & :=\left\{v_{h} \in H(\operatorname{div} ; \Omega):\left.v_{h}\right|_{K} \in V_{h}^{\mathrm{BDM}}(K), \forall K \in \mathcal{T}\right\} \tag{18}
\end{align*}
$$

There exists a constant $\beta>0$, independent of $h$, such that for all $q_{h} \in Q_{h}$

$$
\begin{equation*}
\beta\left\|q_{h}\right\|_{0, \Omega} \leq \sup _{\mathbf{v}_{h} \in V_{h}^{\star \mathrm{BDM}}} \frac{b_{1}\left(q_{h}, \mathbf{v}_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{v}} \tag{19}
\end{equation*}
$$

Proof See [23, Lemma 4.4].
Definition 1 (BDM lifting operator) Let $L^{\mathrm{BDM}}: P_{k}(\partial K) \rightarrow\left[P_{k}(K)\right]^{d}$ be a BDM local lifting of the normal trace defined via the BDM interpolant 3, Example 2.5.1] with zero on the interior, which has the properties:

$$
\begin{equation*}
\left(L^{\mathrm{BDM}} \bar{q}_{h}\right) \cdot n=\bar{q}_{h} \quad \text { and } \quad\left\|L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{K} \leq c h_{K}^{1 / 2}\left\|\bar{q}_{h}\right\|_{\partial K} \quad \forall \bar{q}_{h} \in P_{k}(\partial K), \tag{20}
\end{equation*}
$$

where the inequality follows by a scaling argument.
It follows then by an inverse estimate that

$$
\begin{equation*}
\left\|\nabla L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{K}^{2} \leq c h_{K}^{-2}\left\|L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{K}^{2} \leq c h_{K}^{-1}\left\|\bar{q}_{h}\right\|_{\partial K}^{2}, \tag{21}
\end{equation*}
$$

and by the trace inequality that

$$
\begin{equation*}
\left\|L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{\partial K}^{2} \leq c h_{K}^{-1}\left\|L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{K}^{2} \leq c\left\|\bar{q}_{h}\right\|_{\partial K}^{2} \tag{22}
\end{equation*}
$$

which yields

$$
\begin{equation*}
\left\|\nabla L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{K}^{2}+\frac{\alpha_{v}}{h_{K}}\left\|L^{\mathrm{BDM}} \bar{q}_{h}\right\|_{\partial K}^{2} \leq c h_{K}^{-1}\left\|\bar{q}_{h}\right\|_{\partial K}^{2} \tag{23}
\end{equation*}
$$

Lemma 3 (Stability of $b_{2}$ ) There exists a constant $\bar{\beta}>0$, independent of $h$, such that for all $\bar{q}_{h} \in \bar{Q}_{h}$

$$
\begin{equation*}
\bar{\beta}\left\|\bar{q}_{h}\right\|_{p} \leq \sup _{\mathbf{v}_{h} \in V_{h}^{\star}} \frac{b_{2}\left(\bar{q}_{h}, \mathbf{v}_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{v}} \tag{24}
\end{equation*}
$$

Proof Summing over all cells and by definition of the norm $\|(\cdot, \cdot)\|_{v}$ in eq. (5),

$$
\begin{equation*}
\left\|\left(L^{\mathrm{BDM}} \bar{q}_{h}, 0\right)\right\|_{v} \leq c \sum_{K \in \mathcal{T}} h_{K}^{-1 / 2}\left\|\bar{q}_{h}\right\|_{\partial K} . \tag{25}
\end{equation*}
$$

Using the above, we have:

$$
\begin{align*}
\sup _{\mathbf{v}_{h} \in V_{h}^{\star}} \frac{\sum_{K \in \mathcal{T}} \int_{\partial K} v_{h} \cdot n \bar{q}_{h} \mathrm{~d} s}{\left\|\mathbf{v}_{h}\right\|_{v}} & \geq \frac{\sum_{K \in \mathcal{T}} \int_{\partial K} \bar{q}_{h}^{2} \mathrm{~d} s}{\left\|\left(L^{\mathrm{BDM}} \bar{q}_{h}, 0\right)\right\|_{v}} \\
& \geq c \frac{\sum_{K \in \mathcal{T}}\left\|\bar{q}_{h}\right\|_{\partial K}^{2}}{\sum_{K \in \mathcal{T}} h_{K}^{-1 / 2}\left\|\bar{q}_{h}\right\|_{\partial K}}  \tag{26}\\
& \geq c h_{\min }^{1 / 2} \sum_{K \in \mathcal{T}}\left\|\bar{q}_{h}\right\|_{\partial K} \\
& \geq c c_{*}^{1 / 2}\left\|\bar{q}_{h}\right\|_{p}
\end{align*}
$$

where $c_{*}=h_{\text {min }} / h_{\text {max }}$.
We can now prove the main stability result.
Proof (Proof of lemma 11) Using theorem 1, let $b_{1}(\cdot, \cdot)$ and $b_{2}(\cdot, \cdot)$ be defined as in eq. (15), let $U=V_{h}^{\star}, P_{1}=Q_{h}$ and $P_{2}=\bar{Q}_{h}$. Furthermore, note that $Z_{b_{2}}=$ $V_{h}^{\star \text { BDM }} \subset V_{h}^{\star}$. The conditions in item 2 were proven in lemmas 2and 3 respectively. By equivalence of items 1 and 2 we obtain

$$
\begin{equation*}
\sup _{\mathbf{v}_{h} \in V_{h}^{\star}} \frac{b_{1}\left(q_{h}, \mathbf{v}_{h}\right)+b_{2}\left(\bar{q}_{h}, \mathbf{v}_{h}\right)}{\left\|\mathbf{v}_{h}\right\|_{v}} \geq c\left(\left\|q_{h}\right\|_{0, \Omega}+\left\|\bar{q}_{h}\right\|_{p}\right) \quad\left(q_{h}, \bar{q}_{h}\right) \in Q_{h} \times \bar{Q}_{h} \tag{27}
\end{equation*}
$$

from which eq. (16) follows.

### 2.4 Reduced problem

In practice, a reduced global problem is solved in which $u_{h}$ is eliminated cellwise. We present the reduced problem in a variational setting here for later use in constructing preconditioners. To formulate a reduced problem, we first introduce local solvers.

Definition 2 (Local solver) On an element $K$, consider the 'local' bilinear and linear forms:

$$
\begin{equation*}
a_{K}\left(v_{h}, w_{h}\right):=\left(\nabla v_{h}, \nabla w_{h}\right)_{K}-\left\langle\partial_{n} v_{h}, w_{h}\right\rangle_{\partial K}-\left\langle v_{h}, \partial_{n} w_{h}\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\langle v_{h}, w_{h}\right\rangle_{\partial K} \tag{28}
\end{equation*}
$$

and

$$
\begin{align*}
& L_{K}\left(w_{h}\right):=\left(s, w_{h}\right)_{K}-\left\langle\partial_{n} w_{h}, \bar{m}_{h}\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\langle w_{h}, \bar{m}_{h}\right\rangle_{\partial K} \\
&+\left(\nabla \cdot w_{h}, r_{h}\right)_{K}-\left\langle w_{h} \cdot n, \bar{r}_{h}\right\rangle_{\partial K} \tag{29}
\end{align*}
$$

The function $v_{h}^{L}\left(\bar{m}_{h}, r_{h}, \bar{r}_{h}, s\right) \in V_{h}$ is such that its restriction to element $K$ satisfies the local problem: given $s \in\left[L^{2}(\Omega)\right]^{d}$ and $\left(\bar{m}_{h}, r_{h}, \bar{r}_{h}\right) \in \bar{V}_{h} \times Q_{h} \times \bar{Q}_{h}$

$$
\begin{equation*}
a_{K}\left(v_{h}^{L}, w_{h}\right)=L_{K}\left(w_{h}\right) \quad \forall w_{h} \in V(K) . \tag{30}
\end{equation*}
$$

where $V(K):=\left[P_{k}(K)\right]^{d}$ the polynomial space in which the velocity is approximated on a cell.

We next state the weak formulation of the Stokes problem in which $u_{h}$ is eliminated from eq. (9) by using the local solver to express the velocity field and the velocity test function on cells, and phrasing the problem in terms of the pressure trial/test function on cells and the interface functions.
Lemma 4 (Weak formulation of the reduced Stokes problem) Suppose $\left(\mathbf{u}_{h}, \mathbf{p}_{h}\right) \in$ $X_{h}^{\star}$ satisfy eq. (9) and $f \in\left[L^{2}(\Omega)\right]^{d}$. The velocity field $u_{h}$ is the sum of the local solutions (from definition 园) $l\left(\bar{u}_{h}, \mathbf{p}_{h}\right):=v_{h}^{L}\left(\bar{u}_{h}, p_{h}, \bar{p}_{h}, 0\right)$ and $u_{h}^{f}:=v_{h}^{L}(0,0,0, f)$ :

$$
\begin{equation*}
u_{h}=u_{h}^{f}+l\left(\bar{u}_{h}, \mathbf{p}_{h}\right) . \tag{31}
\end{equation*}
$$

Furthermore, $\left(\bar{u}_{h}, \mathbf{p}_{h}\right) \in \bar{V}_{h} \times Q_{h}^{\star}$ satisfies

$$
\begin{equation*}
\mathcal{B}_{h}\left(\left(\bar{u}_{h}, \mathbf{p}_{h},\right),\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right)=\mathcal{L}_{h}\left(\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right) \quad \forall\left(\bar{w}_{h}, \mathbf{q}_{h}\right) \in \bar{V}_{h} \times Q_{h}^{\star}, \tag{32}
\end{equation*}
$$

where

$$
\begin{align*}
\mathcal{B}_{h}\left(\left(\bar{v}_{h}, \mathbf{r}_{h}\right),\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right): & :=a_{h}\left(\left(l\left(\bar{v}_{h}, \mathbf{r}_{h}\right), \bar{v}_{h}\right),\left(l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), \bar{w}_{h}\right)\right) \\
& +b_{h}\left(\mathbf{r}_{h},\left(l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), \bar{w}_{h}\right)\right)+b_{h}\left(\mathbf{q}_{h},\left(l\left(\bar{v}_{h}, \mathbf{r}_{h}\right), \bar{v}_{h}\right)\right) \tag{33}
\end{align*}
$$

and

$$
\begin{equation*}
\mathcal{L}_{h}\left(\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right):=\left(l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), f\right)_{\mathcal{T}} \tag{34}
\end{equation*}
$$

where $l\left(\bar{v}_{h}, \mathbf{r}_{h}\right):=l\left(\bar{v}_{h}\right)+l\left(\mathbf{r}_{h}\right)$, and $l\left(\bar{v}_{h}\right):=v_{h}^{L}\left(\bar{v}_{h}, 0,0,0\right)$ and $l\left(\mathbf{r}_{h}\right):=v_{h}^{L}\left(0, r_{h}, \bar{r}_{h}, 0\right)$ (by definition 园).

Proof Equation (31) follows by eq. (9a), linearity of the problem and definition 2 (local solver).

We next prove eq. (32). Note that $l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)$, restricted to the cell $K$, satisfies a form of the local problem in eq. (30), and $u_{h}^{f}$, restricted to the cell $K$, satisfies a form of the local problem. Combining the two local problems (the former with $u_{h}^{f}$ in the test function slot, and the latter with $l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)$ in the test function slot), and summing over the cells in the triangulation,

$$
\begin{align*}
&-\left\langle\partial_{n} u_{h}^{f}, \bar{w}_{h}\right\rangle_{\partial \mathcal{T}}+\alpha h_{K}^{-1}\left\langle u_{h}^{f}, \bar{w}_{h}\right\rangle_{\partial \mathcal{T}} \\
&=\left(f, l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right)_{\mathcal{T}}-\left(\nabla \cdot u_{h}^{f}, q_{h}\right)_{\mathcal{T}}+\left\langle u_{h}^{f} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} \tag{35}
\end{align*}
$$

The lifted function $l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)$, restricted to the cell $K$, satisfies the local problem, which with $l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)$ in the test function slot reads:

$$
\begin{align*}
& \left(\nabla l\left(\bar{u}_{h}, \mathbf{p}_{h}\right), \nabla l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right)_{K}-\left\langle\partial_{n} l\left(\bar{u}_{h}, \mathbf{p}_{h}\right), l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right\rangle_{\partial K} \\
& \quad-\left\langle\partial_{n} l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)-\bar{u}_{h}\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\langle l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)-\bar{u}_{h}, l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right\rangle_{\partial K} \\
& \quad-\left(\nabla \cdot l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), p_{h}\right)_{K}+\left\langle l\left(\bar{w}_{h}, \mathbf{q}_{h}\right) \cdot n, \bar{p}_{h}\right\rangle_{\partial K}=0 . \tag{36}
\end{align*}
$$

Substituting eq. (31) into eq. (9), with $\mathbf{v}_{h}=\left(0, \bar{w}_{h}\right)$ in eq. (9a),

$$
\begin{align*}
\left\langle\partial_{n} l\left(\bar{u}_{h}, \mathbf{p}_{h}\right), \bar{w}_{h}\right\rangle_{\partial \mathcal{T}}-\alpha h_{K}^{-1}\left\langle l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)-\bar{u}_{h}, \bar{w}_{h}\right\rangle_{\partial \mathcal{T}} & =-\left\langle\partial_{n} u_{h}^{f}, \bar{w}_{h}\right\rangle_{\partial \mathcal{T}}+\alpha h_{K}^{-1}\left\langle u_{h}^{f}, \bar{w}_{h}\right\rangle_{\partial \mathcal{T}},  \tag{37a}\\
\left(q_{h}, \nabla \cdot l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)\right)_{\mathcal{T}}-\left\langle l\left(\bar{u}_{h}, \mathbf{p}_{h}\right) \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} & =-\left(q_{h}, \nabla \cdot u_{h}^{f}\right)_{\mathcal{T}}+\left\langle u_{h}^{f} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}} \tag{37b}
\end{align*}
$$

Summing eq. (36) over all cells and adding to the left-hand side of eq. (37a), and using eq. (35) to replace the right-hand side of eq. (37a), we have:

$$
\begin{align*}
& \left(\nabla l\left(\bar{u}_{h}, \mathbf{p}_{h}\right), \nabla l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right)_{\mathcal{T}}-\left\langle\partial_{n} l\left(\bar{u}_{h}, \mathbf{p}_{h}\right), l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)-\bar{w}_{h}\right\rangle_{\partial \mathcal{T}} \\
& \quad-\left\langle\partial_{n} l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)-\bar{u}_{h}\right\rangle_{\partial \mathcal{T}} \\
& +\alpha h_{K}^{-1}\left\langle l\left(\bar{u}_{h}, \mathbf{p}_{h}\right)-\bar{u}_{h}, l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)-\bar{w}_{h}\right\rangle_{\partial \mathcal{T}}-\left(\nabla \cdot l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), p_{h}\right)_{\mathcal{T}} \\
& +\left\langle l\left(\bar{w}_{h}, \mathbf{q}_{h}\right) \cdot n, \bar{p}_{h}\right\rangle_{\partial \mathcal{T}}+\left(\nabla \cdot u_{h}^{f}, q_{h}\right)_{\mathcal{T}}-\left\langle u_{h}^{f} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}}=\left(f, l\left(\bar{w}_{h}, \mathbf{q}_{h}\right)\right)_{\mathcal{T}} \tag{38}
\end{align*}
$$

Equation (32) follows after using eq. (37b).
By definition 2, consider $l\left(\bar{v}_{h}\right):=v_{h}^{L}\left(\bar{v}_{h}, 0,0,0\right)$ and $l\left(\mathbf{r}_{h}\right):=v_{h}^{L}\left(0, r_{h}, \bar{r}_{h}, 0\right)$. By linearity, it follows that $l\left(\bar{v}_{h}, \mathbf{r}_{h}\right)=l\left(\bar{v}_{h}\right)+l\left(\mathbf{r}_{h}\right)$. Note also

$$
\begin{align*}
& a_{h}\left(\left(l\left(\bar{v}_{h}, \mathbf{r}_{h}\right), \bar{v}_{h}\right),\left(l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), \bar{w}_{h}\right)\right)=\underbrace{a_{h}\left(\left(l\left(\bar{v}_{h}\right), \bar{v}_{h}\right),\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right)}_{\bar{a}_{h}\left(\bar{v}_{h}, \bar{w}_{h}\right)} \\
& +a_{h}\left(\left(l\left(\bar{v}_{h}\right), \bar{v}_{h}\right),\left(l\left(\mathbf{q}_{h}\right), \bar{w}_{h}\right)\right)+a_{h}\left(\left(l\left(\mathbf{r}_{h}\right), \bar{v}_{h}\right),\left(l\left(\bar{w}_{h}, \mathbf{q}_{h}\right), \bar{w}_{h}\right)\right) \tag{39}
\end{align*}
$$

where $l\left(\bar{w}_{h}\right):=v_{h}^{L}\left(\bar{w}_{h}, 0,0,0\right), l\left(\mathbf{q}_{h}\right):=v_{h}^{L}\left(0, q_{h}, \bar{q}_{h}, 0\right)$ and $l\left(\bar{w}_{h}, \mathbf{q}_{h}\right):=l\left(\bar{w}_{h}\right)+$ $l\left(\mathbf{q}_{h}\right)$. The following result for $\bar{a}_{h}(\cdot, \cdot)$ will be useful in analyzing preconditioners.

Lemma 5 (Equivalence of norm induced by $\bar{a}_{h}(\cdot, \cdot)$ ) There exist positive constants $C_{1}$ and $C_{2}$ independent of $h_{K}$ such that

$$
\begin{equation*}
C_{1}\left\|\bar{w}_{h}\right\|_{h}^{2} \leq \bar{a}_{h}\left(\bar{w}_{h}, \bar{w}_{h}\right) \leq C_{2}\left\|\bar{w}_{h}\right\|_{h}^{2} \quad \forall \bar{w}_{h} \in \bar{V}_{h} . \tag{40}
\end{equation*}
$$

Proof From boundedness and coercivity of $a_{h}(\cdot, \cdot)$ (see eq. (12)), for sufficiently large $\alpha$,

$$
\begin{equation*}
c_{a}^{s}\left\|\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right\|_{v}^{2} \leq \bar{a}_{h}\left(\bar{w}_{h}, \bar{w}_{h}\right) \leq c_{a}^{b}\left\|\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right\|_{v}^{2} \tag{41}
\end{equation*}
$$

We therefore need to demonstrate the equivalence

$$
\begin{equation*}
c_{1}\left\|\bar{w}_{h}\right\|_{h} \leq\left\|\left(l\left(\bar{w}_{h}\right), \bar{w}_{h}\right)\right\|_{v} \leq c_{2}\left\|\bar{w}_{h}\right\|_{h} . \tag{42}
\end{equation*}
$$

We first consider the lower bound in eq. (42). Note that

$$
\begin{equation*}
h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \leq h_{K}^{-1 / 2}\left\|\bar{w}_{h}-l\left(\bar{w}_{h}\right)\right\|_{\partial K}+h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} . \tag{43}
\end{equation*}
$$

Defining

$$
\begin{equation*}
M_{K}\left(l\left(\bar{w}_{h}\right)\right):=\frac{1}{|K|} \int_{K} l\left(\bar{w}_{h}\right) \mathrm{d} s, \tag{44}
\end{equation*}
$$

then

$$
\begin{align*}
& h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \\
& \quad \leq h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\|_{\partial K}+h_{K}^{-1 / 2}\left\|M_{K}\left(l\left(\bar{w}_{h}\right)\right)-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \\
& \quad=h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\|_{\partial K}+h_{K}^{-1 / 2}\left\|m_{K}\left(M_{K}\left(l\left(\bar{w}_{h}\right)\right)-\bar{w}_{h}\right)\right\|_{\partial K}  \tag{45}\\
& \quad \leq h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\|_{\partial K}+h_{K}^{-1 / 2}\left\|M_{K}\left(l\left(\bar{w}_{h}\right)\right)-\bar{w}_{h}\right\|_{\partial K} \\
& \quad \leq 2 h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\|_{\partial K}+h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K},
\end{align*}
$$

where the second inequality is by [4, Eq. (10.6.11)]. By a trace inequality and a (scaled) Friedrich's inequality [4, Lemma 4.3.14],

$$
\begin{equation*}
2 h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\|_{\partial K} \leq 2 h_{K}^{-1}\left\|l\left(\bar{w}_{h}\right)-M_{K}\left(l\left(\bar{w}_{h}\right)\right)\right\| \leq c\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K} \tag{46}
\end{equation*}
$$

Combining eqs. (43), (45) and (46),

$$
\begin{equation*}
h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \leq c\left(\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}+h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K}\right) . \tag{47}
\end{equation*}
$$

The lower bound in eq. (42) follows after squaring, applying Young's inequality, and summing over all cells.

We next consider the upper bound in eq. (42). By definition of $l\left(\bar{w}_{h}\right)$ and considering that $m_{K}\left(\bar{w}_{h}\right)$ is constant on a cell,

$$
\begin{align*}
& \left(\nabla l\left(\bar{w}_{h}\right), \nabla w_{h}\right)_{K}-\left\langle\partial_{n} l\left(\bar{w}_{h}\right), w_{h}\right\rangle_{\partial K}-\left\langle l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right), \partial_{n} w_{h}\right\rangle_{\partial K} \\
& +\alpha h_{K}^{-1}\left\langle l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right), w_{h}\right\rangle_{\partial K}=-\left\langle\partial_{n} w_{h}, \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \\
& +\alpha h_{K}^{-1}\left\langle w_{h}, \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \quad \forall w_{h} \in V(K) \tag{48}
\end{align*}
$$

Setting $w_{h}=l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right)$, then

$$
\begin{align*}
&\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}^{2}-2\langle \left.\partial_{n} l\left(\bar{w}_{h}\right), l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \\
&+\alpha h_{K}^{-1}\left\langle l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right), l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \\
&=-
\end{aligned} \begin{aligned}
& \left\langle\partial_{n} l\left(\bar{w}_{h}\right), \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \\
&  \tag{49}\\
& \quad+\alpha h_{K}^{-1}\left\langle l\left(\bar{w}_{h}\right)-m_{K}\left(\bar{w}_{h}\right), \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K},
\end{align*}
$$

which can be manipulated into

$$
\begin{align*}
& \left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}^{2}+2\left\langle\partial_{n} l\left(\bar{w}_{h}\right), \bar{w}_{h}-w_{h}^{\bar{w}}\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K}^{2} \\
& \quad=\left\langle\partial_{n} w_{h}^{\bar{w}}, \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\langle\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right), \bar{w}_{h}-l\left(\bar{w}_{h}\right)\right\rangle_{\partial K} . \tag{50}
\end{align*}
$$

Considering the left-hand side of eq. (50), by coercivity of $a_{h}(\cdot, \cdot)$,

$$
\begin{align*}
& c_{1}\left(\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}^{2}+\alpha h_{K}^{-1}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K}^{2}\right) \\
& \quad \leq\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}^{2}+2\left\langle\partial_{n} l\left(\bar{w}_{h}\right), \bar{w}_{h}-l\left(\bar{w}_{h}\right)\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K}^{2} . \tag{51}
\end{align*}
$$

For the right-hand side of eq. (50), by Cauchy-Schwarz and a trace inequality,

$$
\begin{align*}
&\left\langle\partial_{n} l\left(\bar{w}_{h}\right), \bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\rangle_{\partial K}+\alpha h_{K}^{-1}\left\langle\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right), \bar{w}_{h}-l\left(\bar{w}_{h}\right)\right\rangle_{\partial K} \\
& \leq h_{K}^{1 / 2}\left\|\partial_{n} l\left(\bar{w}_{h}\right)\right\|_{\partial K} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \\
& \quad+\alpha^{1 / 2} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \alpha^{1 / 2} h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K} \\
& \leq\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \\
& \quad+\alpha^{1 / 2} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} \alpha^{1 / 2} h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K} \\
& \leq\left(\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}+\alpha^{1 / 2} h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K}\right)\left(\alpha^{1 / 2} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K}\right) . \tag{52}
\end{align*}
$$

Combining eqs. (50) to (52),

$$
\begin{equation*}
\left\|\nabla l\left(\bar{w}_{h}\right)\right\|_{K}+\alpha^{1 / 2} h_{K}^{-1 / 2}\left\|l\left(\bar{w}_{h}\right)-\bar{w}_{h}\right\|_{\partial K} \leq c \alpha^{1 / 2} h_{K}^{-1 / 2}\left\|\bar{w}_{h}-m_{K}\left(\bar{w}_{h}\right)\right\|_{\partial K} . \tag{53}
\end{equation*}
$$

The upper bound in eq. (42) follows after squaring, application of Young's inequality, and summing over all elements.

## 3 Preconditioning

We present now the main results, namely preconditioners for the hybridized discontinuous Galerkin discretization of the Stokes equations. We consider first a preconditioner for the full problem (no static condensation), and then the system with the cell-wise velocity degrees-of-freedom eliminated locally. As mentioned in the introduction, we do not consider the case where the cell-wise pressure degrees-of-freedom are also eliminated locally as this complicates preconditioning and the reduction in size of the global systems through eliminating the pressure is modest. Preconditioning a system with both the velocity and pressure degrees-of-freedom eliminated cell-wise is an interesting technical question.
3.1 The full discrete Stokes problem

Let $u \in \mathbb{R}^{n_{u}}$ be the vector of discrete velocity with respect to the basis for $V_{h}$, and $p \in N^{n_{p}}=\left\{q \in \mathbb{R}^{n_{p}} \mid q \neq 1\right\}$ be the vector of the discrete pressure with respect to the basis for $Q_{h}$. Furthermore, let $\bar{u} \in \mathbb{R}^{\bar{n}_{u}}$ and $\bar{p} \in \mathbb{R}^{\bar{n}_{p}}$ be the vectors of discrete velocity and pressure associated with the spaces $\bar{V}_{h}$ and $\bar{Q}_{h}$, respectively. The discrete problem in eq. (9) can be expressed as the system of linear equations:

$$
\left[\begin{array}{cc}
A & B^{T}  \tag{54}\\
B & 0
\end{array}\right]\left[\begin{array}{l}
U \\
P
\end{array}\right]=\left[\begin{array}{l}
L \\
0
\end{array}\right], \quad \text { with } \quad U:=\left[\begin{array}{l}
u \\
\bar{u}
\end{array}\right], \quad P:=\left[\begin{array}{l}
p \\
\bar{p}
\end{array}\right], \quad L:=\left[\begin{array}{l}
L_{u} \\
L_{\bar{u}}
\end{array}\right],
$$

and where $A$ and $B$ are the matrices obtained from the discretization of the bilinear forms $a_{h}(\cdot, \cdot)$ and $b_{h}(\cdot, \cdot)$, defined by eq. (10). The matrices $A$ and $B$ are block matrices:

$$
A:=\left[\begin{array}{cc}
A_{u u} & A_{\bar{u} u}^{T}  \tag{55}\\
A_{\bar{u} u} & A_{\bar{u} \bar{u}}
\end{array}\right] \quad \text { and } \quad B:=\left[\begin{array}{ll}
B_{p u} & 0 \\
B_{\bar{p} u} & 0
\end{array}\right],
$$

where $A_{u u}, A_{\bar{u} u}$ and $A_{\bar{u} \bar{u}}$ are the matrices obtained from the discretization of $a_{h}((\cdot, 0),(\cdot, 0)), a_{h}((\cdot, 0),(0, \cdot))$ and $a_{h}((0, \cdot),(0, \cdot))$, respectively, and $B_{p u}$ and $B_{\bar{p} u}$ are the matrices obtained from the discretization of $b_{h}((\cdot, 0),(\cdot, 0))$ and $b_{h}((0, \cdot),(\cdot, 0))$, respectively.

We also introduce 'cell' and 'facet' pressure mass matrices, $M$ and $\bar{M}$, which are obtained from the discretization of

$$
\begin{equation*}
\left(q_{h}, p_{h}\right)_{\mathcal{T}}:=\sum_{K \in \mathcal{T}} \int_{K} q_{h} p_{h} \mathrm{~d} x \quad \text { and } \quad\left\langle\bar{q}_{h}, \bar{p}_{h}\right\rangle_{p}:=\sum_{K \in \mathcal{T}} h_{K} \int_{\partial K} \bar{q}_{h} \bar{p}_{h} \mathrm{~d} s \tag{56}
\end{equation*}
$$

respectively, and note that

$$
\begin{equation*}
\left\|q_{h}\right\|_{\Omega}^{2}=q^{T} M q, \quad\left\|\bar{q}_{h}\right\|_{p}^{2}=\bar{q}^{T} \bar{M} \bar{q} \tag{57}
\end{equation*}
$$

Defining $\mathcal{M}:=\operatorname{bdiag}(M, \bar{M})$ and $Q:=\left[\begin{array}{ll}q^{T} & \bar{q}^{T}\end{array}\right]^{T}$,

$$
\begin{equation*}
\left\|\mathbf{q}_{h}\right\|_{p}^{2}=Q^{T} \mathcal{M} Q=q^{T} M q+\bar{q}^{T} \bar{M} \bar{q} \tag{58}
\end{equation*}
$$

Lemma 6 (Spectral equivalence between the mass matrix and the Schur complement) Let $A$ and $B$ be the matrices given in eq. (55) and let $\mathcal{M}$ be defined as in eq. (58). Let $\beta_{p}$ and $c_{b}^{b}$ be the constants given in lemma 1 and eq. (13), respectively, and let $c_{a}^{b}$ and $c_{a}^{s}$ be the constants given in eq. (12). The following holds:

$$
\begin{equation*}
\frac{\beta_{p}}{\sqrt{c_{a}^{b}}} \leq \frac{Q^{T} B A^{-1} B^{T} Q}{Q^{T} \mathcal{M} Q} \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}} \tag{59}
\end{equation*}
$$

Proof Stability of $b_{h}$ (see lemma (1) and equivalence of $a_{h}$ with $\|\mid \cdot\|_{v}$ in eq. imply

$$
\begin{equation*}
\frac{\beta_{p}}{\sqrt{c_{a}^{b}}} \leq \sup _{\mathbf{v}_{h} \in V_{h}^{\star}} \frac{b_{h}\left(\mathbf{q}_{h}, \mathbf{v}_{h}\right)}{a_{h}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)^{1 / 2}\left\|\mathbf{q}_{h}\right\|_{p}} \tag{60}
\end{equation*}
$$

Letting $V=\left[v^{T} \bar{v}^{T}\right]^{T}$, with $v \in \mathbb{R}^{n_{u}}$ and $\bar{v} \in \mathbb{R}^{\bar{n}_{u}}$ and $Q=\left[q^{T} \bar{q}^{T}\right]^{T}$, with $q \in N^{n_{p}}$ and $\bar{q} \in \mathbb{R}^{\bar{n}_{q}}$, we can express eq. (60) in matrix form:

$$
\begin{equation*}
\frac{\beta_{p}}{\sqrt{c_{a}^{b}}} \leq \min _{Q} \max _{V \neq 0} \frac{Q^{T} B V}{\left(V^{T} A V\right)^{1 / 2}\left(Q^{T} \mathcal{M} Q\right)^{1 / 2}} \tag{61}
\end{equation*}
$$

which is equivalent to

$$
\begin{equation*}
\frac{\beta_{p}}{\sqrt{c_{a}^{b}}} \leq \min _{Q} \frac{Q^{T} B A^{-1} B^{T} Q}{Q^{T} \mathcal{M} Q} \tag{62}
\end{equation*}
$$

(see 20, Section 3]), proving the lower bound in eq. (59).
For the upper bound, from eqs. (12) and (13) note that:

$$
\begin{equation*}
\left|b_{h}\left(\mathbf{q}_{h}, \mathbf{v}_{h}\right)\right| \leq c_{b}^{b}\left\|\mathbf{v}_{h}\right\|_{v}\left\|\mathbf{q}_{h}\right\|_{p} \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}} a_{h}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)^{1 / 2}\left\|\mathbf{q}_{h}\right\|_{p} \tag{63}
\end{equation*}
$$

The result follows after dividing both sides of eq. (63) by $\bar{a}_{h}\left(\mathbf{v}_{h}, \mathbf{v}_{h}\right)^{1 / 2}\left\|\mathbf{q}_{h}\right\|_{p}$ and expressing in matrix form.

Lemma 6 can be used to formulate a preconditioner for the discrete problem in eq. (54).

Lemma 7 (An optimal preconditioner for the full discrete Stokes problem) Let $A$ and $B$ be the matrices given in eq. (551) and let $\mathcal{M}$ be defined as in eq. (58). Let $R$ be an operator that is spectrally equivalent to $A$. For the preconditioned system

$$
\mathbb{P}^{-1} \mathbb{A} \mathbb{U}=\mathbb{P}^{-1} \mathbb{F} \quad \leftrightarrow \quad\left[\begin{array}{cc}
R & 0  \tag{64}\\
0 & \mathcal{M}
\end{array}\right]^{-1}\left[\begin{array}{cc}
A & B^{T} \\
B & 0
\end{array}\right]\left[\begin{array}{l}
U \\
P
\end{array}\right]=\left[\begin{array}{cc}
R & 0 \\
0 & \mathcal{M}
\end{array}\right]^{-1}\left[\begin{array}{l}
L \\
0
\end{array}\right]
$$

there exist positive constants $C_{1}, C_{2}, C_{3}, C_{4}$, independent of $h$, such that the negative and positive eigenvalues of $\mathbb{P}^{-1} \mathbb{A}$ satisfy $\lambda \in\left[-C_{1},-C_{2}\right]$ and $\lambda \in\left[C_{3}, C_{4}\right]$, respectively.

Proof By lemma 6, $\mathcal{M}$ is spectrally equivalent to the negative Schur complement $B A^{-1} B^{T}$ of $\mathbb{A}$. Furthermore, since $B$ is full rank (by the discrete inf-sup condition lemma (1) and since $A$ is symmetric positive definite, the result follows by direct application of [20, Theorem 5.2].

### 3.2 Preconditioners for the statically condensed Stokes problem

The 'full' system considered in section 3.1 is not the system we wish to solve in practice. We wish to eliminate locally the cell-wise velocity degrees-of-freedom via static condensation and precondition the resulting reduced system. We now consider the elimination of $u$ in eq. (64) to obtain a linear system only for $\bar{u}, p$ and $\bar{p}$.

Separating the degrees-of-freedom associated with $V_{h}$ from those associated with the Lagrange multipliers, $\bar{V}_{h} \times Q_{h}^{\star}$, we write eq. (54) as

$$
\left[\begin{array}{cc}
A_{u u} & \mathrm{~B}^{T}  \tag{65}\\
\mathrm{~B} & \mathrm{C}
\end{array}\right]\left[\begin{array}{l}
u \\
\mathrm{U}
\end{array}\right]=\left[\begin{array}{c}
L_{u} \\
\mathrm{~L}
\end{array}\right], \quad \text { with } \quad \mathrm{U}:=\left[\begin{array}{c}
\bar{u} \\
p \\
\bar{p}
\end{array}\right] \quad \mathrm{L}:=\left[\begin{array}{c}
L_{\bar{u}} \\
0 \\
0
\end{array}\right],
$$

and where

$$
\mathrm{B}:=\left[\begin{array}{l}
A_{\bar{u} u}  \tag{66}\\
B_{p u} \\
B_{\bar{p} u}
\end{array}\right], \quad \mathrm{C}:=\left[\begin{array}{ccc}
A_{\bar{u} \bar{u}} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] .
$$

Note that $A_{u u}$ is a block diagonal matrix (one block per cell). Using $u=A_{u u}^{-1}\left(L_{u}-\mathrm{B}^{T} \mathrm{U}\right)$, we eliminate $u$ from eq. (65). This results in a reduced system for U only,

$$
\left[\begin{array}{cc}
\bar{A} & \bar{B}^{T}  \tag{67}\\
\bar{B} & \bar{C}
\end{array}\right]\left[\begin{array}{l}
\bar{u} \\
P
\end{array}\right]=\left[\begin{array}{l}
\bar{L} \\
\bar{G}
\end{array}\right],
$$

where $\bar{A}=-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}}$ and where

$$
\bar{B}=\left[\begin{array}{l}
-B_{p u} A_{u u}^{-1} A_{\bar{u} u}^{T}  \tag{68}\\
-B_{\bar{p} u} A_{u u}^{-1} A_{\bar{u} u}^{T}
\end{array}\right], \quad \bar{C}=\left[\begin{array}{ll}
-B_{p u} A_{u u}^{-1} B_{p u}^{T} & -B_{p u} A_{u u}^{-1} B_{\bar{p} u}^{T} \\
-B_{\bar{p} u} A_{u u}^{-1} B_{p u}^{T} & -B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}
\end{array}\right] .
$$

The Schur complement of the block matrix in eq. (67) is given by $\bar{S}=-\bar{B} \bar{A}^{-1} \bar{B}^{T}+$ $\bar{C}$. It is easy to show (by direct computation) that $\bar{S}=-B A^{-1} B^{T}$, with $A$ and $B$ the matrices given in eq. (55). An immediate consequence of lemma 6 therefore is the following corollary.
Corollary 1 (Spectral equivalence between the mass matrix and the Schur complement of the statically condensed linear system) Let $\bar{A}, \bar{B}$ and $\bar{C}$ be the matrices given in eq. (67) and let $\mathcal{M}$ be defined as in eq. (58). Let $\beta_{p}$ and $c_{b}^{b}$ be the constants given in, lemma $\square$ and eq. (13), respectively, and let $c_{a}^{b}$ and $c_{a}^{s}$ be the constants given in eq. (12). The following holds:

$$
\begin{equation*}
\frac{\beta_{p}}{\sqrt{c_{a}^{b}}} \leq \frac{Q^{T}\left(\bar{B} \bar{A}^{-1} \bar{B}^{T}-\bar{C}\right) Q}{Q^{T} \mathcal{M} Q} \leq \frac{c_{b}^{b}}{\sqrt{c_{a}^{s}}} . \tag{69}
\end{equation*}
$$

This corollary can now be used to develop a preconditioner for the statically condensed linear system in eq. (67).

Theorem 2 (An optimal preconditioner for the statically condensed discrete Stokes problem based on mass matrices) Let $\bar{A}$ and $\bar{B}$ be the matrices given in eq. (67) and let $\mathcal{M}$ be defined as in eq. (58). Let $\bar{R}$ be an operator that is spectrally equivalent to $\bar{A}$. Consider the preconditioned system

$$
\overline{\mathbb{P}}_{\mathcal{M}}^{-1} \overline{\mathbb{A}} \overline{\mathrm{U}}=\overline{\mathbb{P}}_{\mathcal{M}}^{-1} \overline{\mathbb{F}} \quad \leftrightarrow \quad\left[\begin{array}{cc}
\bar{R} & 0  \tag{70}\\
0 & \mathcal{M}
\end{array}\right]^{-1}\left[\begin{array}{cc}
\bar{A} & \bar{B}^{T} \\
\bar{B} & \bar{C}
\end{array}\right]\left[\begin{array}{c}
\bar{u} \\
P
\end{array}\right]=\left[\begin{array}{cc}
\bar{R} & 0 \\
0 & \mathcal{M}
\end{array}\right]^{-1}\left[\begin{array}{c}
\bar{L} \\
\bar{G}
\end{array}\right] .
$$

There exist positive constants $C_{1}, C_{2}, C_{3}, C_{4}$, independent of $h$, such that the negative and positive eigenvalues of $\overline{\mathbb{P}}_{\mathcal{M}}^{-1} \overline{\mathbb{A}}$ satisfy $\lambda \in\left[-C_{1},-C_{2}\right]$ and $\lambda \in\left[C_{3}, C_{4}\right]$, respectively.
Proof By corollary $1 . \mathcal{M}$ is spectrally equivalent to the Schur complement $S=$ $-\bar{B} \bar{A}^{-1} \bar{B}^{T}+\bar{C}$ of $\overline{\mathbb{A}}$. Furthermore, the Schur complement is invertible. To see this, note that $S=-\bar{B} \bar{A}^{-1} \bar{B}^{T}+\bar{C}=-B A^{-1} B^{T}$, where $A$ and $B$ are the matrices given in eq. (55). The operator $A$ is symmetric positive definite, and $B$ is full rank by lemma hence $S$ is invertible. Since $\bar{A}$ is symmetric positive definite by lemma 5 the result then follows by direct application of [20, Theorem 5.2].

The preconditioner in theorem 2 is constructed based on the fact that $\mathcal{M}$ is spectrally equivalent to the Schur complement of the statically condensed linear system, as described by corollary 1 In fact, $\mathcal{M}$ in eq. (70) can be replaced by any spectrally equivalent operator $\mathcal{C}$, since $\mathcal{C}$ would then also be spectrally equivalent to the Schur complement of the statically condensed linear system. A particularly interesting choice for $\mathcal{C}$ is discussed in the following.

Theorem 3 (An optimal preconditioner for the statically condensed discrete Stokes problem based on element matrices) Let $\bar{A}, \bar{B}$ and $\bar{C}$ be the matrices given in eq. (67) and let $\mathcal{C}$ be the negative block-diagonal of $\bar{C}$ :

$$
\mathcal{C}=\left[\begin{array}{cc}
B_{p u} A_{u u}^{-1} B_{p u}^{T} & 0  \tag{71}\\
0 & B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}
\end{array}\right] .
$$

Let $\bar{R}$ be an operator that is spectrally equivalent to $\bar{A}$. Consider the preconditioned system

$$
\overline{\mathbb{P}}_{\mathcal{C}}^{-1} \overline{\mathbb{A}} \overline{\mathrm{U}}=\overline{\mathbb{P}}_{\mathcal{C}}^{-1} \overline{\mathbb{F}} \quad \leftrightarrow \quad\left[\begin{array}{ll}
\bar{R} & 0  \tag{72}\\
0 & \mathcal{C}
\end{array}\right]^{-1}\left[\begin{array}{cc}
\bar{A} & \bar{B}^{T} \\
\bar{B} & \bar{C}
\end{array}\right]\left[\begin{array}{l}
\bar{u} \\
P
\end{array}\right]=\left[\begin{array}{cc}
\bar{R} & 0 \\
0 & \mathcal{C}
\end{array}\right]^{-1}\left[\begin{array}{l}
\bar{L} \\
\bar{G}
\end{array}\right]
$$

There exist positive constants $C_{1}, C_{2}, C_{3}, C_{4}$, independent of $h$, such that the negative and positive eigenvalues of $\overline{\mathbb{P}}_{\mathcal{C}}^{-1} \overline{\mathbb{A}}$ satisfy $\lambda \in\left[-C_{1},-C_{2}\right]$ and $\lambda \in\left[C_{3}, C_{4}\right]$, respectively.

Proof It suffices to prove that $\mathcal{C}$ and $\mathcal{M}$ are spectrally equivalent. By minor modification of the proof of lemma 6 bysing the spectral equivalence of $a_{h}^{u u}\left(v_{h}, v_{h}\right)$ with $\left\|v_{h}\right\|_{D G}^{2}$ (see eq. (75)) and the inf-sup conditions (see eqs. (77) and (78)), it can by shown that $B_{p u} A_{u u}^{-1} B_{p u}^{T}$ and $M$ are spectrally equivalent and that $B_{\bar{p} u} A_{u u}^{-1} B_{\bar{p} u}^{T}$ and $\bar{M}$ are spectrally equivalent. It follows that $\mathcal{C}$ and $\mathcal{M}$ are spectrally equivalent.

### 3.3 Characterization of $\bar{A}$

Theorems 2 and 3 define optimal preconditioners for the statically condensed discrete Stokes problem provided we have an operator $\bar{R}$ that is spectrally equivalent to $\bar{A}$. To help in finding a suitable $\bar{R}$, we first consider the properties of $\bar{A}$. The operator $\bar{A}$ is obtained from the discretization of $\bar{a}_{h}\left(\bar{u}_{h}, \bar{v}_{h}\right)$ in eq. (39). By lemma 5 we know that $\bar{A}$ is spectrally equivalent to the norm $\left\|\|\cdot\|_{h}^{2}\right.$. As discussed, in for example [16], $\|\cdot \cdot\|_{h}$ is a $H^{1}$-like norm and the near-null space of $\bar{A}$ is spanned by constant functions. This is a condition to successfully apply multigrid-type solvers to $\bar{A}$. This motivates the use of multigrid for the operator $\bar{R}$ that appears in theorems 2 and 3

### 3.4 Block symmetric Gauss-Seidel preconditioners

Theorems 2and 3introduce two block-diagonal preconditioners. In practice, we see that the rates of convergence of preconditioned iterative methods using the block Jacobi-type preconditioners is typically improved upon by adding off-diagonal blocks to the preconditioner. We therefore also consider block symmetric GaussSeidel type preconditioners.

Let $\overline{\mathbb{A}}$ be the system matrix defined in eq. (67), $\mathcal{P}_{D}$ the block-diagonal of $\overline{\mathbb{A}}$ and $\mathcal{P}_{L}$ a strictly lower triangular block matrix such that

$$
\begin{equation*}
\overline{\mathbb{A}}=\mathcal{P}_{L}+\mathcal{P}_{D}+\mathcal{P}_{L}^{T} \tag{73}
\end{equation*}
$$

Furthermore, let $\mathcal{P}_{M}=\operatorname{bdiag}\left(-A_{\bar{u} u} A_{u u}^{-1} A_{\bar{u} u}^{T}+A_{\bar{u} \bar{u}},-M,-\bar{M}\right)$. The block symmetric Gauss-Seidel type preconditioners we consider in section 4 are:

$$
\begin{equation*}
\overline{\mathbb{P}}_{\mathcal{C}}^{S G S}=\left(\mathcal{P}_{L}+\mathcal{P}_{D}\right) \mathcal{P}_{D}^{-1}\left(\mathcal{P}_{L}^{T}+\mathcal{P}_{D}\right), \quad \overline{\mathbb{P}}_{\mathcal{M}}^{S G S}=\left(\mathcal{P}_{L}+\mathcal{P}_{M}\right) \mathcal{P}_{M}^{-1}\left(\mathcal{P}_{L}^{T}+\mathcal{P}_{M}\right) \tag{74}
\end{equation*}
$$

In the numerical examples, the inverse of the first block of $\mathcal{P}_{D}$ and $\mathcal{P}_{M}$ will be replaced by $\bar{R}^{-1}$.

## 4 Numerical example

We now verify numerically the performance of the preconditioners introduced in theorems 2and 3, and the symmetric block Gauss-Seidel preconditioner in eq. (74). We use a preconditioned MINRES solver, with AMG (four multigrid V-cycles) for the operator $\bar{R}^{-1}$. The inverse of the pressure mass-matrix $\mathcal{M}$, and the spectrally equivalent operator $\mathcal{C}$, are also approximated by four AMG V-cycles. In both cases one application, pre and post, of a Gauss-Seidel smoother is used. The MINRES iterations are terminated once the relative true residual reaches a tolerance of $10^{-8}$. We consider unstructured simplicial meshes and unstructured quadrilateral and hexahedral meshes. For simplex cells, we use a quadratic polynomial approximation for $u_{h}, \bar{u}_{h}$ and $\bar{p}_{h}$, and a linear polynomial approximation for $p_{h}$. For meshes with quadrilateral cells, we use a bi-quadratic polynomial approximation for $u_{h}$, quadratic approximation of $\bar{u}_{h}$ and $\bar{p}_{h}$, and a bilinear polynomial approximation for $p_{h}$. For meshes with hexahedral cells, we use a tri-quadratic polynomial approximation for $u_{h}$, bi-quadratic approximation of $\bar{u}_{h}$ and $\bar{p}_{h}$, and a tri-linear polynomial approximation for $p_{h}$. The stabilization parameter is taken as $\alpha=24$ in 2D and $\alpha=40$ in 3D. The formulation has been implemented in MFEM 10] with solver support from PETSc [1, 2]. We use classical algebraic multigrid via the BoomerAMG library [12].

We consider lid-driven cavity flow in a square, $\Omega=[-1,1]^{2}$, and a cube, $\Omega=[0,1]^{3}$. Dirichlet boundary conditions are imposed on $\partial \Omega$. In two dimensions, $u=\left(1-x_{1}^{4}, 0\right)$ on the boundary $x_{2}=1$ and the zero velocity vector on remaining boundaries. In three dimensions we impose $u=\left(1-\tau_{1}^{4},\left(1-\tau_{2}^{4}\right) / 10,0\right)$, with $\tau_{i}=2 x_{i}-1$, on the boundary $x_{3}=1$ and the zero velocity vector on remaining boundaries.

Table 1 presents the iteration counts for MINRES to converge for different levels of refinement on simplicial meshes, and table 2presents the iteration counts for the quadrilateral and hexahedral mesh cases. It is clear that the iteration count does not grow with problem size in all cases. Note that the diagonal preconditioners based on $\mathcal{C}$, i.e., using only the blocks available from the system matrix $\overline{\mathbb{A}}$ in eq. (67), outperform the preconditioners based on $\mathcal{M}$ consisting of the element pressure mass-matrix $M$ and the scaled facet pressure mass-matrix $\bar{M}$. In the case of the symmetric block Gauss-Seidel preconditioners, there is no gain in using $\mathcal{C}$ over $\mathcal{M}$.

We have observed that switching from left-preconditioned MINRES to rightpreconditioned GMRES can improve the iteration count substantially. For example, in the case of the $\overline{\mathbb{P}}_{\mathcal{C}}^{S G S}$ preconditioner for a three-dimensional simplicial grid with 1789952 DOFs, the iteration count for right-preconditioned GMRES is only 37 (compared with 153 iterations for left-preconditioned MINRES).

Two dimensions

| DOFs | $\overline{\mathbb{P}}_{\mathcal{M}}$ | $\overline{\mathbb{P}}_{\mathcal{M}}^{S G S}$ | $\overline{\mathbb{P}}_{\mathcal{C}}$ | $\overline{\mathbb{P}}_{\mathcal{C}}^{S G S}$ |
| :---: | :---: | :---: | :---: | :---: |
| 12012 | 136 | 73 | 94 | 89 |
| 47256 | 131 | 72 | 96 | 98 |
| 187440 | 134 | 69 | 96 | 102 |
| 746592 | 128 | 63 | 97 | 96 |

Three dimensions

| DOFs | $\mathbb{P}_{\mathcal{M}}$ | $\overline{\mathbb{P}}_{\mathcal{M}}^{S G S}$ | $\mathbb{P}_{\mathcal{C}}$ | $\mathbb{P}_{\mathcal{C}}^{S G S}$ |
| :---: | :---: | :---: | :---: | :---: |
| 30128 | 230 | 150 | 122 | 139 |
| 229504 | 259 | 159 | 145 | 151 |
| 1789952 | 258 | 138 | 166 | 153 |

Table 1: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem in two and three dimensions using unstructured simplicial meshes.

Two dimensions

| DOFs | $\mathbb{P}_{\mathcal{M}}$ | $\mathbb{P}_{M}^{S G S}$ | $\mathbb{P}_{\mathcal{C}}$ | $\mathbb{P}_{\mathcal{C}}^{S G S}$ |
| :---: | :---: | :---: | :---: | :---: |
| 10956 | 109 | 56 | 84 | 79 |
| 43032 | 104 | 52 | 80 | 73 |
| 170544 | 104 | 47 | 80 | 68 |
| 679008 | 98 | 42 | 81 | 75 |

Three dimensions

| DOFs | $\mathbb{P}_{\mathcal{M}}$ | $\overline{\mathbb{P}}_{\boldsymbol{M}}^{S G S}$ | $\mathbb{P}_{\mathcal{C}}$ | $\overline{\mathbb{P}}_{\mathcal{C}}^{S G S}$ |
| :---: | :---: | :---: | :---: | :---: |
| 9152 | 130 | 79 | 88 | 82 |
| 66304 | 123 | 70 | 87 | 93 |
| 502784 | 114 | 57 | 85 | 78 |

Table 2: Iteration counts for preconditioned MINRES for the relative true residual to reach a tolerance of $10^{-8}$ for the lid-driven cavity problem in two and three dimensions using unstructured quadrilateral and structured hexahedral meshes.

## 5 Conclusions

We have developed, analyzed and numerically tested two new block-diagonal preconditioners for the statically condensed linear system for a hybridized discontinuous Galerkin method for the Stokes equations. In particular, we proved and showed numerically that the preconditioners are optimal in that preconditioned systems can be solved to a specified tolerance in an iteration count that is independent of the problem size. This makes the preconditioner suitable for very large systems, and especially for problems in which pointwise satisfaction of the continuity equation is important since the considered method has this valuable property. Discretizations of the Stokes problem using a hybridized discontinuous Galerkin method permit static condensation; cell degrees-of-freedom can be eliminated locally, resulting in significantly reduced number of globally coupled
degrees-of-freedom. This does however complicate the analysis, and our analysis addresses the form and structure of a condensed problem.

## A Auxiliary results

We provide here some auxiliary results used in analyzing the preconditioners.
Defining $a_{h}^{u u}\left(u_{h}, v_{h}\right):=a_{h}\left(\left(u_{h}, 0\right),\left(v_{h}, 0\right)\right)$, since $\left\|v_{h}\right\|_{D G}=\left\|\left(v_{h}, 0\right)\right\|_{v}$, a consequence of eq. (12) is:

$$
\begin{equation*}
c_{a}^{s}\left\|v_{h}\right\|_{D G}^{2} \leq a_{h}^{u u}\left(v_{h}, v_{h}\right) \leq c_{a}^{b}\left\|v_{h}\right\|_{D G}^{2} \tag{75}
\end{equation*}
$$

Applying [11, Proposition 10] to a single cell $K$, the following inf-sup condition holds:

$$
\begin{equation*}
\beta_{D G}^{K}\left\|q_{h}\right\|_{K} \leq \sup _{v_{h} \in V_{h}(K)} \frac{\left(q_{h}, \nabla \cdot v_{h}\right)_{K}}{\left\|v_{h}\right\|_{D G(K)}} \quad \forall q_{h} \in P_{k-1}(K) \tag{76}
\end{equation*}
$$

where $\beta_{D G}^{K}>0$ is a constant independent of $h, V_{h}(K):=\left[P_{k}(K)\right]^{d}$ and $\left\|v_{h}\right\|_{D G(K)}^{2}:=$ $\left\|\nabla v_{h}\right\|_{K}^{2}+\alpha h_{K}^{-1}\left\|v_{h}\right\|_{\partial K}^{2}$. It follows that

$$
\begin{equation*}
\beta_{D G}\left\|q_{h}\right\|_{\Omega} \leq \sup _{v_{h} \in V_{h}} \frac{\left(q_{h}, \nabla \cdot v_{h}\right)_{\mathcal{T}}}{\left\|v_{h}\right\|_{D G}} \tag{77}
\end{equation*}
$$

where $\beta_{D G}:=\min _{K \in \mathcal{T}} \beta_{D G}^{K}$. Since $\left\|v_{h}\right\|_{D G}=\left\|\left(v_{h}, 0\right)\right\|_{v}$, it is easy to see from eq. (26) that

$$
\begin{equation*}
\bar{\beta}_{D G}\left\|\bar{q}_{h}\right\|_{p} \leq \sup _{v_{h} \in V_{h}} \frac{\left\langle v_{h} \cdot n, \bar{q}_{h}\right\rangle_{\partial \mathcal{T}}}{\left\|v_{h}\right\|_{D G}} \tag{78}
\end{equation*}
$$

where $\bar{\beta}_{D G}>0$ is a constant independent of $h$.

## References

1. Balay S, Abhyankar S, Adams MF, Brown J, Brune P, Buschelman K, Dalcin L, Eijkhout V, Gropp WD, Kaushik D, Knepley MG, McInnes LC, Rupp K, Smith BF, Zampini S, Zhang H, Zhang H (2016) PETSc users manual. Tech. Rep. ANL-95/11 - Revision 3.7, Argonne National Laboratory, URL http://www.mcs.anl.gov/petsc
2. Balay S, Abhyankar S, Adams MF, Brown J, Brune P, Buschelman K, Dalcin L, Eijkhout V, Gropp WD, Kaushik D, Knepley MG, McInnes LC, Rupp K, Smith BF, Zampini S, Zhang H, Zhang H (2016) PETSc Web page. http://www.mcs.anl.gov/petsc
3. Boffi D, Brezzi F, Fortin M (2013) Mixed Finite Element Methods and Applications, Springer Series in Computational Mathematics, vol 44. Springer-Verlag Berlin Heidelberg
4. Brenner SC, Scott L (2008) The Mathematical Theory of Finite Element Methods, 3rd edn. Springer
5. Cesmelioglu A, Cockburn B, Nguyen NC, Peraire J (2013) Analysis of HDG methods for Oseen equations. J Sci Comput 55:392-431, URL https://doi.org/10.1007/s10915-012-9639-y
6. Cesmelioglu A, Cockburn B, Qiu W (2017) Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations. Math Comp 86:1643-1670, URL https://doi.org/10.1090/mcom/3195
7. Cockburn B, Gopalakrishnan J (2009) The derivation of hybridizable discontinuous Galerkin methods for Stokes flow. SIAM J Numer Anal 47(2):1092-1125, URL https://doi.org/10.1137/080726653
8. Cockburn B, Sayas FJ (2014) Divergence-conforming HDG methods for Stokes flows. Math Comp 83:1571-1598, URL https://doi.org/10.1090/S0025-5718-2014-02802-0
9. Cockburn B, Gopalakrishnan J, Nguyen NC, Peraire J, Sayas FJ (2011) Analysis of HDG methods for Stokes flow. Math Comp 80(274):723-760, URL http://dx.doi.org/10.1090/S0025-5718-2010-02410-X
10. Dobrev VA, Kolev TV, et al (2018) MFEM: Modular finite element methods. http://mfem.org
11. Hansbo P, Larson MG (2002) Discontinuous Galerkin methods for incompressible and nearly incompressible elasticity by Nitsche's method. Comput Methods Appl Mech Engrg 191:1895-1908, URL http://dx.doi.org/10.1016/S0045-7825(01) 00358-9
12. Henson VE, Yang UM (2002) BoomerAMG: A parallel algebraic multigrid solver and preconditioner. Appl Numer Math 41(1):155-177, URL http://dx.doi.org/10.1016/S0168-9274(01)00115-5
13. Howell JS, Walkington NJ (2011) Inf-sup conditions for twofold saddle point problems. Numer Math 118:663-693, URL http://dx.doi.org/10.1007/s00211-011-0372-5
14. John V, Linke A, Merdon C, Neilan M, Rebholz LG (2017) On the divergence constraint in mixed finite element methods for incompressible flows. SIAM Rev 59(3):492-544, URL http://doi.org/10.1137/15M1047696
15. Labeur RJ, Wells GN (2012) Energy stable and momentum conserving hybrid finite element method for the incompressible Navier-Stokes equations. SIAM J Sci Comput 34(2):A889-A913, URL http://dx.doi.org/10.1137/100818583
16. Lee CH, Vassilevski PS (2017) Parallel solver for $H$ (div) problems using hybridization and AMG. In: Domain Decomposition Methods in Science and Engineering XXIII, Springer, pp 69-80, URL http://dx.doi.org/10.1007/978-3-319-52389-7_6
17. Lehrenfeld C, Schöberl J (2016) High order exactly divergence-free hybrid discontinuous Galerkin methods for unsteady incompressible flows. Comput Methods Appl Mech Engrg 307:339-361, URL http://dx.doi.org/10.1016/j.cma.2016.04.025
18. Nguyen NC, Peraire J, Cockburn B (2010) A hybridizable discontinuous galerkin method for stokes flow. Comput Methods Appl Mech Engrg 199(9-12):582-597, URL http://dx.doi.org/10.1016/j.cma.2009.10.007
19. Nguyen NC, Peraire J, Cockburn B (2011) An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations. J Comput Phys 230(4):1147-1170, URL http://dx.doi.org/10.1016/j.jcp.2010.10.032
20. Pestana J, Wathen AJ (2015) Natural preconditioning and iterative methods for saddle point systems. SIAM Review 57(1):71-91, URL http://dx.doi.org/10.1137/130934921
21. Qiu W, Shi K (2016) A superconvergent HDG method for the incompressible NavierStokes equations on general polyhedral meshes. IMA J Numer Anal 36(4):1943-1967, URL http://dx.doi.org/10.1093/imanum/drv067
22. Rhebergen S, Cockburn B (2012) A space-time hybridizable discontinuous galerkin method for incompressible flows on deforming domains. J Comput Phys 231(11):4185-4204, URL http://dx.doi.org/10.1016/j.jcp.2012.02.011
23. Rhebergen S, Wells GN (2017) Analysis of a hybridized/interface stabilized finite element method for the Stokes equations. SIAM J Numer Anal 55(4):1982-2003, URL http://doi.org/10.1137/16M1083839
24. Rhebergen S, Wells GN (2018) A hybridizable discontinuous Galerkin method for the Navier-Stokes equations with pointwise divergence-free velocity field. J Sci Comput URL https://doi.org/10.1007/s10915-018-0671-4
25. Wells GN (2011) Analysis of an interface stabilized finite element method: the advection-diffusion-reaction equation. SIAM J Numer Anal 49(1):87-109, URL http://dx.doi.org/10.1137/090775464

[^0]:    SR gratefully acknowledges support from the Natural Sciences and Engineering Research Council of Canada through the Discovery Grant program (RGPIN-05606-2015) and the Discovery Accelerator Supplement (RGPAS-478018-2015).

    Sander Rhebergen
    Department of Applied Mathematics, University of Waterloo, Waterloo N2L 3G1, Canada
    E-mail: srheberg@uwaterloo.ca
    ORCID: 0000-0001-6036-0356
    Garth N. Wells
    Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom
    E-mail: gnw20@cam.ac.uk
    ORCID: 0000-0001-5291-7951

