

# A FINITE ELEMENT METHOD WITH STRONG MASS CONSERVATION FOR BIOT'S LINEAR CONSOLIDATION MODEL

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**Abstract.** An  $H(\text{div})$  conforming finite element method for solving the linear Biot equations is analyzed. Formulations for the standard mixed method are combined with formulation of interior penalty discontinuous Galerkin method to obtain a consistent scheme. Optimal convergence rates are obtained.

**1. Introduction.** In this article, we present a new finite element discretization of a linear model for poroelasticity [6]. The main features of our approach are a consistent coupling of fluid and solid velocity without projection and consistent approximation rates for both velocity fields and the fluid pressure. Thus, our scheme is robust with respect to fluid and solid compressibility manifested by the storage coefficient  $c_s$  and the Biot-Willis constant  $\alpha$ , and we obtain optimal convergence rates in  $L^2$  for pressure and velocity. We achieve this by using a standard mixed formulation based on  $H^{\text{div}}$ -conforming finite element spaces with matching pressure for the fluid velocity and by using the same vector space combined with discontinuous Galerkin flux terms for  $H^1$ -consistency.

Already in 1994, Murad and Loula [12] analyze the case with  $c_s = 0$  (incompressible fluid). They use  $H^1$ -conforming finite elements for displacement and fluid pressure, and obtain estimates of Taylor-Hood type, that is, for pressure shape functions of degree  $k - 1$  and displacement of degree  $k$ , they have balanced approximation in  $L^2$  for strain and pressure of order  $h^k$ . Assuming additional regularity, duality yields that the displacement converges of order  $h^{k+1}$ , while by taking derivatives, the seepage velocity is of order  $h^{k-1}$ . It is this gap in approximation, we are overcoming with our method.

In [13], Oyarzua and Ruiz-Baier introduce a “total” pressure  $\phi = p - \lambda \nabla \cdot u$  in order to treat the coupling between solid and fluid in a more robust way. Since they compute the pressure  $p$  as well, this amounts to adding a variable for the dilation  $\nabla \cdot u$ . They obtain for a Taylor-Hood approximation of degree  $k/k - 1$  of the displacement/total pressure pair and a pressure approximation of degree  $k$  an energy estimate of order  $k$  involving  $H^1$ -norms of the displacement and pressure and the  $L^2$ -norm of the total pressure. Thus, assuming elliptic regularity, the  $L^2$ -error of the displacement is only one order better than that of the fluid velocity. A similar gap can be observed in [20], where the error of the displacement gradient is in balance with the seepage velocity. The discretization there is more similar to ours though, since it uses Raviart-Thomas elements for the seepage velocity. Different from here, a nonconforming element is used there for the solid displacement. Estimates of the same kind were obtained in [14, 15] for continuous and discontinuous Galerkin approximation of the solid displacement, respectively, but under the restrictive assumption  $c_s > 0$ , which excludes incompressible fluids.

In [21], a mixed method involving discretization of pressure, seepage velocity, “total”

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stress, and displacement is used. The finite elements are Raviart-Thomas pairs for velocity and pressure and Arnold-Winther pairs for stress and displacement. It is to our knowledge the only other result which produces equal order approximation in  $L^2$  for velocity and displacement, if matching polynomial degrees are chosen. Compared to the method proposed here, introducing a discretization of the total stress increases the number of degrees of freedom considerably. In addition, the optimal error estimate in  $L^\infty(L^2)$  there is only obtained for  $c_s > 0$ , while it deteriorates to  $L^2(L^2)$  for  $c_s = 0$ , while our analysis does not suffer from this problem and holds in every timestep. Finally, robustness with respect to all involved parameters of discretizations based on Raviart-Thomas pairs is discussed in [10], and their analysis of the system to be solved in a single time step applies to our method as well. Not using our assumption 4.3 below, they choose the order of the displacement space higher than that of the seepage velocity space.

The remainder of this article is organized as follows: in Section 2, we denote Biot's consolidation equations in displacement, pressure, and seepage velocity variables. Then, in section 3, we state a semidiscrete scheme and present its error analysis in Section 4. A simple time discretization and its analysis are provided in Section 5, and we conclude with numerical tests in Section 6.

**2. Model problem.** The linear Biot system coupling the deformation  $\mathbf{u}$  of the porous media, the fluid pressure  $p$ , and the discharge or seepage velocity  $\mathbf{w}$  of the fluid is written as:

$$\frac{\partial}{\partial t}(c_s p + \alpha \nabla \cdot \mathbf{u}) + \nabla \cdot \mathbf{w} = f_1, \quad \text{in } \Omega \times (0, T), \quad (2.1)$$

$$K^{-1} \mathbf{w} = -\nabla p, \quad \text{in } \Omega \times (0, T), \quad (2.2)$$

$$-\nabla \cdot (\boldsymbol{\sigma} - \alpha p \mathbf{I}) = \mathbf{f}_2, \quad \text{in } \Omega \times (0, T). \quad (2.3)$$

The constant  $\alpha$  is called the Biot-Willis constant [5], which represents unaccounted volume changes due to a third phase, for instant small air inclusions in soil. It takes a value very close to one. The constant  $c_s$  represents the constrained specific storage coefficient (see [19] and references therein) and is related to compressibility of the fluid. Therefore, it is close to zero in many applications. The permeability  $K$  is a symmetric positive definite matrix. We assume here that the effective stress tensor satisfies Hooke's law:

$$\boldsymbol{\sigma} = \lambda(\nabla \cdot \mathbf{u}) \mathbf{I} + 2\mu \boldsymbol{\varepsilon}(\mathbf{u}),$$

where

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

The system is completed by initial conditions

$$p(0) = p_0, \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in } \Omega. \quad (2.4)$$

such that equation (2.3) is satisfied at  $t = 0$ . No initial condition on  $\mathbf{w}$  is required since it is only coupled algebraically. In practice, the initial pressure is experimentally measured and the displacement  $\mathbf{u}_0$  is obtained by solving (2.3).

The boundary of the domain is decomposed into two pairs of disjoint sets:

$$\partial\Omega = \Gamma_{pD} \cup \Gamma_{pN} = \Gamma_{\mathbf{u}D} \cup \Gamma_{\mathbf{u}N},$$

with

$$\Gamma_{pD} \cap \Gamma_{pN} = \Gamma_{uD} \cap \Gamma_{uN} = \emptyset.$$

We prescribe the pressure and velocity on the boundary

$$p = p_D, \quad \text{on } \Gamma_{pD}, \quad (2.5)$$

$$\mathbf{w} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma_{pN}, \quad (2.6)$$

and we prescribe the displacement and total normal stress

$$\mathbf{u} = \mathbf{u}_D, \quad \text{on } \Gamma_{uD}, \quad (2.7)$$

$$(\boldsymbol{\sigma} - \alpha p \mathbf{I}) \mathbf{n} = \boldsymbol{\sigma}_N, \quad \text{on } \Gamma_{uN}. \quad (2.8)$$

Throughout the paper, the unit normal (resp. tangential) vector to the boundary  $\partial\Omega$  is denoted by  $\mathbf{n}$  (resp.  $\boldsymbol{\tau}$ ). We remark that the boundary condition  $\mathbf{w} \cdot \mathbf{n} = 0$  can be changed to the inhomogeneous boundary condition  $\mathbf{w} \cdot \mathbf{n} = g$ . In that case, the datum  $g$  needs to be lifted following a standard technical argument. Furthermore, the deformation may admit more complex boundary conditions, see for instance the numerical experiments. We make the following assumptions:

1. Neither  $\Gamma_{pN} = \partial\Omega$ , nor is  $\mathbf{u} \cdot \mathbf{n}$  prescribed on the whole boundary. This is a technical assumption which guarantees that neither  $\nabla \cdot \mathbf{w}$ , nor  $\nabla \cdot \mathbf{u}$  are forced to have mean value zero.
2. The boundary condition on  $\mathbf{u}$  itself is sufficient to admit Korn's inequality

$$\|\nabla \mathbf{u}\|_{\Omega} \leq C \|\epsilon(\mathbf{u})\|_{\Omega},$$

where  $\|\cdot\|_{\Omega}$  denotes the  $L^2$  norm over  $\Omega$ . In particular, the boundary conditions must exclude solid translations and rotations of the whole domain.

**3. Continuous-in-time Scheme.** Let  $\mathbb{T}_h$  be a shape regular family of conforming subdivisions of  $\Omega$  into simplices, parallelograms or parallelepipeds. Denote by  $h_T$  the diameter of an element  $T$  and denote by  $h$  the maximum diameter over all mesh elements. Denote by  $\Gamma_i$  the set of faces that are interior to  $\Omega$ . For all  $t \geq 0$ , we seek a solution  $(p_h, \mathbf{w}_h, \mathbf{u}_h)$  in  $Q_h \times \mathbf{W}_h \times \mathbf{V}_h$ . The pair  $(\mathbf{W}_h, Q_h)$  is the usual pair of a divergence-conforming velocity space  $\mathbf{W}_h \subset H_{0, \Gamma_{pN}}^{\text{div}}(\Omega)$  and its corresponding pressure space  $Q_h \subset L^2(\Omega)$ . We denote

$$\mathbf{W} = H_{0, \Gamma_{pN}}^{\text{div}}(\Omega) = \{\mathbf{z} \in H^{\text{div}}(\Omega) : \mathbf{z} \cdot \mathbf{n} = 0 \text{ on } \Gamma_{pN}\},$$

and

$$\mathbf{V} = \{\mathbf{v} \in H^{\text{div}}(\Omega) : \mathbf{v} \cdot \mathbf{n} = \mathbf{0} \text{ on } \Gamma_{uD}\}.$$

We use the notation  $\|\cdot\|_{\mathcal{O}}$  for the  $L^2$  norm on any domain  $\mathcal{O}$ . The  $L^2$  inner-product on  $\mathcal{O}$  is denoted by  $(\cdot, \cdot)_{\mathcal{O}}$ . The space  $\mathbf{V}_h$  is a finite-dimensional subspace of  $\mathbf{V} \cap H^1(\mathbb{T}_h)$ , where  $H^1(\mathbb{T}_h)$  is the broken Sobolev space. We denote by  $k$  the polynomial degree for the space  $Q_h$ . The space  $\mathbf{W}_h$  only differs from  $\mathbf{V}_h$  by the location of the boundary conditions, and thus has the same order as  $\mathbf{V}_h$ . We also assume that the spaces  $\mathbf{W}_h$  and  $Q_h$  satisfy:

$$\nabla \cdot \mathbf{W}_h = Q_h. \quad (3.1)$$

Therefore we also have

$$\nabla \cdot \mathbf{V}_h = Q_h. \quad (3.2)$$

Next we introduce an approximation operator  $\pi_h$  satisfying for all  $\mathbf{z} \in \mathbf{W} + \mathbf{V}$

$$(\nabla \cdot \pi_h(\mathbf{z}), q) = (\nabla \cdot \mathbf{z}, q), \quad \forall q \in Q_h, \quad (3.3)$$

$$\|\pi_h(\mathbf{z}) - \mathbf{z}\|_{H^r(T)} \leq Ch_T^{k+1-r} |\mathbf{z}|_{H^{k+1}(T)}, \quad \forall T \in \mathbb{T}_h, \quad 0 \leq r \leq k, \quad (3.4)$$

$$\|\nabla \cdot (\pi_h(\mathbf{z}) - \mathbf{z})\|_{L^2(T)} \leq Ch_T^{k+1} |\nabla \cdot \mathbf{z}|_{H^{k+1}(T)}, \quad \forall T \in \mathbb{T}_h. \quad (3.5)$$

Since the spaces  $\mathbf{W}$  and  $\mathbf{V}$  differ because of the location of the boundary conditions, we also require that

$$\begin{aligned} \forall \mathbf{z} \in \mathbf{W}, \quad \pi_h(\mathbf{z}) &\in \mathbf{W}_h, \\ \forall \mathbf{z} \in \mathbf{V}, \quad \pi_h(\mathbf{z}) &\in \mathbf{V}_h. \end{aligned}$$

We now introduce jump  $[\cdot]$  and average  $\{\cdot\}$  of a scalar function  $\phi$  across an interior face  $F$ . We first associate with each face  $F$  in  $\Gamma_i$  a unit normal vector  $\mathbf{n}_F$ , and we denote by  $T_-$  and  $T_+$  the elements that share  $F$ , such that  $\mathbf{n}_F$  points from  $T_-$  to  $T_+$ . We then define

$$[\phi] = \phi|_{T_-} - \phi|_{T_+}, \quad \{\phi\} = \frac{1}{2}(\phi|_{T_-} + \phi|_{T_+}).$$

Jump and average of vector function  $\phi$  are defined component-wise. The  $L^2$  inner-product on an open domain  $\mathcal{O}$  is denoted by  $(\cdot, \cdot)_{\mathcal{O}}$ . We will also use the following notation for the inner-products on elements and faces:

$$\begin{aligned} (\phi, \psi)_{\mathbb{T}_h} &= \sum_{T \in \mathbb{T}_h} (\phi, \psi)_T, \quad (\phi, \psi)_{\Gamma_i} = \sum_{F \in \Gamma_i} (\phi, \psi)_F, \\ (\phi, \psi)_{\Gamma_{\text{uD}}} &= \sum_{F \in \Gamma_{\text{uD}}} (\phi, \psi)_F, \quad (\phi, \psi)_{\Gamma_{\text{uN}}} = \sum_{F \in \Gamma_{\text{uN}}} (\phi, \psi)_F. \end{aligned}$$

The discretization of the operator  $-2\nabla \cdot \epsilon(\mathbf{u})$  in the nonconforming space  $\mathbf{V}$  follows the interior penalty (SIPG) method [1, 16] with the mesh dependent form:

$$\begin{aligned} d_h(\mathbf{u}, \mathbf{v}) &= 2(\epsilon(\mathbf{u}), \epsilon(\mathbf{v}))_{\mathbb{T}_h} + \frac{\gamma}{h}([\mathbf{u}], [\mathbf{v}])_{\Gamma_i} \\ &\quad - 2(\{\epsilon(\mathbf{u})\mathbf{n}_F\}, [\mathbf{v}])_{\Gamma_i} - 2(\{\epsilon(\mathbf{v})\mathbf{n}_F\}, [\mathbf{u}])_{\Gamma_i} \\ &\quad + \frac{\gamma}{h}(\mathbf{u}, \mathbf{v})_{\Gamma_{\text{uD}}} - 2(\epsilon(\mathbf{u})\mathbf{n}, \mathbf{v})_{\Gamma_{\text{uD}}} - 2(\epsilon(\mathbf{v})\mathbf{n}, \mathbf{u})_{\Gamma_{\text{uD}}}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}. \end{aligned}$$

The parameter  $\gamma > 0$  is the penalty parameter, chosen large enough to ensure coercivity of the bilinear form  $d_h(\cdot, \cdot)$ . Since the space  $\mathbf{V}$  is  $H^{\text{div}}$ -conforming, no penalty formulation for the term  $\nabla \nabla \cdot \mathbf{u}$  is needed. Accordingly, we define the bilinear form

$$a_h(\mathbf{u}, \mathbf{v}) = \mu d_h(\mathbf{u}, \mathbf{v}) + \lambda(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{\Omega}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}.$$

From the equalities of these spaces, we immediately deduce the inf-sup conditions in [7] and [9, 18]:

$$\forall q \in Q_h \quad \exists \mathbf{z} \in \mathbf{W}_h : \quad \nabla \cdot \mathbf{z} = q \quad \wedge \quad \|\mathbf{z}\|_{H^{\text{div}}(\Omega)} \leq \frac{1}{\beta_{\mathbf{W}}} \|q\|_{\Omega} \quad (3.6)$$

$$\forall q \in Q_h \quad \exists \mathbf{v} \in \mathbf{V}_h : \quad \nabla \cdot \mathbf{v} = q \quad \wedge \quad \|\mathbf{v}\|_{1,h} \leq \frac{1}{\beta_{\mathbf{V}}} \|q\|_{\Omega} \quad (3.7)$$

The semi-discrete scheme is: for all  $t > 0$  find  $(p_h(t), \mathbf{w}_h(t), \mathbf{u}_h(t)) \in Q_h \times \mathbf{W}_h \times \mathbf{V}_h$  such that

$$(\partial_t(c_s p_h + \alpha \nabla \cdot \mathbf{u}_h), q)_\Omega + (\nabla \cdot \mathbf{w}_h, q)_\Omega = (f_1, q)_\Omega, \quad \forall q \in Q_h, \quad (3.8a)$$

$$(K^{-1} \mathbf{w}_h, \mathbf{z})_\Omega - (p_h, \nabla \cdot \mathbf{z})_\Omega = -(p_D, \mathbf{z} \cdot \mathbf{n})_{\Gamma_{pD}}, \quad \forall \mathbf{z} \in \mathbf{W}_h, \quad (3.8b)$$

$$a_h(\mathbf{u}_h, \mathbf{v}) - \alpha(p_h, \nabla \cdot \mathbf{v})_\Omega = \mathcal{R}(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad (3.8c)$$

where

$$\mathcal{R}(\mathbf{v}) = (\mathbf{f}_2, \mathbf{v})_\Omega + (\boldsymbol{\sigma}_N, \mathbf{v})_{\Gamma_{uN}} - 2\mu(\boldsymbol{\varepsilon}(\mathbf{v})\mathbf{n}, \mathbf{u}_D)_{\Gamma_{uD}} + \frac{\gamma}{h}(\mathbf{u}_D, \mathbf{v})_{\Gamma_{uD}}.$$

We have the following initial conditions for pressure  $p_h(0) \in Q_h$  and for displacement  $\mathbf{u}_h(0) \in \mathbf{V}_h$

$$\begin{aligned} (p_h(0), q) &= (p_0, q), & \forall q \in Q_h, \\ a_h(\mathbf{u}_h(0), \mathbf{v}) &= a_h(\mathbf{u}_0, \mathbf{v}), & \forall \mathbf{v} \in \mathbf{V}_h. \end{aligned} \quad (3.8d)$$

We first note that the scheme (3.8a–d) is consistent:

LEMMA 3.1. *Let  $(p, \mathbf{u}, \mathbf{w})$  be the solution to (2.1)–(2.8), and assume  $\mathbf{u}(t) \in H^{3/2+\epsilon}(\Omega)$  for all  $t$  and for some positive  $\epsilon$ . Then, it satisfies the equations (3.8a)–(3.8d).*

*Proof.* The consistency of equation (3.8) without the pressure term for solutions  $u \in H^{3/2+\epsilon}(\Omega)$  of equation (2.3) was established in [17, Lemma 2.1]. Since  $\mathbf{V}_h \subset H^{\text{div}}(\Omega)$ , the discretization of  $(p_h, \nabla \cdot \mathbf{v})$  is conforming. Thus, we obtain consistency of the momentum equation (3.8c) with (2.3) and of the compatibility condition (3.8d) with (2.4).

The mixed finite element discretization (3.8a–b) of (2.1), (2.2) is conforming and thus straightforward [7].  $\square$

We next state the coercivity of the bilinear form  $d(\cdot, \cdot)$ , the proof of which depends on Korn's inequality for discontinuous spaces [8] and can be found in [9]

LEMMA 3.2. *Assume  $\gamma$  is large enough. There is a positive constant  $\kappa$  independent of  $h$  (and  $\lambda, \mu, \alpha, c_s$ ) such that:*

$$\kappa \|\mathbf{v}_h\|_{1,h}^2 \leq d_h(\mathbf{v}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (3.9)$$

The norm  $\|\cdot\|_{1,h}$  is defined as:

$$\|\mathbf{v}_h\|_{1,h} = \left( \sum_{T \in \mathbb{T}_h} \|\nabla \mathbf{v}_h\|_T^2 + \sum_{F \in \Gamma_i} \frac{\gamma}{h} \|[\mathbf{v}_h]\|_F^2 + \sum_{F \in \Gamma_{uD}} \frac{\gamma}{h} \|\mathbf{v}_h\|_F^2 \right)^{1/2}, \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

As a corollary of Lemma 3.2, we have

$$\kappa \mu \|\mathbf{v}_h\|_{1,h}^2 + \lambda \|\nabla \cdot \mathbf{v}_h\|_\Omega^2 \leq a_h(\mathbf{v}_h, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \quad (3.10)$$

We follow [20] and apply the theory of differential algebraic equations to the solution of the semidiscrete problem. To this end, we need the following lemma:

LEMMA 3.3. *A differential algebraic equation of the form*

$$\mathbf{E} \partial_t \mathbf{x}(t) + \mathbf{A} \mathbf{x}(t) = \mathbf{q}(t)$$

with  $\mathbf{A}, \mathbf{E} \in \mathbb{R}^{m \times m}$  and  $\mathbf{q}(t) \in \mathbb{R}^m$  is solvable, if and only if the matrix pencil  $\sigma \mathbf{E} + \mathbf{A}$  is regular, that is, there is a value  $\sigma \neq 0$ , such that  $\sigma \mathbf{E} + \mathbf{A}$  is an invertible matrix. This lemma can be found in [11, Theorem 2.4]. Solvable DAE have the property, that initial value problems are uniquely solvable, if the initial condition is compatible with the algebraic constraints. Thus, it remains to verify that the semidiscrete system (3.8a–c) meets the assumptions of this lemma. Obviously, these equations constitute a finite dimensional linear system of equations with  $\mathbf{E}$  corresponding to the time derivative part. Thus, it remains to show the following lemma.

LEMMA 3.4. *For any  $\sigma > 0$ , the system*

$$(\sigma(c_s p_h + \alpha \nabla \cdot \mathbf{u}_h), q) + (\nabla \cdot \mathbf{w}_h, q) = 0, \quad \forall q \in Q_h, \quad (3.11a)$$

$$(K^{-1} \mathbf{w}_h, \mathbf{z}) - (p_h, \nabla \cdot \mathbf{z}) = 0, \quad \forall \mathbf{z} \in \mathbf{W}_h, \quad (3.11b)$$

$$a_h(\mathbf{u}_h, \mathbf{v}) - \alpha(p_h, \nabla \cdot \mathbf{v}) = 0, \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad (3.11c)$$

has the unique solution  $(p_h, \mathbf{w}_h, \mathbf{u}_h) = \mathbf{0}$ .

*Proof.* Choosing test functions  $q = p_h$ ,  $\mathbf{z} = \mathbf{w}_h$ , and  $\mathbf{v} = \sigma \mathbf{u}_h$  and adding the three equations, we obtain

$$\sigma c_s \|p_h\|_{\Omega}^2 + \|K^{-1/2} \mathbf{w}_h\|_{\Omega}^2 + \sigma \mu d_h(\mathbf{u}_h, \mathbf{u}_h) + \sigma \lambda \|\nabla \cdot \mathbf{u}_h\|_{\Omega}^2 = 0.$$

This, combined with the coercivity of  $d_h(\cdot, \cdot)$ , yields  $\mathbf{w}_h = \mathbf{0}$  and  $\mathbf{u}_h = \mathbf{0}$  and concludes the proof for  $c_s \neq 0$ . For  $c_s = 0$ , we choose in (3.11b) according to the inf-sup condition a test function  $\mathbf{z} \neq \mathbf{0}$  with  $\nabla \cdot \mathbf{z} = p_h$ . Thus,  $p_h = 0$ .  $\square$

Thus, together with the previous lemma, our DAE is solvable. This lemma indeed proved that there is not only one  $\sigma$  for which the problem is solvable, but that it is solvable for all positive  $\sigma$ . While such a strong statement is not needed here, it is the core of the proof of well-definedness of time stepping schemes below.

**4. A priori error estimates for continuous-in-time scheme.** In this section, we state our theoretical results. The proofs are given in the rest of the paper. We begin with a simple lemma on mass conservation, which motivated us to choose this method. It turns out that mass conservation is achieved pointwisely by this method.

LEMMA 4.1. *Let the spaces  $Q_h$ ,  $\mathbf{V}_h$ , and  $\mathbf{W}_h$  be divergence conforming as in equations (3.1) and (3.2). Then, for any  $\sigma > 0$ , the solution  $(p_h, \mathbf{w}_h, \mathbf{u}_h)$  of the system (3.11) obey the pointwise mass conservation equation*

$$\sigma(c_s p_h + \alpha \nabla \cdot \mathbf{u}_h) + \nabla \cdot \mathbf{w}_h = 0, \quad \forall x \in T, \forall T \in \mathbb{T}_h. \quad (4.1)$$

*Proof.* We denote

$$r_h = \sigma(c_s p_h + \alpha \nabla \cdot \mathbf{u}_h) + \nabla \cdot \mathbf{w}_h.$$

Because of assumptions (3.1) and (3.2), the quantity  $r_h$  belongs to  $Q_h$ . We test equation (3.11a) with  $r_h$  to obtain the result.  $\square$

Next, we investigate the elastic subproblem. Let  $\tilde{\mathbf{u}}(t) \in \mathbf{V}_h$  be the projection of  $\mathbf{u}(t)$  onto  $\mathbf{V}_h$  with respect to the linear elasticity operator, namely for any  $t > 0$  let  $\tilde{\mathbf{u}}(t)$  satisfy

$$a_h(\tilde{\mathbf{u}}(t), \mathbf{v}) = a_h(\mathbf{u}(t), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (4.2)$$

From the coercivity of  $a_h(\cdot, \cdot)$ , it is easy to see that  $\tilde{\mathbf{u}}(t)$  exists and is unique. By adaptation of [9, Theorem 8] to the Raviart-Thomas element and by the standard duality argument, we have

PROPOSITION 4.2. *There is a constant  $C$  independent of  $h, \lambda, \mu, \alpha, c_s$  such that*

$$\|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)\|_{1,h}^2 \leq Ch^{2k} |\mathbf{u}(t)|_{H^{k+1}(\Omega)}^2, \quad (4.3)$$

$$\|\tilde{\mathbf{u}}(t) - \mathbf{u}(t)\|_{\Omega}^2 \leq Ch^{2k+2} |\mathbf{u}(t)|_{H^{k+1}(\Omega)}^2, \quad (4.4)$$

$$\|\partial_t(\tilde{\mathbf{u}}(t) - \mathbf{u}(t))\|_{1,h}^2 \leq Ch^{2k} |\partial_t \mathbf{u}(t)|_{H^{k+1}(\Omega)}^2. \quad (4.5)$$

We have furthermore observed in experiments, that the divergence is converging optimally. These experiments included rectangular meshes with local refinement. Currently, there is no proof for this fact, and it may be due to superconvergence effects related to the meshes we used. Following [2], we do not expect this to hold on general quadrilateral meshes. Nevertheless, we would like to present an analysis using this fact alongside standard convergence. Accordingly, we will use at some point:

ASSUMPTION 4.3. *There is a constant  $C_{\mu,\lambda}$  that is independent of  $h, \alpha, c_s$  such that*

$$\|\nabla \cdot (\tilde{\mathbf{u}}(t) - \mathbf{u}(t))\|_{\Omega}^2 \leq C_{\mu,\lambda} h^{2k+2} |\nabla \cdot \mathbf{u}(t)|_{H^{k+1}(\Omega)}^2. \quad (4.6)$$

This assumption would naturally imply

$$\|\nabla \cdot \partial_t(\tilde{\mathbf{u}}(t) - \mathbf{u}(t))\|_{\Omega}^2 \leq C_{\mu,\lambda} h^{2k+2} |\nabla \cdot \partial_t \mathbf{u}(t)|_{H^{k+1}(\Omega)}^2. \quad (4.7)$$

Now we are ready to state our first main theorem:

THEOREM 4.4. *There is a constant  $C$  independent of  $h, \lambda, \mu, \alpha, c_s$  such that*

$$\forall t > 0 \quad \mu \|\mathbf{u}_h(t) - \mathbf{u}(t)\|_{1,h}^2 \leq Ch^{2k} (\mathcal{M} + \mu \|\mathbf{u}(t)\|_{H^{k+1}(\Omega)}^2),$$

and

$$c_s \|p_h(t) - p(t)\|_{\Omega}^2 + \|K^{-1/2}(\mathbf{w}_h - \mathbf{w})\|_{L^2(0,t;L^2(\Omega))}^2 \leq C \epsilon_{div}(h)^2 (\mathcal{M} + c_s \|p(t)\|_{H^{k+1}(\Omega)}^2),$$

where  $\epsilon_{div}(h) = h^k$  and

$$\mathcal{M} = \alpha^2 \|\partial_t \nabla \cdot \mathbf{u}\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|\mathbf{w}\|_{L^2(0,T;H^{k+1}(\Omega))}^2.$$

If in addition Assumption 4.3 holds, we have  $\epsilon_{div}(h) = h^{k+1}$  and there is a constant  $C_{\mu,\lambda}$  independent of  $h, \alpha, c_s$  such that

$$\lambda \|\nabla \cdot (\mathbf{u}_h(t) - \mathbf{u}(t))\|_{\Omega}^2 \leq C_{\mu,\lambda} h^{2k+2} (\mathcal{M} + \lambda \|\nabla \cdot \mathbf{u}(t)\|_{H^{k+1}(\Omega)}^2).$$

**4.1. Proof of Theorem 4.4.** We decompose the numerical error into an approximation error and a discrete error. For all  $t > 0$ , choose  $\tilde{\mathbf{u}}(t) \in \mathbf{V}_h$  the  $a_h(\cdot, \cdot)$ -orthogonal projection of  $\mathbf{u}(t)$  satisfying (4.2). Denote the Fortin projection  $\tilde{\mathbf{w}}(t) = \pi_h \mathbf{w}(t) \in \mathbf{W}_h$  and let  $\tilde{p}(t)$  be the  $L^2$  projection of  $p(t)$  in  $Q_h$ :

$$(p(t) - \tilde{p}(t), q_h) = 0, \quad \forall q_h \in Q_h. \quad (4.8)$$

This implies that

$$\left( \frac{\partial p}{\partial t}(t) - \frac{\partial \tilde{p}}{\partial t}(t), q_h \right) = 0, \quad \forall q_h \in Q_h. \quad (4.9)$$

We have the following approximation error bound for the pressure:

$$\|\tilde{p}(t) - p(t)\|_{\Omega} \leq Ch^{k+1}|p(t)|_{H^{k+1}(\Omega)}. \quad (4.10)$$

Let us prove a lemma on the error  $p_h - \tilde{p}$ .

LEMMA 4.5. *There is a constant  $C$  independent of  $h, \mu, \lambda, \alpha, c_s$  such that*

$$\|p_h(t) - \tilde{p}(t)\|_{\Omega} \leq C\|\mathbf{w}_h(t) - \mathbf{w}(t)\|_{\Omega}, \quad \forall t > 0. \quad (4.11)$$

*Proof.* The error equation is

$$(K^{-1}(\mathbf{w}_h - \mathbf{w}), \mathbf{z})_{\Omega} - (p_h - p, \nabla \cdot \mathbf{z})_{\Omega} = 0, \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

Equivalently,

$$(p_h - \tilde{p}, \nabla \cdot \mathbf{z})_{\Omega} = (K^{-1}(\mathbf{w}_h - \mathbf{w}), \mathbf{z})_{\Omega} + (p - \tilde{p}, \nabla \cdot \mathbf{z})_{\Omega}, \quad \forall \mathbf{z} \in \mathbf{W}_h.$$

Using properties (4.8) and (3.1), we have

$$(p_h - \tilde{p}, \nabla \cdot \mathbf{z})_{\Omega} = (K^{-1}(\mathbf{w}_h - \mathbf{w}), \mathbf{z})_{\Omega} \leq \|K^{-\frac{1}{2}}(\mathbf{w}_h - \mathbf{w})\| \|\mathbf{z}\|, \quad \forall \mathbf{z} \in \mathbf{W}_h. \quad (4.12)$$

Since the pair  $(\mathbf{W}_h, Q_h)$  satisfies the inf-sup condition (3.6), we can choose a test function  $\mathbf{z}$  with  $\nabla \cdot \mathbf{z} = p_h - \tilde{p}$  and obtain

$$\|p_h - \tilde{p}\|_{\Omega}^2 \leq \|K^{-\frac{1}{2}}(\mathbf{w}_h - \mathbf{w})\| \frac{1}{\beta_{\mathbf{W}}} \|p_h - \tilde{p}\|_{\Omega},$$

which proves the result.  $\square$

We now write the system of error equations:

$$\begin{aligned} & (c_s \partial_t (p_h - \tilde{p}) + \alpha \partial_t \nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}}) + \nabla \cdot (\mathbf{w}_h - \tilde{\mathbf{w}}), q) \\ & = (c_s \partial_t (p - \tilde{p}) + \alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}) + \nabla \cdot (\mathbf{w} - \tilde{\mathbf{w}}), q), \end{aligned} \quad (4.13)$$

$$(K^{-1}(\mathbf{w}_h - \tilde{\mathbf{w}}), \mathbf{z}) - (p_h - \tilde{p}, \nabla \cdot \mathbf{z}) = (K^{-1}(\mathbf{w} - \tilde{\mathbf{w}}), \mathbf{z}) - (p - \tilde{p}, \nabla \cdot \mathbf{z}), \quad (4.14)$$

$$a_h(\mathbf{u}_h - \tilde{\mathbf{u}}, \mathbf{v}) - \alpha(p_h - \tilde{p}, \nabla \cdot \mathbf{v}) = a_h(\mathbf{u} - \tilde{\mathbf{u}}, \mathbf{v}) - \alpha(p - \tilde{p}, \nabla \cdot \mathbf{v}). \quad (4.15)$$

Next we choose  $q = p_h - \tilde{p}$ ,  $\mathbf{z} = \mathbf{w}_h - \tilde{\mathbf{w}}$  and  $\mathbf{v} = \partial_t(\mathbf{u}_h - \tilde{\mathbf{u}})$  in (4.13), (4.14) and (4.15) respectively. We add the resulting equations and obtain:

$$\begin{aligned} & \frac{c_s}{2} \frac{d}{dt} \|p_h - \tilde{p}\|_{\Omega}^2 + \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 + \frac{\mu}{2} \frac{d}{dt} d_h(\mathbf{u}_h - \tilde{\mathbf{u}}, \mathbf{u}_h - \tilde{\mathbf{u}}) + \frac{\lambda}{2} \frac{d}{dt} \|\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{\Omega}^2 \\ & = (c_s \partial_t (p - \tilde{p}) + \alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), p_h - \tilde{p}) + (\nabla \cdot (\mathbf{w} - \tilde{\mathbf{w}}), p_h - \tilde{p}) \\ & \quad + (K^{-1}(\mathbf{w} - \tilde{\mathbf{w}}), \mathbf{w}_h - \tilde{\mathbf{w}}) - (p - \tilde{p}, \nabla \cdot (\mathbf{w}_h - \tilde{\mathbf{w}})) \\ & \quad + a_h(\mathbf{u} - \tilde{\mathbf{u}}, \partial_t(\mathbf{u}_h - \tilde{\mathbf{u}})) - \alpha(p - \tilde{p}, \nabla \cdot \partial_t(\mathbf{u}_h - \tilde{\mathbf{u}})). \end{aligned} \quad (4.16)$$

Using (4.2) we have

$$a_h(\mathbf{u} - \tilde{\mathbf{u}}, \partial_t(\mathbf{u}_h - \tilde{\mathbf{u}})) = 0.$$

Using property (3.3) of the Fortin interpolation, we have

$$(\nabla \cdot (\mathbf{w} - \tilde{\mathbf{w}}), p_h - \tilde{p}) = 0.$$

Using properties (4.8), (3.1) and (3.2) we have

$$(p - \tilde{p}, \nabla \cdot (\mathbf{w}_h - \tilde{\mathbf{w}})) = 0,$$

$$\alpha(p - \tilde{p}, \nabla \cdot \partial_t(\mathbf{u}_h - \tilde{\mathbf{u}})) = 0.$$

Using property (4.9), we have

$$(c_s \partial_t(p - \tilde{p}), p_h - \tilde{p}) = 0.$$

Thus, equation (4.16) reduces to

$$\begin{aligned} & \frac{c_s}{2} \frac{d}{dt} \|p_h - \tilde{p}\|_{\Omega}^2 + \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 + \frac{\mu}{2} \frac{d}{dt} d_h(\mathbf{u}_h - \tilde{\mathbf{u}}, \mathbf{u}_h - \tilde{\mathbf{u}}) + \frac{\lambda}{2} \frac{d}{dt} \|\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{\Omega}^2 \\ & = (\alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), p_h - \tilde{p}) + (K^{-1}(\mathbf{w} - \tilde{\mathbf{w}}), \mathbf{w}_h - \tilde{\mathbf{w}}). \end{aligned}$$

The second term on the right-hand side is easily bounded by approximation bounds

$$\begin{aligned} (K^{-1}(\mathbf{w} - \tilde{\mathbf{w}}), \mathbf{w}_h - \tilde{\mathbf{w}}) & \leq \|K^{-1/2}(\mathbf{w} - \tilde{\mathbf{w}})\|_{\Omega}^2 + \frac{1}{4} \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 \\ & \leq Ch^{2k+2} \|\mathbf{w}\|_{H^{k+1}(\Omega)}^2 + \frac{1}{4} \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2. \end{aligned}$$

For the first term in the right-hand side we use Lemma 4.5

$$\begin{aligned} (\alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), p_h - \tilde{p}) & \leq C \|\alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega} \|\mathbf{w} - \mathbf{w}_h\|_{\Omega} \\ & \leq C \|\alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega} (\|\mathbf{w} - \tilde{\mathbf{w}}\|_{\Omega} + \|\tilde{\mathbf{w}} - \mathbf{w}_h\|_{\Omega}), \end{aligned}$$

which yields with approximation results

$$\begin{aligned} & (\alpha \partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}}), p_h - \tilde{p}) \\ & \leq C \alpha^2 \|\partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega}^2 + Ch^{2k+2} \|\mathbf{w}\|_{H^{k+1}(\Omega)}^2 + \frac{1}{4} \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2. \end{aligned}$$

Therefore the error bound becomes

$$\begin{aligned} & \frac{c_s}{2} \frac{d}{dt} \|p_h - \tilde{p}\|_{\Omega}^2 + \frac{1}{2} \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 + \frac{\mu}{2} \frac{d}{dt} d_h(\mathbf{u}_h - \tilde{\mathbf{u}}, \mathbf{u}_h - \tilde{\mathbf{u}}) + \frac{\lambda}{2} \frac{d}{dt} \|\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{\Omega}^2 \\ & \leq C \alpha^2 \|\partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega}^2 + Ch^{2k+2} \|\mathbf{w}\|_{H^{k+1}(\Omega)}^2. \end{aligned}$$

Multiply by 2, integrate from  $\tau = 0$  to  $\tau = t$  and remark that  $p_h(0) = \tilde{p}(0)$  and  $\mathbf{u}_h(0) = \tilde{\mathbf{u}}(0)$ :

$$\begin{aligned} c_s \|p_h - \tilde{p}\|_{\Omega}^2 + \int_0^t \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 d\tau + \mu d_h(\mathbf{u}_h - \tilde{\mathbf{u}}, \mathbf{u}_h - \tilde{\mathbf{u}}) + \lambda \|\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{\Omega}^2 \\ \leq C \alpha^2 \int_0^t \|\partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega}^2 d\tau + Ch^{2k+2} \int_0^t \|\mathbf{w}\|_{H^{k+1}(\Omega)}^2 d\tau. \end{aligned}$$

Thus we have using (3.9)

$$\begin{aligned} c_s \|p_h - \tilde{p}\|_{\Omega}^2 + \int_0^t \|K^{-1/2}(\mathbf{w}_h - \tilde{\mathbf{w}})\|_{\Omega}^2 d\tau + \kappa\mu \|\mathbf{u}_h - \tilde{\mathbf{u}}\|_{1,h}^2 + \lambda \|\nabla \cdot (\mathbf{u}_h - \tilde{\mathbf{u}})\|_{\Omega}^2 \\ \leq C\alpha^2 \int_0^t \|\partial_t \nabla \cdot (\mathbf{u} - \tilde{\mathbf{u}})\|_{\Omega}^2 d\tau + Ch^{2k+2} \int_0^t \|\mathbf{w}\|_{H^{k+1}(\Omega)}^2 d\tau. \end{aligned}$$

We then conclude using (4.3) or Assumption (4.7), triangle inequalities and approximation bounds. In the case  $c_s = 0$ , the estimate above does not yield an estimate for the pressure. This can be recovered by Lemma 4.5, such that in addition to the estimate above, the pressure is bounded by (4.11).

**5. Discrete-in-time Scheme.** Let  $\Delta t > 0$  denote the time step, and define  $t^n = n\Delta t$  for  $n \in \mathbb{N}$ . We use a first order in time Euler scheme and seek  $(p_h^{n+1}, \mathbf{w}_h^{n+1}, \mathbf{u}_h^{n+1}) \in \mathbf{Q}_h \times \mathbf{W}_h \times \mathbf{V}_h$  such that for all  $n \geq 0$

$$\left( \frac{1}{\Delta t} (c_s p_h^{n+1} + \alpha \nabla \cdot \mathbf{u}_h^{n+1}), q \right) + (\nabla \cdot \mathbf{w}_h^{n+1}, q) = \mathcal{R}_p^{n+1}(q), \quad \forall q \in Q_h \quad (5.1a)$$

$$(K^{-1} \mathbf{w}_h^{n+1}, \mathbf{z}) - (p_h^{n+1}, \nabla \cdot \mathbf{z}) = (p_D^{n+1}, \mathbf{z} \cdot \mathbf{n})_{\Gamma_{pD}}, \quad \forall \mathbf{z} \in \mathbf{W}_h, \quad (5.1b)$$

$$a_h(\mathbf{u}_h^{n+1}, \mathbf{v}) - \alpha (p_h^{n+1}, \nabla \cdot \mathbf{v}) = \mathcal{R}_u^{n+1}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad (5.1c)$$

where the linear functions in the right-hand sides are

$$\begin{aligned} \mathcal{R}_p^{n+1}(q) &= (f_1^{n+1}, q) + \left( \frac{1}{\Delta t} (c_s p_h^n + \alpha \nabla \cdot \mathbf{u}_h^n), q \right), \\ \mathcal{R}_u^{n+1}(\mathbf{v}) &= (\mathbf{f}_2^{n+1}, \mathbf{v}) + (\boldsymbol{\sigma}_N^{n+1}, \mathbf{v})_{\Gamma_{uN}} - 2\mu(\varepsilon(\mathbf{v}), \mathbf{n}, \mathbf{u}_D^{n+1})_{\Gamma_{uD}} + \frac{\gamma}{h} (\mathbf{u}_D^{n+1}, \mathbf{v}_{\Gamma_{uD}}), \end{aligned}$$

with initial conditions:

$$(p_h^0, q) = (p_0, q) \quad \forall q \in Q_h, \quad (5.2)$$

$$a_h(\mathbf{u}_h^0, \mathbf{v}) = a_h(\mathbf{u}_0, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (5.3)$$

The short-hand notation  $\mathbf{u}_D^n, f_1^n, \mathbf{f}_2^n$  and  $\boldsymbol{\sigma}_N^n$  is used for the functions  $\mathbf{u}_D, f_1, \mathbf{f}_2$  and  $\boldsymbol{\sigma}$  evaluated at  $t^n$ .

**LEMMA 5.1 (Existence and uniqueness).** *There exists an unique solution  $p_h^n, \mathbf{w}_h^n, \mathbf{u}_h^n$  satisfying (5.1a–c) for all  $n \geq 0$ .*

*Proof.* The proof follows closely the proof for existence and uniqueness in the semi-discrete section. First, we note that the discrete initial conditions are the same as (3.8d) and thus compatible with the momentum equation (5.1c) at  $t^0$ .

Assume now the solution at time  $t_n, n \geq 0$  has been computed. Since the problem (5.1a–c) is linear and finite dimensional, it suffices to show uniqueness. Thus, assume the right hand side in (5.1a–c) is zero. Then, we have the situation of Lemma 3.4 with  $\sigma = 1/\Delta t$  in equations (3.11a–c). Thus,  $p^{n+1}, \mathbf{w}^{n+1}$ , and  $\mathbf{u}^{n+1}$  are well-defined.  $\square$

**THEOREM 5.2.** *Let Assumption 4.3 hold. Then, there is a constant  $C$  independent of*

$h, \mu, \lambda, \alpha, c_s$  such that for all  $m \geq 1$

$$c_s \|p_h^m - p(t^m)\|_{\Omega}^2 \leq Ch^{2k+2} \left( \mathcal{M}_h^2 + c_s \|p(t^m)\|_{H^{k+1}(\Omega)}^2 \right) + C\Delta t^2 \mathcal{M}_t^2, \quad (5.4)$$

$$\mu \|\mathbf{u}_h^m - \mathbf{u}(t^m)\|_{1,h}^2 \leq Ch^{2k+2} \mathcal{M}_h^2 + C\Delta t^2 \mathcal{M}_t^2 + \mu Ch^{2k} \|\mathbf{u}(t^m)\|_{H^{k+1}(\Omega)}^2, \quad (5.5)$$

$$\lambda \|\nabla \cdot (\mathbf{u}_h^m - \mathbf{u}(t^m))\|_{\Omega}^2 \leq Ch^{2k+2} \mathcal{M}_h^2 + C\Delta t^2 \mathcal{M}_t^2 + \lambda C_{\lambda, \mu} h^{2k+2} \|\nabla \cdot \mathbf{u}(t^m)\|_{H^{k+1}(\Omega)}^2, \quad (5.6)$$

$$\Delta t \sum_{n=0}^{m-1} \|K^{-1/2}(\mathbf{w}_h^{n+1} - \mathbf{w}(t^{n+1}))\|_{\Omega}^2 \leq Ch^{2k+2} \mathcal{M}_h^2 + C\Delta t^2 \mathcal{M}_t^2. \quad (5.7)$$

where

$$\begin{aligned} \mathcal{M}_h^2 &= \alpha^2 \|\partial_t \mathbf{u}\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + \|\mathbf{w}\|_{L^2(0,T;H^{k+1}(\Omega))}^2 \\ \mathcal{M}_t^2 &= c_s^2 \|p_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \alpha^2 \|\mathbf{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2 \end{aligned}$$

Note that the theorem holds even without Assumption (4.3), but with reduced convergence orders for  $\mathbf{w}$  and  $p$ , similarly to Theorem 4.4.

*Proof.* Error analysis follows closely the one at the continuous-in-time level. We can choose  $\tilde{\mathbf{w}}^n$  such that

$$\tilde{\mathbf{w}}^n = \pi_h \mathbf{w}(t^n), \quad n \geq 1$$

We also can choose  $\tilde{\mathbf{u}}^n$  such that

$$a_h(\tilde{\mathbf{u}}^n, \mathbf{v}) = a_h(\mathbf{u}(t^n), \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h, \quad \forall n \geq 0 \quad (5.8)$$

Using Proposition 4.2 and Assumption 4.3, we have the following a priori error bounds, for any  $n \geq 0$ :

$$\|\tilde{\mathbf{u}}^n - \mathbf{u}(t^n)\|_{1,h}^2 \leq Ch^{2k} \|\mathbf{u}(t^n)\|_{H^{k+1}(\Omega)}^2, \quad (5.9)$$

$$\|\tilde{\mathbf{u}}^n - \mathbf{u}(t^n)\|_{\Omega}^2 + \|\nabla \cdot (\tilde{\mathbf{u}}^n - \mathbf{u}(t^n))\|_{\Omega}^2 \leq C_{\mu, \lambda} h^{2k+2} \|\mathbf{u}(t^n)\|_{H^{k+1}(\Omega)}^2. \quad (5.10)$$

We can also choose  $\tilde{p}$  to be the  $L^2$  projection of  $p$  in  $Q_h$ :

$$(p(t^n) - \tilde{p}^n, q_h) = 0, \quad \forall q_h \in Q_h, \quad \forall n \geq 0. \quad (5.11)$$

We decompose the errors as follows:

$$\begin{aligned} \mathbf{w}_h^n - \mathbf{w}(t^n) &= \chi_{\mathbf{w}}^n - \eta_{\mathbf{w}}^n, & \chi_{\mathbf{w}}^n &= \mathbf{w}_h^n - \tilde{\mathbf{w}}^n, & \eta_{\mathbf{w}}^n &= \mathbf{w}(t^n) - \tilde{\mathbf{w}}^n, \\ \mathbf{u}_h^n - \mathbf{u}(t^n) &= \chi_{\mathbf{u}}^n - \eta_{\mathbf{u}}^n, & \chi_{\mathbf{u}}^n &= \mathbf{u}_h^n - \tilde{\mathbf{u}}^n, & \eta_{\mathbf{u}}^n &= \mathbf{u}(t^n) - \tilde{\mathbf{u}}^n, \\ p_h^n - p(t^n) &= \chi_p^n - \eta_p^n, & \chi_p^n &= p_h^n - \tilde{p}^n, & \eta_p^n &= p(t^n) - \tilde{p}^n. \end{aligned}$$

Using Taylor approximation, we have

$$\frac{p(t^{n+1}) - p(t^n)}{\Delta t} = \frac{\partial p}{\partial t}(t^{n+1}) + \Delta t \rho_{p,n+1},$$

and

$$\frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} = \frac{\partial \mathbf{u}}{\partial t}(t^{n+1}) + \Delta t \rho_{\mathbf{u},n+1}.$$

with

$$\|\rho_{p,n+1}\|_{\Omega} \leq C \|p_{tt}\|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))}, \quad \|\rho_{\mathbf{u},n+1}\|_{\Omega} \leq C \|\mathbf{u}_{tt}\|_{L^\infty(t_n, t_{n+1}; L^2(\Omega))} \quad (5.12)$$

Error equations become:

$$\begin{aligned} & \left( \frac{1}{\Delta t} (c_s(\chi_p^{n+1} - \chi_p^n) + \alpha \nabla \cdot (\boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n)), q \right) + (\nabla \cdot \boldsymbol{\chi}_{\mathbf{w}}^{n+1}, q) = \\ & \left( \frac{1}{\Delta t} (c_s(\eta_p^{n+1} - \eta_p^n) + \alpha \nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n)), q \right)_{\Omega} + (\nabla \cdot \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, q) \\ & \quad + \Delta t (c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}, q), \quad \forall q \in Q_h, \end{aligned} \quad (5.13)$$

$$(K^{-1} \boldsymbol{\chi}_{\mathbf{w}}^{n+1}, \mathbf{z}) - (\chi_p^{n+1}, \nabla \cdot \mathbf{z}) = (K^{-1} \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \mathbf{z}) - (\eta_p^{n+1}, \nabla \cdot \mathbf{z}), \quad \forall \mathbf{z} \in \mathbf{W}_h, \quad (5.14)$$

$$a_h(\boldsymbol{\chi}_{\mathbf{u}}^{n+1}, \mathbf{v}) - (\alpha \chi_p^{n+1}, \nabla \cdot \mathbf{v}) = a_h(\boldsymbol{\eta}_{\mathbf{u}}^{n+1}, \mathbf{v}) - (\alpha \eta_p^{n+1}, \nabla \cdot \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h. \quad (5.15)$$

We choose  $q = \chi_p^{n+1}$  in (5.13) and  $\mathbf{z} = \boldsymbol{\chi}_{\mathbf{w}}^{n+1}$  in (5.14), and add the two resulting equations:

$$\begin{aligned} & \left( \frac{1}{\Delta t} (c_s(\chi_p^{n+1} - \chi_p^n) + \alpha \nabla \cdot (\boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n)), \chi_p^{n+1} \right) + \|K^{-1/2} \boldsymbol{\chi}_{\mathbf{w}}^{n+1}\|_{\Omega}^2 \\ & = \left( \frac{1}{\Delta t} (c_s(\eta_p^{n+1} - \eta_p^n) + \alpha \nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n)), \chi_p^{n+1} \right)_{\Omega} + (\nabla \cdot \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \chi_p^{n+1}) \\ & \quad + (K^{-1} \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \boldsymbol{\chi}_{\mathbf{w}}^{n+1}) - (\eta_p^{n+1}, \nabla \cdot \boldsymbol{\chi}_{\mathbf{w}}^{n+1}) + \Delta t (c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}, \chi_p^{n+1}). \end{aligned} \quad (5.16)$$

Next we select the test function  $\mathbf{v}$  in (5.15)

$$\mathbf{v} = \frac{1}{\Delta t} (\boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n),$$

and add the resulting equation to (5.16):

$$\begin{aligned} & \left( \frac{1}{\Delta t} (c_s(\chi_p^{n+1} - \chi_p^n)), \chi_p^{n+1} \right) + \|K^{-1/2} \boldsymbol{\chi}_{\mathbf{w}}^{n+1}\|_{\Omega}^2 + \frac{1}{\Delta t} a_h(\boldsymbol{\chi}_{\mathbf{u}}^{n+1}, \boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n) \\ & = \left( \frac{1}{\Delta t} c_s(\eta_p^{n+1} - \eta_p^n), \chi_p^{n+1} \right) + \left( \frac{1}{\Delta t} \alpha \nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n), \chi_p^{n+1} \right) \\ & \quad + (\nabla \cdot \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \chi_p^{n+1}) + (K^{-1} \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \boldsymbol{\chi}_{\mathbf{w}}^{n+1}) - (\eta_p^{n+1}, \nabla \cdot \boldsymbol{\chi}_{\mathbf{w}}^{n+1}) + \Delta t (c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}, \chi_p^{n+1}) \\ & \quad + \frac{1}{\Delta t} a_h(\boldsymbol{\eta}_{\mathbf{u}}^{n+1}, \boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n) - (\alpha \eta_p^{n+1}, \nabla \cdot \frac{1}{\Delta t} (\boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n)) \\ & = T_1 + \dots + T_8. \end{aligned} \quad (5.17)$$

Because of (3.1) and the definition of the  $L^2$  projection (see (5.11)), the terms  $T_1, T_5$  and  $T_8$  vanish. Because of (3.3), the term  $T_3$  is zero. Finally, because of (5.8), the term  $T_7$  also vanishes. Therefore (5.17) simplifies to:

$$\begin{aligned} & \left( \frac{1}{\Delta t} (c_s(\chi_p^{n+1} - \chi_p^n)), \chi_p^{n+1} \right) + \|K^{-1/2} \boldsymbol{\chi}_{\mathbf{w}}^{n+1}\|_{\Omega}^2 + \frac{1}{\Delta t} a_h(\boldsymbol{\chi}_{\mathbf{u}}^{n+1}, \boldsymbol{\chi}_{\mathbf{u}}^{n+1} - \boldsymbol{\chi}_{\mathbf{u}}^n) \\ & = \frac{\alpha}{\Delta t} (\nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n), \chi_p^{n+1}) + (K^{-1} \boldsymbol{\eta}_{\mathbf{w}}^{n+1}, \boldsymbol{\chi}_{\mathbf{w}}^{n+1}) + \Delta t (c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}, \chi_p^{n+1}). \end{aligned} \quad (5.18)$$

Lemma 4.5 is valid at the discrete level:

$$\|p_h^n - \tilde{p}^n\|_\Omega \leq C\|\mathbf{w}_h^n - \mathbf{w}(t^n)\|_\Omega, \quad \forall n \geq 1.$$

This means that

$$\|\chi_p^n\|_\Omega \leq C(\|\chi_{\mathbf{w}}^n\|_\Omega + \|\boldsymbol{\eta}_{\mathbf{w}}^n\|_\Omega), \quad \forall n \geq 1.$$

Therefore this implies

$$\begin{aligned} & \frac{c_s}{2\Delta t} (\|\chi_p^{n+1}\|_\Omega^2 - \|\chi_p^n\|_\Omega^2) + \frac{1}{2} \|K^{-1/2}\chi_{\mathbf{w}}^{n+1}\|_\Omega^2 + \frac{\kappa}{2\Delta t} \mu (\|\chi_{\mathbf{u}}^{n+1}\|_{1,h}^2 - \|\chi_{\mathbf{u}}^n\|_{1,h}^2) \\ & \quad + \lambda \frac{1}{2\Delta t} (\|\nabla \cdot \chi_{\mathbf{u}}^{n+1}\|_\Omega^2 - \|\nabla \cdot \chi_{\mathbf{u}}^n\|_\Omega^2) \\ & \leq C\|\boldsymbol{\eta}_{\mathbf{w}}^{n+1}\|_\Omega^2 + C\frac{\alpha^2}{\Delta t^2} \|\nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n)\|_\Omega^2 + C\Delta t^2 \|c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}\|_\Omega^2. \end{aligned}$$

We multiply the above inequality by  $2\Delta t$ , sum from  $n = 0$  to  $n = m - 1$  and remark that  $\chi_p^0 = 0$  and  $\chi_{\mathbf{u}}^0 = \mathbf{0}$ :

$$\begin{aligned} & c_s \|\chi_p^m\|_\Omega^2 + \Delta t \sum_{n=0}^{m-1} \|K^{-1/2}\chi_{\mathbf{w}}^{n+1}\|_\Omega^2 + \kappa \mu \|\chi_{\mathbf{u}}^m\|_{1,h}^2 + \lambda \|\nabla \cdot \chi_{\mathbf{u}}^m\|_\Omega^2 \\ & \leq C\alpha^2 \Delta t \sum_{n=0}^{m-1} \left\| \frac{1}{\Delta t} \nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n) \right\|_\Omega^2 + C\Delta t \sum_{n=0}^{m-1} \|\boldsymbol{\eta}_{\mathbf{w}}^{n+1}\|_\Omega^2 \\ & \quad + \Delta t^3 \sum_{n=0}^{m-1} \|c_s \rho_{p,n+1} + \alpha \rho_{\mathbf{u},n+1}\|_\Omega^2. \end{aligned} \quad (5.19)$$

From the approximation bounds (5.10), we have

$$\begin{aligned} \Delta t \sum_{n=0}^{m-1} \left\| \frac{1}{\Delta t} \nabla \cdot (\boldsymbol{\eta}_{\mathbf{u}}^{n+1} - \boldsymbol{\eta}_{\mathbf{u}}^n) \right\|_\Omega^2 & \leq Ch^{2k+2} \Delta t \sum_{n=0}^{m-1} \left\| \frac{\mathbf{u}(t^{n+1}) - \mathbf{u}(t^n)}{\Delta t} \right\|_{H^{k+1}(\Omega)}^2 \\ & \leq Ch^{2k+2} \Delta t \sum_{n=0}^{m-1} \left\| \frac{\partial \mathbf{u}}{\partial t}(t^{*,n}) \right\|_{H^{k+1}(\Omega)}^2 \leq Ch^{2k+2} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^2(0,T;H^{k+1}(\Omega))}^2. \end{aligned}$$

Similarly we obtain

$$\Delta t \sum_{n=0}^{m-1} \|\boldsymbol{\eta}_{\mathbf{w}}^{n+1}\|_\Omega^2 \leq Ch^{2k+2} \Delta t \sum_{n=0}^{m-1} \|\mathbf{w}(t^{n+1})\|_{H^{k+1}(\Omega)}^2 \leq Ch^{2k+2} \|\mathbf{w}\|_{L^2(0,T;H^{k+1}(\Omega))}^2.$$

Finally using (5.12), we obtain:

$$\begin{aligned} & c_s \|\chi_p^m\|_\Omega^2 + \Delta t \sum_{n=0}^{m-1} \|K^{-1/2}\chi_{\mathbf{w}}^{n+1}\|_\Omega^2 + \kappa \mu \|\chi_{\mathbf{u}}^m\|_{1,h}^2 + \lambda \|\nabla \cdot \chi_{\mathbf{u}}^m\|_\Omega^2 \\ & \leq C\alpha^2 h^{2k+2} \left\| \frac{\partial \mathbf{u}}{\partial t} \right\|_{L^2(0,T;H^{k+1}(\Omega))}^2 + Ch^{2k+2} \|\mathbf{w}\|_{L^2(0,T;H^{k+1}(\Omega))}^2 \\ & \quad + C\Delta t^2 (c_s^2 \|p_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2 + \alpha^2 \|\mathbf{u}_{tt}\|_{L^\infty(0,T;L^2(\Omega))}^2). \end{aligned} \quad (5.20)$$

The final results are obtained by triangle inequalities and approximation bounds.  $\square$

**REMARK 5.3.** Let  $\tilde{f}_1$  denote the  $L^2$  projection of  $f_1$  onto  $Q_h$ . The discrete pressure and displacement satisfy the conservation property, pointwisely:

$$\frac{1}{\Delta t} (c_s p_h^{n+1} + \alpha \nabla \cdot \mathbf{u}_h^{n+1}) - \frac{1}{\Delta t} (c_s p_h^n + \alpha \nabla \cdot \mathbf{u}_h^n) = \tilde{f}_1^{n+1}, \quad \forall x \in T, \quad \forall T \in \mathbb{T}_h.$$

**6. Numerical experiments.** For our numerical experiments, we follow the approach in [4] to construct an exact solution to equations (2.1)–(2.3). Differing from their results, we construct smooth solutions in order to verify the expected convergence orders. To this end, we let  $\mu = 1$  and  $K = \mathbf{I}$ , which corresponds to the nondimensionalization in [4] and does not restrict generality of our results. Furthermore, we consider only incompressible fluids, that is,  $c_s = 0$ . Further, we take  $\alpha = 1$ . We choose  $\Omega = (0, 1)^2$  with boundary conditions

$$\left. \begin{aligned} \partial_n(\mathbf{u} \cdot \mathbf{n}) &= 0 \\ \mathbf{u} \times \mathbf{n} &= 0 \\ p &= 0 \end{aligned} \right\} \text{ on } \partial\Omega. \quad (6.1)$$

Thus, the deformation can only be in normal direction on each boundary, and the pressure is prescribed. The seepage velocity at the boundary is free. Let  $\phi(x, y) = \sin(2\pi x)\sin(2\pi y)$  and choose as right hand side in (2.1)

$$f_1(x, y, t) = \phi(x, y) \sin(2\pi t). \quad (6.2)$$

With the auxiliary function

$$\psi(t) = \frac{1}{64\pi^4 + 4\pi^2} (8\pi^2 \sin(2\pi t) - 2\pi \cos(2\pi t) + 2\pi e^{-8\pi^2 t}), \quad (6.3)$$

we obtain the solutions

$$\begin{aligned} p(x, y, t) &= \psi(t)\phi(x, y), \\ \mathbf{w}(x, y, t) &= \psi(t)\nabla\phi(x, y), \\ \mathbf{u}(x, y, t) &= \frac{\psi(t)}{8\pi^2}\nabla\phi(x, y). \end{aligned} \quad (6.4)$$

We discretize  $\Omega$  by a sequence of Cartesian meshes, such that  $\mathbb{T}_0$  is the mesh consisting of the single square  $\Omega$ . The mesh  $\mathbb{T}_\ell$  is defined recursively by dividing every square of  $\mathbb{T}_{\ell-1}$  into four congruent squares. Thus,  $\mathbb{T}_\ell$  consists of  $4^\ell$  mesh cells with sides of length  $2^{-\ell}$ . Figure 6.1 shows the solution at time  $t = 0.5$  with considerably enlarged deformations and seepage velocity arrows.

In Figure 6.2, we display different norms of the errors of  $\mathbf{u}$ ,  $\mathbf{w}$ , and  $p$ , respectively. Note that all  $L^2$ -errors as well as the quadratic errors of the divergences are of second order, while the errors of the gradients are first order, confirming our theoretical results and the assumption on the divergence error, respectively. In Figure 6.3, we show the same results for elements of one polynomial order higher. The results exhibit again the expected convergence orders. Details of the discretization and the time steps chosen can be found in Table 6.1. Due to the low accuracy of the Euler scheme analyzed above, computations were performed with the  $\theta$ -scheme, which reads for a general spatial operator  $F$ :

$$u_{n+1} + \theta \Delta t F(u_{n+1}) = u_n - (1 - \theta) \Delta t F(u_n).$$

A value of  $\theta = 0.5$  yields the second-order Crank-Nicolson method. We chose  $\theta = 0.501$  such that the scheme is strongly A-stable. While it is only first order, its error constant is much smaller than for the backward Euler scheme. In any case, we chose time steps sufficiently small such that further reduction did not improve significant digits of the error.

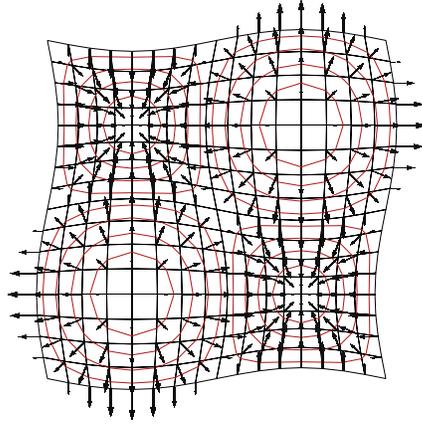


FIG. 6.1. The seepage velocity  $\mathbf{w}$  (arrows) and the pressure  $p$  (isolines) on the mesh deformed by  $\mathbf{u}$  (arrows and deformations not in scale)

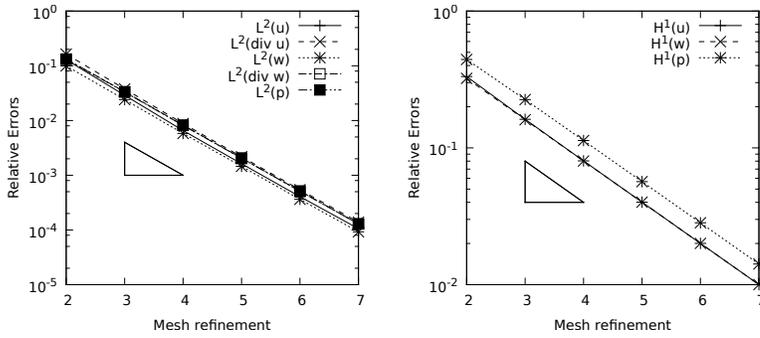


FIG. 6.2. Relative errors for  $RT_1/Q_1$  elements. The triangle on the left indicates second order convergence, the one on the right first order.

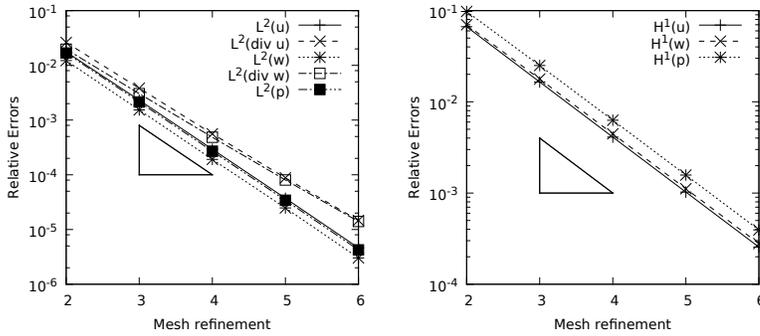


FIG. 6.3. Relative errors for  $RT_2/Q_2$  elements. The triangle on the left indicates third order convergence, the one on the right second order.

			$RT_1$		$RT_2$	
$\ell$	$h$	cells	$\Delta t$	dofs	$\Delta t$	dofs
2	$1/4$	16	0.08	352	0.02	768
3	$1/8$	64	0.04	1344	0.006	2976
4	$1/16$	256	0.02	5248	0.002	11712
5	$1/32$	1024	0.01	20736	0.0007	46464
6	$1/64$	4096	0.005	82432	0.0003	185088
7	$1/128$	16384	0.002	328704	0.0001	738816

TABLE 6.1

Additional data on the discretization.

$c_s$	$\alpha$	$\lambda$	$\ \Delta m(0.5)\ _{L^2(\Omega)}$
0	1	1	8.55e-17
0	0.9	1	7.36e-17
0.1	0.9	1	7.66e-17
0.1	0.9	1000	3.19e-14

TABLE 6.2

Verification of mass balance for various parameters.  $h = 1/8$ ,  $\Delta t = 1/10$ . Right hand side  $f_1$  as in equation (6.2).

Finally, we verify the mass conservation of the method. Given  $u_h(0) = 0$ , exact mass conservation implies that at time  $t > 0$  there holds

$$\Delta m(t) := c_s p(t) + \alpha \nabla \cdot \mathbf{u}(t) - \int_0^t [\nabla \cdot \mathbf{w}(s) - \tilde{f}_1(s)] ds = 0,$$

for the continuous in time scheme. Here,  $\tilde{f}_1$  is the  $L^2$ -projection of  $f_1$  into the discrete pressure space  $Q_h$ . Discretely in time, this identity still holds, if we replace the integral by the quadrature rule consistent with the timestepping scheme. In particular, the equality holds independent of approximation quality, such that we test it on very coarse meshes and with coarse time steps. In Table 6.2, we show results, where we vary the parameters of the equation. In particular,  $c_s \neq 0$  allows for compressible fluids and  $\alpha \neq 1$  for some slack in the mass balance between solid and fluid. Nevertheless, all norms are within machine accuracy, confirming our claim.

**7. Conclusions.** We presented a discretization scheme for Biot's consolidation model which provides pointwise mass balance. It is based on a superapproximation assumption on the divergence of the Hdiv-DG discretization of the elasticity subproblem. The approximations of displacement and seepage velocity, respectively, are of equal order.

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