# Acceleration of weak Galerkin methods for the Laplacian eigenvalue problem 

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#### Abstract

Recently, we proposed a weak Galerkin finite element method for the Laplace eigenvalue problem. In this paper, we present two-grid and two-space skills to accelerate the weak Galerkin method. By choosing parameters properly, the two-grid and two-space weak Galerkin method not only doubles the convergence rate, but also maintains the asymptotic lower bounds property of the weak Galerkin method. Some numerical examples are provided to validate our theoretical analysis.


Keywords. weak Galerkin finite element method, eigenvalue problem, two-grid method, two-space method, lower bound.

AMS Subject Classification: Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45, 35J50, 35J35

## 1 Introduction

The eigenvalue problem arises from many branches of mathematics and physics, including quantum mechanics, fluid mechanics, stochastic process and structural mechanics. A variety of applications of eigenvalue problem, especially the Laplacian eigenvalue problem, are surveyed by a recent SIAM review paper [11]. Many numerical methods have been developed for the Laplacian eigenvalue problem, such as finite difference methods [30] and finite element methods [2, 3, 6].

The finite element method is one of the efficient approaches for the Laplacian eigenvalue problem for its simplicity and adaptivity on triangular meshes. Due to the minimum-maximum principle, the conforming finite element method always gives the upper bounds for the Laplacian eigenvalues. In order to get accurate intervals for eigenvalues, it is necessary to have lower bounds of eigenvalues. There are mainly two ways, the post-processing method [5, 15, 17, 18, 17, 26, 27] and the nonconforming finite element method [1, 13, 19, 39]. Some specific nonconforming finite element methods provide asymptotic lower bounds for eigenvalues without solving an auxiliary problem, while it seems difficult to construct a high order nonconforming element.

[^0]Recently, a new method for solving the partial differential equations, named the weak Galerkin (WG) method, has been developed. The WG method was first introduced in [29] for the second order elliptic equation, and was soon applied to many types of partial differential equations, such as the parabolic equation [16], the biharmonic equation [22, 23, 38, the Brinkman equation [21], and the Maxwell equation [24]. In [32], the Laplacian eigenvalue problem was investigated by the WG method. An astonishing feature is: it offers asymptotic lower bounds for the Laplacian eigenvalues on polygonal meshes by employing high order polynomial elements. Comparing with the boundary value problems, the eigenvalue problem is more difficult to solve since it is actually a special nonlinear equation. Solving eigenvalue problems need more computational work and memory than solving corresponding boundary value problems. So how to accelerate the solving speed is a necessary and important topic in computational mathematics.

Two-grid and two-space methods are both efficient numerical methods for nonlinear problems. The main idea is to approximate a large nonlinear system by solving a small nonlinear system and a large linear system, and thereby to reduce the computational cost. The two-grid method was first introduced in [33] to solve a semilinear second order elliptic problem. Soon it was adopted for different kinds of PDEs 20, 34, 36. The eigenvalue problem can also be viewed as a nonlinear problem, the corresponding two-grid method was studied in 35, and some variations have been developed later, such as the shiftedinverse power method [14, 37, some applications have also been developed for Stokes [7, 10, 31] and Maxwell eigenvalue problems [40], the second order elliptic eigenvalue problems by the mixed finite element methods [8, Bose-Einstein problems [12]. The two-space method is proposed for the biharmoinc eigenvalue problem by the nonconforming finite element methods. Then it is adopted for the Laplacian eigenvalue problems by the conforming finite element methods [25] and Stokes eigenvalue problems [7].

In this paper, we apply the two-grid [35] and two-space methods to accelerate the WG method for the Laplacian eigenvalue problems. In this way, the computing complexity of the WG method can be reduced greatly. Another important nice feature is: by choosing the mesh sizes properly, the two-grid WG method can still provide lower bounds for the Laplacian eigenvalues. Rigorous theoretical analysis will be given for the proposed method, and numerical examples will be provided as well.

An outline of the paper goes as follows. In Section 2, we introduce the WG method for the eigenvalue problem and the corresponding basic error estimates. In Section 3, we give the $H^{-1}$ error estimate for the WG method, which plays an important role in the analysis. The two-grid method will be introduced and analyzed in Section 4. Section 5 is devoted to the two-space method. In Section 6, some numerical examples are presented to validate our theoretical analysis. Some concluding remarks are given in the final section.

## 2 A standard discretization of weak Galerkin scheme

In this section, we state some notation in this paper, introduce the standard WG scheme for Laplacian eigenvalue problem briefly and present some results from 32. Throughout this paper, we always use $C$ to represent a constant independent of mesh sizes $H$ and $h$, which may have different values according to the occurrence. The symbol $a \lesssim b$ stands for $a \leq C b$ for some constant $C$.

In this paper, for simplicity, we consider the following Laplacian eigenvalue problem: Find $(\lambda, u)$ such that

$$
\left\{\begin{array}{rlrl}
-\Delta u & =\lambda u, & & \text { in } \Omega  \tag{2.1}\\
u & =0, & \text { on } \partial \Omega \\
\int_{\Omega} u^{2} & =1, &
\end{array}\right.
$$

where $\Omega$ is a polygon region in $\mathbb{R}^{d}(d=2,3)$.
The standard Sobolev space notation are also used in this paper. Let $D$ be any open bounded domain with Lipschitz continuous boundary in $\mathbb{R}^{d}(d=2,3)$. We use the standard definition for the Sobolev space $H^{s}(D)$ and their associated inner products $(\cdot, \cdot)_{s, D}$, norms $\|\cdot\|_{s, D}$, and seminorms $|\cdot|_{s, D}$ for any
$s \geq 0$. For example, for any integer $s \geq 0$, the seminorm $|\cdot|_{s, D}$ is given by

$$
|v|_{s, D}=\left(\sum_{|\alpha|=s} \int_{D}\left|\partial^{\alpha} v\right|^{2} \mathrm{~d} D\right)^{\frac{1}{2}}
$$

with the usual multi-index notation

$$
\alpha=\left(\alpha_{1}, \cdots, \alpha_{d}\right), \quad|\alpha|=\alpha_{1}+\cdots+\alpha_{d}, \quad \partial^{\alpha}=\prod_{j=1}^{d} \partial_{x_{j}}^{\alpha_{j}}
$$

The Sobolev norm $\|\cdot\|_{m, D}$ is given by

$$
\|v\|_{m, D}=\left(\sum_{j=0}^{m}|v|_{j, D}^{2}\right)^{\frac{1}{2}}
$$

The space $H^{0}(D)$ coincides with $L^{2}(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_{D}$ and $(\cdot, \cdot)_{D}$, respectively. When $D=\Omega$, we shall drop the subscript $D$ in the norm and in the inner product notation.

Let $\mathcal{T}_{h}$ be a partition of the domain $\Omega$, and the elements in $\mathcal{T}_{h}$ are polygons satisfying the regular assumptions specified in [29]. Denote by $\mathcal{E}_{h}$ the edges in $\mathcal{T}_{h}$, and by $\mathcal{E}_{h}^{0}$ the interior edges $\mathcal{E}_{h} \backslash \partial \Omega$. For each element $T \in \mathcal{T}_{h}, h_{T}$ represents the diameter of $T$, and $h=\max _{T \in \mathcal{T}_{h}} h_{T}$ denotes the mesh size.

Now we introduce a WG scheme for the eigenvalue problem 2.1). For a given integer $k \geq 1$, define the WG finite element space

$$
V_{h}=\left\{v=\left(v_{0}, v_{b}\right):\left.v_{0}\right|_{T} \in P_{k}(T),\left.v_{b}\right|_{e} \in P_{k-1}(e), \forall T \in \mathcal{T}_{h}, e \in \mathcal{E}_{h}, \text { and } v_{b}=0 \text { on } \partial \Omega\right\} .
$$

For each weak function $v \in V_{h}$, we can define its weak gradient $\nabla_{w} v$ by distribution element-wisely as follows.

Definition 2.1. [28] For each $v \in V_{h},\left.\nabla_{w} v\right|_{T}$ is the unique polynomial in $\left[P_{k-1}(T)\right]^{d}$ satisfying

$$
\begin{equation*}
\left(\nabla_{w} v, \mathbf{q}\right)_{T}=-\left(v_{0}, \nabla \cdot \mathbf{q}\right)_{T}+\left\langle v_{b}, \mathbf{q} \cdot \mathbf{n}\right\rangle_{\partial T}, \quad \forall \mathbf{q} \in\left[P_{k-1}(T)\right]^{d} \tag{2.2}
\end{equation*}
$$

where $\mathbf{n}$ denotes the outward unit normal vector.
For the aim of analysis, some projection operators are also employed in this paper. Let $Q_{0}$ denote the $L^{2}$ projection from $L^{2}(T)$ onto $P_{k}(T), Q_{b}$ denote the $L^{2}$ projection from $L^{2}(e)$ onto $P_{k-1}(e)$, and $\mathbb{Q}_{h}$ denote the $L^{2}$ projection from $\left[L^{2}(T)\right]^{d}$ onto $\left[P_{k-1}(T)\right]^{d}$. Combining $Q_{0}$ and $Q_{b}$ together, we can define $Q_{h}=\left\{Q_{0}, Q_{b}\right\}$, which is a projection from $H_{0}^{1}(\Omega)$ onto $V_{h}$.

Now we define three bilinear forms on $V_{h}$ for any $v, w \in V_{h}$,

$$
\begin{aligned}
s(v, w) & =\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1+\varepsilon}\left\langle Q_{b} v_{0}-v_{b}, Q_{b} w_{0}-w_{b}\right\rangle_{\partial T} \\
a_{s}(v, w) & =\left(\nabla_{w} v, \nabla_{w} w\right)+s(v, w) \\
b_{w}(v, w) & =\left(v_{0}, w_{0}\right)
\end{aligned}
$$

where $0 \leq \varepsilon<1$ is a constant [32]. Define the following norm on $V_{h}$ that

$$
\|v\|^{2}=a_{s}(v, v), \quad \forall v \in V_{h} .
$$

For the simplicity of notation, we introduce a semi-norm $\|\cdot\|_{b}$ by

$$
\|v\|_{b}^{2}:=b_{w}(v, v), \quad \forall v \in V_{h} .
$$

With these preparations we can give the following WG algorithm.

Weak Galerkin Algorithm 1. [32] Find $\left(\lambda_{h}, u_{h}\right) \in \mathbb{R} \times V_{h}$ such that $\left\|u_{h}\right\|_{b}=1$ and

$$
\begin{equation*}
a_{s}\left(u_{h}, v_{h}\right)=\lambda_{h} b_{w}\left(u_{h}, v_{h}\right), \quad \forall v_{h} \in V_{h} \tag{2.3}
\end{equation*}
$$

Denote $V_{0}=H_{0}^{1}(\Omega)$, and define the sum space $V=V_{0}+V_{h}$. Now we introduce the following semi-norm on $V$ that

$$
\|w\|_{V}^{2}=\sum_{T \in \mathcal{T}_{h}}\left(\left\|\nabla w_{0}\right\|_{T}^{2}+h_{T}^{-1}\left\|Q_{b} w_{0}-w_{b}\right\|_{\partial T}^{2}\right)
$$

$\|\cdot\|_{V}$ indeed defines a norm on $V[32]$. For the analysis in this paper, we still need to introduce the dual norm of $\|\cdot\|_{V}$ as follows

$$
\left\|v_{h}\right\|_{-V}=\sup _{w \in V,\|w\|_{V} \neq 0} \frac{b_{w}\left(v_{h}, w\right)}{\|w\|_{V}}
$$

For the standard WG scheme, the following convergence theorem holds true, and which also gives a lower bound estimate.

Theorem 2.1. [32] Suppose $\lambda_{j, h}$ is the $j$-th eigenvalue of 2.3) and $u_{j, h}$ is the corresponding eigenfunction. There exists an exact eigenfunction $u_{j}$ corresponding to the $j$-th exact eigenvalue $\lambda_{j}$ such that the following error estimates hold

$$
\begin{align*}
& C h^{2 k}\left\|u_{j}\right\|_{k+1} \leq \lambda_{j}-\lambda_{j, h} \leq C h^{2 k-2 \varepsilon}\left\|u_{j}\right\|_{k+1}  \tag{2.4}\\
& \left\|u_{j}-u_{j, h}\right\|_{V} \leq C h^{k-\varepsilon}\left\|u_{j}\right\|_{k+1}  \tag{2.5}\\
& \left\|u_{j}-u_{j, h}\right\|_{b} \leq C h^{k+1-\varepsilon}\left\|u_{j}\right\|_{k+1} \tag{2.6}
\end{align*}
$$

when $u_{j} \in H^{k+1}(\Omega)$ and $h$ is small enough.

## 3 Error estimate in negative norm

In this section, we shall analysis the $\|\cdot\|_{-V}$ error estimate for the WG scheme 2.3). First, we need to establish the $\|\cdot\|_{-V}$ error estimate for the corresponding boundary value problem. Consider the Poisson equation

$$
\left\{\begin{align*}
-\Delta u=f, & \text { in } \Omega  \tag{3.1}\\
u=0, & \text { on } \partial \Omega
\end{align*}\right.
$$

where $\Omega$ is a polygon or polyhedra in $\mathbb{R}^{d}(d=2,3)$.
The WG method is adopted to solve equation 3.1. For analysis, we define the following norm

$$
\|v\|_{-1}=\sup _{w \in V_{h}, w \neq 0} \frac{b_{w}(v, w)}{\|w\|}
$$

It is easy to check that $\|\cdot\|_{V}$ is equivalent to $\|\cdot\|_{1}$ on the space $H_{0}^{1}(\Omega)$. The relationship between $\|\cdot\|_{V}$ and $\|\cdot\| \|$ has been discussed in [32], which is presented as follows.

Lemma 3.1. [32] There exist two constants $C_{1}$ and $C_{2}$ such that the following inequalities hold for any $w \in V_{h}$

$$
\begin{equation*}
C_{1}\|w\| \leq\|w\|_{V} \leq C_{2} h^{-\frac{\varepsilon}{2}}\|w\| \tag{3.2}
\end{equation*}
$$

The WG method for the boundary value problem (3.1) can be described as follows:

Weak Galerkin Algorithm 2. Find $u_{h} \in V_{h}$ such that

$$
\begin{equation*}
a_{s}\left(u_{h}, v\right)=b_{w}(f, v), \quad \forall v \in V_{h} \tag{3.3}
\end{equation*}
$$

Suppose $u$ is the exact solution for (3.1) and $u_{h}$ is the corresponding numerical solution of (3.3). Denote by $e_{h}$ the error that

$$
e_{h}=Q_{h} u-u_{h}=\left\{Q_{0} u-u_{0}, Q_{b} u-u_{b}\right\}
$$

Then $e_{h}$ satisfies the following equation.
Lemma 3.2. [32] Let $e_{h}$ be the error of the weak Galerkin scheme (3.3). Then we have

$$
\begin{equation*}
a_{s}\left(e_{h}, v\right)=\ell(u, v), \quad \forall v \in V_{h} \tag{3.4}
\end{equation*}
$$

where

$$
\ell(u, v)=\sum_{T \in \mathcal{T}_{h}}\left\langle\left(\nabla u-\mathbb{Q}_{h} \nabla u\right) \cdot \mathbf{n}, v_{0}-v_{b}\right\rangle_{\partial T}+s\left(Q_{h} u, v\right) .
$$

Moreover, we have

$$
\begin{equation*}
a_{s}\left(Q_{h} u, v\right)=\ell(u, v)+b_{w}(f, v), \quad \forall v \in V_{h} \tag{3.5}
\end{equation*}
$$

Theorem 3.1. [32] Assume the exact solution $u$ of (3.1) satisfies $u \in H^{k+1}(\Omega)$ and $u_{h}$ is the numerical solution of the $W G$ scheme (3.3). Then the following error estimate holds true,

$$
\begin{equation*}
\left\|Q_{h} u-u_{h}\right\| \leq C h^{k-\frac{\varepsilon}{2}}\|u\|_{k+1} \tag{3.6}
\end{equation*}
$$

Now, we come to estimate the error $e_{h}$ in the norm $\|\cdot\|_{-1}$. We suppose the partition $\mathcal{T}_{h}$ is a triangulation, instead of an arbitrary polytopal mesh. The idea is to introduce a continuous interpolation for $v_{h} \in V_{h}$. To this end, we define $\mathcal{N}_{T}$ as the vertices of the element $T \in \mathcal{T}_{h}$. Here, the notation $V_{h}^{C}$ is used to denote the conforming linear finite element space [4, 9. We need to define an interpolation operator $\Pi_{h}: V_{h} \rightarrow V_{h}^{C} \subset V_{0}$ as follows. For each node $A$ in $\mathcal{T}_{h}$, let

$$
K(A):=\bigcup_{A \in \mathcal{N}_{T}} T
$$

and $N_{A}$ is the number of elements in $K(A)$. Then, for any $v_{h} \in V_{h}$, the value of $\Pi_{h} v_{h}$ at the node $A$ is defined by

$$
\left(\Pi_{h} v_{h}\right)(A)=\left.\frac{1}{N_{A}} \sum_{T \in K(A)} v_{0}\right|_{T}(A)
$$

Then the function $\Pi_{h} v_{h} \in V_{h}^{C}$ is determined by its nodal values and the basis for the space $V_{h}^{C}$.
Lemma 3.3. For any $v_{h} \in V_{h}$, we have the following estimate

$$
\begin{equation*}
\left|\Pi_{h} v_{h}\right|_{1} \lesssim\left\|v_{h}\right\|_{V} \tag{3.7}
\end{equation*}
$$

Proof. For any $T \in \mathcal{T}_{h}$, define

$$
K(T):=\bigcup_{\mathcal{N}_{T} \cap \mathcal{N}_{T}^{\prime} \neq \varnothing} T^{\prime} .
$$

We only need to prove that

$$
\begin{equation*}
\left|\Pi_{h} v_{h}\right|_{1, T} \lesssim\left\|v_{h}\right\|_{V, K(T)} \tag{3.8}
\end{equation*}
$$

since summing (3.8) over $T \in \mathcal{T}_{h}$ can lead to the desired result (3.7).
Define $\widehat{T}$ the reference element and $F_{T}: T \rightarrow \widehat{T}$ the affine isomorphism. Denote $K(\widehat{T})=F_{T}(K(T))$. It follows from the regularity assumption of the mesh that $K(\widehat{T})$ is also of unit size. Then we define the following Banach spaces

$$
\widehat{V}_{h}=\left\{\widehat{v}_{h}: \widehat{v}_{h}=v_{h}\left(F_{T}^{-1}(\widehat{x})\right)=v_{h} \circ F_{T}^{-1},\left.\quad \forall v_{h} \in V_{h}\right|_{K(T)}\right\}
$$

and $M=\left\{\widehat{v}_{h}: \widehat{v}_{h} \in \widehat{V}_{h}\right.$ and $\left.\int_{\widehat{T}} \widehat{v}_{h} d \widehat{T}=0\right\}$. Obviously the complement of $M$ in $\widehat{V}_{h}$ with the $L^{2}$ inner product is $M^{\perp}=\left\{\widehat{v}_{h}: \widehat{v}_{h} \in \widehat{V}_{h}\right.$ and $\widehat{v}_{h}$ is a constant on $\left.K(\widehat{T})\right\}$. We also define the interpolation operator $\widehat{\Pi}_{T}:=\Pi_{h} \circ F_{T}^{-1}$ on $K(\widehat{T})$ corresponding the operator $\Pi_{h}$ on $K(T)$. Notice that $\left|\widehat{\Pi}_{T} \widehat{v}_{h}\right|_{1, \widehat{T}}$ defines a seminorm on $M$ and $\left\|\widehat{v}_{h}\right\|_{V, K(\widehat{T})}$ defines a norm on $M$. From the equivalence of norms on finite dimensional Banach spaces, we obtain

$$
\left|\widehat{\Pi}_{T} \widehat{v}_{h}\right|_{1, \widehat{T}} \lesssim\left\|\widehat{v}_{h}\right\|_{V, K(\widehat{T})}, \quad \forall \widehat{v}_{h} \in M
$$

Furthermore, since $\left|\widehat{\Pi}_{T} \widehat{v}_{h}\right|_{1, \widehat{T}}=\left\|\widehat{v}_{h}\right\|_{V, K(\widehat{T})}=0$ for all $\widehat{v}_{h} \in M^{\perp}$, we have

$$
\left|\widehat{\Pi}_{T} \widehat{v}_{h}\right|_{1, \widehat{T}} \lesssim\left\|\widehat{v}_{h}\right\|_{V, K(\widehat{T})}, \quad \forall \widehat{v}_{h} \in \widehat{V}_{h} .
$$

From the property of affine isomorphism, the following inequalities hold

$$
\left|\Pi_{h} v_{h}\right|_{1, T} \lesssim h_{T}^{\frac{d}{2}-1}\left|\widehat{\Pi}_{T} \widehat{v}_{h}\right|_{1, \widehat{T}} \lesssim h_{T}^{\frac{d}{2}-1}\left\|\widehat{v}_{h}\right\|_{V, K(\widehat{T})} \lesssim\left\|v_{h}\right\|_{V, K(T)}
$$

Then the proof is completed.
Lemma 3.4. For any $v_{h} \in V_{h}$, we have the following estimate

$$
\left\|v_{h}-\Pi_{h} v_{h}\right\|_{b} \lesssim h\left\|v_{h}\right\|_{V}
$$

Proof. Similarly to the proof of Lemma 3.3 , we only need to prove that

$$
\begin{equation*}
\left\|v_{0}-\Pi_{h} v_{h}\right\|_{T} \lesssim h\left\|v_{h}\right\|_{V, K(T)}, \quad \forall T \in \mathcal{T}_{h} \tag{3.9}
\end{equation*}
$$

First, on the element $T$, we have the following estimates

$$
\begin{align*}
\left\|\Pi_{1, T} v_{0}-\Pi_{h} v_{h}\right\|_{T}^{2} & =\int_{T} \sum_{A_{i} \in \mathcal{N}_{T}}\left|v_{0}\left(A_{i}\right)-\Pi_{h} v_{h}\left(A_{i}\right)\right|^{2} \varphi_{i}^{2} d T \\
& \lesssim h^{d} \sum_{A_{i} \in \mathcal{N}_{T}}\left|v_{0}\left(A_{i}\right)-\Pi_{h} v_{h}\left(A_{i}\right)\right|^{2} \tag{3.10}
\end{align*}
$$

where $\varphi_{i}$ is the linear Lagrange basis function corresponding to $A_{i}$ and $\Pi_{1, T}$ is the linear Lagrange interpolation for the finite element space $V_{h}^{C}$ on the element $T$.

For each node $A_{i}$, denote $\left\{T_{1}, T_{2}, \cdots, T_{N_{i}}\right\}$ the elements in $K\left(A_{i}\right)$ in counter-clock order. From the definition of $\Pi_{h} v_{h}$ we can obtain

$$
\begin{align*}
\left|v_{0}\left(A_{i}\right)-\Pi_{h} v\left(A_{i}\right)\right| & \left.\leq \frac{1}{N_{i}} \sum_{j=1}^{N_{i}}\left|v_{0}\right|_{T}\left(A_{i}\right)-\left.v_{0}\right|_{T_{j}}\left(A_{i}\right) \right\rvert\, \\
& \left.\leq \frac{1}{N_{i}} \sum_{j=1}^{N_{i}} \sum_{k=1}^{j}\left|v_{0}\right|_{T_{k}}\left(A_{i}\right)-\left.v_{0}\right|_{T_{k-1}}\left(A_{i}\right) \right\rvert\, \tag{3.11}
\end{align*}
$$

From the $L^{\infty}-L^{2}$ inverse inequality, it follows that

$$
\begin{equation*}
\left|v_{0}\right|_{T_{k}}\left(A_{i}\right)-\left.v_{0}\right|_{T_{k-1}}\left(A_{i}\right) \left\lvert\, \lesssim h^{-\frac{d-1}{2}}\left\|\llbracket v_{0} \rrbracket\right\|_{e}\right. \tag{3.12}
\end{equation*}
$$

where $e$ is the edge between $T_{k}$ and $T_{k-1}$.
Combining (3.10-(3.12) and the definition of the norm $\|\cdot\|_{V}$ leads to the following estimates

$$
\left\|\Pi_{1, T} v_{0}-\Pi_{h} v_{h}\right\|_{T}^{2} \lesssim h \sum_{e \in K(T)}\left\|\llbracket v_{0} \rrbracket\right\|_{e}^{2} \lesssim h \sum_{T \in K(T)}\left\|v_{0}-v_{b}\right\|_{\partial T}^{2} \lesssim h^{2}\left\|v_{h}\right\|_{V, K(T)}^{2}
$$

Together with $\left\|v_{0}-\Pi_{1, T} v_{0}\right\|_{T}^{2} \lesssim h^{2}\left\|\nabla v_{0}\right\|_{T}^{2}$, we can obtain the desired result 3.9$)$ easily and the proof is completed.

Lemma 3.5. For any $\varphi \in V_{h}$, there exists $\widetilde{\varphi} \in V_{0}$ such that

$$
\begin{equation*}
\|\widetilde{\varphi}\|_{1} \lesssim\|\varphi\|_{V} \quad \text { and } \quad\|\widetilde{\varphi}-\varphi\|_{b} \lesssim h\|\widetilde{\varphi}\|_{1} \tag{3.13}
\end{equation*}
$$

The proof can be given easily by combining Lemmas 3.3. Lemma 3.4, and taking $\widetilde{\varphi}=\Pi_{h} \varphi$ which is a function in $V_{0}$.

In order to deduce the error estimate in $\|\cdot\|_{-V}$, we define the following dual problem

$$
\left\{\begin{align*}
-\Delta \psi=\widetilde{\varphi}, & \text { in } \Omega  \tag{3.14}\\
\psi=0, & \text { on } \partial \Omega
\end{align*}\right.
$$

where $\widetilde{\varphi} \in V_{0}$.
Theorem 3.2. Assume $u \in H^{k+1}(\Omega)$ is the exact solution of (3.1) and $u_{h}$ is the numerical solution of the $W G$ scheme (3.3). If the solution $\psi$ of the dual problem (3.14) has $H^{3}(\Omega)$-regularity and $k \geq 2$, the following estimate holds true

$$
\begin{equation*}
\left\|Q_{h} u-u_{h}\right\|_{-1} \leq C h^{k+2-\varepsilon}\|u\|_{k+1} \tag{3.15}
\end{equation*}
$$

Proof. Denote $e_{h}=Q_{h} u-u_{h}$. We choose $\phi \in V_{h}$ and $\widetilde{\phi} \in V_{0}$ such that $\|\phi\|=1,\left\|e_{h}\right\|=b_{w}\left(e_{h}, \phi\right)$, and $\widetilde{\phi}$ satisfies the estimates in 3.13. From Lemma 3.2, we have

$$
\begin{equation*}
a_{s}\left(Q_{h} \psi, v\right)=\ell(\psi, v)+b_{w}(\varphi, v), \quad \forall v \in V_{h} . \tag{3.16}
\end{equation*}
$$

Taking $v=Q_{h} \psi$ in (3.4) and $v=e_{h}$ in (3.16), and subtracting (3.4) from 3.16, we have

$$
\left(e_{0}, \widetilde{\varphi}\right)=\ell\left(u, Q_{h} \psi\right)-\ell\left(\psi, e_{h}\right)
$$

Since $\psi \in H^{3}(\Omega), k>2$, and $u \in H^{k+1}(\Omega)$, the following estimates hold

$$
\begin{align*}
\ell\left(\psi, e_{h}\right)= & \sum_{T \in \mathcal{T}_{h}}\left\langle\left(\nabla \psi-\mathbb{Q}_{h} \nabla \psi\right) \cdot \mathbf{n}, e_{0}-e_{b}\right\rangle_{\partial T} \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1+\varepsilon}\left\langle Q_{b} Q_{0} \psi-Q_{b} \psi, Q_{b} e_{0}-e_{b}\right\rangle_{\partial T} \\
\leq & C h^{2-\frac{\varepsilon}{2}}\|\psi\|_{3}\left\|e_{h}\right\|,  \tag{3.17}\\
\ell\left(u, Q_{h} \psi\right)= & \sum_{T \in \mathcal{T}_{h}}\left\langle\left(\nabla u-\mathbb{Q}_{h} \nabla u\right) \cdot \mathbf{n}, Q_{0} \psi-Q_{b} \psi\right\rangle_{\partial T} \\
& +\sum_{T \in \mathcal{T}_{h}} h_{T}^{-1+\varepsilon}\left\langle Q_{b} Q_{0} u-Q_{b} u, Q_{b} Q_{0} \psi-Q_{b} \psi\right\rangle_{\partial T}
\end{align*}
$$

$$
\begin{equation*}
\leq C h^{k+2-\varepsilon}\|\psi\|_{3}\|u\|_{k+1} \tag{3.18}
\end{equation*}
$$

Thus, combining $(3.17)-3.18$ and Lemma 3.5 leads to

$$
\begin{aligned}
\left\|e_{h}\right\|_{-1} & =b\left(e_{h}, \varphi\right) \leq\left(e_{0}, \tilde{\varphi}\right)+\left\|e_{0}\right\|\|\varphi-\tilde{\varphi}\| \\
& \leq C h^{k+2-\varepsilon}\|\tilde{\varphi}\|_{1}\|u\|_{k+1} \\
& \leq C h^{k+2-\varepsilon}\|u\|_{k+1},
\end{aligned}
$$

which completes the proof.
From Lemma 3.1 and Theorem 3.2, we have the following corollary.
Corollary 3.1. Under the conditions of Theorem 3.2, the following estimate holds true

$$
\begin{equation*}
\left\|Q_{h} u-u_{h}\right\|_{-V} \leq C h^{k+2-\varepsilon}\|u\|_{k+1} \tag{3.19}
\end{equation*}
$$

Here, we shall also give the estimate for the projection error $\left\|u-Q_{h} u\right\|_{-V}$.
Lemma 3.6. When $u \in H^{k+1}(\Omega)$, the following estimate holds true

$$
\left\|u-Q_{h} u\right\|_{-V} \leq C h^{k+2}\|u\|_{k+1}
$$

Proof. From the definition, we know there exists $v \in V$ such that

$$
\left\|u-Q_{h} u\right\|_{-V}=\frac{b_{w}\left(u-Q_{h} u, v\right)}{\|v\|_{V}} .
$$

Since $V=H_{0}^{1}(\Omega) \oplus\left(V_{h} \backslash H_{0}^{1}(\Omega)\right), v$ can be decomposed as $v=v_{1}+v_{2}$, where $v_{1} \in H_{0}^{1}(\Omega), v_{2} \in V_{h} \backslash H_{0}^{1}(\Omega)$. It follows that

$$
\begin{aligned}
\left\|u-Q_{h} u\right\|_{-V} & =\frac{b_{w}\left(u-Q_{h} u, v\right)}{\|v\|_{V}} \\
& =\frac{b_{w}\left(u-Q_{h} u, v_{1}\right)}{\|v\|_{V}}+\frac{b_{w}\left(u-Q_{h} u, v_{2}\right)}{\|v\|_{V}} \\
& \leq \frac{b_{w}\left(u-Q_{h} u, v_{1}\right)}{\left\|v_{1}\right\|_{V}} \\
& \leq C \frac{b_{w}\left(u-Q_{h} u, v_{1}-Q_{h} v_{1}\right)}{\left\|v_{1}\right\|_{1}} \\
& \leq C h^{k+2}\|u\|_{k+1}
\end{aligned}
$$

where we used the following error estimates for the projection operator $Q_{h}$

$$
\left\|u-Q_{h} u\right\|_{b} \leq C h^{k+1}\|u\|_{k+1} \quad \text { and } \quad\left\|v_{1}-Q_{h} v_{1}\right\|_{b} \leq C h\left\|v_{1}\right\|_{1}
$$

Then the proof is completed.
Combining Corollary 3.1 with Lemma 3.6, we have the following error estimate result for the boundary value problem (3.1).
Theorem 3.3. Under the conditions of Theorem 3.2, the following estimate holds true

$$
\begin{equation*}
\left\|u-u_{h}\right\|_{-V} \leq C h^{k+2-\varepsilon}\|u\|_{k+1} \tag{3.20}
\end{equation*}
$$

From the Babuška's theory and the results in [32], the conclusion of Theorem 3.3 can be extended to the eigenvalue problem which means we have the following error estimate and the proof is similar to [32, Section 4].

Theorem 3.4. Suppose $\lambda_{j, h}$ is the $j$-th eigenvalue of (2.3) and $u_{j, h}$ is the corresponding eigenfunction. Then there exists an exact eigenfunction $u_{j}$ corresponding to the $j$-th exact eigenvalue of (2.1) such that the following error estimate holds

$$
\begin{equation*}
\left\|u_{j}-u_{j, h}\right\|_{-V} \leq C h^{k+2-\varepsilon}\left\|u_{j}\right\|_{k+1} \tag{3.21}
\end{equation*}
$$

where $u_{j} \in H^{k+1}(\Omega), k \geq 2$, and $h$ is sufficiently small.

## 4 A two-grid scheme

In this section, we propose a two-grid WG scheme for the eigenvalue problem, and give the corresponding analysis for the convergence and efficiency of this scheme. Here, we drop the subscript $j$ to denote a certain eigenvalue of problem 2.1.

Weak Galerkin Algorithm 3. Step 1: Generate a coarse grid $\mathcal{T}_{H}$ on the domain $\Omega$ and solve the following eigenvalue problem on the coarse grid $\mathcal{T}_{H}$ :

Find $\left(\lambda_{H}, u_{H}\right) \in \mathbb{R} \times V_{H}$, such that

$$
a_{s}\left(u_{H}, v_{H}\right)=\lambda_{H} b_{w}\left(u_{H}, v_{H}\right), \quad \forall v_{H} \in V_{H}
$$

Step 2: Refine the coarse grid $\mathcal{T}_{H}$ to obtain a finer grid $\mathcal{T}_{h}$ and solve one single linear problem on the fine grid $\mathcal{T}_{h}$ :

Find $\tilde{u}_{h} \in V_{h}$ such that

$$
a_{s}\left(\tilde{u}_{h}, v_{h}\right)=\lambda_{H} b_{w}\left(u_{H}, v_{h}\right), \quad \forall v_{h} \in V_{h}
$$

Step 3: Calculate the Rayleigh quotient for $\tilde{u}_{h}$

$$
\tilde{\lambda}_{h}=\frac{a_{s}\left(\tilde{u}_{h}, \tilde{u}_{h}\right)}{b_{w}\left(\tilde{u}_{h}, \tilde{u}_{h}\right)} .
$$

Finally, we obtain the eigenpair approximation $\left(\tilde{\lambda}_{h}, \tilde{u}_{h}\right)$.
First, we need the following discrete Poincaré's inequality for the WG method, which has been proved in [29].

Lemma 4.1. The discrete Poincaré-type inequality holds true on $V_{h}$, i.e.

$$
\left\|v_{h}\right\| \lesssim\left\|v_{h}\right\|, \quad \forall v_{h} \in V_{h}
$$

From Theorem 2.1. suppose the eigenfunction $u$ is smooth enough and we have the following estimate immediately,

$$
h^{2 k} \lesssim \lambda-\lambda_{h} \lesssim h^{2 k-2 \varepsilon} .
$$

For simplicity, here and hereafter, we assume the concerned eigenvalues are simple. In order to estimate $\left|\lambda-\tilde{\lambda}_{h}\right|$, we just need to estimate $\left|\lambda_{h}-\tilde{\lambda}_{h}\right|$.

Lemma 4.2. Suppose $\left(\tilde{\lambda}_{h}, \tilde{u}_{h}\right)$ is calculated by Algorithm 3 and $\left(\lambda_{h}, u_{h}\right)$ satisfies 2.3). Then the following estimate holds

$$
\begin{equation*}
\left|\tilde{\lambda}_{h}-\lambda_{h}\right| \lesssim\left\|\tilde{u}_{h}-u_{h}\right\|^{2} \tag{4.1}
\end{equation*}
$$

Proof. From 2.3) and Lemma 4.1, we have

$$
\begin{aligned}
& \left(\tilde{\lambda}_{h}-\lambda_{h}\right) b_{w}\left(\tilde{u}_{h}, \tilde{u}_{h}\right)=a_{s}\left(\tilde{u}_{h}, \tilde{u}_{h}\right)-\lambda_{h} b_{w}\left(\tilde{u}_{h}, \tilde{u}_{h}\right) \\
= & a_{s}\left(\tilde{u}_{h}-u_{h}, \tilde{u}_{h}-u_{h}\right)+2 a_{s}\left(u_{h}, \tilde{u}_{h}\right)-a_{s}\left(u_{h}, u_{h}\right) \\
& -\lambda_{h} b_{w}\left(\tilde{u}_{h}-u_{h}, \tilde{u}_{h}-u_{h}\right)-2 \lambda_{h} b_{w}\left(u_{h}, \tilde{u}_{h}\right)+\lambda_{h} b_{w}\left(u_{h}, u_{h}\right) \\
= & a_{s}\left(\tilde{u}_{h}-u_{h}, \tilde{u}_{h}-u_{h}\right)-\lambda_{h} b_{w}\left(\tilde{u}_{h}-u_{h}, \tilde{u}_{h}-u_{h}\right) \\
\lesssim & \left\|\tilde{u}_{h}-u_{h}\right\|^{2},
\end{aligned}
$$

which completes the proof.
Lemma 4.3. Under the conditions of Lemma 4.2. the following estimate holds true

$$
\begin{equation*}
\left\|\tilde{u}_{h}-u_{h}\right\| \lesssim H^{2 k-2 \varepsilon}+H^{k+\gamma-\varepsilon}, \quad \text { when } h<H . \tag{4.2}
\end{equation*}
$$

Here and hereafter $\gamma$ is defined as follows

$$
\gamma=\left\{\begin{array}{cc}
1, \quad \text { when the solution of dual problem }  \tag{4.3}\\
\psi \in H^{2}(\Omega) \text { or } k=1, \\
2-\frac{\varepsilon}{2}, & \text { when the solution of dual problem } \\
\psi \in H^{3}(\Omega) \text { and } k>1
\end{array}\right.
$$

Proof. For all $v_{h} \in V_{h}$, from equation 2.3), Theorems 2.1, and 3.4 we can obtain

$$
\begin{aligned}
& a_{s}\left(\tilde{u}_{h}-u_{h}, v_{h}\right)=a_{s}\left(\tilde{u}_{h}, v_{h}\right)-a_{s}\left(u_{h}, v_{h}\right) \\
= & \lambda_{H} b_{w}\left(u_{H}, v_{h}\right)-\lambda_{h} b_{w}\left(u_{h}, v_{h}\right) \\
= & \lambda_{H} b_{w}\left(u_{H}, v_{h}\right)-\lambda_{H} b_{w}\left(u_{h}, v_{h}\right)+\lambda_{H} b_{w}\left(u_{h}, v_{h}\right)-\lambda_{h} b_{w}\left(u_{h}, v_{h}\right) \\
= & \lambda_{H} b_{w}\left(u_{H}-u_{h}, v_{h}\right)+\left(\lambda_{H}-\lambda_{h}\right) b_{w}\left(u_{h}, v_{h}\right) \\
= & \lambda_{H} b_{w}\left(u_{H}-u, v_{h}\right)+\lambda_{H} b_{w}\left(u-u_{h}, v_{h}\right) \\
& +\left(\lambda_{H}-\lambda\right) b_{w}\left(u_{h}, v_{h}\right)+\left(\lambda-\lambda_{h}\right) b_{w}\left(u_{h}, v_{h}\right) .
\end{aligned}
$$

If $k=1$ or the solution of the dual problem 3.14 has the regularity $\psi \in H^{2}(\Omega)$, we have

$$
\begin{align*}
a_{s}\left(\tilde{u}_{h}-u_{h}, v_{h}\right) & \lesssim\left(\left\|u-u_{H}\right\|+\left\|u-u_{h}\right\|\right)\left\|v_{h}\right\|_{b}+\left(\left|\lambda_{H}-\lambda\right|+\left|\lambda_{h}-\lambda\right|\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(H^{k+1-\varepsilon}+h^{k+1-\varepsilon}\right)\left\|v_{h}\right\|_{b}+\left(H^{2 k-2 \varepsilon}+h^{2 k-2 \varepsilon}\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(H^{k+1-\varepsilon}+H^{2 k-2 \varepsilon}\right)\left\|v_{h}\right\| . \tag{4.4}
\end{align*}
$$

If $k>1$ and the solution of the dual problem 3.14 has the regularity $\psi \in H^{3}(\Omega)$, the following estimates hold

$$
\begin{align*}
a_{s}\left(\tilde{u}_{h}-u_{h}, v_{h}\right) & \lesssim\left(\left\|u-u_{H}\right\|_{-V}+\left\|u-u_{h}\right\|_{-V}\right)\left\|v_{h}\right\|_{V}+\left(\left|\lambda_{H}-\lambda\right|+\left|\lambda_{h}-\lambda\right|\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(H^{k+2-\varepsilon}+h^{k+2-\varepsilon}\right)\left\|v_{h}\right\|_{V}+\left(H^{2 k-2 \varepsilon}+h^{2 k-2 \varepsilon}\right)\left\|v_{h}\right\| \\
& \lesssim\left(H^{k+2-\frac{3}{2} \varepsilon}+H^{2 k-2 \varepsilon}\right)\left\|v_{h}\right\| . \tag{4.5}
\end{align*}
$$

From (4.4)-(4.5) and taking $v_{h}=\tilde{u}_{h}-u_{h}$, we can obtain the desired result (4.2) and the proof is completed.

From Lemmas 4.2 and 4.3 , the convergence of $\left|\lambda_{h}-\tilde{\lambda}_{h}\right|$ follows immediately.
Lemma 4.4. Suppose $\left(\tilde{\lambda}_{h}, \tilde{u}_{h}\right)$ is calculated by Algorithm 3 and $\left(\lambda_{h}, u_{h}\right)$ satisfies (2.3). Then the following estimate holds

$$
\begin{equation*}
\left|\tilde{\lambda}_{h}-\lambda_{h}\right| \lesssim H^{4 k-4 \varepsilon}+H^{2 k+2 \gamma-2 \varepsilon} . \tag{4.6}
\end{equation*}
$$

With Lemmas 4.3 and 4.4 , we arrive at the following convergence theorem.
Theorem 4.1. Suppose $\left(\tilde{\lambda}_{h}, \tilde{u}_{h}\right)$ is calculated by Algorithm3, $h<H$ and the exact eigenfunctions of 2.1) have $H^{k+1}(\Omega)$-regularity. Then there exists an exact eigenpair $(\lambda, u)$ such that the following estimates hold true

$$
\begin{align*}
\left\|Q_{h} u-\tilde{u}_{h}\right\| & \lesssim H^{\bar{k}}+h^{k-\varepsilon}  \tag{4.7}\\
\left|\tilde{\lambda}_{h}-\lambda\right| & \lesssim H^{2 \bar{k}}+h^{2 k-2 \varepsilon} \tag{4.8}
\end{align*}
$$

where $\bar{k}=\min \{2 k-2 \varepsilon, k+\gamma-\varepsilon\}$.
From Theorem 2.1 and Lemma 4.4, we can get the following lower bound estimate.
Theorem 4.2. Suppose the conditions of Theorem 4.1 hold and let $\bar{k}=\min \{2 k-2 \varepsilon, k+\gamma-\varepsilon\}$ and $\delta$ be a positive number. If $H^{2 \bar{k}} \leq C h^{2 k+\delta}$, then we have

$$
\lambda-\tilde{\lambda}_{h} \geq 0
$$

when $H$ and $h$ are sufficiently small.
Proof. From Theorem 2.1 we have

$$
\lambda-\lambda_{h} \geq C h^{2 k}
$$

According to Lemma 4.4 the following estimates hold

$$
\left|\tilde{\lambda}_{h}-\lambda_{h}\right| \leq C H^{2 \bar{k}} \leq C h^{2 k+\delta}
$$

When $h$ is sufficiently small, it follows that

$$
\begin{aligned}
\lambda-\tilde{\lambda}_{h} & =\lambda-\lambda_{h}+\lambda_{h}-\tilde{\lambda}_{h} \geq \lambda-\lambda_{h}-\left|\tilde{\lambda}_{h}-\lambda_{h}\right| \\
& \geq C h^{2 k}-C h^{2 k+\delta} \geq 0
\end{aligned}
$$

which completes the proof.

## 5 A two-space scheme

In this section, we shall give a two-space WG scheme for problem 2.1, where different polynomial spaces are employed on the same mesh.

Denote the finite element spaces

$$
\begin{aligned}
& V_{h}^{1}=\left\{v=\left(v_{0}, v_{b}\right):\left.v_{0}\right|_{T} \in P_{k_{1}}(T),\left.v_{b}\right|_{e} \in P_{k_{1}-1}(e), \forall T \in \mathcal{T}_{h}, e \in \mathcal{E}_{h}, \text { and } v_{b}=0 \text { on } \partial \Omega\right\} \\
& V_{h}^{2}=\left\{v=\left(v_{0}, v_{b}\right):\left.v_{0}\right|_{T} \in P_{k_{2}}(T),\left.v_{b}\right|_{e} \in P_{k_{2}-1}(e), \forall T \in \mathcal{T}_{h}, e \in \mathcal{E}_{h}, \text { and } v_{b}=0 \text { on } \partial \Omega\right\}
\end{aligned}
$$

Denote $\left(\lambda_{h}^{2}, u_{h}^{2}\right)$ the numerical solution of the standard WG scheme, which satisfies

$$
\begin{equation*}
a_{s}\left(u_{h}^{2}, v_{h}\right)=\lambda_{h}^{2} b_{w}\left(u_{h}^{2}, v_{h}\right), \quad \forall v_{h} \in V_{h}^{2} \tag{5.1}
\end{equation*}
$$

In the two-space method, the spaces $V_{h}^{1}$ and $V_{h}^{2}$ are defined on the same mesh, with different degrees of polynomials.

Weak Galerkin Algorithm 4. Step 1: Solve the eigenvalue problem in the space $V_{h}^{1}$ :
Find $\left(\lambda_{h}^{1}, u_{h}^{1}\right) \in \mathbb{R} \times V_{h}^{1}$ such that

$$
a_{s}\left(u_{h}^{1}, v_{h}\right)=\lambda_{h}^{1} b_{w}\left(u_{h}^{1}, v_{h}\right), \quad \forall v_{h} \in V_{h}^{1}
$$

Step 2: Solve one single linear problem in the space $V_{h}^{2}$ :
Find $\hat{u}_{h} \in V_{h}^{2}$ such that

$$
a_{s}\left(\hat{u}_{h}, v_{h}\right)=\lambda_{h}^{1} b_{w}\left(u_{h}^{1}, v_{h}\right), \quad \forall v_{h} \in V_{h}^{2}
$$

Step 3: Calculate the Rayleigh quotient

$$
\hat{\lambda}_{h}=\frac{a_{s}\left(\hat{u}_{h}, \hat{u}_{h}\right)}{b_{w}\left(\hat{u}_{h}, \hat{u}_{h}\right)} .
$$

Finally, we obtain the eigenpair approximation $\left(\hat{\lambda}_{h}, \hat{u}_{h}\right)$.
The proof is similar to the two-grid algorithm, and we just need to interpret Lemmas 4.2 and 4.3 into the two-space case.
Lemma 5.1. Suppose $\left(\hat{\lambda}_{h}, \hat{u}_{h}\right)$ is calculated by Algorithm 4 and $\left(\lambda_{h}^{2}, u_{h}^{2}\right)$ satisfies 5.1). Then the following estimate holds

$$
\begin{equation*}
\left|\hat{\lambda}_{h}-\lambda_{h}^{2}\right| \lesssim\left\|\hat{u}_{h}-u_{h}^{2}\right\|^{2} \tag{5.2}
\end{equation*}
$$

Lemma 5.2. Suppose $\left(\hat{\lambda}_{h}, \hat{u}_{h}\right)$ is calculated by Algorithm 4. $\left(\lambda_{h}^{2}, u_{h}^{2}\right)$ satisfies 5.1) and $k_{1}<k_{2}$. When the exact solution $u \in H^{k_{2}+1}(\Omega)$, the following estimate holds true

$$
\begin{equation*}
\left\|\hat{u}_{h}-u_{h}^{2}\right\| \lesssim h^{2 k_{1}-2 \varepsilon}+h^{k_{1}+\gamma-\varepsilon} \tag{5.3}
\end{equation*}
$$

Proof. For all $v_{h} \in V_{h}^{2}$, from formula 5.1, Theorems 2.1 and 3.4, we can obtain

$$
\begin{aligned}
& a_{s}\left(\hat{u}_{h}^{2}-u_{h}^{2}, v_{h}\right)=a_{s}\left(\hat{u}_{h}^{2}, v_{h}\right)-a_{s}\left(u_{h}^{2}, v_{h}\right) \\
= & \lambda_{h}^{1} b_{w}\left(u_{h}^{1}, v_{h}\right)-\lambda_{h}^{2} b_{w}\left(u_{h}^{2}, v_{h}\right) \\
= & \lambda_{h}^{1} b_{w}\left(u_{h}^{1}, v_{h}\right)-\lambda_{h}^{1} b_{w}\left(u_{h}^{2}, v_{h}\right)+\lambda_{h}^{1} b_{w}\left(u_{h}^{2}, v_{h}\right)-\lambda_{h}^{2} b_{w}\left(u_{h}^{2}, v_{h}\right) \\
= & \lambda_{h}^{1} b_{w}\left(u_{h}^{1}-u_{h}^{2}, v_{h}\right)+\left(\lambda_{h}^{1}-\lambda_{h}^{2}\right) b_{w}\left(u_{h}^{2}, v_{h}\right) \\
= & \lambda_{h}^{1} b_{w}\left(u_{h}^{1}-u, v_{h}\right)+\lambda_{h}^{1} b_{w}\left(u-u_{h}^{2}, v_{h}\right) \\
& +\left(\lambda_{h}^{1}-\lambda\right) b_{w}\left(u_{h}^{2}, v_{h}\right)+\left(\lambda-\lambda_{h}^{2}\right) b_{w}\left(u_{h}^{2}, v_{h}\right)
\end{aligned}
$$

If $k_{1}=1$ or the solution of the dual problem 3.14 has the regularity $\psi \in H^{2}(\Omega)$, we have

$$
\begin{align*}
a_{s}\left(\hat{u}_{h}^{2}-u_{h}^{2}, v_{h}\right) & \lesssim\left(\left\|u-u_{h}^{1}\right\|+\left\|u-u_{h}^{2}\right\|\right)\left\|v_{h}\right\|_{b}+\left(\left|\lambda_{h}^{1}-\lambda\right|+\left|\lambda_{h}^{2}-\lambda\right|\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(h^{k_{1}+1-\varepsilon}+h^{k_{2}+1-\varepsilon}\right)\left\|v_{h}\right\|_{b}+\left(h^{2 k_{1}-2 \varepsilon}+h^{2 k_{2}-2 \varepsilon}\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(h^{2 k_{1}-2 \varepsilon}+h^{k_{1}+1-\varepsilon}\right)\left\|v_{h}\right\| . \tag{5.4}
\end{align*}
$$

If $k_{1}>1$ and the solution of the dual problem $(3.14)$ has the regularity $\psi \in H^{3}(\Omega)$, the following estimates hold

$$
\begin{align*}
a_{s}\left(\hat{u}_{h}^{2}-u_{h}^{2}, v_{h}\right) & \lesssim\left(\left\|u-u_{h}^{1}\right\|_{-V}+\left\|u-u_{h}^{2}\right\|_{-V}\right)\left\|v_{h}\right\|_{V}+\left(\left|\lambda_{h}^{1}-\lambda\right|+\left|\lambda_{h}^{2}-\lambda\right|\right)\left\|v_{h}\right\|_{b} \\
& \lesssim\left(h^{k_{1}+2-\varepsilon}+h^{k_{2}+2-\varepsilon}\right)\left\|v_{h}\right\|_{V}+\left(h^{2 k_{1}-2 \varepsilon}+h^{2 k_{2}-2 \varepsilon}\right)\left\|v_{h}\right\| \\
& \lesssim\left(h^{2 k_{1}-2 \varepsilon}+h^{k_{1}+2-\frac{3}{2} \varepsilon}\right)\left\|v_{h}\right\| . \tag{5.5}
\end{align*}
$$

From (5.4)- 5.5 and taking $v_{h}=\hat{u}_{h}-u_{h}^{2}$, we can obtain the desired result 5.3 and the proof is completed.

With Lemmas 5.1 and 5.2 , we can get the following estimate easily.
Theorem 5.1. Suppose $\left(\hat{\lambda}_{h}, \hat{u}_{h}\right)$ is calculated by Algorithm 4, $(\lambda, u)$ satisfies (2.1) and $k_{1} \leq k_{2}$. When the exact solution $u \in H^{k_{2}+1}(\Omega)$, the following estimate holds true

$$
\left|\hat{\lambda}_{h}-\lambda\right| \lesssim h^{\hat{k}}+h^{2 k_{2}-2 \varepsilon}
$$

where $\hat{k}=\min \left\{4 k_{1}-4 \varepsilon, 2 k_{1}+2 \gamma-2 \varepsilon\right\}$.
From Theorem 2.1 and Lemma 5.2, we can get the following lower bound estimate.
Theorem 5.2. Suppose the assumptions of Theorem 5.1 hold true and let $\hat{k}=\min \left\{4 k_{1}-4 \varepsilon, 2 k_{1}+2 \gamma-2 \varepsilon\right\}$ and $\delta$ be a positive number. If $\hat{k}>2 k_{2}$, we have

$$
\lambda-\hat{\lambda}_{h} \geq 0
$$

when $h$ is sufficiently small.
Proof. From Theorem 2.1, we have

$$
\lambda-\lambda_{h} \geq C h^{2 k_{2}}
$$

According to Lemma 5.2 the following inequalities hold

$$
\left|\hat{\lambda}_{h}-\lambda_{h}\right| \leq C h^{\hat{k}} \leq C h^{2 k_{2}+\delta}
$$

When $h$ is sufficiently small, it follows that

$$
\begin{aligned}
\lambda-\hat{\lambda}_{h} & =\lambda-\lambda_{h}+\lambda_{h}-\hat{\lambda}_{h} \geq \lambda-\lambda_{h}-\left|\hat{\lambda}_{h}-\lambda_{h}\right| \\
& \geq C h^{2 k_{2}}-C h^{2 k_{2}+\delta} \geq 0
\end{aligned}
$$

which completes the proof.

## 6 Numerical Experiments

In this section, we present two numerical examples of Algorithms 3 and 4 to check the efficiencies of Algorithms 3 and 4 for the eigenvalue problem (2.1).

### 6.1 Two-gird method

In the first example, we consider the problem 2.1 on the unit square $\Omega=(0,1)^{2}$. It is known that the eigenvalue problem has the following eigenpairs

$$
\lambda=\left(m^{2}+n^{2}\right) \pi^{2}, \quad u=\sin (m \pi x) \sin (n \pi y),
$$

where $m, n$ are arbitrary positive integers. The first four different eigenvalues are $\lambda_{1}=2 \pi^{2}, \lambda_{2}=5 \pi^{2}$, $\lambda_{3}=8 \pi^{2}$ and $\lambda_{4}=10 \pi^{2}$, where algebraic or geometric multiplicities for $\lambda_{1}$ and $\lambda_{3}$ are 1 and for $\lambda_{2}$ and $\lambda_{4}$ are both 2 .

The uniform mesh is applied in the following examples, $H$ and $h$ denote mesh sizes. Numerical results for different choices of the parameter $\varepsilon$ and the degree $k$ of polynomial are presented. The corresponding numerical results are showed in Tables 1.6. In Tables 1.2, the polynomial degree $k=1$, and $\varepsilon$ is set to be 0 and 0.1 , separately. From Theorem 4.1, we know the convergence order for eigenvalue approximation is $2-2 \varepsilon$ which is shown from the numerical results included in Tables 1 for $\varepsilon=0$ and 2 for $\varepsilon=0.1$. In

Table 1: The eigenvalue errors $\lambda-\tilde{\lambda}_{h}$ for Example 1 with $k=1, \varepsilon=0$.

| $H$ | $1 / 4$ | $1 / 8$ | $1 / 16$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 16$ | $1 / 64$ | $1 / 256$ |
| $\lambda_{1}-\tilde{\lambda}_{1, h}$ | $2.1554 \mathrm{e}-1$ | $1.3006 \mathrm{e}-2$ | $8.0627 \mathrm{e}-4$ |
| order |  | 4.0507 | 4.0118 |
| $\lambda_{2}-\tilde{\lambda}_{2, h}$ | $1.3687 \mathrm{e}+0$ | $8.2219 \mathrm{e}-2$ | $4.8684 \mathrm{e}-3$ |
| order |  | 4.0572 | 4.0780 |
| $\lambda_{3}-\tilde{\lambda}_{3, h}$ | $1.3687 \mathrm{e}+0$ | $7.8229 \mathrm{e}-2$ | $4.8240 \mathrm{e}-3$ |
| order |  | 4.1290 | 4.0194 |
| $\lambda_{4}-\tilde{\lambda}_{4, h}$ | $3.1148 \mathrm{e}+0$ | $2.0798 \mathrm{e}-1$ | $1.2318 \mathrm{e}-2$ |
| order |  | 3.9046 | 4.0776 |
| $\lambda_{5}-\tilde{\lambda}_{5, h}$ | $4.3750 \mathrm{e}+0$ | $3.0337 \mathrm{e}-1$ | $1.8860 \mathrm{e}-2$ |
| order |  | 3.8501 | 4.0077 |
| $\lambda_{6}-\tilde{\lambda}_{6, h}$ | $4.0896 \mathrm{e}+0$ | $3.0980 \mathrm{e}-1$ | $1.8206 \mathrm{e}-2$ |
| order |  | 3.7225 | 4.0889 |

Table 2: The errors for the eigenvalue approximation $\tilde{\lambda}_{h}$ for Example 1 with $k=1, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 8$ | $1 / 16$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 16$ | $1 / 64$ | $1 / 256$ |
| $\lambda_{1}-\tilde{\lambda}_{1, h}$ | $2.7369 \mathrm{e}-1$ | $1.8802 \mathrm{e}-2$ | $1.3194 \mathrm{e}-3$ |
| order |  | 3.8636 | 3.8330 |
| $\lambda_{2}-\tilde{\lambda}_{2, h}$ | $1.7347 \mathrm{e}+0$ | $1.1960 \mathrm{e}-1$ | $8.0009 \mathrm{e}-3$ |
| order |  | 3.8584 | 3.9019 |
| $\lambda_{3}-\tilde{\lambda}_{3, h}$ | $1.7347 \mathrm{e}+0$ | $1.1339 \mathrm{e}-1$ | $7.9261 \mathrm{e}-3$ |
| order |  | 3.9353 | 3.8386 |
| $\lambda_{4}-\tilde{\lambda}_{4, h}$ | $3.9686 \mathrm{e}+0$ | $3.0206 \mathrm{e}-1$ | $2.0167 \mathrm{e}-2$ |
| order |  | 3.7157 | 3.9048 |
| $\lambda_{5}-\tilde{\lambda}_{5, h}$ | $5.7315 \mathrm{e}+0$ | $4.4163 \mathrm{e}-1$ | $3.1474 \mathrm{e}-2$ |
| order |  | 3.6980 | 3.8106 |
| $\lambda_{6}-\tilde{\lambda}_{6, h}$ | $5.1859 \mathrm{e}+0$ | $4.5160 \mathrm{e}-1$ | $2.9977 \mathrm{e}-2$ |
| order |  | 3.5215 | 3.9131 |

Tables 3.6, the polynomial degree $k=2$ and $\varepsilon=0.1$. The mesh size $h$ is selected to be $H^{2}$ in Tables 3.4, and $H^{\frac{3}{2}}$ in Tables 546 . The convergence orders for the eigenvalues, the trip-bar norm of eigenfunctions are presented. The convergence orders in Tables 3/4 coincide with that pblackicted in Theorem 4.1. Since the choices of $k=2, \varepsilon=0.1$ and $h=H^{2}$ or $h=H^{\frac{3}{2}}$ do not satisfy the condition of Theorem 4.2, it is not surprising that the eigenvalue approximations $\tilde{\lambda}_{j, h}(j=1, \ldots, 6)$ are not the lower bounds of the corresponding exact eigenvalues (see Tables 3 and 5 ).

Furthermore, the choice of $\varepsilon$ can really affect the convergence order which means the error estimates in (4.2), (4.6), 4.7), and (4.8) are reasonable.

### 6.2 Two-space method

In the second example, the analytic solution is the same as (6.1). The polynomials of degree $k_{1}=1, k_{2}=2$ and $k_{1}=2, k_{2}=3$ are employed in $V_{h}^{1}$ and $V_{h}^{2}$, respectively. The parameter $\varepsilon$ is chosen to be 0.2 . The results are listed in Figures 112 for the case $k_{1}=1, k_{2}=2$ and $\varepsilon=0.2$ and Tables 344 for $k_{1}=2, k_{2}=3$ and $\varepsilon=0.2$.

Table 3: The eigenvalue errors $\lambda_{h}-\tilde{\lambda}_{h}$ for Example 1 with $k=2, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 8$ | $1 / 16$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 16$ | $1 / 64$ | $1 / 256$ |
| $\lambda_{1}-\tilde{\lambda}_{1, h}$ | $3.2380 \mathrm{e}-4$ | $1.8403 \mathrm{e}-6$ | $8.5157 \mathrm{e}-9$ |
| order |  | 7.4590 | 7.7556 |
| $\lambda_{2}-\tilde{\lambda}_{2, h}$ | $-6.9598 \mathrm{e}-2$ | $-2.2124 \mathrm{e}-4$ | $-8.7753 \mathrm{e}-7$ |
| order |  | 8.2973 | 7.9779 |
| $\lambda_{3}-\tilde{\lambda}_{3, h}$ | $-6.5952 \mathrm{e}-2$ | $-1.8022 \mathrm{e}-4$ | $-7.0453 \mathrm{e}-7$ |
| order |  | 8.5155 | 7.9989 |
| $\lambda_{4}-\tilde{\lambda}_{4, h}$ | $-1.5619 \mathrm{e}+0$ | $-5.5200 \mathrm{e}-3$ | $-2.1488 \mathrm{e}-5$ |
| order |  | 8.1445 | 8.0050 |
| $\lambda_{5}-\tilde{\lambda}_{5, h}$ | $-2.8009 \mathrm{e}+0$ | $-1.0533 \mathrm{e}-2$ | $-3.7459 \mathrm{e}-5$ |
| order |  | 8.0549 | 8.1354 |
| $\lambda_{6}-\tilde{\lambda}_{6, h}$ | $-5.3281 \mathrm{e}-1$ | $-1.8470 \mathrm{e}-3$ | $-5.4252 \mathrm{e}-6$ |
| order |  | 8.1723 | 8.4113 |

Table 4: The eigenfunction errors for Example 1 with $k=2, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 8$ | $1 / 16$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 16$ | $1 / 64$ | $1 / 256$ |
| $\left\\|Q_{h} u_{1}-\tilde{u}_{1, h}\right\\|$ | $4.2020 \mathrm{e}-2$ | $2.5571 \mathrm{e}-3$ | $1.6773 \mathrm{e}-4$ |
| order |  | 4.0385 | 3.9303 |
| $\left\\|Q_{h} u_{2}-\tilde{u}_{2, h}\right\\|$ | $3.4429 \mathrm{e}-1$ | $1.9352 \mathrm{e}-2$ | $1.2275 \mathrm{e}-3$ |
| order |  | 4.1531 | 3.9786 |
| $\left\\|Q_{h} u_{3}-\tilde{u}_{3, h}\right\\|$ | $3.4382 \mathrm{e}-1$ | $1.9351 \mathrm{e}-2$ | $1.2275 \mathrm{e}-3$ |
| order |  | 4.1512 | 3.9786 |
| $\left\\|Q_{h} u_{4}-\tilde{u}_{4, h}\right\\|$ | $2.1940 \mathrm{e}+0$ | $1.3372 \mathrm{e}-1$ | $8.3730 \mathrm{e}-3$ |
| order |  | 4.0363 | 3.9973 |
| $\left\\|Q_{h} u_{5}-\tilde{u}_{5, h}\right\\|$ | $2.7400 \mathrm{e}+0$ | $1.6506 \mathrm{e}-1$ | $9.9455 \mathrm{e}-3$ |
| order |  | 4.0531 | 4.0528 |
| $\left\\|Q_{h} u_{6}-\tilde{u}_{6, h}\right\\|$ | $2.7293 \mathrm{e}+0$ | $1.6504 \mathrm{e}-1$ | $9.9455 \mathrm{e}-3$ |
| order |  | 4.0476 | 4.0526 |

Table 5: The eigenvalue errors $\lambda_{h}-\tilde{\lambda}_{h}$ for Example 1 with $k=2, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 16$ | $1 / 64$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 8$ | $1 / 64$ | $1 / 512$ |
| $\lambda_{1}-\tilde{\lambda}_{1, h}$ | $1.3127 \mathrm{e}-2$ | $4.1262 \mathrm{e}-6$ | $2.5784 \mathrm{e}-9$ |
| order |  | 5.8177 | 5.3221 |
| $\lambda_{2}-\tilde{\lambda}_{2, h}$ | $1.3204 \mathrm{e}-1$ | $7.3767 \mathrm{e}-5$ | $2.3883 \mathrm{e}-8$ |
| order |  | 5.4029 | 5.7964 |
| $\lambda_{3}-\tilde{\lambda}_{3, h}$ | $8.1757 \mathrm{e}-2$ | $5.4651 \mathrm{e}-5$ | $1.8027 \mathrm{e}-8$ |
| order |  | 5.2734 | 5.7829 |
| $\lambda_{4}-\tilde{\lambda}_{4, h}$ | $-1.0040 \mathrm{e}+0$ | $2.4182 \mathrm{e}-4$ | $8.0721 \mathrm{e}-8$ |
| order |  | 6.0098 | 5.7744 |
| $\lambda_{5}-\tilde{\lambda}_{5, h}$ | $-1.7621 \mathrm{e}+0$ | $4.8559 \mathrm{e}-4$ | $1.5872 \mathrm{e}-7$ |
| order |  | 5.9126 | 5.7895 |
| $\lambda_{6}-\tilde{\lambda}_{6, h}$ | $1.0614 \mathrm{e}-1$ | $5.1734 \mathrm{e}-4$ | $1.5931 \mathrm{e}-7$ |
| order |  | 3.8403 | 5.8325 |

Table 6: The eigenfunction errors for Example 1 with $k=2, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 16$ | $1 / 64$ |
| :---: | :---: | :---: | :---: |
| $h$ | $1 / 8$ | $1 / 64$ | $1 / 512$ |
| $\left\\|\left\\|Q_{h} u_{1}-\tilde{u}_{1, h}\right\\|\right.$ | $1.2856 \mathrm{e}-1$ | $1.9489 \mathrm{e}-3$ | $3.3757 \mathrm{e}-5$ |
| order |  | 3.0218 | 2.9257 |
| $\left\\|\left\\|Q_{h} u_{2}-\tilde{u}_{2, h}\right\\|\right.$ | $6.8967 \mathrm{e}-1$ | $7.9977 \mathrm{e}-3$ | $1.3471 \mathrm{e}-4$ |
| order |  | 3.2151 | 2.9458 |
| $\left\\|\left\\|Q_{h} u_{3}-\tilde{u}_{3, h}\right\\|\right.$ | $6.8499 \mathrm{e}-1$ | $7.9974 \mathrm{e}-3$ | $1.3471 \mathrm{e}-4$ |
| order |  | 3.2102 | 2.9458 |
| $\left\\|Q_{h} u_{4}-\tilde{u}_{4, h}\right\\|$ | $2.4166 \mathrm{e}+0$ | $1.8197 \mathrm{e}-2$ | $2.7328 \mathrm{e}-4$ |
| order |  | 3.5265 | 3.0286 |
| $\left\\|\left\\|Q_{h} u_{5}-\tilde{u}_{5, h}\right\\|\right.$ | $3.0868 \mathrm{e}+0$ | $2.5302 \mathrm{e}-2$ | $3.8628 \mathrm{e}-4$ |
| order |  | 3.4653 | 3.0167 |
| $\left\\|Q_{h} u_{6}-\tilde{u}_{6, h}\right\\|$ | $3.0298 \mathrm{e}+0$ | $2.5299 \mathrm{e}-2$ | $3.8628 \mathrm{e}-4$ |
| order |  | 3.4520 | 3.0167 |



Figure 1: The eigenvalue errors $\lambda_{h}-\hat{\lambda}_{h}$ for Example 2 with $k_{1}=1, k_{2}=2$ and $\varepsilon=0.2$.

The convergence orders shown in Figures 14 are consistent with the results in Theorems 5.1 and 5.2 , Even the choices of $k_{1}=1, k_{2}=2$ and $\varepsilon=0.2$ do not satisfy the condition of Theorem 5.2 , the eigenvalue approximations by the two-space method are still the lower bounds of the exact eigenvalues.

### 6.3 L-shape

In the third example, we consider the problem 2.1 on the L-shape domain $\Omega=(-1,1)^{2} /[0,1)^{2}$. Since the exact eigenvalues are unknown. We only check the eigenvalues $\tilde{\lambda}_{j, h}(j=1, \ldots, 6)$. The corresponding numerical results are shown in Table 7. From Table 7, we find that the two-grid method defined in Algorithm 3 is accurate and provides lower bounds.

## 7 Concluding remarks and ongoing work

In this paper, we propose and analyze the two-grid and two-space schemes for the eigenvalue problem by the WG method. Based on our analysis, the eigenpair approximations by the two-grid and two-space methods possess the same reasonable accuracy as the direct WG approximations, but the calculation cost


Figure 2: The errors for the eigenfunction approximations $\left\|u_{h}-\hat{u}_{h}\right\|$ for Example 2 with $k_{1}=1, k_{2}=2$ and $\varepsilon=0.2$.


Figure 3: The eigenvalue errors $\lambda_{h}-\hat{\lambda}_{h}$ for Example 2 with $k_{1}=2, k_{2}=3$ and $\varepsilon=0.2$.
is significantly reduced. From the numerical examples, we also find that the eigenvalue approximations by the two-grid method have the same lower bound property as the direct WG approximations, if we choose the grid or space properly.

In the future work, we are going to study the shift-inverse power method and multigrid method for the Laplacian eigenvalue problem, and other kinds of eigenvalue problems, such as biharmonic eigenvalue problems and Stokes eigenvalue problems.

## References

[1] Maria G. Armentano and Ricardo G. Duran. Asymptotic lower bounds for eigenvalues by nonconforming finite element methods. ETNA, Electron. Trans. Numer. Anal., 17:93-101, 2004.
[2] I. Babuska and J. Osborn. Handbook of Numerical Analysis, Vol II, Part1. Elsevier Science Publishers, North-Holland, 1991.


Figure 4: The errors for the eigenfunction approximations $\left\|u_{h}-\hat{u}_{h}\right\|$ for Example 2 with $k_{1}=2, k_{2}=3$ and $\varepsilon=0.2$.

Table 7: The errors for the eigenvalue approximations $\lambda_{h}-\tilde{\lambda}_{h}$ for Example 3 with $k=2, \varepsilon=0.1$.

| $H$ | $1 / 4$ | $1 / 16$ | $1 / 64$ | Trend |
| :---: | :---: | :---: | :---: | :---: |
| $h$ | $1 / 8$ | $1 / 64$ | $1 / 512$ |  |
| $\tilde{\lambda}_{1, h}$ | 9.6152615304 | 9.6383056544 | 9.6396344695 | $\nearrow$ |
| $\tilde{\lambda}_{2, h}$ | 15.1905227597 | 15.1972465939 | 15.1972519114 | $\nearrow$ |
| $\tilde{\lambda}_{3, h}$ | 19.7262367431 | 19.7392046788 | 19.7392088004 | $\nearrow$ |
| $\tilde{\lambda}_{4, h}$ | 29.4848098848 | 29.5214666179 | 29.5214811041 | $\nearrow$ |
| $\tilde{\lambda}_{5, h}$ | 31.7992824737 | 31.9091062924 | 31.9124163062 | $\nearrow$ |
| $\tilde{\lambda}_{6, h}$ | 41.3482606285 | 41.4717757164 | 41.4743429661 | $\nearrow$ |

[3] Daniele Boffi. Finite element approximation of eigenvalue problems. Acta Numer., 19:1-120, may 2010.
[4] Susanne C. Brenner and L. Ridgway Scott. The Mathematical Theory of Finite Element Methods, volume 15 of Texts in Applied Mathematics. Springer New York, New York, NY, 2008.
[5] C. Carstensen and J. Gedicke. Guaranteed lower bounds for eigenvalues. Math. Comp., 83(290):26052629, 2014.
[6] Francoise Chatelin. Spectral approximation of linear operators. In Spectr. Approx. Linear Oper., pages xxvii +458 . Society for Industrial and Applied Mathematics, jan 2011.
[7] Hongtao Chen, Shanghui Jia, and Hehu Xie. Postprocessing and higher order convergence for the mixed finite element approximations of the Stokes eigenvalue problems. Appl. Math., 54(3):237-250, jun 2009.
[8] Hongtao Chen, Shanghui Jia, and Hehu Xie. Postprocessing and higher order convergence for the mixed finite element approximations of the eigenvalue problem. Appl. Numer. Math., 61(4):615-629, 2011.
[9] Philippe G Ciarlet. The finite element method for elliptic problems, volume 40 of Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002.
[10] Xinlong Feng, Zhifeng Weng, and Hehu Xie. Acceleration of two-grid stabilized mixed finite element method for the Stokes eigenvalue problem. Appl. Math., 59(6):615-630, dec 2014.
[11] D. S. Grebenkov and B.-T. Nguyen. Geometrical structure of Laplacian eigenfunctions. SIAM Rev., 55(4):601-667, jan 2013.
[12] Patrick Henning, Axel Målqvist, and Daniel Peterseim. Two-level discretization techniques for ground state computations of Bose-Einstein condensates. SIAM J. Numer. Anal., 52(4):1525-1550, jan 2014.
[13] Jun Hu, Yunqing Huang, and Qun Lin. Lower bounds for eigenvalues of elliptic operators: By nonconforming finite element methods. J. Sci. Comput., 61(1):196-221, 2014.
[14] Xiaozhe Hu and Xiaoliang Cheng. Acceleration of a two-grid method for eigenvalue problems. Math. Comput., 80(275):1287-1301, feb 2011.
[15] Mats G Larson. A posteriori and a priori error analysis for finite element approximations of selfadjoint elliptic eigenvalue problems. SIAM J. Numer. Anal., 38(2):608-625, jan 2000.
[16] Qiaoluan Li and Junping Wang. Weak Galerkin finite element methods for parabolic equations. Numer. Methods Partial Differ. Equ., 29(6):1-21, 2013.
[17] Xuefeng Liu. A framework of verified eigenvalue bounds for self-adjoint differential operators. Appl. Math. Comput., 267:341-355, 2015.
[18] Xuefeng Liu and Shin'ichi Oishi. Verified eigenvalue evaluation for the Laplacian over polygonal domains of arbitrary shape. SIAM J. Numer. Anal., 51(3):1634-1654, may 2013.
[19] FuSheng Luo, Qun Lin, and HeHu Xie. Computing the lower and upper bounds of Laplace eigenvalue problem: by combining conforming and nonconforming finite element methods. Sci. China Math., 55(5):1069-1082, 2012.
[20] Martine Marion and Jinchao Xu. Error estimates on a new nonlinear Galerkin method based on two-grid finite elements. SIAM J. Numer. Anal., 32(4):1170-1184, aug 1995.
[21] Lin Mu, Junping Wang, and Xiu Ye. A stable numerical algorithm for the Brinkman equations by weak Galerkin finite element methods. J. Comput. Phys., 273:327-342, sep 2014.
[22] Lin Mu, Junping Wang, and Xiu Ye. Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes. Numer. Methods Partial Differ. Equ., 30(3):1003-1029, may 2014.
[23] Lin Mu, Junping Wang, Xiu Ye, and Shangyou Zhang. A C0-weak Galerkin finite element method for the biharmonic equation. J. Sci. Comput., 59(2):473-495, may 2014.
[24] Lin Mu, Junping Wang, Xiu Ye, and Shangyou Zhang. A weak Galerkin finite element method for the Maxwell equations. J. Sci. Comput., 65(1):363-386, oct 2015.
[25] Milena R Racheva and Andrey B Andreev. Superconvergence postprocessing for eigenvalues. Comput. Methods Appl. Math., 2(2):171-185, 2002.
[26] Tomáš Vejchodský. Computing upper bounds on friedrichs' constant. in Applications of Mathematics 2012, J. Brandts, J. Chleboun, S. Korotov, K. Segeth, J. Šístek, and T. Vejchodský, eds., Institute of Mathematics, ASCR, Prague, 2012, pages 278-289, 2012.
[27] Ivana Šebestov́ and Tomáš Vejchodský. Two-sided bounds for eigenvalues of differential operators with applications to friedrichs, poincaré, trace, and similar constants. SIAM J. Numer. Anal., 52(1):308-329, 2014.
[28] Junping Wang and Xiu Ye. A weak Galerkin finite element method for second-order elliptic problems. J. Comput. Appl. Math., 241(1):103-115, mar 2013.
[29] Junping Wang and Xiu Ye. A weak Galerkin mixed finite element method for second order elliptic problems. Math. Comput., 83(289):2101-2126, may 2014.
[30] H. F. Weinberger. Lower bounds for higher eigenvalues by finite difference methods. Pacific J. Math., 8(2):339-368, 1958.
[31] Hehu Xie and Xiaobo Yin. Acceleration of stabilized finite element discretizations for the Stokes eigenvalue problem. Adv. Comput. Math., 41(4):799-812, aug 2015.
[32] Hehu Xie, Qilong Zhai, and Ran Zhang. The weak Galerkin method for eigenvalue problems. arXiv Prepr. arXiv1508.05304, aug 2015.
[33] Jinchao Xu. A novel two-grid method for semilinear elliptic equations. SIAM J. Sci. Comput., 15(1):231-237, 1994.
[34] Jinchao Xu. Two-grid discretization techniques for linear and nonlinear PDEs. SIAM J. Numer. Anal., 33(5):1759-1777, 1996.
[35] Jinchao Xu and Aihui Zhou. A two-grid discretization scheme for eigenvalue problems. Math. Comput., 70(233):17-26, aug 1999.
[36] Jinchao Xu and Aihui Zhou. Local and parallel finite element algorithms based on two-grid discretizations. Math. Comput., 69(231):881-909, 2000.
[37] Yidu Yang and Hai Bi. Two-grid finite element discretization schemes based on shifted-inverse power method for elliptic eigenvalue problems. SIAM J. Numer. Anal., 49(4):1602-1624, jan 2011.
[38] Ran Zhang and Qilong Zhai. A weak Galerkin finite element scheme for the biharmonic equations by using polynomials of reduced order. J. Sci. Comput., 64(2):559-585, aug 2015.
[39] Zhimin Zhang, Yidu Yang, and Zhen Chen. Eigenvalue approximation from below by Wilson's element. Math. Numer. Sin., 29(3):319-321, 2007.
[40] J. Zhou, X. Hu, L. Zhong, S. Shu, and L. Chen. Two-Grid methods for Maxwell eigenvalue problems. SIAM J. Numer. Anal., 52(4):2027-2047, jan 2014.


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