Optimal order of uniform convergence for finite element method on Bakhvalov-type meshes[☆]

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Abstract

We propose a new analysis of convergence for a kth order $(k \ge 1)$ finite element method, which is applied on Bakhvalov-type meshes to a singularly perturbed two-point boundary value problem. A novel interpolant is introduced, which has a simple structure and is easy to generalize. By means of this interpolant, we prove an optimal order of uniform convergence with respect to the perturbation parameter. Numerical experiments illustrate these theoretical results.

Keywords: Singular perturbation, Convection–diffusion equation, Finite element method, Bakhvalov mesh, Uniform convergence

1. Introduction

We consider the two-point boundary value problem

$$Lu := -\varepsilon u'' - b(x)u' + c(x)u = f(x) \quad \text{in } \Omega := (0,1), \quad u(0) = u(1) = 0, \quad (1)$$

where ε is a positive parameter, b, c and f are sufficiently smooth functions such that $b(x) \ge \beta > 1$ on $\overline{\Omega}$ and

$$c(x) + \frac{1}{2}b'(x) \ge \gamma > 0 \quad \text{on } \bar{\Omega}$$
⁽²⁾

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with some constants β and γ . The condition (2) ensures that the boundary value problem has a unique solution. In the cases of interest the diffusion parameter ε can be arbitrarily small and satisfies $0 < \varepsilon \ll 1$. Thus this problem is *singuarly perturbed* and its solution typically features a boundary layer of width $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ at x = 0 (see [12]).

Solutions to singularly perturbed problems are characterized by the presence of boundary or interior *layers*, where solutions change rapidly. Numerical solutions of these problems are of significant mathematical interest. Classical numerical methods are often inappropriate, because in practice it is very unlikely that layers are fully resolved by common meshes. Hence specialised numerical methods are designed to compute accurate approximate solutions in an efficient way. For example, standard numerical methods on layer-adapted meshes, which are fine in layer regions and standard outside, are commonly used; see [12, 9] and many references therein. On these meshes, classical numerical methods are *uniformly* convergent with respect to the singular perturbation parameter; see [7]. Among them, there are two kinds of popular grids: Bakhvalov-type meshes (B-meshes) and Shishkin-type meshes (S-meshes); see [7].

The accuracy of finite difference methods on these locally refined meshes has been extensively studied and sharp error estimations have been derived (see [9, 5, 7]). For instance, in [7] the author presented convergence rates of $\mathcal{O}(N^{-1})$ and $\mathcal{O}(N^{-1} \ln N)$ for a first-order upwind difference scheme on Bakhvalov grid [1] and Shishkin grid [14], respectively, where N is the number of mesh intervals in each coordinate direction. Usually, the performance of B-meshes is superior to that of S-meshes. This advantage is more and more obvious when higher-order schemes are used. Besides, the width of the mesh subdomain used to resolve the layer is $\mathcal{O}(\varepsilon \ln(1/\varepsilon))$ for B-meshes and $\mathcal{O}(\varepsilon \ln N)$ for S-meshes. The former is independent of the mesh parameter N and this property will be important under certain circumstances.

For finite element methods, the development of numerical theories on Bmeshes is completely different from one on S-meshes. On standard Shishkin meshes Stynes and O'Riordan [15] derived a sharp uniform convergence in the energy norm for finite element method. Henceforward numerous articles deal with uniform convergence of finite element methods on S-meshes; see e.g. [12, 11, 18, 19, 8, 16, 13] and the references therein. However, it is still open for the optimal uniform convergence of finite element methods on B-meshes (see [11, Question 4.1] for more details).

This dilemma arises from the fact that the standard Lagrange interpolant does not work for uniform convergence of finite element methods on B-meshes. More specifically, the Lagrange interpolant cannot provide enough stability in L^2 norm on a special mesh interval, which lies in the fine part and is adjacent to the coarse part of B-meshes. In [10] and [2] a quasi-interpolant is used and provides enough stability for the optimal uniform convergence. Unfortunately, in both articles the analysis is limited to one dimension and linear finite element. It is hard to extend the analysis to higher dimensions or higher-order finite elements for singularly perturbed problems.

In this contribution we will study the optimal uniform convergence of a kth order ($k \ge 1$) finite element method on Bakhvalov-type meshes. A novel interpolant is constructed by redefining the standard Lagrange interpolant to the solution. This interpolant has a simple structure and it can also be applied to higher-dimensional problems in a straightforward way. By means of this novel function, we prove the optimal order of uniform convergence in a standard way.

The rest of the paper is organized as follows. In Section 2 we describe our regularity on the solution u to (1), introduce two Bakhvalov-type meshes and define the finite element method. Some preliminary results for the subsequent analysis are also derived in this section. In Section 3 we construct and analyze an interpolant Πu for the uniform convergence on B-meshes. In Section 4 uniform convergence is obtained by means of the interpolant Πu and careful derivations of the convective term in the bilinear form. In Section 5, numerical results illustrate our theoretical bounds.

We use the standard Sobolev spaces $W^{m,p}(D)$, $H^m(D) = W^{m,2}(D)$, $H_0^m(D)$ for nonnegative integers m and $1 \le p \le \infty$. Here D is any measurable subset of Ω . We denote by $|\cdot|_{W^{m,p}(D)}$ and $||\cdot||_{W^{m,p}(D)}$ the semi-norms and the norms in $W^{m,p}(D)$, respectively. On $H^m(D)$, $|\cdot|_{m,D}$ and $||\cdot||_{m,D}$ are the usual Sobolev semi-norm and norm. Denote by $||\cdot||_{L^p(D)}$ the norms in the Lebesgue spaces $L^p(D)$. We use the notation $(\cdot, \cdot)_D$ and $||\cdot||_D$ for the $L^2(D)$ -inner product and the $L^2(D)$ -norm, respectively. When $D = \Omega$ we drop the subscript D from the notation for simplicity. Throughout the article, all constants C and C_i are independent of ε and the mesh parameter N; unsubscripted constants C are generic and may take different values in different formulas while subscripted constants C_i are fixed.

2. Regularity, Bakhvalov mesh and finite element method

2.1. Regularity of the solution

Information about higher-order derivatives of the solution u of (1) are usually needed by uniform convergence of finite element methods. Such estimations appeared in [12, Lemma 1.9] and are reproduced in the following lemma.

Lemma 1. Let k be some positive integer. Assume that (2) holds true and b, c, f are sufficiently smooth. The solution u of (1) can be decomposed into

$$u = S + E, (3)$$

where the smooth part S and the layer part E satisfy LS = f and LE = 0, respectively. Furthermore, one has

$$|S^{(l)}(x)| \le C, \quad |E^{(l)}(x)| \le C\varepsilon^{-l} \exp\left(-\frac{x}{\varepsilon}\right) \quad \text{for } 0 \le l \le k+1.$$
 (4)

Note k depends on the regularity of the coefficients, in particular (4) holds for any $q \in \mathbb{N}$ if $b, c, f \in C^{\infty}[0, 1]$.

2.2. Bakhvalov mesh

Bakhvalov mesh first appeared in [1] and is constructed according to layer functions like E in Lemma 1. Its mesh generating function is piecewise and belongs to C^1 . Its breakpoint, which separates the mesh generating function, must be solved by a nonlinear equation and usually is not explicitly known (see [12, Part I $\S2.4.1$]).

In this article, we focus on two Bakhvalov-type meshes introduced in [10] and [5, 6]. Their breakpoints are known already, and both mesh generating functions do not belong to C^1 any longer. In [10] the Bakhvalov mesh is defined by

$$x = \psi(t) = \begin{cases} -\sigma \varepsilon \ln(1 - 2(1 - \varepsilon)t) & \text{for } t \in [0, 1/2], \\ 1 - d(1 - t) & \text{for } t \in (1/2, 1], \end{cases}$$
(5)

where $\sigma \ge k + 1$ with some positive integer k and d is used to ensure the continuity of $\psi(t)$ at t = 1/2. The mesh generating function in [5, 6] is defined by

$$x = \varphi(t) = \begin{cases} -\sigma \varepsilon \ln(1 - 2t) & \text{for } t \in [0, \vartheta], \\ 1 - d_1(1 - t) & \text{for } t \in (\vartheta, 1], \end{cases}$$
(6)

where $\sigma \ge k+1$, $\vartheta = 1/2 - C_1 \varepsilon$ with some positive constant C_1 independent of ε and N, d_1 is chosen so that $\varphi(t)$ is continuous at $t = \vartheta$. The original Bakhvalov mesh can be recovered from (6) by setting $\vartheta = 1/2 - \mathcal{C}(\varepsilon)\varepsilon$, where

$$0 < C_2 \le \mathcal{C}(\varepsilon) \le C_3. \tag{7}$$

For technical reasons, we assume $C_1 \leq 1/(\varepsilon N)$ and therefore $1/2 - N^{-1} \leq \vartheta < 1/2$. We also assume that $\varepsilon \leq N^{-1}$ in our analysis, as is generally the case in practice. If $\varepsilon > N^{-1}$, one sets $\psi(t) = \varphi(t) = t$, which generate uniform meshes.

Assume that N/2 is a positive integer and define the mesh points $x_i = \psi(i/N)$ or $x_i = \varphi(i/N)$ for i = 0, 1, ..., N. For both Bakhvalov meshes one usually has $x_{N/2} \leq 1/2$. Denote an arbitrary subinterval $[x_i, x_{i+1}]$ by I_i , its length by $h_i = x_{i+1} - x_i$ and a generic subinterval by I.

2.3. The finite element method

The weak form of problem (1) is to find $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega), \tag{8}$$

where $a(u, v) := \varepsilon(u', v') - (bu', v) + (cu, v)$. Note that the variational formulation

(8) has a unique solution by means of the Lax-Milgram lemma.

Define the ${\cal C}^0$ finite element space on the Bakhvalov meshes

$$V^N = \{ w \in C(\bar{\Omega}) : w(0) = w(1) = 0, w |_{I_i} \in P_k(I_i) \text{ for } i = 0, \dots, N-1 \}.$$

The finite element method for (8) reads as

$$a(u^N, v^N) = (f, v^N) \quad \forall v^N \in V^N.$$
(9)

The natural norm associated with $a(\cdot, \cdot)$ is defined by

$$||v||_{\varepsilon} := \left\{ \varepsilon |v|_1^2 + ||v||^2 \right\}^{1/2} \quad \forall v \in H^1(\Omega).$$

Using (2), it is easy to see that one has the coercivity

$$a(v^N, v^N) \ge \alpha \|v^N\|_{\varepsilon}^2 \quad \text{for all } v^N \in V^N$$
(10)

with $\alpha = \min\{1, \gamma\}$. It follows that u^N is well defined by (9) (see [3] and references therein).

2.4. Preliminary results of Bakhvalov meshes

In this subsection, we present some important properties of the Bakhvalov meshes and the layer function E, which are necessary for our uniform convergence.

We present some properties about the step sizes of Bakhavlov meshes as follows.

Lemma 2. For Bakhvalov mesh (5), one has

$$h_0 \le h_1 \le \ldots \le h_{N/2-2},\tag{11}$$

$$\frac{1}{4}\sigma\varepsilon \le h_{N/2-2} \le \sigma\varepsilon,\tag{12}$$

$$\frac{1}{2}\sigma\varepsilon \le h_{N/2-1} \le 2\sigma N^{-1},\tag{13}$$

$$N^{-1} \le h_i \le 2N^{-1} \quad N/2 \le i \le N - 1.$$
 (14)

On Bakhvalov mesh (6), bounds analogous to (11)-(14) also hold.

Proof. We just consider Bakhvalov mesh (5) and the other mesh can be similarly analyzed. Recalling that $x_{N/2} \leq 1/2$ and the Bakhvalov mesh separates $[x_{N/2}, 1]$ into N/2 uniform subintervals, one obtains (14). For $0 \leq i \leq N/2 - 1$, one has

$$h_i = x_{i+1} - x_i = \int_{i/N}^{(i+1)/N} \sigma \varepsilon \frac{2(1-\varepsilon)}{1-2(1-\varepsilon)t} \mathrm{d}t$$

and

$$\sigma \varepsilon \frac{2(1-\varepsilon)}{1-2(1-\varepsilon)iN^{-1}}N^{-1} \le h_i \le \sigma \varepsilon \frac{2(1-\varepsilon)}{1-2(1-\varepsilon)(i+1)N^{-1}}N^{-1}.$$
 (15)

From (15), we can prove (11), (12) and (13) easily.

We collect some bounds of the layer function E and the function $e^{-x/\varepsilon}$ on the Bakhvalov meshes in the following lemma.

Lemma 3. On Bakhvalov meshes (5) and (6), one has

$$|E(x_{N/2-1})| \le CN^{-\sigma}, \quad |E(x_{N/2})| \le C\varepsilon^{\sigma}, \tag{16}$$

$$||E||_{I_{N/2-1}} + \varepsilon ||E'||_{I_{N/2-1}} \le C\varepsilon^{1/2} N^{-\sigma}, \tag{17}$$

$$||E'||_{[x_{N/2},x_N]} \le C\varepsilon^{\sigma-1/2}.$$
 (18)

For $0 \leq i \leq N/2 - 2$ and $0 \leq \mu \leq \sigma$, we have

$$h_i^{\mu} \max_{x_i \le x \le x_{i+1}} e^{-x/\varepsilon} = h_i^{\mu} e^{-x_i/\varepsilon} \le C \varepsilon^{\mu} N^{-\mu}.$$
 (19)

Proof. We just consider Bakhvalov mesh (5) and the mesh (6) can be similarly analyzed.

Recalling $\varepsilon \leq N^{-1}$, we prove (16), (17) and (18) directly from (4). Let $0 \leq i \leq N/2 - 2$. From (5) one has

$$-\sigma\varepsilon\ln(1-2(1-\varepsilon)i/N) = x_i \le x \le x_{i+1} = -\sigma\varepsilon\ln(1-2(1-\varepsilon)(i+1)/N),$$

and for $x \in [x_i, x_{i+1}]$

$$(1 - 2(1 - \varepsilon)(i + 1)/N)^{\sigma} \le e^{-x/\varepsilon} \le e^{-x_i/\varepsilon} = (1 - 2(1 - \varepsilon)i/N)^{\sigma}.$$
 (20)

From (15) and (20), we have

$$\begin{split} h_{i}^{\mu} \max_{x_{i} \leq x \leq x_{i+1}} e^{-x/\varepsilon} &\leq C_{1}^{*} \varepsilon^{\mu} N^{-\mu} (1 - 2(1 - \varepsilon)i/N)^{\sigma} (1 - 2(1 - \varepsilon)(i + 1)/N)^{-\mu} \\ &\leq C_{1}^{*} \varepsilon^{\mu} N^{-\mu} (1 - 2(1 - \varepsilon)i/N)^{\sigma-\mu} \left(\frac{1 - 2(1 - \varepsilon)i/N}{1 - 2(1 - \varepsilon)(i + 1)/N} \right)^{\mu} \\ &\leq C_{1}^{*} C_{2}^{*} C_{3}^{*} \varepsilon^{\mu} N^{-\mu}, \\ \text{where } C_{1}^{*} &= (2\sigma(1 - \varepsilon))^{\mu} \leq (2\sigma)^{\mu} \text{ and for } 0 \leq i \leq N/2 - 2 \end{split}$$

$$C_2^* = (1 - 2(1 - \varepsilon)i/N)^{\sigma - \mu} \le 1, \ C_3^* = \left(\frac{1 - 2(1 - \varepsilon)i/N}{1 - 2(1 - \varepsilon)(i + 1)/N}\right)^{\mu} \le 2^{\mu}.$$

Thus (19) is proved.

3. Interpolation operator and interpolation errors

Now a new interpolation operator is introduced, which is used for our uniform convergence. Set $x_{i+j/k} := x_i + (j/k)h_i$ for i = 0, 1, ..., N - 1 and j = 1, ..., k - 1. For any $v \in C^0(\overline{\Omega})$ its Lagrange interpolant $v^I \in V^N$ on each Bakhvalov mesh is defined by

$$v^{I} = \sum_{i=0}^{N} v(x_{i})\theta_{i}(x) + \sum_{i=0}^{N-1} \sum_{j=1}^{k-1} v(x_{i+j/k})\theta_{i+j/k}(x),$$

where $\theta_i(x)$, $\theta_{i+j/k}(x)$ is the piecewise kth order Lagrange basis function satisfying the well-known delta properties associated with the nodes x_i and $x_{i+j/k}$, respectively. For the solution u to (1), recall (3) in Lemma 1 and define the interpolant Πu by

$$\Pi u = S^I + \pi E,\tag{21}$$

where S^{I} is the Lagrange interpolant to S and

$$(\pi E)(x) = \sum_{i=0, i \neq N/2-1}^{N} E(x_i)\theta_i(x) + \sum_{i=0, i \neq N/2-1}^{N-1} \sum_{j=1}^{k-1} v(x_{i+j/k})\theta_{i+j/k}(x).$$
(22)

Define

$$(\mathcal{P}E)(x) = E(x_{N/2-1})\theta_{N/2-1}(x) + \sum_{j=1}^{k-1} E(x_{N/2-1+j/k})\theta_{N/2-1+j/k}(x), \quad (23)$$

and clearly we have

$$(\pi E)(x) = E^{I} - (\mathcal{P}E)(x), \quad \Pi u = u^{I} - (\mathcal{P}E)(x), \quad (24)$$

$$\pi E|_{[x_0, x_{N/2-2}] \cup [x_{N/2}, x_N]} = E^I|_{[x_0, x_{N/2-2}] \cup [x_{N/2}, x_N]},$$
(25)
$$\Pi u \in V^N.$$

Interpolation theories in Sobolev spaces [4, Theorem 3.1.4] tell us that

$$\|v - v^{I}\|_{W^{l,q}(I_{i})} \le Ch_{i}^{k+1-l+1/q-1/p} |v|_{W^{k+1,p}(I_{i})},$$
(26)

for all $v \in W^{k+1,p}(I)$, where i = 0, 1, ..., N - 1, l = 0, 1 and $1 \le p, q \le \infty$.

Lemma 4. On Bakhvalov meshes (5) and (6), one has

$$||E - E^{I}||_{L^{\infty}(\Omega)} + ||S - S^{I}||_{L^{\infty}(\Omega)} + ||u - u^{I}||_{L^{\infty}(\Omega)} \le CN^{-(k+1)},$$
(27)

$$||E - E^{I}|| + ||S - S^{I}|| + ||u - u^{I}|| \le CN^{-(k+1)},$$
(28)

$$\|E^{I}\|_{I_{N/2-1}} \le Ch_{N/2-1}^{1/2} N^{-\sigma}, \|E^{I}\|_{[x_{N/2}, x_{N}]} \le C\varepsilon^{\sigma},$$
(29)

$$||E - E^I||_{\varepsilon} + ||u - u^I||_{\varepsilon} \le CN^{-k}, \tag{30}$$

$$\|(\mathcal{P}E)(x)\|_{\varepsilon} \le CN^{-\sigma},\tag{31}$$

where $(\mathcal{P}E)(x)$ is defined in (23).

Proof. We just consider Bakhvalov mesh (5) and mesh (6) can be similarly analyzed.

From (26) and (4), for $0 \le i \le N/2 - 2$ one has

$$||E - E^{I}||_{L^{\infty}(I_{i})} \leq Ch_{i}^{k+1}|E|_{W^{k+1,\infty}(I_{i})}$$

$$\leq C\varepsilon^{-(k+1)}h_{i}^{k+1}e^{-x_{i}/\varepsilon} \leq CN^{-(k+1)},$$
(32)

where we have used (19) with $\mu = k + 1$ and $\sigma \ge k + 1$. For $N/2 - 1 \le i \le N - 1$ we have

$$||E - E^{I}||_{L^{\infty}(I_{i})} \le ||E||_{L^{\infty}(I_{i})} + ||E^{I}||_{L^{\infty}(I_{i})} \le Ce^{-x_{i}/\varepsilon} \le CN^{-\sigma}.$$
 (33)

Collecting (32), (33) and noting $\sigma \ge k+1$, we prove $||E-E^I||_{L^{\infty}(\Omega)} \le CN^{-(k+1)}$. Lemma 2, (26) and (4) yield $||S-S^I||_{L^{\infty}(\Omega)} \le CN^{-(k+1)}$. From (3) we prove (27). The bound (28) can be easily obtained from (27) and Hölder inequalities. From (4) and direct calculations one can easily prove (29).

Now we are ready to analyze $||E - E^I||_{\varepsilon}$. First we decompose $\varepsilon ||(E - E^I)'||^2$ into the following two parts

$$\varepsilon \| (E - E^{I})' \|^{2} = \varepsilon \sum_{i=0}^{N/2-2} \| (E - E^{I})' \|_{I_{i}}^{2} + \varepsilon \| (E - E^{I})' \|_{[x_{N/2-1}, x_{N}]}^{2}$$

$$=: \mathcal{S}_{1} + \mathcal{S}_{2}.$$
(34)

From (26), (4), (19) with $\mu = (2k+1)/2$ and $\sigma \ge k+1$, we have

$$S_{1} \leq C\varepsilon \sum_{i=0}^{N/2-2} h_{i}^{2k} |E|_{k+1,I_{i}}^{2} \leq C\varepsilon \sum_{i=0}^{N/2-2} h_{i}^{2k} \int_{x_{i}}^{x_{i+1}} \varepsilon^{-2(k+1)} e^{-2x/\varepsilon} dx$$
$$\leq C\varepsilon \sum_{i=0}^{N/2-2} h_{i}^{2k} \varepsilon^{-2(k+1)} e^{-2x_{i}/\varepsilon} h_{i} \leq C\varepsilon \sum_{i=0}^{N/2-2} \varepsilon^{-2(k+1)} \left(h_{i}^{(2k+1)/2} e^{-x_{i}/\varepsilon} \right)^{2} (35)$$
$$\leq C\varepsilon \sum_{i=0}^{N/2-2} \varepsilon^{-2(k+1)} \varepsilon^{2k+1} N^{-(2k+1)} \leq CN^{-2k}.$$

From a triangle inequality, (13), (14), (17), (18), inverse inequality [4, Theorem 3.2.6] and (29), one has

$$S_{2} \leq C\varepsilon \left(\|E'\|_{[x_{N/2-1},x_{N}]}^{2} + \|(E^{I})'\|_{[x_{N/2-1},x_{N/2}]}^{2} + \|(E^{I})'\|_{[x_{N/2},x_{N}]}^{2} \right)$$

$$\leq C\varepsilon (\varepsilon^{-1}N^{-2\sigma} + h_{N/2-1}^{-2} \|E^{I}\|_{[x_{N/2},x_{N}]}^{2} + N^{2} \|E^{I}\|_{[x_{N/2},x_{N}]}^{2} \right)$$

$$\leq CN^{-2\sigma} + C\varepsilon^{2\sigma+1}N^{2}.$$
(36)

Substituting (35), (36) into (34) and recalling $\varepsilon \leq N^{-1}$ and $\sigma \geq k+1$, we obtain

 $\varepsilon \| (E - E^I)' \|^2 \le C N^{-2k}$

and prove $||E - E^I||_{\varepsilon} \leq CN^{-k}$ from (28). From (26) and Lemma 2, one can easily prove $||S - S^I||_{\varepsilon} \leq C(\varepsilon^{1/2}N^{-(k-1/2)} + N^{-(k+1/2)})$. A triangle inequality yields $||u - u^I||_{\varepsilon} \leq CN^{-k}$. Thus (30) is proved.

Now we consider (31). Direct calculations yield

$$\begin{aligned} \|(\mathcal{P}E)(x)\|_{\varepsilon} \leq & |E(x_{N/2-1})| \|\theta_{N/2-1}(x)\|_{\varepsilon} + \sum_{j=1}^{k-1} |E(x_{N/2-1+j/k})| \|\theta_{N/2-1+j/k}(x)\|_{\varepsilon} \\ \leq & CN^{-\sigma} \sum_{j=0}^{k-1} \|\theta_{N/2-1+j/k}(x)\|_{\varepsilon} \leq CN^{-\sigma}, \end{aligned}$$

where we have used (23), (16), (12) and (13).

4. Uniform convergence

Introduce $\chi := \Pi u - u^N$. From (10), the Galerkin orthogonality, (3), (21), (24) and integration by parts for $\int_0^1 b(\pi E - E)' \chi dx$, one has

$$\begin{aligned} \alpha \|\chi\|_{\varepsilon}^{2} &\leq a(\chi,\chi) = a(\Pi u - u,\chi) \\ &= \varepsilon \int_{0}^{1} (u^{I} - u)'\chi' \mathrm{d}x - \varepsilon \int_{0}^{1} (\mathcal{P}E)'\chi' \mathrm{d}x \\ &+ \int_{0}^{1} b(\pi E - E)\chi' \mathrm{d}x - \int_{0}^{1} b(S^{I} - S)'\chi \mathrm{d}x \\ &+ \int_{0}^{1} b'(\pi E - E)\chi \mathrm{d}x + \int_{0}^{1} c(u^{I} - u)\chi \mathrm{d}x - \int_{0}^{1} c(\mathcal{P}E)\chi \mathrm{d}x \\ &=: \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V} + \mathrm{VI} + \mathrm{VII}. \end{aligned}$$

$$(37)$$

In the following we will analyze the terms in the right-hand side of (37). Hölder inequalities yield

$$(\mathbf{I} + \mathbf{V}\mathbf{I}) + (\mathbf{I}\mathbf{I} + \mathbf{V}\mathbf{I}\mathbf{I}) \le C \|u - u^{I}\|_{\varepsilon} \|\chi\|_{\varepsilon} + C \|\omega_{E}\|_{\varepsilon} \|\chi\|_{\varepsilon} \le CN^{-k} \|\chi\|_{\varepsilon}, \quad (38)$$

where (30) and (31) have been used. From (26) and (4), one has $||(S^I - S)'|| \le CN^{-k}$ and $||\pi E - E|| \le CN^{-(k+1)}$ from (28) and (31). Consequently we obtain

$$IV + V \le C(\|(S^{I} - S)'\| + \|\pi E - E\|)\|\chi\| \le CN^{-k}\|\chi\|.$$
(39)

We put the arguments for III in the following lemma.

Lemma 5. Let the mesh $\{x_i\}$ be either the Bakhvalov mesh (5) or the Bakhvalov mesh (6). Let πE be defined in (22). Then one has

$$|\mathrm{III}| = \left| \int_0^1 b(\pi E - E) \chi' \mathrm{d}x \right| \le C N^{-k} \|\chi\|_{\varepsilon}.$$
 (40)

Proof. According to (25), the term $(b(\pi E - E), \chi')$ is separated into three parts as follows:

$$\int_{0}^{1} b(\pi E - E)\chi' dx = \int_{x_{0}}^{x_{N/2-2}} b(E^{I} - E)\chi' dx + \int_{x_{N/2-2}}^{x_{N/2}} b(\pi E - E)\chi' dx + \int_{x_{N/2}}^{x_{N}} b(E^{I} - E)\chi' dx \quad (41) =:\mathcal{I}_{1} + \mathcal{I}_{2} + \mathcal{I}_{3}.$$

From Hölder inequalities, (26), (19) with $\mu = k + 1$ and $\sigma \ge k + 1$, we obtain

$$\begin{aligned} |\mathcal{I}_{1}| &\leq C \sum_{i=0}^{N/2-3} \int_{x_{i}}^{x_{i+1}} |E^{I} - E| |\chi'| dx \\ &\leq C \sum_{i=0}^{N/2-3} \|E^{I} - E\|_{L^{\infty}(I_{i})} \|\chi'\|_{L^{1}(I_{i})} \\ &\leq C \sum_{i=0}^{N/2-3} h_{i}^{k+1} \varepsilon^{-(k+1)} e^{-x_{i}/\varepsilon} \cdot h_{i}^{1/2} \|\chi'\|_{I_{i}} \leq C \varepsilon^{1/2} \sum_{i=0}^{N/2-3} N^{-(k+1)} \|\chi'\|_{I_{i}} \quad (42) \\ &\leq C \varepsilon^{1/2} \left(\sum_{i=0}^{N/2-3} N^{-2(k+1)} \right)^{1/2} \left(\sum_{i=0}^{N/2-3} \|\chi'\|_{I_{i}}^{2} \right)^{1/2} \\ &\leq C N^{-(k+1/2)} \|\chi\|_{\varepsilon}, \end{aligned}$$

where (11) and (12) have been used.

From Hölder inequalities and inverse inequalities, one has

$$\begin{aligned} |\mathcal{I}_{3}| &\leq C \|E^{I} - E\|_{[x_{N/2}, x_{N}]} \|\chi'\|_{[x_{N/2}, x_{N}]} \\ &\leq CN^{-(k+1)} \cdot N \|\chi\|_{[x_{N/2}, x_{N}]} \leq CN^{-k} \|\chi\|, \end{aligned}$$
(43)

where (28) has been used.

Now we analyze the term \mathcal{I}_2 . Note $\pi E = E^I - E(x_{N/2-1})\theta_{N/2-1}(x)$ on $[x_{N/2-2}, x_{N/2-1}]$ and one has

$$\begin{aligned} \left| \int_{x_{N/2-2}}^{x_{N/2-1}} b(\pi E - E) \chi' dx \right| \\ \leq C \int_{x_{N/2-2}}^{x_{N/2-1}} |E^{I} - E| |\chi'| dx + C|E(x_{N/2-1})| \int_{x_{N/2-2}}^{x_{N/2-1}} |\theta_{N/2-1}\chi'| dx \\ \leq C \left(\|E^{I} - E\|_{L^{\infty}(I_{N/2-2})} + |E(x_{N/2-1})| \right) \|\chi'\|_{L^{1}(I_{N/2-2})} \\ \leq C(h_{N/2-2}^{k+1} \varepsilon^{-(k+1)} e^{-x_{N/2-2}/\varepsilon} + N^{-\sigma}) \cdot h_{N/2-2}^{1/2} \|\chi'\|_{I_{N/2-2}} \\ \leq C(N^{-(k+1)} + N^{-\sigma}) \|\chi\|_{\varepsilon, I_{N/2-2}} \end{aligned}$$
(44)

where (22), Hölder inequalities, (26), (19) with $\mu = k + 1$ and $\sigma \ge k + 1$, (12) have been used. On $[x_{N/2-1}, x_{N/2}]$, we have $\pi E = E(x_{N/2})\theta_{N/2}(x)$ from (22)

and

$$\left| \int_{x_{N/2-1}}^{x_{N/2}} b(\pi E - E) \chi' dx \right| \\
\leq C |E(x_{N/2})| \int_{x_{N/2-1}}^{x_{N/2}} |\theta_{N/2}(x)| |\chi'| dx + C \int_{x_{N/2-1}}^{x_{N/2}} |E| |\chi'| dx \\
\leq C \left(\varepsilon^{\sigma} \|\theta_{N/2}\|_{I_{N/2-1}} + C \|E\|_{I_{N/2-1}} \right) \|\chi'\|_{I_{N/2-1}} \\
\leq C \left(\varepsilon^{\sigma} h_{N/2-1}^{1/2} + \varepsilon^{1/2} N^{-\sigma} \right) \|\chi'\|_{I_{N/2-1}} \\
\leq C (\varepsilon^{\sigma-1/2} N^{-1/2} + N^{-\sigma}) \|\chi\|_{\varepsilon, I_{N/2-1}},$$
(45)

where Hölder inequalities, (16), (17) and (13) have been used. From (44) and (45) we prove

$$|\mathcal{I}_2| \le C N^{-(k+1)} \|\chi\|_{\varepsilon},\tag{46}$$

where $\varepsilon \leq N^{-1}$ and $\sigma \geq k+1$ have been used. Substituting (42), (43) and (46) into (41), we are done.

Now we are in a position to present the main result.

Theorem 1. Let the mesh $\{x_i\}$ be either Bakhvalov mesh (5) or Bakhvalov mesh (6) with $\sigma \ge k + 1$. Let u and u^N be the solutions of (1) and (9), respectively. Then one has

$$\|u - u^N\|_{\varepsilon} \le CN^{-k}.$$
(47)

Proof. Substituting (38), (39) and (40) into (37), we obtain $\|\Pi u - u^N\|_{\varepsilon} \leq CN^{-k}$. From (24) and (31) we have $\|u^I - u^N\|_{\varepsilon} \leq \|\Pi u - u^N\|_{\varepsilon} + \|(\mathcal{P}E)\|_{\varepsilon} \leq CN^{-k}$. From a triangle inequality and (30), one has

$$\|u - u^N\|_{\varepsilon} \le \|u - u^I\|_{\varepsilon} + \|u^I - u^N\|_{\varepsilon} \le CN^{-k}.$$

Thus we are done.

Remark 1. For the original Bakhvalov mesh [1], Theorem 1 also holds true because of the property (7). The analysis is similar to one on the mesh (6).

5. Numerical experiments

We now present the results of some numerical experiments in order to illustrate the conclusions of Theorem 1, and to check if they are sharp. All calculations were carried out by using Intel Visual FORTRAN 11 and the discrete problems were solved by the LU factorization.

The following boundary value problem is considered

$$-\varepsilon u'' - (3-x)u' + u = f(x) \quad \text{in } \Omega = (0,1),$$

$$u(0) = u(1) = 0,$$

(48)

where the right-hand side f is chosen such that

$$u(x) = (1-x)(1-e^{-2x/\varepsilon}) = 1-x-e^{-2x/\varepsilon}+xe^{-2x/\varepsilon}.$$
(49)

is the exact solution. The solution (49) exhibits typical boundary layer behavior.

For our numerical experiments we consider $\varepsilon = 10^{-4}, 10^{-5}, \dots, 10^{-9}, k = 1, 2, 3, 4$ and $N = 8, 16, \dots$. For both Bakhvalov meshes (5) and (6) we take $\sigma = k + 1$. Set $C_1 = 5(k + 1)/4$ in (6).

We estimate the uniform errors for a fixed N by taking the maximum error over a wide range of ε , namely

$$e^N := \max_{\varepsilon = 10^{-4}, 10^{-5}, \dots, 10^{-9}} \|u - u^N\|_{\varepsilon}$$

Rates of convergence \boldsymbol{r}_e^N are computed by means of the formula

$$r_e^N = \log_2(e^N/e^{2N}).$$

The numerical results are presented in Tables 1 and 2. The errors e^N and the convergence rates r_e^N are in accordance with Theorem 1 and illustrate its sharpness. Moreover, in Tables 1 and 2 we can observe that Bakhvalov mesh (5) gives almost the same performance as Bakhvalov mesh (6).

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	k = 1				k = 2			
	B-mesh (5)		B-mesh (6)		B-mesh (5)		B-mesh (6)	
Ν	e^N	r_e^N	e^N	r_e^N	e^N	r_e^N	e^N	r_e^N
8	0.338E + 00	1.02	0.339E-01	1.02	$0.103E{+}00$	2.00	$0.103E{+}00$	2.00
16	$0.167E{+}00$	1.00	0.167E + 00	1.00	0.257E-01	2.00	0.257 E-01	2.00
32	0.834E-01	1.00	0.834 E-01	1.00	0.642E-02	2.00	0.642E-02	2.00
64	0.417E-01	1.00	0.417E-01	1.00	0.160E-02	2.00	0.160E-02	2.00
128	0.208E-01	1.00	0.208E-01	1.00	0.401E-03	2.00	0.401E-03	2.00
256	0.104E-01	1.00	0.104 E-01	1.00	0.100E-03	1.99	0.100E-03	1.99
512	0.521E-02	1.00	0.521 E-02	1.00	0.251E-04	2.00	0.251E-04	2.00
1024	0.260E-02	1.00	0.260E-02	1.00	0.627E-05	2.00	0.627 E-05	2.00
2048	0.130E-02		0.130E-02	—	0.157E-05		0.157E-05	

Table 1: Errors and convergence rates for problem (48)

Table 2: Errors and convergence rates for problem (48)

	k = 3				k = 4			
	B-mesh (5)		B-mesh (6)		B-mesh (5)		B-mesh (6)	
N	e^N	r_e^N	e^N	r_e^N	e^N	r_e^N	e^N	r_e^N
8	0.301E-01	2.94	0.301E-01	2.94	0.898E-02	3.88	0.898E-02	3.88
16	0.393E-02	2.99	0.393E-02	2.99	0.609E-03	3.97	0.609E-03	3.97
32	0.496E-03	3.00	0.496E-03	3.00	0.388E-04	3.99	0.388E-04	3.99
64	0.622 E-04	3.00	0.622 E-04	3.00	0.244 E-05	4.00	0.244 E-05	4.00
128	0.778 E-05	3.00	0.778 E-05	3.00	0.153E-06	4.00	0.153E-06	4.00
256	0.973E-06	3.00	0.973E-06	3.00	0.954 E-08	4.00	0.954 E-08	4.00
512	0.122E-06	3.00	0.122E-06	3.00	0.596E-09	3.84	0.596E-09	3.79
1024	0.152E-07		0.152E-07	—	0.415E-10		0.430E-10	

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