The operator-splitting method for Cahn-Hilliard is stable

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Abstract

We prove energy stability of a standard operator-splitting method for the Cahn-Hilliard equation. We establish uniform bound of Sobolev norms of the numerical solution and convergence of the splitting approximation. This is the first unconditional energy stability result for the operatorsplitting method for the Cahn-Hilliard equation. Our analysis can be extended to many other models.

1 Introduction

We consider numerical solutions of the Cahn-Hilliard ([2]) equation:

$$\begin{cases} \partial_t u = \Delta(-\nu\Delta u + f(u)), & (t, x) \in (0, \infty) \times \Omega, \\ u\Big|_{t=0} = u_0, \end{cases}$$
 (1.1)

where u = u(t, x) is a real-valued function corresponding to the concentration difference in a binary system. The parameter $\nu > 0$ is usually called the mobility coefficient which is taken to a constant here for simplicity. The nonlinear term f(u) is derived from a standard double well potential, namely:

$$f(u) = u^3 - u = F'(u), \quad F(u) = \frac{1}{4}(u^2 - 1)^2.$$

Due to this specific choice of equal-well double potential, the minima of the potential are located at $u = \pm 1$ which corresponds different phases or states. The

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length scale of the transitional region is usual proportional to $\sqrt{\nu}$. In this note we take the spatial domain Ω in (1.1) as the two-dimensional 2π -periodic torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2 = [-\pi,\pi]^2$. Our analysis extends to other physical dimensions $d \leq 3$ but we choose the prototypical case d=2 to simplify the presentation. For simplicity we consider mean zero initial data, that is

$$\int_{\mathbb{T}^2} u_0(x)dx = 0. \tag{1.2}$$

For smooth solutions, there is the mass conservation law

$$\frac{d}{dt}M(t) = \frac{d}{dt} \int_{\Omega} u(t,x)dx \equiv 0. \tag{1.3}$$

It follows that $u(t,\cdot)$ has zero mean for all t>0. For the class of mean-zero functions with suitable regularity, one can employ the operator $|\nabla|^s$ for s<0 as the Fourier multiplier $|k|^s \cdot 1_{k\neq 0}$. The system (1.1) naturally arises as a gradient flow of a Ginzburg-Landau type energy functional $\mathcal{E}(u)$ in H^{-1} , namely

$$\partial_t u = -\frac{\delta \mathcal{E}}{\delta u}\Big|_{H^{-1}} = \Delta(\frac{\delta \mathcal{E}}{\delta u}),$$
 (1.4)

where $\frac{\delta \mathcal{E}}{\delta u}\Big|_{H^{-1}}$, $\frac{\delta \mathcal{E}}{\delta u}$ denote the standard variational derivatives in H^{-1} and L^2 respectively, and

$$\mathcal{E}(u) = \int_{\Omega} \left(\frac{1}{2} \nu |\nabla u|^2 + F(u) \right) dx = \int_{\Omega} \left(\frac{1}{2} \nu |\nabla u|^2 + \frac{1}{4} (u^2 - 1)^2 \right) dx. \tag{1.5}$$

For smooth solutions, the fundamental energy conservation law takes the form

$$\frac{d}{dt}\mathcal{E}(u(t)) + \||\nabla|^{-1}\partial_t u\|_2^2 = \frac{d}{dt}\mathcal{E}(u(t)) + \int_{\Omega} |\nabla(-\nu\Delta u + f(u))|^2 dx = 0. \quad (1.6)$$

It follows that

$$\mathcal{E}(u(t)) \le \mathcal{E}(u(s)), \quad \forall t \ge s;$$
 (1.7a)

$$\|\nabla u(t)\|_{2} \le \sqrt{\frac{2}{\nu}} \mathcal{E}(u(t)) \le \sqrt{\frac{2}{\nu}} \mathcal{E}(u_{0}), \qquad \forall t > 0.$$
 (1.7b)

In particular, one obtains a priori \dot{H}^1 -norm control of the solution for all t > 0. Since the scaling-critical space for CH is L^2 in 2D, the global wellposedness and regularity for H^1 -initial data follows easily.

For $\tau > 0$, we let $S_L(\tau) = e^{-\tau \nu \Delta^2}$ be the exact solution operator to the linear equation:

$$\partial_t u = -\nu \Delta^2 u. \tag{1.8}$$

We define $S_N(\tau): w \to u$ as the solution operator to the following problem:

$$\frac{u-w}{\tau} = \Delta(w^3 - w). \tag{1.9}$$

In yet other words,

$$u = S_N(\tau)w = w + \tau \Delta(w^3 - w).$$
 (1.10)

This is one of the simplest discretization on the timer interval $[0, \tau]$ for the exact problem

$$\begin{cases} \partial_t u = \Delta(u^3 - u), & t > 0; \\ u\big|_{t=0} = w. \end{cases}$$
 (1.11)

By using the operator-splitting, the solution of the original equation from time t to time $t + \tau$ is approximated as

$$u(t+\tau,x) \approx \Big(S_L(\tau)S_N(\tau)u\Big)(t,x).$$
 (1.12)

The main purpose of this note is establish stability of the above operator-splitting algorithm. Prior to our work, there were very few rigorous results on the analysis of the operator-splitting type algorithms for the Cahn-Hilliard equation and similar models. In [24], Weng, Zhai and Feng considered a viscous Cahn-Hilliard model of the form

$$(1 - \alpha)\partial_t u = \Delta(-\epsilon^2 \Delta u + f(u) + \alpha \partial_t u), \tag{1.13}$$

where $0 < \alpha < 1$. They considered a fast explicit Strang splitting and established stability and convergence under the assumption that $A = \|\nabla u^{\text{num}}\|_{\infty}^2$, $B = \|u^{\text{num}}\|_{\infty}^2$ are bounded, and satisfy a technical condition 6A + 8 - 24B > 0 (see Theorem 1 on pp. 7 of [24]), where u^{num} denotes the numerical solution. In [23], Gidey and Reddy considered a convective Cahn-Hilliard model of the form

$$\partial_t u - \gamma \nabla \cdot \mathbf{h}(u) + \epsilon^2 \Delta^2 u = \Delta(f(u)),$$
 (1.14)

where $\mathbf{h}(u) = \frac{1}{2}(u^2, u^2)$. They considered operator-splitting of (1.14) into hyperbolic part, nonlinear diffusion part and diffusion part respectively, and obtained various conditional results concerning certain weak solutions. In [22], Cheng, Kurganov, Qu and Tang considered the Strang splitting for the Cahn-Hilliard equation and molecular beam epitaxy type models. Some conditional results were given in [22] but rigorous analysis of energy stability has remained open. The purpose of this note is to establish a new theoretical framework for the rigorous analysis of energy stability and higher-order Sobolev-norm stability for the operator-splitting method applied to these difficult equations. Our first result establishes uniform Sobolev control of the numerical solution for all time.

Theorem 1.1. Let $\nu > 0$ and consider the two-dimensional periodic torus $\mathbb{T}^2 = [-\pi, \pi]^2$. Assume the initial data $u^0 \in H^{k_0}(\mathbb{T}^2)$ $(k_0 \ge 1 \text{ is an integer})$ and has mean zero. Let $\tau > 0$ and define

$$u^{n+1} = S_L(\tau)S_N(\tau)u^n, \quad n \ge 0.$$
 (1.15)

There exists a constant $\tau_* > 0$ depending only on $||u^0||_{H^1}$ and ν , such that if $0 < \tau < \tau_*$, then

$$\sup_{n>0} \|u^n\|_{H^{k_0}} \le A_1 < \infty, \tag{1.16}$$

where $A_1 > 0$ depends on $(\|u^0\|_{H^{k_0}}, \nu, k_0)$.

Remark 1.1. Similar statements also hold if we consider $u^{n+1} = S_N(\tau)S_L(\tau)u^n$. Theorem 1.1 is a special case of Theorem 3.2 in Section 3.

Our second result establishes the convergence of the operator splitting approximation.

Theorem 1.2 (Convergence of the splitting approximation). Assume the initial data $u^0 \in H^8(\mathbb{T}^2)$ with mean zero. Let u^n be defined as in Theorem 1.1. Let u be the exact PDE solution to (1.1) corresponding to initial data u^0 . Let $0 < \tau < \tau_*$ as in Theorem 1.1. Then for any T > 0, we have

$$\sup_{n \ge 1, n\tau \le T} \|u^n - u(n\tau, \cdot)\|_{L^2(\mathbb{T}^2)} \le C \cdot \tau, \tag{1.17}$$

where C > 0 depends on $(\nu, ||u^0||_{H^8}, T)$.

Remark 1.2. The regularity assumption on initial data can be lowered but we shall not dwell on this issue here for simplicity of presentation. One can also work out the convergence in higher Sobolev norms. We shall not pursue this issue here.

The rest of this note is organized as follows. In Section 2 we set up the notation and collect some preliminary lemmas. In Section 3 we analyze in detail the propagator $S_L(\tau)S_N(\tau)$ and prove Theorem 3.2. Theorem 1.1 follows as a special case of Theorem 3.2. In Section 4 we give the proof of Theorem 1.2.

2 Notation and preliminaries

For any two positive quantities X and Y, we shall write $X \lesssim Y$ or $Y \gtrsim X$ if $X \leq CY$ for some constant C > 0 whose precise value is unimportant. We shall write $X \sim Y$ if both $X \lesssim Y$ and $Y \lesssim X$ hold. We write $X \lesssim_{\alpha} Y$ if the constant

C depends on some parameter α . We shall write X = O(Y) if $|X| \lesssim Y$ and $X = O_{\alpha}(Y)$ if $|X| \lesssim_{\alpha} Y$.

We shall denote $X \ll Y$ if $X \leq cY$ for some sufficiently small constant c. The smallness of the constant c is usually clear from the context. The notation $X \gg Y$ is similarly defined. Note that our use of \ll and \gg here is different from the usual Vinogradov notation in number theory or asymptotic analysis.

For any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we denote $|x| = |x|_2 = \sqrt{x_1^2 + \dots + x_d^2}$, and $|x|_{\infty} = \max_{1 \le j \le d} |x_j|$. Also occasionally we use the Japanese bracket notation: $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$.

We denote by $\mathbb{T}^d = [-\pi, \pi]^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$ the usual 2π -periodic torus. For $1 \leq p \leq \infty$ and any function $f: x \in \mathbb{T}^d \to \mathbb{R}$, we denote the Lebesgue L^p -norm of f as

$$||f||_{L_x^p(\mathbb{T}^d)} = ||f||_{L^p(\mathbb{T}^d)} = ||f||_p.$$

If $(a_j)_{j\in I}$ is a sequence of complex numbers and I is the index set, we denote the discrete l^p -norm as

$$\|(a_j)\|_{l_j^p(j\in I)} = \|(a_j)\|_{l^p(I)} = \begin{cases} \left(\sum_{j\in I} |a_j|^p\right)^{\frac{1}{p}}, & 0
$$(2.1)$$$$

For example, $\|\hat{f}(k)\|_{l_k^2(\mathbb{Z}^d)} = \left(\sum_{k\in\mathbb{Z}^d} |\hat{f}(k)|^2\right)^{\frac{1}{2}}$. If $f = (f_1, \dots, f_m)$ is a vector-valued function, we denote $|f| = \sqrt{\sum_{j=1}^m |f_j|^2}$, and $\|f\|_p = \|(\sum_{j=1}^m f_j^2)^{\frac{1}{2}}\|_p$. We use similar convention for the corresponding discrete l^p norms for the vector-valued case.

We use the following convention for the Fourier transform pair:

$$\hat{f}(k) = \int_{\mathbb{T}^d} f(x)e^{-ik\cdot x}dx, \quad f(x) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \hat{f}(k)e^{ik\cdot x}, \tag{2.2}$$

and denote for $0 \leq s \in \mathbb{R}$,

$$||f||_{\dot{H}^s} = ||f||_{\dot{H}^s(\mathbb{T}^d)} = ||\nabla|^s f||_{L^2(\mathbb{T}^d)} \sim ||k|^s \hat{f}(k)||_{l^2_{L}(\mathbb{Z}^d)}, \tag{2.3a}$$

$$||f||_{H^s} = \sqrt{||f||_2^2 + ||f||_{\dot{H}^s}^2} \sim ||\langle |k| \rangle^s \hat{f}(k)||_{l_k^2(\mathbb{Z}^d)}.$$
(2.3b)

Lemma 2.1. Let $d \leq 3$ and $\beta > 0$. Consider on the torus $\mathbb{T}^d = [-\pi, \pi]^d$,

$$K(x) = \mathcal{F}^{-1}(e^{-\beta|k|^4}) = e^{-\beta\Delta^2}\delta_0,$$
 (2.4)

where δ_0 is the periodic Dirac comb. Then for any $1 \leq p \leq \infty$,

$$||K||_{L^p(\mathbb{T}^d)} \le c_{d,p} \left(1 + \beta^{-d(\frac{1}{4} - \frac{1}{4p})}\right),$$
 (2.5)

where $c_{d,p} > 0$ depends only on d and p. Define

$$\widetilde{K} = \mathcal{F}^{-1}(e^{-\beta|k|^4} 1_{k \neq 0}).$$
 (2.6)

Then

$$\|\widetilde{K}\|_{L^p(\mathbb{T}^d)} \le \widetilde{c}_{d,p} \,\beta^{-d(\frac{1}{4} - \frac{1}{4p})},$$
 (2.7)

where $\tilde{c}_{d,p} > 0$ depends only on d and p.

Remark 2.1. Define $K_w(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-\beta|\xi|^4} d\xi$. By the usual Poisson summation formula, it is not difficult to check that

$$K(x) = \sum_{l \in \mathbb{Z}^d} K_w(x + 2\pi l). \tag{2.8}$$

This identity will be used below without explicit mentioning. We note that a formal proof of (2.8) may proceed as follows.

$$(2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} e^{-\beta |k|^4} e^{ik \cdot x}$$

$$= (2\pi)^{-d} \sum_{k \in \mathbb{Z}^d} \int_{\mathbb{R}^d} K_w(y) e^{ik \cdot (x-y)} dy$$

$$= \int_{\mathbb{R}^d} K_w(y) \sum_{l \in \mathbb{Z}^d} \delta(x - y - 2\pi l) dy = \sum_{l \in \mathbb{Z}^d} K_w(x + 2\pi l).$$

The above formal computation can be justified by the usual limiting process. We omit the details.

Proof of Lemma 2.1. Define

$$K_1(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} e^{-|\xi|^4} d\xi.$$

It is easy to check that $|K_1(x)| \lesssim \langle x \rangle^{-10}$ and $K_1 \in L^1_x(\mathbb{R}^d)$ for $d \leq 3$. Now note that for $d \leq 3$, if $|x|_{\infty} \leq \pi$, then $|x| \leq \sqrt{d}\pi \leq \sqrt{3}\pi$. Thus if $|l| \geq 400$, then

$$\pi|l| \le |x + 2\pi l| \le 4\pi|l|, \qquad \forall |x|_{\infty} \le \pi.$$

It follows that for all $1 \le p \le \infty$ and $|l| \ge 400$,

$$\|\langle \beta^{-\frac{1}{4}}(x+2\pi l)\rangle^{-10}\|_{L_x^p(|x|_{\infty}\leq \pi)} \lesssim \langle \beta^{-\frac{1}{4}}\pi |l|\rangle^{-10}.$$

Clearly then

$$||K||_{L^{p}(\mathbb{T}^{d})} \leq \beta^{-\frac{d}{4}} \sum_{l \in \mathbb{Z}^{d}} ||K_{1}(\beta^{-\frac{1}{4}}(x+2\pi l))||_{L^{p}_{x}(|x|_{\infty} \leq \pi)}$$

$$\lesssim \beta^{-\frac{d}{4}} \sum_{|l| \leq 400} ||K_{1}(\beta^{-\frac{1}{4}}(x+2\pi l))||_{p} + \sum_{|l| > 400} \beta^{-\frac{d}{4}} \langle \beta^{-\frac{1}{4}} \pi |l| \rangle^{-10}$$

$$\lesssim \beta^{-d(\frac{1}{4} - \frac{1}{4p})} + 1. \tag{2.9}$$

Now we consider the estimate for $\widetilde{K}(x) = K(x) - \frac{1}{(2\pi)^d}$. Obviously by using the previous bound we have $\|\widetilde{K}\|_1 \lesssim \|K\|_1 + 1 \lesssim 1$. Alternatively one can compute

$$\|\tilde{K}\|_{L_x^1(\mathbb{T}^d)} \lesssim 1 + \|\sum_{l \in \mathbb{Z}^d} \beta^{-\frac{d}{4}} |K_1(\beta^{-\frac{1}{4}}(x+2\pi l))|\|_{L_x^1(\mathbb{T}^d)} \lesssim 1 + \|K_1\|_{L_x^1(\mathbb{R}^d)} \lesssim 1.$$

We bound the L^2 norm as

$$\|\widetilde{K}\|_{L_x^2(\mathbb{T}^d)} \lesssim \|e^{-\beta|k|^4}\|_{l_k^2(0\neq k\in\mathbb{Z}^d)} \lesssim \beta^{-\frac{d}{8}}.$$

Similarly,

$$\|\widetilde{K}\|_{L_x^{\infty}(\mathbb{T}^d)} \lesssim \|e^{-\beta|k|^4}\|_{l_k^1(0\neq k\in\mathbb{Z}^d)} \lesssim \beta^{-\frac{d}{4}}.$$

By using interpolation we then get the L^p estimate.

Lemma 2.2. Let d=2 and $\nu>0$. Let $\tau>0$. Then for any $g\in L^4(\mathbb{T}^2)$ with zero mean, we have

$$||e^{-\nu\tau\Delta^2}g||_{\infty} \le C_1(\nu\tau)^{-\frac{1}{8}}||g||_4;$$
 (2.10)

For any $g_1 \in L^{\frac{4}{3}}(\mathbb{T}^2)$, we have

$$\|\tau \Delta e^{-\nu \tau \Delta^2} g_1\|_{\infty} \le C_2 \tau (\nu \tau)^{-\frac{7}{8}} \|g_1\|_{\frac{4}{3}}.$$
 (2.11)

In the above $C_1 > 0$, $C_2 > 0$ are absolute constants.

Proof. Denote $\beta = \nu \tau$. The first inequality follows from Lemma 2.1 (see the bound for \tilde{K} therein). For the second inequality denote

$$K_{\beta} = \mathcal{F}^{-1} \left(\beta^{\frac{1}{2}} |k|^2 e^{-\beta |k|^4} \right).$$
 (2.12)

We then have
$$||K_{\beta}||_{L_{x}^{4}(\mathbb{T}^{2})} \lesssim ||\widehat{K_{\beta}}||_{l_{x}^{\frac{4}{3}}(\mathbb{Z}^{2})} \lesssim \beta^{-\frac{3}{8}}.$$

Lemma 2.3. Let $d \geq 1$. If $E_p = \int_{\mathbb{T}^d} \frac{1}{4} (v^2 - 1)^2 dx$, then

$$||v||_{L^4(\mathbb{T}^d)} \lesssim 1 + E_p^{\frac{1}{4}}, \qquad ||v^3 - v||_{L^{\frac{4}{3}}(\mathbb{T}^d)} \lesssim E_p^{\frac{1}{2}}(1 + E_p^{\frac{1}{4}}).$$
 (2.13)

Proof. Obvious. For the second inequality, note that $\|(v^2-1)v\|_{\frac{4}{3}} \leq \|v^2-1\|_2\|v\|_4$.

3 Analysis of the propagator $S_L(\tau)S_N(\tau)$

In this section we analyze in detail the propagator $S_L(\tau)S_N(\tau)$. If $u = S_L(\tau)S_N(\tau)w$, then

$$u = e^{-\tau\nu\Delta^2} \left(w + \tau\Delta(w^3 - w) \right). \tag{3.1}$$

Denote

$$E_1(w) = \frac{1}{2\tau} \||\nabla|^{-1} (e^{\tau\nu\Delta^2} - 1)^{\frac{1}{2}} w\|_2^2 + \frac{1}{4} \int_{\mathbb{T}^2} (w^2 - 1)^2 dx.$$
 (3.2)

Theorem 3.1 (One-step energy stability). Suppose w has mean zero and $E_1(w)$ is finite. We have

$$E_1(u) - E_1(w) + \left(\frac{1}{2} + \sqrt{\frac{2\nu}{\tau}}\right) \|u - w\|_2^2 \le \|u - w\|_2^2 \cdot \frac{3}{2} \max\{\|u\|_{\infty}^2, \|w\|_{\infty}^2\}.$$
(3.3)

Proof. Recall $f(w) = w^3 - w$. We rewrite (3.1) as

$$\frac{u-w}{\tau} + \frac{e^{\tau\nu\Delta^2}u - u}{\tau} = \Delta(f(w)). \tag{3.4}$$

Taking the L^2 inner product with $(-\Delta)^{-1}(u-w)$ on both sides and applying the identity $b\cdot(b-a)=\frac{1}{2}(|b|^2-|a|^2+|b-a|^2)$, we get

$$\frac{1}{\tau} \||\nabla|^{-1} (u - w)\|_{2}^{2} + \frac{1}{2} (\|Tu\|_{2}^{2} - \|Tw\|_{2}^{2} + \|T(u - w)\|_{2}^{2})$$

$$= (\Delta(f(w)), (-\Delta)^{-1} (u - w)), \tag{3.5}$$

where $T = |\nabla|^{-1} \tau^{-\frac{1}{2}} (e^{\tau \nu \Delta^2} - 1)^{\frac{1}{2}}$. Clearly

$$(\Delta(f(w)), (-\Delta)^{-1}(u-w)) = -(f(w), u-w).$$

Introduce the auxiliary function g(s)=F(w+s(u-w)), where $F(z)=\frac{1}{4}(z^2-1)^2$. By using the Taylor expansion $g(1)=g(0)+g'(0)+\int_0^1g''(s)(1-s)ds$, we get

$$F(u) = F(w) + f(w)(u - w) - \frac{1}{2}(u - w)^{2} + (u - w)^{2} \int_{0}^{1} \tilde{f}'(w + s(u - w))(1 - s)ds,$$
(3.6)

where $\tilde{f}(z) = z^3$ and $\tilde{f}'(z) = 3z^2$ for $z \in \mathbb{R}$. From this it is easy to see that

$$-(f(w), u-w)$$

$$\leq F(w) - F(u) - \frac{1}{2} \|u - w\|_{2}^{2} + \|u - w\|_{2}^{2} \cdot \frac{3}{2} \max\{\|u\|_{\infty}^{2}, \|w\|_{\infty}^{2}\}. \tag{3.7}$$

Thus

$$E_{1}(u) - E_{1}(w) + \frac{1}{\tau} \||\nabla|^{-1}(u - w)\|_{2}^{2} + \frac{1}{2} \|T(u - w)\|_{2}^{2} + \frac{1}{2} \|u - w\|_{2}^{2}$$

$$\leq \|u - w\|_{2}^{2} \cdot \frac{3}{2} \max\{\|u\|_{\infty}^{2}, \|w\|_{\infty}^{2}\}.$$
(3.8)

Now observe that for $\xi \neq 0$,

$$\frac{1}{\tau|\xi|^2} + \frac{e^{\tau\nu|\xi|^4} - 1}{2\tau|\xi|^2} = \frac{e^{\tau\nu|\xi|^4} + 1}{2\tau|\xi|^2} \ge \frac{2 + \tau\nu|\xi|^4}{2\tau|\xi|^2} \ge \sqrt{\frac{2\nu}{\tau}}.$$
 (3.9)

It follows that

$$\frac{1}{\tau} \||\nabla|^{-1} (u - w)\|_{2}^{2} + \frac{1}{2} \|T(u - w)\|_{2}^{2} \ge \sqrt{\frac{2\nu}{\tau}} \|u - w\|_{2}^{2}.$$
 (3.10)

The desired inequality follows easily.

Lemma 3.1. We have

$$||u||_{\infty} \le c_1 \cdot (\nu \tau)^{-\frac{1}{8}} (1 + E_1(w)^{\frac{1}{4}}) + c_1 \cdot \tau(\nu \tau)^{-\frac{7}{8}} E_1(w)^{\frac{1}{2}} (1 + E_1(w)^{\frac{1}{4}}), \quad (3.11)$$

where $c_1 > 0$ is an absolute constant. Assume

$$||w||_{\infty} \le \alpha_1(\nu\tau)^{-\frac{1}{8}} + \alpha_2\tau(\nu\tau)^{-\frac{7}{8}},\tag{3.12}$$

for some constants α_1 , α_2 satisfying

$$\alpha_1 \ge c_1 (1 + E_1(w)^{\frac{1}{4}}), \qquad \alpha_2 \ge c_1 \cdot E_1(w)^{\frac{1}{2}} (1 + E_1(w)^{\frac{1}{4}}).$$
 (3.13)

Define $\alpha_* = \max\{\alpha_1, \alpha_2\}$. If

$$0 < \tau < c \cdot \min\{\alpha_*^{-8}, \alpha_*^{-\frac{8}{3}}\}\nu^3, \tag{3.14}$$

where c > 0 is a sufficiently small absolute constant, then

$$E_1(u) \le E_1(w). (3.15)$$

Proof. The bound (3.11) follows from Lemma 2.2 and Lemma 2.3. To show (3.15), by Theorem 3.1, we only need to check the inequality

$$\frac{1}{2} + \sqrt{\frac{2\nu}{\tau}} \ge \frac{3}{2} \cdot \max\{\|u\|_{\infty}^2, \|w\|_{\infty}^2\}. \tag{3.16}$$

It amounts to checking the inequalities

$$\sqrt{\frac{2\nu}{\tau}} \gg \alpha_*^2 (\nu \tau)^{-\frac{1}{4}}, \quad \sqrt{\frac{2\nu}{\tau}} \gg \alpha_*^2 \tau^2 (\nu \tau)^{-\frac{7}{4}}.$$
 (3.17)

The result is obvious.

The following lemma shows that the energy $E_1(w)$ is well-defined.

Lemma 3.2. Suppose $u^0 \in H^1(\mathbb{T}^2)$ and has mean zero. Set $w = S_L(\tau)S_N(\tau)u^0$. Then

$$E_1(w) \le c_0^{(1)} (1 + \nu + \nu^{-1})^4 (1 + \|u^0\|_{H^1}^3)^4;$$

$$\|w\|_{\infty} \le c_0^{(2)} (\nu \tau)^{-\frac{1}{8}} (1 + \nu^{-1}) (\|u^0\|_{H^1} + \|u^0\|_{H^1}^3), \tag{3.18}$$

where $c_0^{(1)} > 0$, $c_0^{(2)} > 0$ are absolute constants.

Proof. First we note that

$$\frac{1}{4} \int (w^2 - 1)^2 dx \lesssim 1 + \|w\|_4^4 \lesssim 1 + \|w\|_{H^1}^4. \tag{3.19}$$

Since $w = S_L(\tau)(u^0 + \tau \Delta(f(u^0)))$, it follows that (below $\overline{f(u^0)}$ denotes the average of $f(u^0)$ on \mathbb{T}^2)

$$||w||_{H^{1}} \lesssim ||u^{0}||_{H^{1}} + ||\tau\Delta|\nabla|e^{-\nu\tau\Delta^{2}}(f(u^{0}))||_{2}$$

$$\lesssim ||u^{0}||_{H^{1}} + ||\tau|\nabla|^{3.5}e^{-\nu\tau\Delta^{2}}(f(u^{0}) - \overline{f(u^{0})})||_{\frac{4}{3}}$$

$$\lesssim ||u^{0}||_{H^{1}} + \nu^{-1}(||u^{0}||_{H^{1}} + ||u^{0}||_{H^{1}}^{3}). \tag{3.20}$$

Write $w = S_L(\frac{\tau}{2})g$, where $g = S_L(\frac{\tau}{2})(u^0 + \tau \Delta(f(u^0)))$. By a similar estimate as above, we have

$$||g||_{H^1} \lesssim ||u^0||_{H^1} + \nu^{-1}(||u^0||_{H^1} + ||u^0||_{H^1}^3).$$
 (3.21)

Clearly

$$\frac{1}{2\tau}(|\nabla|^{-2}(e^{\tau\nu\Delta^{2}}-1)w,w) = \nu(\frac{1-e^{-\tau\nu\Delta^{2}}}{2\nu\tau\Delta^{2}}|\nabla|g,|\nabla|g)
\lesssim \nu||g||_{H^{1}}^{2}.$$
(3.22)

The desired bound on $E_1(w)$ easily follows.

For the L^{∞} -bound, we note that by Lemma 2.2,

$$||w||_{\infty} = ||e^{-\frac{\tau}{2}\nu\Delta^2}g||_{\infty} \lesssim (\nu\tau)^{-\frac{1}{8}}||g||_4 \lesssim (\nu\tau)^{-\frac{1}{8}}||g||_{H^1}.$$
(3.23)

Theorem 1.1 is a simplified version of the following Theorem.

Theorem 3.2. Suppose $u^0 \in H^1(\mathbb{T}^2)$ and has mean zero. Define $u^1 = S_L(\tau)S_N(\tau)u^0$. Then

$$E_1(u^1) \le c_0^{(1)} (1 + \nu + \nu^{-1})^4 (1 + ||u^0||_{H^1}^3)^4;$$

$$||u^1||_{\infty} \le c_0^{(2)} (\nu \tau)^{-\frac{1}{8}} (1 + \nu^{-1}) (||u^0||_{H^1} + ||u^0||_{H^1}^3),$$
(3.24)

where $c_0^{(1)} > 0$, $c_0^{(2)} > 0$ are absolute constants and we recall

$$E_1(u^1) = \frac{1}{2\tau} |||\nabla|^{-1} (e^{\tau\nu\Delta^2} - 1)^{\frac{1}{2}} u^1||_{L^2(\mathbb{T}^2)}^2 + \frac{1}{4} \int_{\mathbb{T}^2} ((u^1)^2 - 1)^2 dx.$$
 (3.25)

Set

$$\alpha = \max \left\{ c_1 (1 + E_1(u^1)^{\frac{1}{4}}), \ c_1 E_1(u^1)^{\frac{1}{2}} (1 + E_1(u^1)^{\frac{1}{4}}), \ c_0^{(2)} (1 + \nu^{-1}) (\|u^0\|_{H^1} + \|u^0\|_{H^1}^3) \right\},$$
(3.26)

where c_1 is the same absolute constant in (3.11). Define the iterates

$$u^{n+1} = S_L(\tau)S_N(\tau)u^n, \quad n \ge 1.$$
 (3.27)

If $0 < \tau < \tau_* = c \cdot \min\{\alpha^{-8}, \alpha^{-\frac{8}{3}}\}\nu^3$ where c > 0 is a sufficiently small absolute constant, then it holds that

$$E_1(u^{n+1}) \le E_1(u^n), \quad \forall n \ge 1;$$
 (3.28)

$$\sup_{n\geq 0} \|u^n\|_{\infty} \leq \alpha(\nu\tau)^{-\frac{1}{8}} + \alpha\tau(\nu\tau)^{-\frac{7}{8}}.$$
 (3.29)

Furthermore, if $u^0 \in H^{k_0}(\mathbb{T}^2)$ for some integer $k_0 \geq 2$, then we also have the uniform H^{k_0} bound:

$$\sup_{n\geq 0} \|u^n\|_{H^{k_0}(\mathbb{T}^2)} \leq B_1 < \infty, \tag{3.30}$$

where $B_1 > 0$ depends on $(\|u^0\|_{H^{k_0}(\mathbb{T}^2)}, \nu, k_0)$.

Proof. The estimates of u^1 follows from Lemma 3.2. The energy decay and L^{∞} bound on u^n follows from Lemma 3.1 and an induction argument. For (3.30), we note that

$$u^{n+1} = S_L(\tau)S_N(\tau)u^n$$

$$= S_L(\tau)(u^n + \tau\Delta(f(u^n)))$$

$$= S_L(\tau)(S_L(\tau)u^{n-1} + \tau\Delta(f(u^{n-1}))) + \tau S_L(\tau)\Delta(f(u^n))$$

$$= \cdots$$

$$= S_L((n+1)\tau)u^0 + \tau \sum_{k=0}^n S_L((k+1)\tau)\Delta(f(u^{n-k})). \tag{3.31}$$

The desired estimate then follows from the above using smoothing estimates (cf. [17]). We omit the details.

4 Proof of Theorem 1.2

In this section we complete the proof of Theorem 1.2. For convenience we shall set $\nu = 1$. Since $u^{n+1} = S_L(\tau)S_N(\tau)u^n$, we have

$$u^{n+1} = e^{-\tau \Delta^2} u^n + \tau e^{-\tau \Delta^2} \Delta(f(u^n)). \tag{4.1}$$

We rewrite the above as

$$u^{n+1} = (1 + \tau \Delta^2)^{-1} u^n + \tau (1 + \tau \Delta^2)^{-1} \Delta (f(u^n)) + (1 + \tau \Delta^2)^{-1} g^n, \tag{4.2}$$

where

$$g^{n} = (1 + \tau \Delta^{2}) \Big((e^{-\tau \Delta^{2}} - (1 + \tau \Delta^{2})^{-1}) u^{n} + \tau (e^{-\tau \Delta^{2}} - (1 + \tau \Delta^{2})^{-1}) \Delta(f(u^{n})) \Big).$$
(4.3)

Lemma 4.1. For some absolute constant $d_1 > 0$, it holds that

$$||g^n||_2 \le d_1 \tau^2 \cdot (||u^n||_{H^8} + ||u^n||_{H^8}^3). \tag{4.4}$$

Proof. Obvious.

Rewrite (4.2) as

$$\frac{u^{n+1} - u^n}{\tau} = -\Delta^2 u^{n+1} + \Delta(f(u^n)) + g^n. \tag{4.5}$$

Note that $\sup_{n\geq 0} \|u^n\|_{H^8} \lesssim 1$. With the help of Lemma 4.1, the proof of Theorem 1.2 then follows from Proposition 4.1 of [13].

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