# PRECONDITIONED LEGENDRE SPECTRAL GALERKIN METHODS FOR THE NON-SEPARABLE ELLIPTIC EQUATION 

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#### Abstract

The Legendre spectral Galerkin method of self-adjoint second order elliptic equations usually results in a linear system with a dense and ill-conditioned coefficient matrix. In this paper, the linear system is solved by a preconditioned conjugate gradient (PCG) method where the preconditioner $M$ is constructed by approximating the variable coefficients with a ( $T+1$ )-term Legendre series in each direction to a desired accuracy. A feature of the proposed PCG method is that the iteration step increases slightly with the size of the resulting matrix when reaching a certain approximation accuracy. The efficiency of the method lies in that the system with the preconditioner $M$ is approximately solved by a one-step iterative method based on the $\operatorname{ILU}(0)$ factorization. The $\operatorname{ILU}(0)$ factorization of $M \in \mathbb{R}^{(N-1)^{d} \times(N-1)^{d}}$ can be computed using $\mathcal{O}\left(T^{2 d} N^{d}\right)$ operations, and the number of nonzeros in the factorization factors is of $\mathcal{O}\left(T^{d} N^{d}\right), d=1,2,3$. To further speed up the PCG method, an algorithm is developed for fast matrix-vector multiplications by the resulting matrix of Legendre-Galerkin spectral discretization, without the need to explicitly form it. The complexity of the fast matrix-vector multiplications is of $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$. As a result, the PCG method has a $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$ total complexity for a $d$ dimensional domain with $(N-1)^{d}$ unknows, $d=1,2,3$. Numerical examples are given to demonstrate the efficiency of proposed preconditioners and the algorithm for fast matrix-vector multiplications.


## 1. INTRODUCTION

Spectral methods are an important tool in engineer and scientific computing for solving differential equations due to their high order of accuracy; see [3, 23, 10, 24] and the references therein. However, for problems with general variable coefficients, spectral methods lead to a linear system with a dense and ill-conditioned matrix. Moreover, the dense matrix is usually not explicitly available, since it is costly to form it. In practice, it becomes rather prohibitive to solve the linear system by a direct solver or even an iterative method without preconditioning for the multi-dimensional cases, when the size of the matrix is large.

Over the years there has been intensive research on the spectral collocation method for solving problems with variable coefficients, since it is easy to implement, once the differentiation matrices are precomputed. One significant attempt is to use a lower-order method (finite differences or finite elements [2, 7, 15, 16, 9]) or integration operator [4, 11, 30] as a preconditioner and to take advantage of the fact that the matrix-vector multiplication from a Fourier- or Chebyshevspectral discretization can be performed in a quasi-optimal complexity. Another approach is the finite element multigrid preconditioning method proposed by Shen et al.[25] for the Chebyshevcollocation approximation of second-order elliptic equations. Although many spectral collocation methods have been applied to numerically solve variable-coefficient differential equations, few efforts are found for spectral Galerkin methods, especially Legendre-Galerkin methods, in literature.

[^0]An early work is the Chebyshev-Legendre Galerkin method for second-order elliptic problems introduced in [21], which is based on the Legendre-Galerkin formulation, and only the coefficients of Legendre expansions and the values at the Chebyshev-Gauss-Lobatto points are used in the computation. A fast direct solver was presented for the Legendre-Galerkin approximation of the two and three dimensional Helmholtz equations by Shen in [27], whose complexity is of $\mathcal{O}\left(N^{d+1}\right)$ in a $d$ dimensional domain. An improved two-dimensional algorithm was constructed by further exploring the matrix structures of the Legendre-Galerkin discretization in [26], whose complexity is of $\mathcal{O}\left(N^{2} \log _{2} N\right)$, which was extended to the Legendre-Galerkin spectral approximation of the three dimensional Helmholtz equation in [1].

The Legendre-Galerkin method of self-adjoint second order elliptic equations leads to symmetric linear systems, but its efficiency is limited by the lack of fast transforms between the physical space (values at the Legendre Gauss points) and the spectral space (coefficients of the Legendre polynomials). In traditional spectral methods, a fast algorithm for Legendre expansions is a procedure to fast evaluate the Legendre expansion at Chebyshev points, and conversely, to fast evaluate the coefficients of the Legendre expansion from the table of its values at the Chebyshev-GaussLobatto points. Recently, a series of work were done for fast discrete Legendre transforms between the values at the Legendre Gauss points and the coefficients of the Legendre polynomials [29, 19, 17, 13]. In particular, an $\mathcal{O}\left(N(\log N)^{2} / \log \log N\right)$ algorithm based on the FFT was proposed in [13] in one dimension for computing the discrete Legendre transform with a degree $N-1$ Legendre expansion at $N$ Legendre points.

The goal of this article is to fast solve the linear system resulting from the Legendre-Galerkin spectral discretization of second order elliptic equations with variable coefficients by the preconditioned conjugate gradient (PCG) method. The novelties of the paper lie in the following three folds:

- Firstly, the preconditioner is constructed by using a truncated Legendre series to approximate the variable coefficients. It is in the case that the iterative step of the PCG method only increase slightly with the size of the resulting matrix. A closely related preconditioner of [21, 20] is to use a constant-coefficient problem to precondition variable-coefficient problems. However, for coefficients with large variations, iterative methods with that preconditioner usually converge very slowly [25].
- Secondly, by means of fast discrete Legendre transforms, an algorithm is developed for fast matrix-vector multiplications by the resulting matrix without the need to explicitly form it. As a result, they can be done in $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$ operations.
- Last but not least, the system with the preconditioner as the coefficient matrix is approximately solved by a one-step iterative method based on the ILU(0) factorization. Thanks to the sparse structure of the preconditioner $M$, the $\operatorname{ILU}(0)$ factorization gives an unit lower triangular matrix $L$ and an upper triangular matrix $U$, where together the $L$ and $U$ matrices have the same number of nonzero elements as the matrix $M$. The complexity essentially depends on the number of nonzeros in $M$, which is of $\mathcal{O}\left(T^{2 d} N^{d}\right)$ with $T$ the cutoff number of the Legendre series in each direction used to approximate the coefficient functions.

The remainder of this article is organized as follows. Some preliminaries are given in Section 2. Section 3 introduces the Legendre-Galerkin method of second-order elliptic equations with non-separable coefficients. In Section 4, the preconditioned conjugate gradient method with
implementation issues is described. In section 5, some numerical experiments are presented to illustrate the efficiency of both the algorithm for fast matrix-vector multiplications and the proposed preconditioner. The conclusion is in the last section.

## 2. Preliminaries

In this section, some properties of Legendre polynomials and a useful transform are presented.
2.1. Legendre polynomials. Denote by $L_{n}(x)$ the Legendre polynomial of degree $n$ which satisfies the following three-term recurrence relation:

$$
\begin{aligned}
L_{0}(x) & =1, \quad L_{1}(x)=x \\
(n+1) L_{n+1}(x) & =(2 n+1) x L_{n}(x)-n L_{n-1}(x), \quad n \geq 1
\end{aligned}
$$

The Legendre polynomials are orthogonal to each other with respect to the uniform weight function,

$$
\int_{-1}^{1} L_{m}(x) L_{n}(x) \mathrm{d} x=\frac{2}{2 m+1} \delta_{m n}, \quad m, n \geq 0
$$

where $\delta_{m n}$ is the Kronecker delta symbol. Moreover, they satisfy the derivative recurrence relation

$$
\begin{equation*}
(2 n+1) L_{n}(x)=L_{n+1}^{\prime}(x)-L_{n-1}^{\prime}(x), \quad n \geq 1, \tag{2.1}
\end{equation*}
$$

and symmetric property

$$
\begin{equation*}
L_{n}(-x)=(-1)^{n} L_{n}(x), \quad L_{n}( \pm 1)=( \pm 1)^{n} . \tag{2.2}
\end{equation*}
$$

Lemma 2.1 ([5]). For $m, n \geq 0$, it holds that

$$
\begin{equation*}
L_{m}(x) L_{n}(x)=\sum_{s=0}^{\min (m, n)} \frac{m+n+\frac{1}{2}-2 s}{m+n+\frac{1}{2}-s} \frac{C_{s} C_{m-s} C_{n-s}}{C_{m+n-s}} L_{m+n-2 s}(x) \tag{2.3}
\end{equation*}
$$

where

$$
C_{r}=\frac{1 \cdot 3 \ldots(2 r-3)(2 r-1)}{r!2^{r}}
$$

2.2. Discrete Legendre transforms. Given $N+1$ values $c_{0}, c_{1}, \cdots, c_{N}$, the backward discrete Legendre transform (BDLT) calculates the discrete sums:

$$
\begin{equation*}
f_{k}=\sum_{n=0}^{N} c_{n} L_{n}\left(x_{k}\right), \quad 0 \leq k \leq N \tag{2.4}
\end{equation*}
$$

where the Legendre-Gauss quadrature nodes $x_{0}, x_{1}, \cdots, x_{N}$ are the roots of $L_{N+1}(x)$. Given $f_{0}, f_{1}, \cdots, f_{N}$, the forward discrete Legendre transform (FDLT) computes $c_{0}, c_{1}, \cdots, c_{N}$, which reads

$$
\begin{equation*}
c_{n}=\frac{2 n+1}{2} \sum_{k=0}^{N} w_{k} f_{k} L_{n}\left(x_{k}\right), \quad 0 \leq n \leq N \tag{2.5}
\end{equation*}
$$

where $w_{0}, w_{1}, \cdots, w_{N}$ are the Legendre-Gauss quadrature weights. Assuming that $\left(L_{n}\left(x_{j}\right)\right)_{j, n=0, \cdots, N}$ have been precomputed, the discrete Legendre transforms (2.4) and (2.5) can be carried out by a standard matrix-vector multiplication routine in about $N^{2}$ flops. In this paper, the algorithm in [13] is used for the fast computation of the discrete Legendre transforms (2.4) and (2.5) which is of $\mathcal{O}\left(N(\log N)^{2} / \log \log N\right)$ complexity, because it has no precomputational cost and only involves the FFT and Taylor approximations.

## 3. THE LEGENDRE-GALERKIN METHOD OF NON-SEPARABLE SECOND ORDER ELLIPTIC EQUATIONS

Consider non-separable second order elliptic equations of the form

$$
\left\{\begin{array}{l}
-\operatorname{div}(\beta(\boldsymbol{x}) \nabla u)+\alpha(\boldsymbol{x}) u=f, \quad \boldsymbol{x} \in \Omega=(-1,1)^{d}  \tag{3.1}\\
\left.u\right|_{\partial \Omega}=0
\end{array}\right.
$$

where $d=1,2,3$, the coefficient functions $\beta(\boldsymbol{x}), \alpha(\boldsymbol{x})$ and $f(\boldsymbol{x})$ are continuous, and $0<b_{1} \leq$ $\beta(\boldsymbol{x}) \leq b_{2}, 0 \leq \alpha(\boldsymbol{x})<a$ in $\Omega$ for some positive constants $b_{1}, b_{2}, a$.

Let $P_{N}$ be the space of polynomials of degree less than or equal to $N$, and

$$
X_{N}=\left\{v \in P_{N}: v( \pm 1)=0\right\}
$$

Denote $X_{N}^{d}=\left(X_{N}\right)^{d}$. Then the Legendre-Galerkin approximation to 3.1) is: Find $u_{N} \in X_{N}^{d}$ such that

$$
\begin{equation*}
\left(\beta(\boldsymbol{x}) \nabla u_{N}, \nabla v_{N}\right)+\left(\alpha(\boldsymbol{x}) u_{N}, v_{N}\right)=\left(f, v_{N}\right), \quad \forall v_{N} \in X_{N}^{d} \tag{3.2}
\end{equation*}
$$

where $(u, v)=\int_{\Omega} u v \mathrm{~d} \boldsymbol{x}$ is the scalar product in $L^{2}(\Omega)$.
Denote

$$
\phi_{k}(x):=L_{k}(x)-L_{k+2}(x)
$$

Due to $(2.2)$, it is easy to know that the function $\phi_{k}(x)$ satisfies the boundary condition of problem (3.1). Hence, the basis functions of the space $X_{N}$ can be chosen as

$$
\phi_{0}(x), \phi_{1}(x), \ldots, \phi_{N-2}(x)
$$

- One dimensional case. Assume $u_{N}=\sum_{n=0}^{N-2} \widehat{u}_{n} \phi_{n}(x)$, and denote

$$
\begin{aligned}
A & =\left[\left(\beta(x) \phi_{k}^{\prime}, \phi_{j}^{\prime}\right)\right]_{0 \leq k, j \leq N-2} \\
B & =\left[\left(\alpha(x) \phi_{k}, \phi_{j}\right)\right]_{0 \leq k, j \leq N-2} \\
\widehat{u} & =\left(\widehat{u}_{0}, \widehat{u}_{1}, \cdots, \widehat{u}_{N-2}\right)^{T} \\
F & =\left(f_{0}, f_{1}, \cdots, f_{N-2}\right)^{T}, \quad f_{k}=\left(f, \phi_{k}\right)
\end{aligned}
$$

- Two dimensional case. The multi-dimensional basis functions are constructed by using the tensor product of one-dimensional basis functions. In two dimensions, they read

$$
\varphi_{k, j}(\boldsymbol{x}):=\phi_{k}(x) \phi_{j}(y), \quad k, j=0,1, \cdots, N-2
$$

Assume $u_{N}=\sum_{k, j=0}^{N-2} \widehat{u}_{k, j} \varphi_{k, j}(\boldsymbol{x})$, and denote

$$
\begin{aligned}
A & =\left[\left(\beta(\boldsymbol{x}) \nabla \varphi_{k, j}, \nabla \varphi_{m, n}\right)\right]_{0 \leq k, j, m, n \leq N-2} \\
B & =\left[\left(\alpha(\boldsymbol{x}) \varphi_{k, j}, \varphi_{m, n}\right)\right]_{0 \leq k, j, m, n \leq N-2} \\
\widehat{u} & =\left(\widehat{u}_{0,0}, \widehat{u}_{1,0}, \ldots, \widehat{u}_{N-2,0} ; \ldots ; \widehat{u}_{0, N-2}, \widehat{u}_{1, N-2}, \ldots, \widehat{u}_{N-2, N-2}\right)^{T} \\
F & =\left(f_{0,0}, f_{1,0}, \cdots, f_{N-2,0} ; \ldots ; f_{0, N-2}, f_{1, N-2}, \ldots, f_{N-2, N-2}\right)^{T} \\
f_{k, j} & =\left(f, \varphi_{k, j}\right)
\end{aligned}
$$

- Three dimensional case. Similarly, the three-dimensional basis functions are as follows

$$
\psi_{i, j, k}(\boldsymbol{x}):=\phi_{i}(x) \phi_{j}(y) \phi_{k}(z), \quad i, k, j=0,1, \cdots, N-2
$$

Assume $u_{N}=\sum_{i, j, k=0}^{N-2} \widehat{u}_{i, j, k} \psi_{i, j, k}(\boldsymbol{x})$, and denote

$$
\begin{aligned}
& A=\left[\left(\beta(\boldsymbol{x}) \nabla \psi_{i, j, k}, \nabla \psi_{m, n, l}\right)\right]_{0 \leq i, k, j, m, n, l \leq N-2} \\
& B=\left[\left(\alpha(\boldsymbol{x}) \psi_{i, j, k}, \psi_{m, n, l}\right)\right]_{0 \leq i, k, j, m, n, l \leq N-2} \\
& \widehat{u}=\left(\widehat{u}_{0}, \widehat{u}_{1}, \ldots, \widehat{u}_{N-2}\right)^{T} \\
& \widehat{u}_{k}=\left(\widehat{u}_{0,0, k}, \widehat{u}_{1,0, k}, \ldots, \widehat{u}_{N-2,0, k} ; \ldots ; \widehat{u}_{0, N-2, k}, \widehat{u}_{1, N-2, k}, \ldots, \widehat{u}_{N-2, N-2, k}\right), \\
& F=\left(f_{0}, f_{1}, \ldots, f_{N-2}\right)^{T} \\
& f_{k}=\left(f_{0,0, k}, f_{1,0, k}, \ldots, f_{N-2,0, k} ; \ldots ; f_{0, N-2, k}, f_{1, N-2, k}, \ldots, f_{N-2, N-2, k}\right), \\
& f_{i, j, k}=\left(f, \psi_{i, j, k}\right)
\end{aligned}
$$

Then the equation (3.2) is equivalent to the following algebraic system:

$$
\begin{equation*}
(A+B) \widehat{u}=F \tag{3.3}
\end{equation*}
$$

For variable coefficients $\alpha(\boldsymbol{x})$ and $\beta(\boldsymbol{x})$, the matrices $A$ and $B$ in equation (3.3) are usually dense and ill-conditioned. Hence, it is prohibitive to use a direct inversion method or an iterative method without preconditioning. Moreover, it is imperative to use an iterative method with a good preconditioner.

## 4. PRECONDITIONED CONJUGATE GRADIENT METHOD

In this article, the symmetric definite linear system (3.3) is solved by the preconditioned conjugate gradient (PCG) method presented in Algorithm 1.

```
Algorithm 1: PCG
    1 Initialize: \(A, B, F\) and initialization vector \(x\), preconditioner \(M\), the maximum loop size
    \(k_{\max }\), stop criteria \(\varepsilon\)
    \(k=0\),
    \(r=F-(A+B) x\)
    While \(\sqrt{r^{T} r}>\varepsilon\|F\|_{2}\) and \(k<k_{\max }\) do
        Solve \(M z=r\),
        \(k=k+1\)
        if \(k=1\)
            \(p=z ; \rho=r^{T} z\)
        else
            \(\tilde{\rho}=\rho ; \rho=r^{T} z ; \beta=\rho / \tilde{\rho} ; p=z+\beta p\)
        end if
        \(w=(A+B) p ; \alpha=\rho / p^{T} w\)
        \(x=x+\alpha p ; r=r-\alpha w\)
    end while
```

In each iteration of the PCG method, it needs to solve a system with $M$ as the coefficient matrix and involves a matrix-vector multiplication. The complexity of these two steps dominates that of the algorithm. In order to accelerate the convergence rate of conjugate gradient type methods, an efficient preconditioner $M$ is needed. Moreover, an accurate numerical solver is needed to solve the preconditioning equation $M z=r$. At last, fast matrix vector multiplications have to be used to further reduce the complexity of the algorithm.
4.1. Proposed preconditioner. A preconditioner is prescribed for the coefficient matrix $A+B$ of the linear system (3.3) in the following way.

Since $\beta(\boldsymbol{x})$ and $\alpha(\boldsymbol{x})$ are continuous, they can be approximated by a finite number of Legendre polynomials to any desired accuracy. That is, for any $\epsilon_{1}>0, \epsilon_{2}>0$, there exists $t_{1}, t_{2} \in \mathbb{N}$ and $p_{t_{1}} \in\left(P_{t_{1}}\right)^{d}, p_{t_{2}} \in\left(P_{t_{2}}\right)^{d}$ such that

$$
\begin{align*}
& \left\|\beta(\boldsymbol{x})-p_{t_{1}}\right\|_{L^{\infty}\left([-1,1]^{d}\right)}<\epsilon_{1},  \tag{4.1}\\
& \left\|\alpha(\boldsymbol{x})-p_{t_{2}}\right\|_{L^{\infty}\left([-1,1]^{d}\right)}<\epsilon_{2} . \tag{4.2}
\end{align*}
$$

It is stressed that $t_{1}$ and $t_{2}$ can be surprisingly small when $\beta(\boldsymbol{x}), \alpha(\boldsymbol{x})$ are analytic or many times differentiable. For practical purposes, the preconditioner $M$ is constructed by replacing $\beta(\boldsymbol{x})$ with $p_{t_{1}}$ in the matrix $A$ and $\alpha(\boldsymbol{x})$ with $p_{t_{2}}$ in the matrix $B$. More precisely,

- $d=1$

$$
\begin{align*}
& p_{t_{1}}(x)=\sum_{t=0}^{t_{1}} \widehat{\beta}_{t} L_{t}(x),  \tag{4.3}\\
& p_{t_{2}}(x)=\sum_{t=0}^{t_{2}} \widehat{\alpha}_{t} L_{t}(x),  \tag{4.4}\\
& M=\left[\left(p_{t_{1}}(x) \phi_{i}^{\prime}(x), \phi_{j}^{\prime}(x)\right)+\left(p_{t_{2}}(x) \phi_{i}(x), \phi_{j}(x)\right)\right]_{0 \leq i, j \leq N-2} . \tag{4.5}
\end{align*}
$$

- $d=2$

$$
\begin{align*}
& p_{t_{1}}(\boldsymbol{x})=\sum_{m, n=0}^{t_{1}} \widehat{\beta}_{m n} L_{m}(x) L_{n}(y)  \tag{4.6}\\
& p_{t_{2}}(\boldsymbol{x})=\sum_{m, n=0}^{t_{2}} \widehat{\alpha}_{m n} L_{m}(x) L_{n}(y),  \tag{4.7}\\
& M=\left[\left(p_{t_{1}}(\boldsymbol{x}) \nabla \varphi_{i, j}, \nabla \varphi_{m, n}\right)+\left(p_{t_{2}}(\boldsymbol{x}) \varphi_{i, j}, \varphi_{m, n}\right)\right]_{0 \leq i, j, m, n \leq N-2} . \tag{4.8}
\end{align*}
$$

- $d=3$

$$
\begin{align*}
& p_{t_{1}}(\boldsymbol{x})=\sum_{m, n, l=0}^{t_{1}} \widehat{\beta}_{m n l} L_{m}(x) L_{n}(y) L_{l}(z)  \tag{4.9}\\
& p_{t_{2}}(\boldsymbol{x})=\sum_{m, n, l=0}^{t_{2}} \widehat{\alpha}_{m n l} L_{m}(x) L_{n}(y) L_{l}(z)  \tag{4.10}\\
& M=\left[\left(p_{t_{1}}(\boldsymbol{x}) \nabla \psi_{i, j, k}, \nabla \psi_{m, n, l}\right)+\left(p_{t_{2}}(\boldsymbol{x}) \psi_{i, j, k}, \psi_{m, n, l}\right)\right]_{0 \leq i, j, k, m, n, l \leq N-2} . \tag{4.11}
\end{align*}
$$

Note that the Legendre expansion coefficients in $p_{t_{1}}(\boldsymbol{x})$ and $p_{t_{2}}(\boldsymbol{x})$ can be calculated respectively in $\mathcal{O}\left(t_{1}^{d}\left(\log t_{1}\right)^{2}\right)$ operations and $\mathcal{O}\left(t_{2}^{d}\left(\log t_{2}\right)^{2}\right)$ operations by means of the fast discrete Legendre transform. Moreover, the preconditioner $M$ for one-dimensional case is a banded matrix with a fixed bandwidth dependent of $t_{1}$ and $t_{2}$ from the following proposition.

Proposition 4.1. Denote

$$
\begin{array}{ll}
M_{1}=\left[\beta_{j i}\right]_{0 \leq i, j \leq N-2}, & \beta_{j i}=\left(p_{t_{1}}(x) \phi_{i}^{\prime}(x), \phi_{j}^{\prime}(x)\right), \\
M_{2}=\left[\alpha_{j i}\right]_{0 \leq i, j \leq N-2}, & \alpha_{j i}=\left(p_{t_{2}}(x) \phi_{i}(x), \phi_{j}(x)\right) .
\end{array}
$$

If $t_{1}$ in (4.3) and $t_{2}$ in (4.4) are fixed, the bandwidth $q_{1}$ of banded matrix $M_{1}$ and the bandwidth $q_{2}$ of banded matrix $M_{2}$ are as follows:

$$
\left.\begin{array}{l}
q_{1}=\left\{\begin{array}{ll}
2 t_{1}-1, & t_{1} \text { even, } \\
2 t_{1}+1, & t_{1} \text { odd },
\end{array} \quad \beta(x)\right. \text { is an odd function; } \\
q_{1}=\left\{\begin{array}{ll}
2 t_{1}+1, & t_{1} \text { even, } \\
2 t_{1}-1, & t_{1} \text { odd, }
\end{array} \quad \beta(x)\right. \text { is an even function; }
\end{array}\right\} \begin{aligned}
& q_{2}=\left\{\begin{array}{ll}
2 t_{2}+3, & t_{1} \text { even, } \\
2 t_{2}+5, & t_{1} \text { odd, }
\end{array} \quad \alpha(x)\right. \text { is an odd function; } \\
& q_{2}=\left\{\begin{array}{ll}
2 t_{2}+5, & t_{2} \text { even, } \\
2 t_{2}+3, & t_{2} \text { odd, }
\end{array} \quad \alpha(x)\right. \text { is an even function }
\end{aligned}
$$

Proof. In the case $d=1$, it follows from (2.3) that both $p_{t_{1}}(x) \phi_{k}^{\prime}(x)$ and $p_{t_{2}}(x) \phi_{k}(x)$ can be represented in Legendre series, i.e.,

$$
\begin{aligned}
& p_{t_{1}}(x) \phi_{k}^{\prime}(x)=(-2 k-3) L_{k+1}(x) \sum_{t=0}^{t_{1}} \widehat{\beta}_{t} L_{t}(x)= \begin{cases}\sum_{j=0}^{t_{1}+k+1} \tilde{\beta}_{j} L_{j}, & t_{1} \geq k+1, \\
\sum_{j=k+1-t_{1}}^{t_{1}+k+1} \tilde{\beta}_{j} L_{j}, & t_{1}<k+1,\end{cases} \\
& p_{t_{2}}(x) \phi_{k}(x)=\left(L_{k}(x)-L_{k+2}(x)\right) \sum_{t=0}^{t_{2}} \widehat{\alpha}_{t} L_{t}(x)= \begin{cases}\sum_{j=0}^{t_{2}+k+2} \tilde{\alpha}_{j} L_{j}, & t_{2} \geq k, \\
\sum_{j=k-t_{2}}^{t_{2}+k+2} \tilde{\alpha}_{j} L_{j}, & t_{2}<k,\end{cases}
\end{aligned}
$$

where $\tilde{\beta}_{j}$ are Legendre expansion coefficients in terms of $C_{j}$ in (2.3) and $\widehat{\beta}_{j}, \tilde{\alpha}_{j}$ are Legendre expansion coefficients in terms of $C_{j}$ in (2.3) and $\widehat{\alpha}_{j}$. Together with parity arguments on $\beta(x)$ and $\alpha(x)$, this leads to the conclusion.

Remark 4.1. The bandwidth of preconditioner $M$ is of $\mathcal{O}(N)$ in the case $d=2$ and of $\mathcal{O}\left(N^{2}\right)$ in the case $d=3$.
4.2. Incomplete LU preconditioning for banded linear systems. Without loss of generality, assume $\alpha(\boldsymbol{x})=0$ in problem (3.1). And the preconditioner $M$ is constructed by using a ( $T+1$ )term Legendre series in each direction to approximate the coefficient function $\beta(\boldsymbol{x})$. In what follows, it is shown that the preconditioning equation $M z=r$ is approximately solved by a one-step iterative process based on the ILU(0) factorization, see for instance, [28, Chapter 10], in $\mathcal{O}\left(T^{2 d} N^{d}\right)$ operations for $d=1,2,3$.

To approximately solve $M z=r$, proceed as follows:
Step 1. Perform the $\operatorname{ILU}(0)$ factorization to obtain a sparse unit lower triangular matrix $L$ and a sparse upper triangular matrix $U$;

Step 2. Solve the unit lower triangular system $L y=r$ by a forward substitution shown in Algorithm 3;
Step 3. Solve the upper triangular system $U z=y$ by a backward substitution shown in Algorithm 4.

For the sparse matrix $M$ whose elements are $m_{i j}, i, j=1, \ldots,(N-1)^{d}$, the incomplete LU factorization process with no fill-in, denoted by ILU(0), is to compute a sparse unit lower triangular matrix $L$ and a sparse upper triangular matrix $U$ so that the elements of $M-L U$ are zeros in the locations of $N Z(M)$, where $N Z(M)$ is the set of pairs $(i, j), 1 \leq i, j \leq(N-1)^{d}$ such that $m_{i j} \neq 0$, and the entries in the extra diagonals in the product $L U$ are called fill-in elements. Due to the fact that fill-in elements are ignored, it is possible to find $L$ and $U$ so that their product is equal to $M$ in the other diagonals. By definition, together the $L$ and $U$ matrices in $\operatorname{ILU}(0)$ have the same number of nonzero elements as the matrix $M$.

```
Algorithm 2: ILU(0)
    \(L(i, j)\) if \(i>j\) and by \(U(i, j)\) otherwise.
    for \(i=2: n\)
        for \(k=1: i-1\) and \((i, k) \in N Z(H)\)
            \(H(i, k)=H(i, k) / H(k, k)\)
            for \(j=k+1: n\) and \((i, j) \in N Z(H)\)
                \(H(i, j)=H(i, j)-H(i, k) \cdot H(k, j)\)
            end for
        end for
    end for
```

    Initialize: Given \(H \in \mathbb{R}^{n \times n}\), the following algorithm computes an unit lower triangular
    matrix \(L\) and an upper triangular matrix \(U\), assuming they exist. \(H(i, j)\) is overwritten by
    ```
Algorithm 3: Forward substitution
    Initialize: Given an unit lower triangular matrix \(L \in \mathbb{R}^{n \times n}\) and a vector \(r \in \mathbb{R}^{n}\), the
    following algorithm computes the linear system \(L y=r\).
    \(y(1)=r(1)\)
    for \(i=2: n\)
        for \(j=1: i-1\) and \((i, j) \in N Z(L)\)
            \(y(i)=y(i)+L(i, j) \cdot y(j)\)
        end for
        \(y(i)=r(i)-y(i)\)
    end for
```

To evaluate the complexity of the one-step process to solve a system with the coefficient matrix $M$, the number of nonzeros in $M$ is considered. Taking two-dimensional problems as an example,

```
Algorithm 4: Backward substitution
    Initialize: Given an upper triangular matrix \(U \in \mathbb{R}^{n \times n}\) and a vector \(y \in \mathbb{R}^{n}\), the following
    algorithm computes the linear system \(U z=y\).
    \(z(n)=y(n) / U(n, n)\)
    for \(i=n-1: 1\)
        for \(j=n: i\) and \((i, j) \in N Z(U)\)
            \(y(i)=y(i)-U(i, j) \cdot z(j)\)
        end for
        \(z(i)=y(i) / U(i, i)\)
    end for
```

the matrix $M$ can be rewritted in the following formulation:

$$
\begin{aligned}
& p_{T}(\boldsymbol{x})=\sum_{t=0}^{T} \sum_{k=0}^{T} \widehat{\beta}_{t k} L_{t}(x) L_{k}(y), \\
& M=\sum_{t=0}^{T} \sum_{k=0}^{T} \widehat{\beta}_{t k}\left[M_{1 y}^{(k)} \otimes S_{1 x}^{(t)}+S_{1 y}^{(k)} \otimes M_{1 x}^{(t)}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
& M_{1 y}^{(k)}=\left[\left(L_{k}(y) \phi_{j}(y), \phi_{n}(y)\right)\right]_{0 \leq j, n \leq N-1}, \\
& S_{1 x}^{(t)}=\left[\left(L_{t}(x) \phi_{i}^{\prime}(x), \phi_{m}^{\prime}(x)\right)\right]_{0 \leq i, m \leq N-1}, \\
& S_{1 y}^{(k)}=\left[\left(L_{k}(y) \phi_{j}^{\prime}(y), \phi_{n}^{\prime}(y)\right)\right]_{0 \leq j, n \leq N-1}, \\
& M_{1 x}^{(t)}=\left[\left(L_{t}(x) \phi_{i}(x), \phi_{m}(x)\right)\right]_{0 \leq i, m \leq N-1} .
\end{aligned}
$$

It follows from Proposition 4.1 that each matrix $M_{1 y}^{(T)}, S_{1 x}^{(T)}, S_{1 y}^{(T)}, M_{1 x}^{(T)}$ has $\mathcal{O}(T N)$ nonzero elements. Thus, the number of nonzeros in $M$ is of $\mathcal{O}\left(T^{2} N^{2}\right)$. Then it is deduced that the number of nonzeros in $M$ for three-dimensional problems is of $\mathcal{O}\left(T^{3} N^{3}\right)$.

From Algorithm 2, the cost of performing the $\operatorname{ILU}(0)$ factorization essentially depends on the number of nonzero elements in $M$, which is of $\mathcal{O}\left(T^{2 d} N^{d}\right)$. And the complexity of performing either the forward substitution in Algorithm 3 or the backward substitution in Algorithm 4 is of $\mathcal{O}\left(T^{d} N^{d}\right)$. As a result, the one-step iterative process to approximately solve the preconditioning equation $M z=r \operatorname{costs} \mathcal{O}\left(T^{2 d} N^{d}\right)$ numerical operations, $d=1,2,3$.
4.3. Fast matrix-vector multiplications. The fast transforms of the Legendre expansions provide the possibility for fast matrix-vector multiplications of vectors by the discrete matrix $A+B$ resulting from the Legendre-Gelerkin method.

Denote $\Lambda=(a, b)^{d}$ and $P_{N}^{d}=\left(P_{N}\right)^{d}$. Define the interpolation operator $I_{N}: C(\Lambda) \rightarrow P_{N}^{d}(\Lambda)$ such that for any $u \in C(\Lambda)$,

$$
\left(I_{N} u\right)(\boldsymbol{x})=u(\boldsymbol{x}), \quad \boldsymbol{x} \in\left\{x_{0}, x_{1}, \cdots, x_{N}\right\}^{d},
$$

where $x_{0}, x_{1}, \cdots, x_{N}$ are the Legendre-Gauss quadrature nodes mentioned above.

Given the coefficient vector $p$ of $u_{N} \in X_{N}^{d}$, the matrix-vector multiplicaton of $(A+B) p$ is performed as follows (with the operation counts of each step in parenthese):
Step 1. Compute the Legendre coefficients of $\nabla u_{N}$ and $u_{N}$ respectively; $\left(\mathcal{O}\left(N^{d}\right)\right)$
Step 2. Perform the BDLT of $\nabla u_{N}$ and $u_{N}$ respectively; $\left(\mathcal{O}\left(N^{d}(\log N)^{2}\right)\right)$
Step 3. Compute $\beta(\boldsymbol{x}) \nabla u_{N}$ and $\alpha(\boldsymbol{x}) u_{N}$ at the Legendre-Gauss quadrature nodes and then the FDLT of $I_{N}\left(\beta(\boldsymbol{x}) \nabla u_{N}\right), I_{N}\left(\alpha(\boldsymbol{x}) u_{N}\right) ;\left(\mathcal{O}\left(N^{d}(\log N)^{2}\right)\right)$
Step 4. Compute the matrix-vector multiplicaton of $(A+B) p .\left(\mathcal{O}\left(N^{d}\right)\right)$
For clarity of presentation, fast matrix-vector multiplicatons are described in details.
One dimensional case. Given $u_{N}=\sum_{k=0}^{N-2} \widehat{u}_{k} \phi_{k}(x)$, the computation

$$
(A \widehat{u})_{j}=\left(I_{N}\left(\beta u_{N}^{\prime}\right), \phi_{j}^{\prime}\right), \quad j=0,1, \cdots, N-2
$$

without explicitly forming the matrix $A$ is presented as follows.
1, Using (2.1) to determine $\left\{\widetilde{u}^{\prime}\right\}$ from

$$
u_{N}^{\prime}(x)=\sum_{k=0}^{N-2} \widehat{u}_{k} \phi_{k}^{\prime}(x)=\sum_{k=0}^{N} \widetilde{u}_{k}^{\prime} L_{k}(x) ;
$$

2, (BDLT) Compute

$$
u_{N}^{\prime}\left(x_{j}\right)=\sum_{k=0}^{N} \widetilde{u}_{k}^{\prime} L_{k}\left(x_{j}\right), \quad j=0,1, \cdots, N
$$

3, (FDLT) Determine $\left\{\widehat{\beta}_{k}\right\}$ from

$$
I_{N}\left(\beta u_{N}^{\prime}\right)\left(x_{j}\right)=\sum_{k=0}^{N} \widehat{\beta}_{k} L_{k}\left(x_{j}\right), \quad j=0,1, \cdots, N
$$

4, For $j=0,1, \cdots, N-2$, compute

$$
(A \widehat{u})_{j}=\left(I_{N}\left(\beta u_{N}^{\prime}\right),-(2 j+3) L_{j+1}(x)\right)=-2 \widehat{\beta}_{j+1}
$$

Similarly, the computation

$$
(B \widehat{u})_{j}=\left(I_{N}\left(\alpha u_{N}\right), \phi_{j}\right), \quad j=0,1, \cdots, N-2
$$

without explicitly forming the matrix $B$ is presented as follows.
1, Determine $\left\{\widehat{u}_{k}^{(1)}\right\}$ from

$$
u_{N}(x)=\sum_{k=0}^{N-2} \widehat{u}_{k} \phi_{k}(x)=\sum_{k=0}^{N} \widehat{u}_{k}^{(1)} L_{k}(x)
$$

2, (BDLT) Compute

$$
u_{N}\left(x_{j}\right)=\sum_{k=0}^{N} \widehat{u}_{k}^{(1)} L_{k}\left(x_{j}\right), \quad j=0,1, \cdots, N
$$

3, (FDLT) Determine $\left\{\widehat{\alpha}_{k}\right\}$ from

$$
I_{N}\left(\alpha u_{N}\right)\left(x_{j}\right)=\sum_{k=0}^{N} \widehat{\alpha}_{k} L_{k}\left(x_{j}\right), \quad j=0,1, \cdots, N
$$

4, For $j=0,1, \cdots, N-2$, compute

$$
(B \widehat{u})_{j}=\left(I_{N}\left(\alpha u_{N}\right), \phi_{j}(x)\right)=\frac{2 \widehat{\alpha}_{j}}{2 j+1}-\frac{2 \widehat{\alpha}_{j+2}}{2 j+5} .
$$

Two dimensional case. Given $u_{N}=\sum_{k, j=0}^{N-2} \widehat{u}_{k j} \varphi_{k, j}(\boldsymbol{x})$, the calculation

$$
(A \widehat{u})_{k j}=\left(I_{N}\left(\beta \nabla u_{N}\right), \nabla \varphi_{k, j}\right), \quad k, j=0,1, \cdots, N-2
$$

without explicitly forming the matrix $A$ is presented below.
1, Using (2.1) to determine $\left\{\widetilde{u}_{k j}^{x}\right\}$ and $\left\{\widetilde{u}_{k j}^{y}\right\}$ from

$$
\begin{aligned}
& \partial_{x} u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \widehat{u}_{k j} \phi_{k}^{\prime}(x) \phi_{j}(y)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{x} L_{k}(x) L_{j}(y), \\
& \partial_{y} u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \widehat{u}_{k j} \phi_{k}(x) \phi_{j}^{\prime}(y)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{y} L_{k}(x) L_{j}(y) ;
\end{aligned}
$$

2, (BDLT) For $m, n=0,1, \cdots, N$, compute

$$
\begin{aligned}
& \partial_{x} u_{N}\left(x_{m}, y_{n}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{x} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right), \\
& \partial_{y} u_{N}\left(x_{m}, y_{n}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{y} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) ;
\end{aligned}
$$

3, (FDLT) Determine $\left\{\widehat{\beta}_{k j}^{x}\right\}$ and $\left\{\widehat{\beta}_{k j}^{y}\right\}$ from

$$
\begin{aligned}
& I_{N}\left(\beta \partial_{x} u_{N}\right)\left(x_{m}, y_{n}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widehat{\beta}_{k j}^{x} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right), \\
& I_{N}\left(\beta \partial_{y} u_{N}\right)\left(x_{m}, y_{n}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widehat{\beta}_{k j}^{y} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) ;
\end{aligned}
$$

4, For $k, j=0,1, \cdots, N-2$, compute

$$
\begin{aligned}
& (A \widehat{u})_{k j}=\left(I_{N}\left(\beta \partial_{x} u_{N}\right), \phi_{k}^{\prime}(x) \phi_{j}(y)\right)+\left(I_{N}\left(\beta \partial_{y} u_{N}\right), \phi_{k}(x) \phi_{j}^{\prime}(y)\right) \\
& =\left(I_{N}\left(\beta \partial_{x} u_{N}\right),(-2 k-3) L_{k+1}(x) \phi_{j}(y)\right)+\left(I_{N}\left(\beta \partial_{y} u_{N}\right),(-2 j-3) \phi_{k}(x) L_{j+1}(y)\right) \\
& =\frac{4}{2 j+5} \widehat{\beta}_{k+1, j+2}^{x}-\frac{4}{2 j+1} \widehat{\beta}_{k+1, j}^{x}+\frac{4}{2 k+5} \widehat{\beta}_{k+2, j+1}^{y}-\frac{4}{2 k+1} \widehat{\beta}_{k, j+1}^{y} .
\end{aligned}
$$

Similarly, the evaluation

$$
(B \widehat{u})_{k j}=\left(I_{N}\left(\alpha u_{N}\right), \varphi_{k, j}\right), \quad k, j=0,1, \cdots, N-2
$$

without explicitly forming the matrix $B$ is shown below.
1, Determine $\left\{\widetilde{u}_{k j}^{(1)}\right\}$ from

$$
u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \widehat{u}_{k j} \phi_{k}(x) \phi_{j}(y)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{(1)} L_{k}(x) L_{j}(y) ;
$$

2, (BDLT) Compute

$$
u_{N}\left(x_{m}, y_{m}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widetilde{u}_{k j}^{(1)} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right), \quad m, n=0,1, \cdots, N
$$

3, (FDLT) Determine $\left\{\widehat{\alpha}_{k j}\right\}$ from

$$
I_{N}\left(\alpha u_{N}\right)\left(x_{m}, y_{n}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \widehat{\alpha}_{k j} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right), \quad m, n=0,1, \cdots, N
$$

4 , For $k, j=0,1, \cdots, N-2$, compute

$$
\begin{aligned}
& (B \widehat{u})_{k j}=\left(I_{N}\left(\alpha u_{N}\right), \phi_{k}(x) \phi_{j}(y)\right) \\
& =\frac{4 \widehat{\alpha}_{k, j}}{(2 k+1)(2 j+1)}-\frac{4 \widehat{\alpha}_{k, j+2}}{(2 k+1)(2 j+5)}-\frac{4 \widehat{\alpha}_{k+2, j}}{(2 k+5)(2 j+1)}+\frac{4 \widehat{\alpha}_{k+2, j+2}}{(2 k+5)(2 j+5)} .
\end{aligned}
$$

Three dimensional case. Given $u_{N}=\sum_{k, j, l=0}^{N-2} \widehat{u}_{k j l} \psi_{k, j, l}(\boldsymbol{x})$, the evaluation

$$
(A \widehat{u})_{k j l}=\left(I_{N}\left(\beta \nabla u_{N}\right), \nabla \psi_{k, j, l}\right), \quad k, j, l=0,1, \cdots, N-2
$$

without explicitly forming the matrix $A$ is presented below.
1, Using (2.1) to determine $\left\{\widetilde{u}_{k j l}^{x}\right\},\left\{\widetilde{u}_{k j l}^{y}\right\}$ and $\left\{\widetilde{u}_{k j l}^{z}\right\}$ from

$$
\begin{aligned}
& \partial_{x} u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \sum_{l=0}^{N-2} \widehat{u}_{k j l} \phi_{k}^{\prime}(x) \phi_{j}(y) \phi_{l}(z)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{x} L_{k}(x) L_{j}(y) L_{l}(z), \\
& \partial_{y} u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \sum_{l=0}^{N-2} \widehat{u}_{k j l} \phi_{k}(x) \phi_{j}^{\prime}(y) \phi_{l}(z)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{y} L_{k}(x) L_{j}(y) L_{l}(z), \\
& \partial_{z} u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \sum_{l=0}^{N-2} \widehat{u}_{k j l} \phi_{k}(x) \phi_{j}(y) \phi_{l}^{\prime}(z)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{z} L_{k}(x) L_{j}(y) L_{l}(z)
\end{aligned}
$$

2, (BDLT) For $m, n, i=0,1, \cdots, N$, compute

$$
\begin{aligned}
& \partial_{x} u_{N}\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{x} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right), \\
& \partial_{y} u_{N}\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{y} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right), \\
& \partial_{z} u_{N}\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{z} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right)
\end{aligned}
$$

3, (FDLT) Determine $\left\{\widehat{\beta}_{k j l}^{x}\right\},\left\{\widehat{\beta}_{k j l}^{y}\right\}$ and $\left\{\widehat{\beta}_{k j l}^{z}\right\}$ from

$$
I_{N}\left(\beta \partial_{x} u_{N}\right)\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widehat{\beta}_{k j l}^{x} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right)
$$

$$
\begin{aligned}
& I_{N}\left(\beta \partial_{y} u_{N}\right)\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widehat{\beta}_{k j l}^{y} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right), \\
& I_{N}\left(\beta \partial_{z} u_{N}\right)\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widehat{\beta}_{k j l}^{z} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right) ;
\end{aligned}
$$

4, For $k, j, l=0,1, \cdots, N-2$, compute

$$
\begin{aligned}
(A \widehat{u})_{k j l}= & \left(I_{N}\left(\beta \partial_{x} u_{N}\right), \phi_{k}^{\prime}(x) \phi_{j}(y) \phi_{l}(z)\right)+\left(I_{N}\left(\beta \partial_{y} u_{N}\right), \phi_{k}(x) \phi_{j}^{\prime}(y) \phi_{l}(z)\right) \\
& +\left(I_{N}\left(\beta \partial_{z} u_{N}\right), \phi_{k}(x) \phi_{j}(y) \phi_{l}^{\prime}(z)\right) \\
= & -\frac{8}{(2 j+1)(2 l+1)} \widehat{\beta}_{k+1, j, l}^{x}+\frac{8}{(2 j+5)(2 l+1)} \widehat{\beta}_{k+1, j+2, l}^{x} \\
& +\frac{8}{(2 j+1)(2 l+5)} \widehat{\beta}_{k+1, j, l+2}^{x}-\frac{8}{(2 j+5)(2 l+5)} \widehat{\beta}_{k+1, j+2, l+2}^{x} \\
& -\frac{8}{(2 k+1)(2 l+1)} \widehat{\beta}_{k, j+1, l}^{y}+\frac{8}{(2 k+5)(2 l+1)} \widehat{\beta}_{k+2, j+1, l}^{y} \\
& +\frac{8}{(2 k+1)(2 l+5)} \widehat{\beta}_{k, j+1, l+2}^{y}-\frac{8}{(2 k+5)(2 l+5)} \widehat{\beta}_{k+2, j+1, l+2}^{y} \\
& -\frac{8}{(2 k+1)(2 j+1)} \widehat{\beta}_{k, j, l+1}^{z}+\frac{8}{(2 k+5)(2 j+1)} \widehat{\beta}_{k+2, j, l+1}^{z} \\
& +\frac{8}{(2 k+1)(2 j+5)} \widehat{\beta}_{k, j+2, l+1}^{z}-\frac{8}{(2 k+5)(2 j+5)} \widehat{\beta}_{k+2, j+2, l+1}^{z} ;
\end{aligned}
$$

Similarly, the computation

$$
(B \widehat{u})_{k j l}=\left(I_{N}\left(\alpha u_{N}\right), \psi_{k, j, l}\right), \quad k, j, l=0,1, \cdots, N-2
$$

without explicitly forming the matrix $B$ is shown below.
1, Determine $\left\{\widetilde{u}_{k j}^{(1)}\right\}$ from

$$
u_{N}=\sum_{k=0}^{N-2} \sum_{j=0}^{N-2} \sum_{l=0}^{N-2} \widehat{u}_{k j l} \phi_{k}(x) \phi_{j}(y) \phi_{l}(z)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{(1)} L_{k}(x) L_{j}(y) L_{l}(z) ;
$$

2, (BDLT) Compute

$$
u_{N}\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widetilde{u}_{k j l}^{(1)} \phi_{k}\left(x_{m}\right) \phi_{j}\left(y_{n}\right) \phi_{l}\left(z_{i}\right), \quad m, n, i=0,1, \cdots, N ;
$$

3, (FDLT) Determine $\left\{\widehat{\alpha}_{k j l}\right\}$ from

$$
I_{N}\left(\alpha u_{N}\right)\left(x_{m}, y_{n}, z_{i}\right)=\sum_{k=0}^{N} \sum_{j=0}^{N} \sum_{l=0}^{N} \widehat{\alpha}_{k j l} L_{k}\left(x_{m}\right) L_{j}\left(y_{n}\right) L_{l}\left(z_{i}\right), \quad m, n, i=0,1, \cdots, N
$$

4, For $k, j, l=0,1, \cdots, N-2$, compute
$(B \widehat{u})_{k j l}=\left(I_{N}\left(\alpha u_{N}\right), \phi_{k}(x) \phi_{j}(y) \phi_{l}(z)\right)$

$$
=-\frac{8 \widehat{\alpha}_{k, j, l+2}}{(2 k+1)(2 j+1)(2 l+5)}+\frac{8 \widehat{\alpha}_{k, j, l}}{(2 k+1)(2 j+1)(2 l+1)}-\frac{8 \widehat{\alpha}_{k, j+2, l}}{(2 k+1)(2 j+5)(2 l+1)}
$$

$$
\begin{aligned}
& +\frac{8 \widehat{\alpha}_{k, j+2, l+2}}{(2 k+1)(2 j+5)(2 l+5)}-\frac{8 \widehat{\alpha}_{k+2, j, l}}{(2 k+5)(2 j+1)(2 l+1)}+\frac{8 \widehat{\alpha}_{k+2, j+2, l}}{(2 k+5)(2 j+5)(2 l+1)} \\
& +\frac{8 \widehat{\alpha}_{k+2, j, l+2}}{(2 k+5)(2 j+1)(2 l+5)}-\frac{8 \widehat{\alpha}_{k+2, j+2, l+2}}{(2 k+5)(2 j+5)(2 l+5)} .
\end{aligned}
$$

Note that the main cost in the above procedure of evaluating $A \widehat{u}$ and $B \widehat{u}$ is the discrete Legendre transforms in steps 2 and 3. The cost for each of steps 1 and 4 is of $\mathcal{O}\left(N^{d}\right)$ flops. In summary, the total cost for evaluating $(A+B) \widehat{u}$ is dominated by several fast discrete legendre transforms, and is of $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$.

## 5. Numerical results

In this section, some numerical experiments are provided to demonstrate the effectiveness of both the proposed preconditioner $M$ and matrix-vector multiplications. Meanwhile, the properties of matrices from the Legendre-Galerkin methods are numerically studied. In particular, a class of coefficient functions with high variations are test. In all numerical experiments, the stopping criterion $\varepsilon=10^{-12}$. All the numerical results are performed on a 3.30 GHz Intel Core 15-4590 desktop computer with 12GB RAM. The code is in MATLAB 2016 b.
5.1. Numerical results for fast matrix-vector multiplications. The first test investigates the time taken to compute one matrix-vector multiplication of a vector by the discretization matrix resulting from the Legendre-Gelerkin method. The vector is generated randomly by the rand() command. For this purpose, the numerical experiments are carried out for different coefficients $\beta(\boldsymbol{x})$ and $\alpha(\boldsymbol{x})$ in one, two and three dimensions.

Table 5.1. CPU time for fast multiplication of matrix $A \in \mathbb{R}^{(N-1) \times(N-1)}$ by any vector.

| $\beta(x)=\left(2 x^{2}+1\right)^{4}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 320 | 640 | 1280 | 2560 | 5120 | 10240 |
| time(s) | 0.0157 | 0.0211 | 0.0306 | 0.0514 | 0.1130 | 0.2349 |
| $\beta(x)=e^{2 x}$ |  |  |  |  |  |  |
| $N$ | 320 | 640 | 1280 | 2560 | 5120 | 10240 |
| time(s) | 0.0184 | 0.0207 | 0.0301 | 0.0522 | 0.1128 | 0.2376 |

The average time of 10 tests of matrix-vector multiplications by matrix $A$ is reported in Table 5.1, Table 5.3, Table 5.5, and by matrix $B$ in Table 5.2, Table 5.4, Table 5.6 respectively. It can be observed that the time scales roughly linearly in the dimension of matrices $A$ and $B$, which is consistent with the discussions in section 4.3.
5.2. Numerical results for the number of iterations. The second test is to demonstrate the effectiveness of proposed preconditioner $M$. To this end, the iteration steps of the PCG method with a constant-coefficient preconditioner (PCG-I) and the PCG method with the proposed preconditioner $M$ (PCG-II) are compared. The preconditioner $M$ is constructed by approximating $\beta(\boldsymbol{x})$ and $\alpha(\boldsymbol{x})$ with a $\left(t_{1}+1\right)$-term Legendre series and a $\left(t_{2}+1\right)$-term Legendre series in each direction

Table 5.2. CPU time for fast multiplication of matrix $B \in \mathbb{R}^{(N-1) \times(N-1)}$ by any vector.

| $\alpha(x)=\left(2 x^{2}+1\right)^{4}$ |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 320 | 640 | 1280 | 2560 | 5120 | 10240 |  |
| time(s) | 0.0178 | 0.0215 | 0.0320 | 0.0529 | 0.1140 | 0.2414 |  |
| $\alpha(x)=e^{2 x}$ |  |  |  |  |  |  |  |
| $N$ | 320 | 640 | 1280 | 2560 | 5120 | 10240 |  |
| time(s) | 0.0202 | 0.0236 | 0.0300 | 0.0518 | 0.1188 | 0.2404 |  |

Table 5.3. CPU time for fast multiplication of matrix $A \in \mathbb{R}^{(N-1)^{2} \times(N-1)^{2}}$ by any vector.

| $\beta(\boldsymbol{x})=\left(2 x^{2}+2 y^{2}+1\right)^{4}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
| time(s) | 0.0596 | 0.2052 | 0.7670 | 2.9913 | 12.2643 | 50.4063 |
| $\beta(\boldsymbol{x})=e^{2(x+y)}$ |  |  |  |  |  |  |
| $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
| time(s) | 0.0575 | 0.2077 | 0.7650 | 2.9453 | 12.0757 | 49.7519 |

Table 5.4. CPU time for fast multiplication of matrix $B \in \mathbb{R}^{(N-1)^{2} \times(N-1)^{2}}$ by any vector.

| $\alpha(\boldsymbol{x})=\left(2 x^{2}+2 y^{2}+1\right)^{4}$ |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
| time(s) | 0.0297 | 0.1046 | 0.3723 | 1.5264 | 6.2888 | 26.0985 |
| $\alpha(\boldsymbol{x})=e^{2(x+y)}$ |  |  |  |  |  |  |
| $N$ | 16 | 32 | 64 | 128 | 256 | 512 |
| time(s) | 0.0281 | 0.1049 | 0.3794 | 1.4797 | 6.0668 | 25.1772 |

respectively. In each iteration of PCG-I, the system with the constant-coefficient preconditioner as the coefficient matrix is solved by direct methods in $\mathcal{O}(N)$ operations for $d=1$ [21] and in $\mathcal{O}\left(N^{d}(\log N)^{d-1}\right)$ operations for $d=2,3[21, ~ 26]$. Numerical results are presented in Table 5.7, 5.8 and 5.9. Test problems are considered as follows:

Example 1. The problem (3.1) in one dimension takes the following coefficients:
(a) $\beta(x)=\left(2 x^{2}+1\right)^{4}$ and $\alpha(x)=\cos (x)$.
(b) $\beta(x)=e^{2 x}$ and $\alpha(x)=0$.

TABLE 5.5. CPU time for fast multiplication of matrix $A \in \mathbb{R}^{(N-1)^{3} \times(N-1)^{3}}$ by any vector.

| $\beta(\boldsymbol{x})=\left(2 x^{2}+2 y^{2}+2 z^{2}+1\right)^{4}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 4 | 8 | 16 | 32 | 64 |
| time(s) | 0.0778 | 0.3989 | 2.5246 | 20.4493 | 166.0483 |
| $\beta(\boldsymbol{x})=e^{2(x+y+z)}$ |  |  |  |  |  |
| $N$ | 4 | 8 | 16 | 32 | 64 |
| time(s) | 0.0770 | 0.3982 | 2.2168 | 16.3830 | 132.7023 |

TABLE 5.6. CPU time for fast multiplication of matrix $B \in \mathbb{R}^{(N-1)^{3} \times(N-1)^{3}}$ by any vector.

| $\alpha(x)=\left(2 x^{2}+2 y^{2}+2 z^{2}+1\right)^{4}$ |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ | 4 | 8 | 16 | 32 | 64 |
| time(s) | 0.0251 | 0.1404 | 0.7706 | 6.2419 | 50.8091 |
| $\alpha(x)=e^{2(x+y+z)}$ |  |  |  |  |  |
| $N$ | 4 | 8 | 16 | 32 | 64 |
| time(s) | 0.0255 | 0.1268 | 0.7127 | 5.7871 | 47.1070 |

Example 2. The coefficients of problem (3.1) in two dimensions are as follows :
(a) $\beta(\boldsymbol{x})=\left(2 x^{2}+2 y^{2}+1\right)^{4}$ and $\alpha(\boldsymbol{x})=\cos (x+y)$.
(b) $\beta(\boldsymbol{x})=e^{2(x+y)}$ and $\alpha(\boldsymbol{x})=0$.

Example 3. The coefficients of problem (3.1) in three dimensions are as follows:
(a) $\beta(\boldsymbol{x})=\left(2 x^{2}+2 y^{2}+2 z^{2}+1\right)^{4}$ and $\alpha(\boldsymbol{x})=\cos (x+y+z)$.
(b) $\beta(\boldsymbol{x})=e^{2(x+y+z)}$ and $\alpha(\boldsymbol{x})=0$.

Table 5.7 reports the results for the one-dimensional problem. Note that the PCG method with the proposed preconditioner $M$ exhibits excellent performance in terms of iteration step over the PCG method with a constant-coefficient preconditioner. Meanwhile, the iteration steps of PCG-II only increase slightly as the discretization parameter $N$ increases. As $\beta(\boldsymbol{x})$ and $\alpha(\boldsymbol{x})$ are approximated by a finite number of Legendre series to a higher accuracy, the iteration steps decrease which indicates that the PCG method converges more quickly. Table 5.8 and 5.9 list the numerical results for the two- and three-dimensional problems respectively. The iteration steps of PCG-II for the 2-D and 3-D cases behave similarly as the 1-D case. All of the examples show that the proposed preconditioner $M$ is very effective for problems with large variations in coefficient functions.

Table 5.7. Iteration counts for Example 1.

| Example 1 (a) |  |  |  |  |  |  |  |
| :---: | :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| $N$ |  | 320 | 640 | 1280 | 2560 | 5120 | 10240 |
| PCG-I | $t_{1}=0, t_{2}=0$ | 130 | 134 | 138 | 142 | 145 | 148 |
|  | $t_{1}=4, t_{2}=2$ | 16 | 17 | 17 | 17 | 18 | 18 |
|  | $t_{1}=6, t_{2}=2$ | 7 | 8 | 8 | 8 | 8 | 8 |
| $N$ |  |  |  |  |  |  |  |
| PCG-I | $t_{1}=0, t_{2}=0$ | 108 | 110 | 113 | 116 | 119 | 121 |
|  | $t_{1}=4, t_{2}=0$ | 11 | 11 | 11 | 12 | 12 | 12 |
|  | $t_{1}=5, t_{2}=0$ | 7 | 7 | 7 | 8 | 8 | 8 |

Table 5.8. Iteration counts for Example 2.

| Example 2 (a) |  |  |  |  |  |  |
| :---: | :--- | :---: | :---: | :---: | :---: | :---: |
| $N$ |  | 40 | 60 | 80 | 100 | 120 |
| PCG-I | $t_{1}=0, t_{2}=0$ | 242 | 281 | 288 | 291 | 292 |
| PCG-II | $t_{1}=4, t_{2}=3$ | 15 | 17 | 19 | 21 | 23 |
|  | $t_{1}=6, t_{2}=3$ | 6 | 7 | 9 | 10 | 10 |
| $N$ |  |  |  |  |  |  |
| $N$ |  | 40 | 60 | 80 | 100 | 120 |
| PCG-I | $t_{1}=0, t_{2}=0$ | 545 | 585 | 602 | 610 | 615 |
| PCG-II | $t_{1}=5, t_{2}=0$ | 14 | 18 | 22 | 26 | 30 |
|  | $t_{1}=7, t_{2}=0$ | 8 | 11 | 13 | 13 | 13 |

Example 4. The PCG method with the proposed preconditioner $M$ can be applied to more general second order problems:

$$
\begin{align*}
& \left\{\begin{array}{l}
-\left(\beta_{1}(x) u_{x}\right)_{x}-\left(\beta_{2}(y) u_{y}\right)_{y}+\alpha(\boldsymbol{x}) u=f, \quad \boldsymbol{x} \in \Omega=(-1,1)^{2}, \\
\left.u\right|_{\partial \Omega}=0,
\end{array}\right.  \tag{5.1}\\
& \left\{\begin{array}{l}
-\left(\beta_{1}(x) u_{x}\right)_{x}-\left(\beta_{2}(y) u_{y}\right)_{y}-\left(\beta_{3}(z) u_{z}\right)_{z}+\alpha(\boldsymbol{x}) u=f, \quad \boldsymbol{x} \in \Omega=(-1,1)^{3}, \\
\left.u\right|_{\partial \Omega}=0 .
\end{array}\right. \tag{5.2}
\end{align*}
$$

Consider the problem (5.1) and (5.2) with the following coefficients:
(a) $\beta_{1}(x)=e^{2 x}, \beta_{2}(y)=\cos (y)$ and $\alpha(\boldsymbol{x})=0$.
(b) $\beta_{1}(x)=e^{2 x}, \beta_{2}(y)=\cos (y), \beta_{3}(z)=\cos (z)$ and $\alpha(\boldsymbol{x})=0$.

Table 5.9. Iteration counts for Example 3.

| Example 3 (a) |  |  |  |  |  |
| :---: | :--- | :---: | :---: | :---: | :---: |
| $N$ |  | 12 | 16 | 20 | 24 |
| PCG-I | $t_{1}=0, t_{2}=0$ | 256 | 366 | 436 | 476 |
| PCG-II | $t_{1}=4, t_{2}=3$ | 15 | 19 | 22 | 23 |
|  | $t_{1}=6, t_{2}=3$ | 7 | 7 | 8 | 9 |
|  | Example 3 (b) |  |  |  |  |  |
| $N$ |  |  |  |  |  |
| PCG-I | $t_{1}=0, t_{2}=0$ | 1484 | 2186 | 2564 | 2744 |
|  | $t_{1}=5, t_{2}=0$ | 7 | 9 | 13 | 16 |
|  | $t_{1}=6, t_{2}=0$ | 5 | 8 | 10 | 12 |

For problems of the form (3.1], Shen in [21] have pointed out that it is efficient to make a change of dependent variable $v=\sqrt{\beta} u$ [6] which reduces (3.1) to the following equation:

$$
\left\{\begin{array}{l}
-\Delta v+p(\boldsymbol{x}) v=q, \quad \boldsymbol{x} \in \Omega=[-1,1]^{d}, d=1,2,3  \tag{5.3}\\
\left.v\right|_{\partial \Omega}=0
\end{array}\right.
$$

where $p(\boldsymbol{x})=\frac{\Delta(\sqrt{\beta})}{\sqrt{\beta}}+\frac{\alpha(\boldsymbol{x})}{\beta}$ and $q(\boldsymbol{x})=\frac{f}{\sqrt{\beta}}$, then the resulting system from the above problem (5.3) can be solved by using a preconditioned conjugate gradient method with a constantcoefficient preconditioner. However, this strategy is limited in the situation such as problem (5.1) and (5.2). In what follows, both PCG-I and PCG-II are performed for the linear systems rising from problem (5.1) and (5.2). The preconditioner $M$ is constructed by approximating $\beta_{j}$ in case (a) with the $t_{1 j}$-term Legendre polynomials, $j=1,2$, and $\beta_{j}$ in case (b) with the $t_{1 j}$-term Legendre polynomials, $j=1,2,3$. Numerical results are shown in Table 5.10 and Table 5.11.

Table 5.10. Iteration counts for Example 4 (a).

| $N$ |  | 40 | 60 | 80 | 100 | 120 |
| :---: | :--- | :---: | :---: | :---: | :---: | :---: |
| PCG-I | $t_{11}=0, t_{12}=0$ | 86 | 89 | 90 | 92 | 92 |
| PCG-II | $t_{11}=4, t_{12}=3$ | 10 | 11 | 12 | 12 | 13 |
|  | $t_{11}=5, t_{12}=3$ | 8 | 10 | 12 | 13 | 13 |

Tables 5.10 and 5.11 show that the strategy that using a constant-coefficient problem precondition variable-coefficient problems is not effective if coefficient functions have large variation over the domain. Further, it indicates the effectiveness of the proposed preconditioner $M$.
5.3. Numerical ranks of off-diagonal blocks of matrices in the Legendre-Galerkin method. A direct spectral method for differential equations with variable coefficients in one dimension was proposed in [22]. The strategy therein is based on the rank structures of the matrices in Fourier-

Table 5.11. Iteration counts for Example 4 (b).

| $N$ |  | 12 | 16 | 20 | 24 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| PCG-I | $t_{11}=0, t_{12}=0, t_{13}=0$ | 84 | 92 | 96 | 98 |
| PCG-II | $t_{11}=4, t_{12}=3, t_{13}=3$ | 9 | 10 | 10 | 11 |
|  | $t_{11}=6, t_{12}=3, t_{13}=3$ | 6 | 6 | 7 | 8 |

and Chebyshev-spectral methods. Numerical ranks of the off-diagonal block $\left.(A)\right|_{1: \frac{N}{2}, \frac{N}{2}+1: \text { end }}$ and $\left.(B)\right|_{1: \frac{N}{2}, \frac{N}{2}+1: \text { end }}$ with different variable coefficients are computed. Here $\left.A\right|_{j, k}$ denotes the $(j, k)$ entry of $A$ and can be similarly understood when $j$ and $k$ are replaced by index sets, which is the same as the notation in [22].

Table 5.12. Numerical ranks of the off-diagonal block $\left.(B)\right|_{1: \frac{N}{2}, \frac{N}{2}+1: \text { end }}$ for $\alpha(x)$, with different sizes $N$ and tolerances $\tau$.

| $N$ |  | 320 | 640 | 1280 | 2560 | 5120 | 10240 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Numerical $\operatorname{rank}($ with $\alpha(x)=\cos (\sin (x))))$ | $\tau=10^{-6}$ | 6 | 6 | 2 | 2 | 2 | 2 |
|  | $\tau=10^{-12}$ | 8 | 8 | 8 | 8 | 8 | 8 |
| Numerical rank $\left(\right.$ with $\left.\alpha(x)=e^{x}\right)$ | $\tau=10^{-6}$ | 3 | 3 | 3 | 3 | 3 | 3 |
|  | $\tau=10^{-12}$ | 5 | 5 | 5 | 5 | 5 | 5 |
| Numerical rank $\left(\right.$ with $\left.\alpha(x)=\frac{1}{100 x^{2}+1}\right)$ | $\tau=10^{-6}$ | 2 | 2 | 2 | 2 | 2 | 2 |
|  | $\tau=10^{-12}$ | 2 | 2 | 2 | 2 | 2 | 2 |

TABLE 5.13. Numerical ranks of the off-diagonal block $\left.(A)\right|_{1: \frac{N}{2}, \frac{N}{2}+1: \text { end }}$ for $\beta(x)$, with different sizes $N$ and tolerances $\tau$.

| $N$ |  | 320 | 640 | 1280 | 2560 | 5120 | 10240 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Numerical $\operatorname{rank}($ with $\beta(x)=\cos (\sin (x))))$ | $\tau=10^{-6}$ | 6 | 6 | 8 | 8 | 8 | 16 |
|  | $\tau=10^{-12}$ | 115 | 291 | 610 | 1260 | 2551 | 5116 |
| Numerical rank $\left(\right.$ with $\left.\beta(x)=e^{x}\right)$ | $\tau=10^{-6}$ | 4 | 4 | 4 | 4 | 5 | 20 |
|  | $\tau=10^{-12}$ | 117 | 292 | 610 | 1260 | 2549 | 5116 |
| Numerical rank $\left(\right.$ with $\left.\beta(x)=\frac{1}{100 x^{2}+1}\right)$ | $\tau=10^{-6}$ | 2 | 2 | 2 | 2 | 2 | 2 |
|  | $\tau=10^{-12}$ | 8 | 65 | 374 | 1105 | 2440 | 5052 |

Table 5.12 indicates that the off-diagonal numerical ranks of the matrix $B$ do not increase with $N$, indicating the low-rank property for all cases. However, when $N$ doubles, the numerical ranks of $A$ increase apparently if the accuracy $\tau$ increases from $10^{-6}$ to $10^{-12}$, as is shown in Table 5.13. Therefore the direct spectral solver based on the rank structures of the coefficient matrices is impracticable for the Legendre-Gelerkin method. Besides, the algorithm of constructing an HSS
approximation to a dense matrix requires considerable programming effort. Moreover, for twoand three-dimensional problems, a simple HSS structure is generally not practical.

## 6. CONCLUSION

An efficient preconditioner $M$ for the PCG method is proposed for the linear system arising from the Legendre-Galerkin method of second-order elliptic equations. Since the iteration step of the PCG method increase slightly as the discretization parameter $N$ increases, matrix-vector multiplications can be evaluated in $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$ operations, and the complexity of approximately solving the system with a preconditioner $M$ is of $\mathcal{O}\left(T^{2 d} N^{d}\right)$, where the preconditioner $M$ is constructed by using the $(T+1)$-term Legendre polynomials in each direction to approximate the variable coefficient functions, the algorithm admits an $\mathcal{O}\left(N^{d}(\log N)^{2}\right)$ computational complexity for $d=1,2,3$ while providing spectral accuracy. Furthermore, numerical results indicate that it is very robust.

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