# A FULLY DISCRETE LOW-REGULARITY INTEGRATOR FOR THE NONLINEAR SCHRÖDINGER EQUATION 

ALEXANDER OSTERMANN AND FANGYAN YAO


#### Abstract

For the solution of the one dimensional cubic nonlinear Schrödinger equation on the torus, we propose and analyse a fully discrete low-regularity integrator. The considered scheme is explicit. Its implementation relies on the fast Fourier transform with a complexity of $\mathcal{O}(N \log N)$ operations per time step, where $N$ denotes the degrees of freedom in the spatial discretisation. We prove that the new scheme provides an $\mathcal{O}\left(\tau^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon}+N^{-\gamma}\right)$ error bound in $L^{2}$ for any initial data in $H^{\gamma}, \frac{1}{2}<\gamma \leq 1$, where $\tau$ denotes the temporal step size. Numerical examples illustrate this convergence behavior.


## 1. Introduction

In this paper, we analyze low-regularity integrators for the cubic nonlinear Schrödinger equation (NLS):

$$
\left\{\begin{array}{l}
i \partial_{t} u(t, x)+\partial_{x x} u(t, x)=|u(t, x)|^{2} u(t, x),  \tag{1.1}\\
u(0, x)=u^{0}(x) .
\end{array}\right.
$$

We consider this problem on the one dimensional torus $\mathbb{T}=(0,2 \pi)$. The function $u: \mathbb{R}^{+} \times \mathbb{T} \rightarrow \mathbb{C}$ is the unknown and $u^{0} \in H^{\gamma}(\mathbb{T}), \gamma \geq 0$ is the given initial data. The NLS equation (1.1) is globally well-posed in $H^{\gamma}(\mathbb{T}), \gamma \geq 0$; see, e.g., [2].

The construction of efficient numerical schemes for dispersive equations has been the subject of much research work. In particular, for the NLS equation, substantial research has been undertaken in numerical analysis. For smooth solutions, many classical numerical methods have been analyzed for numerical discretisation in space and time, for example, finite difference methods [13], operator splitting [1, 4, 9], and exponential integrators [3]. These methods have all their own characteristics. However, they generally require relatively high regularity of the solution. For example, initial data in $H^{\gamma+2}$ are required to obtain first-order convergence in $H^{\gamma}$ for the NLS equation in [9], and initial data in $H^{\gamma+4}$ for second-order convergence.

As mentioned above, if the exact solution is smooth enough, one can rely on classical numerical schemes. In practical application, however, nonsmooth initial data are encountered as well. A typical example is applications in nonlinear optics, where initial data can be corrupted with noise. Therefore, recent attention has been focused on lower regularity requirements. In order to achieve convergence with the lowest possible regularity of the initial data, so-called low-regularity integrator were proposed recently.

[^0]For the NLS equation, Ostermann and Schratz [10] achieved first-order convergence in $H^{\gamma}\left(\mathbb{T}^{d}\right), d \geq 1$ with initial data $H^{\gamma+1}\left(\mathbb{T}^{d}\right)$ by introducing a new exponential-type numerical scheme under the assumption $\gamma>\frac{d}{2}$. Furthermore, the authors obtained the convergence in $H^{\gamma}(\mathbb{T})$ with initial data $H^{\gamma}(\mathbb{T})$ for the quadratic NLS equation in one space dimension. Then Wu and Yao [14] constructed a new first-order scheme for the cubic NLS equation in one space dimension and obtained first-order convergence in $H^{\gamma}(\mathbb{T})$ with initial data $H^{\gamma}(\mathbb{T})$ for $\gamma>\frac{3}{2}$. A second-order scheme was proposed by Knöller, Ostermann and Schratz in [7]. In one space dimension, this scheme requires two additional derivatives of the solution; in higher dimensions, three additional derivatives are necessary. Later, Ostermann, Rousset and Schratz [11, 12] proved convergence in $L^{2}$ for initial data in $H^{s}\left(\mathbb{R}^{d}\right)$ and $H^{s}(\mathbb{T})$ respectively, $0<s \leq 1$ and obtained fractional orders of convergence in a frame work of discrete Bourgain spaces. All results discussed above concern time integration only. Recently, Li and Wu [8] considered a fully discrete low-regularity integrator in one space dimension for the NLS equation and got first-order convergence (up to a logarithmic factor) in both time and space in $L^{2}(\mathbb{T})$ for $H^{1}(\mathbb{T})$ initial data.

The purpose of this article is to construct a fully discrete low-regularity integrator and to prove convergence for initial data $u^{0} \in H^{s}(\mathbb{T}), \frac{1}{2}<s \leq 1$. For the spatial discretisation, we use a Fourier ansatz with frequency truncation. For the temporal discretisation, we employ a careful analysis of the nonlinear dynamics in phase space. In addition, we use harmonic analysis technique in the proof of some technical lemmas.

The rest of this paper is structured as follows. We present the fully discrete lowregularity integrator in section 2 . We state the main convergence result and explain the employed notations. In section 3, we derive the considered scheme and give some technical lemmas, which will be used in the convergence proof. In section 4, we show the $H^{\gamma}$ error bound before the poof of the theorem. Finally we prove stability and the error bound of the fully discrete scheme. In sections 5 and 6 , we report numerical experiments that illustrate our theoretical analysis and we draw some conclusions.

## 2. Notations and main Result

2.1. Some notations. We denote by $\langle\cdot, \cdot\rangle$ the $L^{2}$ inner product on $\mathbb{T}$, that is

$$
\langle f, g\rangle=\int_{\mathbb{T}} f(x) \overline{g(x)} d x, \quad f, g \in L^{2}(\mathbb{T} ; \mathbb{C})
$$

The Fourier transform $\left(\hat{f}_{k}\right)_{k \in \mathbb{Z}}$ of a function $f: \mathbb{T} \rightarrow \mathbb{C}$ is defined by

$$
\hat{f}_{k}=\frac{1}{2 \pi} \int_{\mathbb{T}} e^{-i k x} f(x) d x
$$

The Fourier inversion formula is given by

$$
f(x)=\sum_{k \in \mathbb{Z}} \hat{f}_{k} e^{i k x}
$$

We recall the following properties:

$$
\|f\|_{L^{2}(\mathbb{T})}^{2}=2 \pi \sum_{k \in \mathbb{Z}}\left|\hat{f}_{k}\right|^{2}, \quad f \in L^{2}(\mathbb{T}) ;
$$

$$
\widehat{(f g})_{k}=\sum_{k=k_{1}+k_{2}} \hat{f}_{k_{1}} \hat{g}_{k_{2}}, \quad f, g \in L^{2}(\mathbb{T})
$$

For our convergence analysis, we equip the Sobolev space $H^{s}(\mathbb{T}), s \geq 0$ with the norm

$$
\|f\|_{H^{s}(\mathbb{T})}^{2}=\left\|J^{s} f\right\|_{L^{2}(\mathbb{T})}^{2}=2 \pi \sum_{k \in \mathbb{Z}}\left(1+k^{2}\right)^{s}\left|\hat{f}_{k}\right|^{2}, \quad J^{s}=\left(1-\partial_{x x}\right)^{\frac{s}{2}}
$$

This norm is equivalent with the standard norm of $H^{s}(\mathbb{T})$.
Further, we denote by $\partial_{x}^{-1}$ the operator defined in Fourier space as

$$
\left(\widehat{\partial_{x}^{-1} f}\right)_{k}= \begin{cases}(i k)^{-1} \hat{f}_{k} & \text { if } k \neq 0  \tag{2.1}\\ 0 & \text { if } k=0\end{cases}
$$

For convenience, we introduce some additional notations. First, we define the zero-mode operator by

$$
P_{0} f=\hat{f}_{0}=\frac{1}{2 \pi} \int_{\mathbb{T}} f(x) d x
$$

Furthermore, for any positive integer $N$, we define the projection operators $P_{N}$ and $P_{>N}$ by

$$
\left(\widehat{P_{N} f}\right)_{k}=\left\{\begin{array}{ll}
\hat{f}_{k} & \text { if }|k| \leq N,  \tag{2.2}\\
0 & \text { if }|k|>N ;
\end{array} \quad\left(\widehat{P_{>N} f}\right)_{k}= \begin{cases}\hat{f}_{k} & \text { if }|k|>N \\
0 & \text { if }|k| \leq N\end{cases}\right.
$$

Let $S_{N}$ be the space consisting of all functions $f \in L^{2}(\mathbb{T})$ such that $\hat{f}_{k}=0$ for $|k|>N$. Then, for $f$ and $g \in S_{N}$, the cost of computing the Fourier coefficients of $P_{N}(f g) \in S_{N}$ is $\mathcal{O}(N \log N)$. This can be seen as follows.

Let $I_{2 N}$ be the $(4 N+1)$-point trigonometric interpolation operator

$$
\begin{equation*}
I_{2 N} f(x)=\sum_{k=-2 N}^{2 N} \mathrm{e}^{i k x} \tilde{f}_{k} \tag{2.3}
\end{equation*}
$$

where

$$
\tilde{f}_{k}=\frac{1}{4 N+1} \sum_{k=-2 N}^{2 N} \mathrm{e}^{-i k x_{n}} f\left(x_{n}\right), \quad x_{n}=\frac{2 \pi n}{4 N+1}, \quad n=-2 N, \cdots, 2 N
$$

Consequently, if the Fourier coefficient $\hat{f}_{k}$ of the function $f$ satisfies $\hat{f}_{k}=0$ for $|k|>2 N$, then $I_{2 N} f=f$ and $\tilde{f}_{k}=\hat{f}_{k}$. Furthermore, if $f$ and $g \in S_{N}$, then $\widehat{(f g)_{k}}=0$ for $|k|>2 N$. Hence we get $f g=I_{2 N}(f g)$ from the definition of $I_{2 N}$. Further, the cost of computing the Fourier coefficients of $P_{N} I_{2 N}(f g) \in S_{N}$ is $\mathcal{O}(N \log N)$.
2.2. Numerical method and main result. Let $\tau$ denote the temporal step size and $t_{n}=n \tau$ the corresponding sequence of discretisation points in the time interval $[0, T]$. In section 3.1 below, we construct the fully discrete low-regularity integrator for equation (1.1) as

$$
\begin{equation*}
u_{\tau, N}^{n+1}=\Phi_{\tau, N}\left(u_{\tau, N}^{n}\right) \tag{2.4}
\end{equation*}
$$

with $u_{\tau, N}^{0}=P_{N} I_{2 N} u_{0}$. The numerical flow $\Phi_{\tau, N}$ maps a function $f \in S_{N}$ to $\Phi_{\tau, N}(f) \in$ $S_{N}$. It is defined in the following way:

$$
\begin{align*}
\Phi_{\tau, N}(f)= & \mathrm{e}^{i \tau \partial_{x}^{2}} f+\frac{1}{2} \partial_{x}^{-1} P_{N}\left[\left(\mathrm{e}^{-i \tau \partial_{x}^{2}} \partial_{x}^{-1} \bar{f}\right) \cdot \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left(f^{2}\right)\right]-\frac{1}{2} \mathrm{e}^{i \tau \partial_{x}^{2}} \partial_{x}^{-1} P_{N}\left[\partial_{x}^{-1} \bar{f} \cdot P_{N}\left(f^{2}\right)\right] \\
& -i \tau P_{0}\left[\bar{f} \cdot P_{N}\left(f^{2}\right)\right]-i \tau\left[\mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left(f^{2}\right)-P_{0}\left(f^{2}\right)\right] \cdot P_{0}(\bar{f}) \\
& +\frac{1}{2} \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left[\bar{f} \cdot \mathrm{e}^{-i \tau \partial_{x}^{2}} P_{N}\left(\mathrm{e}^{i \tau \partial_{x}^{2}} \partial_{x}^{-1} f\right)^{2}\right]-\frac{1}{2} \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left[\bar{f} \cdot P_{N}\left(\partial_{x}^{-1} f\right)^{2}\right] \\
& +i \tau \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left[\bar{f} \cdot P_{N}\left(f^{2}\right)\right]-2 i \tau \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left[\bar{f} \cdot P_{N} f\right] \cdot P_{0}(f) \\
& +i \tau \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N} \bar{f} \cdot\left(P_{0}(f)\right)^{2} . \tag{2.5}
\end{align*}
$$

From the discussion in section 2.1, we infer that the initial data $u_{\tau, N}^{0}=P_{N} I_{2 N} u_{0}$ can be computed with FFT with computational cost of $\mathcal{O}(N \log N)$. Furthermore, the terms $\mathrm{e}^{i \tau \partial_{x}^{2}} f$ and $P_{N}(f g)$ also can be computed with FFT for given functions $f, g \in S_{N}$. We note that (2.5) only consists of the above expressions. Therefore, (2.5) can be computed with FFT.

Now, we state the main result of this paper. We show convergence of the fully discrete low-regularity integrator given in (2.4).

Theorem 2.1. Let $u_{\tau, N}^{n}$ be the numerical solution (2.4) of equation (1.1) up to some fixed time $T>0$. Under the assumption that $u^{0} \in H^{\gamma}(\mathbb{T})$ for some $\frac{1}{2}<\gamma \leq 1$, for arbitrary given $\varepsilon>0$, there exist constants $\tau_{0}, C>0$ such that for any step size $0<\tau \leq \tau_{0}$ and all $0 \leq n \tau \leq T$

$$
\begin{equation*}
\left\|u\left(t_{n}, \cdot\right)-u_{\tau, N}^{n}\right\|_{L^{2}} \leq C \tau^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon}+C N^{-\gamma} \tag{2.6}
\end{equation*}
$$

where the constant $\tau_{0}$ depends only on $T$ and $\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$, and the constant $C$ depends only on $T,\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$ and $\varepsilon$.

We write $A \lesssim B$ or $B \gtrsim A$ to express that $A \leq C B$ for some positive constant $C$ which may be different at each constant but is independent of $\tau, N$ and $n$. Further, we write $A \sim B$ for $A \lesssim B \lesssim A$. We write $\mathcal{O}(Y)$ to denote a quantity $X$ that satisfied $|X| \lesssim|Y|$.

Henceforth, we denote by $T_{m}(M ; v)$ the class of functions $f \in L^{2}(\mathbb{T})$ such that

$$
\begin{equation*}
\left|\hat{f}_{k}\right| \lesssim \sum_{k=k_{1}+\cdots+k_{m}}\left|M\left(k, k_{1}, \cdots, k_{m}\right)\right|\left|\hat{v}_{k_{1}}\right| \cdots\left|\hat{v}_{k_{m}}\right|, \quad \text { for all } \quad k \tag{2.7}
\end{equation*}
$$

where $v_{k}$ is the $k$ th Fourier coefficient of $v$.

## 3. The construction of the scheme and some technical lemmas

In this section, we construct the fully discrete low-regularity exponential integrator by frequency truncation and harmonic analysis techniques. Then, we state some lemmas that will be used frequently in section 4 .
3.1. The construction of the scheme. It is known that if $u^{0} \in H^{\gamma}(\mathbb{T}), \gamma \geq 0$, then the NLS equation (1.1) has a unique solution $u \in C\left([0, T] ; H^{\gamma}(\mathbb{T})\right)$; see [2]. Recalling Duhamel's formula, we write

$$
u\left(t_{n+1}\right)=\mathrm{e}^{i \tau \partial_{x}^{2}} u\left(t_{n}\right)-i \int_{0}^{\tau} \mathrm{e}^{i\left(t_{n+1}-\left(t_{n}+s\right)\right) \partial_{x}^{2}}\left[\left|u\left(t_{n}+s\right)\right|^{2} u\left(t_{n}+s\right)\right] d s .
$$

With the twisted variable $v(t)=\mathrm{e}^{-i t \partial_{x}^{2}} u(t)$, the above formula becomes

$$
\begin{equation*}
v\left(t_{n+1}\right)=v\left(t_{n}\right)-i \int_{0}^{\tau} \mathrm{e}^{-i\left(t_{n}+s\right) \partial_{x}^{2}}\left[\left|\mathrm{e}^{i\left(t_{n}+s\right) \partial_{x}^{2}} v\left(t_{n}+s\right)\right|^{2} \mathrm{e}^{i\left(t_{n}+s\right) \partial_{x}^{2}} v\left(t_{n}+s\right)\right] d s \tag{3.1}
\end{equation*}
$$

Applying the Fourier transform, (3.1) can be expressed as

$$
\hat{v}_{k}\left(t_{n+1}\right)=\hat{v}_{k}\left(t_{n}\right)-i \int_{0}^{\tau} \sum_{k=k_{1}+k_{2}+k_{3}} \mathrm{e}^{i\left(t_{n}+s\right) \phi} \widehat{\bar{v}}_{k_{1}}\left(t_{n}+s\right) \hat{v}_{k_{2}}\left(t_{n}+s\right) \hat{v}_{k_{3}}\left(t_{n}+s\right) d s
$$

where $\hat{v}_{k}(t)$ denotes the $k$ th Fourier coefficient of $v(t)$. We also use the phase function

$$
\begin{equation*}
\phi\left(k, k_{1}, k_{2}, k_{3}\right)=k^{2}+k_{1}^{2}-k_{2}^{2}-k_{3}^{2} . \tag{3.2}
\end{equation*}
$$

In order to obtain a first-order approximation, by (3.1), we have for any $s \in[0, \tau]$,

$$
\begin{equation*}
v\left(t_{n}+s\right)-v\left(t_{n}\right) \in \tau T_{3}(1 ; v) \tag{3.3}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\hat{v}_{k}\left(t_{n+1}\right)=\hat{v}_{k}\left(t_{n}\right)+\hat{I}_{1, k}+\hat{\mathcal{R}}_{1, k}(v) \tag{3.4}
\end{equation*}
$$

where

$$
\hat{I}_{1, k}=-i \sum_{k=k_{1}+k_{2}+k_{3}} \int_{0}^{\tau} \mathrm{e}^{i\left(t_{n}+s\right) \phi} d s \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} \quad \text { and } \quad \mathcal{R}_{1}(v) \in \tau^{2} T_{5}(1 ; v) .
$$

Henceforth, we denote $\hat{v}_{k}\left(t_{n}\right)$ by $\hat{v}_{k}$ and the $k$ th Fourier coefficient of $\mathcal{R}_{j}(v)$ for $j \geq 1$ by $\hat{\mathcal{R}}_{j, k}(v)$ for short.

For further approximation, we consider a decomposition into low and high frequencies. In particular, we consider the following two cases: $|k| \leq N$ and $|k|>N$.

Case 1: $|k| \leq N$. We consider only the first term $\hat{I}_{1, k}$ in (3.4) and truncate $\hat{I}_{1, k}$ to the frequency domain $\left|k_{2}+k_{3}\right| \leq N$,

$$
\begin{equation*}
\hat{I}_{1, k}=\mathcal{K}_{k}(v)+\hat{\mathcal{R}}_{2, k}(v), \tag{3.5}
\end{equation*}
$$

where $\mathcal{K}_{k}(v)$ is defined by

$$
\mathcal{K}_{k}(v)=-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\ \mid k_{2}+k_{3} \leq N}} \int_{0}^{\tau} \mathrm{e}^{i\left(t_{n}+s\right) \phi} d s \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} .
$$

The remainder $\hat{\mathcal{R}}_{2, k}(v)$ is given by

$$
\begin{equation*}
\hat{\mathcal{R}}_{2, k}(v)=-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right|>N}} \int_{0}^{\tau} \mathrm{e}^{i\left(t_{n}+s\right) \phi} d s \hat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} . \tag{3.6}
\end{equation*}
$$

Furthermore, from the definition of $T_{m}(M ; v)$ in (2.7), the function $\mathcal{R}_{2}$ satisfies

$$
\mathcal{R}_{2}(v) \in \tau T_{3}\left(1_{\left|k_{2}+k_{3}\right|>N} ; v\right) .
$$

Next we consider the term $\mathcal{K}_{k}(v)$. Note that if $k=k_{1}+k_{2}+k_{3}$, then from (3.2), the following equality holds

$$
\phi\left(k, k_{1}, k_{2}, k_{3}\right)=2 k k_{1}+2 k_{2} k_{3} .
$$

In order to get a first-order scheme, we need to find an appropriate approximation to the exponential $\mathrm{e}^{i s \phi}$. Using the formula

$$
\mathrm{e}^{i s \phi}=\mathrm{e}^{2 i s k k_{1}} \mathrm{e}^{2 i s k_{2} k_{3}}=\mathrm{e}^{2 i s k k_{1}}+\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right)+\left(\mathrm{e}^{2 i s k k_{1}}-1\right)\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right),
$$

we decompose $\mathcal{K}_{k}(v)$ into two terms

$$
\mathcal{K}_{k}(v)=-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right| \leq N}} \int_{0}^{\tau} \mathrm{e}^{i t_{n} \phi}\left[\mathrm{e}^{2 i s k k_{1}}+\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right)\right] d s \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}}+\hat{\mathcal{R}}_{3, k}(v),
$$

where $\hat{\mathcal{R}}_{3, k}(v)$ is defined by

$$
\begin{equation*}
\hat{\mathcal{R}}_{3, k}(v)=-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right| \leq N}} \int_{0}^{\tau} \mathrm{e}^{i t_{n} \phi}\left(\mathrm{e}^{2 i s k k_{1}}-1\right)\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right) d s \hat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} . \tag{3.8}
\end{equation*}
$$

We find that the integral in $\mathcal{K}_{k}(v)$ can be computed easily. Integrating with respect to $s$, we have for any $|k| \leq N$,

$$
\begin{align*}
& \mathcal{K}_{k}(v) \\
= & -i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\
\left|k_{2}+k_{3}\right| \leq N \\
|k| \neq 0,\left|k_{1}\right| \neq 0}} \frac{1}{2 i k k_{1}} \mathrm{e}^{i t_{n} \phi}\left(\mathrm{e}^{2 i \tau k k_{1}}-1\right) \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}}-i \tau \sum_{\substack{0=k_{1}+k_{2}+k_{3} \\
\left|k_{2}+k_{3}\right| \leq N}} \mathrm{e}^{i t_{n}\left(k_{1}^{2}-k_{2}^{2}-k_{3}^{2}\right)} \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} \\
& -i \tau \sum_{\substack{k=k_{2}+k_{3} \\
\left|k_{2}+k_{3}\right| \leq N}} \mathrm{e}^{i t_{n}\left(k^{2}-k_{2}^{2}-k_{3}^{2}\right)} \hat{v}_{k_{2}} \hat{v}_{k_{3}} \widehat{\bar{v}}_{0}+i \tau \sum_{0=k_{2}+k_{3}} \mathrm{e}^{i t_{n}\left(-k_{2}^{2}-k_{3}^{2}\right)} \hat{v}_{k_{2}} \hat{v}_{k_{3}} \widehat{\bar{v}}_{0} \\
& -i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\
\left|k_{2}+k_{3}\right| \leq N \\
\left|k_{2}\right| \neq 0,\left|k_{3}\right| \neq 0}} \frac{1}{2 i k_{2} k_{3}} \mathrm{e}^{i t_{n} \phi}\left(\mathrm{e}^{2 i \tau k_{2} k_{3}}-1\right) \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}}+i \tau \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\
\left|k_{2}+k_{3}\right| \leq N}} \mathrm{e}^{i t_{n} \phi} \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} \\
& -2 i \tau \sum_{\substack{k=k_{1}+k_{2} \\
\left|k_{2}\right| \leq N}} \mathrm{e}^{i t_{n}\left(k^{2}+k_{1}^{2}-k_{2}^{2}\right)} \hat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{0}+i \tau \mathrm{e}^{2 i t_{n} k^{2}} \widehat{\bar{v}}_{k}\left(\hat{v}_{0}\right)^{2}+\hat{\mathcal{R}}_{3, k}(v) . \tag{3.9}
\end{align*}
$$

For specific details of the above formula, we refer to the literature [7, 8].
Case 2: $|k|>N$. Let $\mathcal{R}_{4}(v)$ be the function with Fourier coefficients

$$
\hat{\mathcal{R}}_{4, k}(v)=-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\|k|>N}} \int_{0}^{\tau} \mathrm{e}^{i\left(t_{n}+s\right) \phi} d s \widehat{\bar{v}}_{k_{1}} \hat{v}_{k_{2}} \hat{v}_{k_{3}} .
$$

Then, for $|k|>N$, we get

$$
\begin{equation*}
\hat{I}_{1, k}=\hat{\mathcal{R}}_{4, k}(v) . \tag{3.10}
\end{equation*}
$$

Furthermore, from the definition of $T_{m}$, we find

$$
\mathcal{R}_{4}(v) \in \tau T_{3}\left(1_{|k|>N} ; v\right) .
$$

Hence, putting together (3.4), (3.5), (3.9) and (3.10) yields

$$
\begin{equation*}
v\left(t_{n+1}\right)=\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)+\mathcal{R}_{1}(v)+\mathcal{R}_{2}(v)+\mathcal{R}_{3}(v)+\mathcal{R}_{4}(v), \tag{3.11}
\end{equation*}
$$

where $\Phi_{\tau, N}^{n}$ is given by

$$
\begin{align*}
\Phi_{\tau, N}^{n}(f)= & f+\frac{1}{2} \mathrm{e}^{-i t_{n+1} \partial_{x}^{2}} \partial_{x}^{-1} P_{N}\left[\left(\mathrm{e}^{-i t_{n+1} \partial_{x}^{2}} \partial_{x}^{-1} \bar{f}\right) \cdot \mathrm{e}^{i \tau \partial_{x}^{2}} P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2}\right] \\
& -\frac{1}{2} \mathrm{e}^{-i t_{n} \partial_{x}^{2}} \partial_{x}^{-1} P_{N}\left[\left(\mathrm{e}^{-i t_{n} \partial_{x}^{2}} \partial_{x}^{-1} \bar{f}\right) \cdot P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2}\right] \\
& -i \tau P_{0}\left[\left(\mathrm{e}^{-i t_{n} \partial_{x}^{2}} \bar{f}\right) \cdot P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2}\right] \\
& -i \tau \mathrm{e}^{-i t_{n} \partial_{x}^{2}} P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2} \cdot P_{0}(\bar{f})+i \tau P_{0}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2} \cdot P_{0}(\bar{f}) \\
& +\frac{1}{2} \mathrm{e}^{-i t_{n} \partial_{x}^{2}} P_{N}\left[\left(\mathrm{e}^{-i t_{n} \partial_{x}^{2}} \bar{f}\right) \cdot \mathrm{e}^{-i \tau \partial_{x}^{2}} P_{N}\left(\mathrm{e}^{i t_{n+1} \partial_{x}^{2}} \partial_{x}^{-1} f\right)^{2}\right] \\
& -\frac{1}{2} \mathrm{e}^{-i t_{n} \partial_{x}^{2}} P_{N}\left[\left(\mathrm{e}^{\left.\left.-i t_{n} \partial_{x}^{2} \bar{f}\right) \cdot P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} \partial_{x}^{-1} f\right)^{2}\right]}\right.\right. \\
& +i \tau \mathrm{e}^{-i t_{n} \partial_{x}^{2}} P_{N}\left[\left(\mathrm{e}^{-i t_{n} \partial_{x}^{2}} \bar{f}\right) \cdot P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)^{2}\right] \\
& -2 i \tau \mathrm{e}^{-i t_{n} \partial_{x}^{2}} P_{N}\left[\left(\mathrm{e}^{-i t_{n} \partial_{x}^{2}} \bar{f}\right) \cdot P_{N}\left(\mathrm{e}^{i t_{n} \partial_{x}^{2}} f\right)\right] \cdot P_{0}(f) \\
& +i \tau \mathrm{e}^{-2 i t_{n} \partial_{x}^{2}} P_{N} \bar{f} \cdot\left(P_{0}(f)\right)^{2} . \tag{3.12}
\end{align*}
$$

Accordingly, for given $v^{n} \in S_{N}$, we compute $v^{n+1} \in S_{N}$ by

$$
\begin{equation*}
v^{n+1}=\Phi_{\tau, N}^{n}\left(v^{n}\right), \quad n \geq 0 ; \quad v^{0}=u^{0} . \tag{3.13}
\end{equation*}
$$

This finishes the construction of the numerical scheme (2.4).
3.2. Some technical estimates. We will frequently apply the following inequality, see [6].

Lemma 3.1. (Kato-Ponce inequality, [6]) The following estimates hold:
(i) Let $f, g \in H^{\gamma}$ for some $\gamma>\frac{1}{2}$. Then we have

$$
\left\|J^{\gamma}(f g)\right\|_{L^{2}} \lesssim\|f\|_{H^{\gamma}}\|g\|_{H^{\gamma}}
$$

(ii) Let $f \in H^{\gamma+\gamma_{1}}, g \in H^{\gamma}$ for some $\gamma \geq 0, \gamma_{1}>\frac{1}{2}$. Then we have

$$
\left\|J^{\gamma}(f g)\right\|_{L^{2}} \lesssim\|f\|_{H^{\gamma+\gamma_{1}}}\|g\|_{H^{\gamma}} .
$$

Next we present two specific estimates, which are used in section 4.
Lemma 3.2. The following bounds hold:
(i) Let $v \in L^{\infty}\left((0, T) ; H^{\gamma}\right)$ for some $\gamma>\frac{1}{2}$, and $g \in T_{3}\left(1_{|k|>N} ; v\right)$. Then

$$
\|g\|_{L^{2}} \lesssim N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

(ii) Let $v \in L^{\infty}\left((0, T) ; H^{\gamma}\right)$ for some $\gamma>\frac{1}{2}$, and $g \in T_{3}\left(1_{\left|k_{2}+k_{3}\right|>N} ; v\right)$. Then

$$
\|g\|_{L^{2}} \lesssim N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

(iii) Let $v \in L^{\infty}\left((0, T) ; H^{\gamma}\right)$ for some $\gamma>\frac{1}{2}$, and $g \in T_{m}(1 ; v), m \geq 1$. Then

$$
\|g\|_{H^{\gamma}} \lesssim\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{m} .
$$

Proof. We employ the notation $\hat{V}_{k_{j}}=\left|\hat{v}_{k_{j}}(t)\right|$.
(i) If $g \in T_{3}\left(1_{|k|>N} ; v\right)$, using the definition of $T_{m}(M ; v)$ in (2.7), we have

$$
\left|\hat{g}_{k}\right| \lesssim \begin{cases}\sum_{k=k_{1}+k_{2}+k_{3}} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}} & \text { for } \quad|k|>N  \tag{3.14}\\ 0 & \text { for } \\ |k| \leq N\end{cases}
$$

By Parseval's identity, we get

$$
\|g\|_{L^{2}}^{2}=2 \pi \sum_{k \in \mathbb{Z}}\left|\hat{g}_{k}\right|^{2}
$$

From the above identity and (3.14), the following estimates hold

$$
\begin{aligned}
\|g\|_{L^{2}} & \lesssim\left(\sum_{|k|>N}\left(\sum_{k=k_{1}+k_{2}+k_{3}} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}}\right)^{2}\right)^{\frac{1}{2}} \\
& \lesssim N^{-\gamma}\left(\sum_{|k|>N}\left(\sum_{k=k_{1}+k_{2}+k_{3}}|k|^{\gamma} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}}\right)^{2}\right)^{\frac{1}{2}}
\end{aligned}
$$

where the last estimate holds true because the frequency is limited to $|k|>N$.
For convenience, we employ the notation

$$
\tilde{V}=\sum_{k \in \mathbb{Z}} e^{i k x} \hat{V}_{k}
$$

Then, we have that $\hat{\tilde{V}}_{k}=\hat{V}_{k}=\left|\hat{v}_{k}(t)\right|$ and thus

$$
\begin{align*}
\|\tilde{V}\|_{H^{\gamma}}^{2} & =2 \pi \sum_{k \in \mathbb{Z}}\left(1+k^{2}\right)^{\gamma}\left|\hat{\tilde{V}}_{k}\right|^{2}=2 \pi \sum_{k \in \mathbb{Z}}\left(1+k^{2}\right)^{\gamma}\left|\hat{V}_{k}\right|^{2} \\
& =2 \pi \sum_{k \in \mathbb{Z}}\left(1+k^{2}\right)^{\gamma}\left|\hat{v}_{k}(t)\right|^{2}=\|v\|_{H^{\gamma}}^{2} . \tag{3.15}
\end{align*}
$$

Using Parseval's identity once more we obtain

$$
\|g\|_{L^{2}} \lesssim N^{-\gamma}\left\|\tilde{V}^{3}\right\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)} \lesssim N^{-\gamma}\|\tilde{V}\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3}
$$

Therefore, from (3.15) and the above inequality, we have

$$
\|g\|_{L^{2}} \lesssim N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

(ii) We use the same argument as in (i) and Parseval's identity to get

$$
\begin{aligned}
\|g\|_{L^{2}} & \lesssim\left(\sum_{k \in \mathbb{Z}}\left(\sum_{\substack{k=k_{1}+k_{2}+k_{3} \\
\mid k_{2}+k_{3}>N}} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}}\right)^{2}\right)^{\frac{1}{2}} \\
& \lesssim\left\|\tilde{V} \cdot P_{>N}\left(\tilde{V}^{2}\right)\right\|_{L^{\infty}\left((0, T) ; L^{2}\right)},
\end{aligned}
$$

where the operator $P_{>N}$ is defined in (2.2).
Then, by Hölder's inequality, we have

$$
\|g\|_{L^{2}} \lesssim\|\tilde{V}\|_{L^{\infty}\left((0, T) ; L^{\infty}\right)}\left\|P_{>N}\left(\tilde{V}^{2}\right)\right\|_{L^{\infty}\left((0, T) ; L^{2}\right)}
$$

Note that

$$
P_{>N}\left(\tilde{V}^{2}\right) \in T_{2}\left(1_{|k|>N} ; v\right),
$$

hence we use the result of (i) and (3.15) to obtain

$$
\left\|P_{>N}\left(\tilde{V}^{2}\right)\right\|_{L^{\infty}\left((0, T) ; L^{2}\right)} \lesssim N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{2} .
$$

Now, we can use Sobolev's inequality to finally obtain

$$
\|g\|_{L^{2}} \lesssim N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

(iii) Using the explicit form of $T_{m}(M ; v)$ given in (2.7), we have

$$
\left|\hat{g}_{k}\right| \lesssim \sum_{k=k_{1}+\cdots+k_{m}} \hat{V}_{k_{1}} \cdots \hat{V}_{k_{m}} .
$$

By Parseval's identity, Lemma 3.1 and (3.15) again, we obtain that

$$
\|g\|_{H^{\gamma}} \lesssim\left\|\tilde{V}^{m}\right\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)} \lesssim\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{m} .
$$

This concludes the proof.
Furthermore, from Lemma 3.2, we have for any $\gamma>\frac{1}{2}$,

$$
\begin{align*}
& \left\|\mathcal{R}_{1}(v)\right\|_{L^{2}} \leq\left\|\mathcal{R}_{1}(v)\right\|_{H^{\gamma}} \lesssim \tau^{2}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{5},  \tag{3.16}\\
& \left\|\mathcal{R}_{2}(v)\right\|_{L^{2}} \lesssim \tau N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3}, \quad\left\|\mathcal{R}_{2}(v)\right\|_{H^{\gamma}} \lesssim \tau\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3},  \tag{3.17}\\
& \left\|\mathcal{R}_{4}(v)\right\|_{L^{2}} \lesssim \tau N^{-\gamma}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3}, \quad\left\|\mathcal{R}_{4}(v)\right\|_{H^{\gamma}} \lesssim \tau\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} . \tag{3.18}
\end{align*}
$$

Lemma 3.3. The following estimates hold:
(i) Let $v \in L^{\infty}\left((0, T) ; H^{\gamma}\right)$ for some $1 \geq \gamma>\frac{1}{2}$. Then for any small enough $\varepsilon>0$,

$$
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

(ii) Let $v \in L^{\infty}\left((0, T) ; H^{\gamma}\right)$ for some $\gamma>\frac{1}{2}$. Then,

$$
\left\|\mathcal{R}_{3}(v)\right\|_{H^{\gamma}} \lesssim \tau\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

Proof. (i) As employed in Lemma 3.2, we use the notation $\hat{V}_{k_{j}}=\left|\hat{v}_{k_{j}}(t)\right|$. Note that for $s \in[0, \tau]$,

$$
\left|\mathrm{e}^{2 i s k k_{1}}-1\right| \lesssim \tau^{\frac{\gamma}{2}}|k|^{\frac{\gamma}{2}}\left|k_{1}\right|^{\frac{\gamma}{2}}, \quad\left|\mathrm{e}^{2 i s k_{2} k_{3}}-1\right| \lesssim \tau^{\gamma-\frac{1}{2}-\varepsilon}\left|k_{2}\right|^{\gamma-\frac{1}{2}-\varepsilon}\left|k_{3}\right|^{\gamma-\frac{1}{2}-\varepsilon} .
$$

Therefore, we obtain from (3.8) that

$$
\left|\hat{\mathcal{R}}_{3, k}(v)\right| \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon} \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\ \mid k_{2}+k_{3} \leq \leq N}}|k|^{\frac{\gamma}{2}}\left|k_{1}\right|^{\frac{\gamma}{2}}\left|k_{2}\right|^{\gamma-\frac{1}{2}-\varepsilon}\left|k_{3}\right|^{\gamma-\frac{1}{2}-\varepsilon} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}} .
$$

Based on the relation between the frequencies, we consider three cases.
Case 1: $|k| \lesssim\left|k_{1}\right|$. From the above estimate, we have

$$
\left|\hat{\mathcal{R}}_{3, k}(v)\right| \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon} \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right| \leq N}}\left|k_{1}\right|^{\gamma}\left|k_{2}\right|^{\gamma-\frac{1}{2}-\varepsilon}\left|k_{3}\right|^{\gamma-\frac{1}{2}-\varepsilon} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}} .
$$

We denote as before

$$
\tilde{V}=\sum_{k \in \mathbb{Z}} e^{i k x} \hat{V}_{k}
$$

Then from (3.15), we have $\|\tilde{V}\|_{H^{\gamma}}=\|v\|_{H^{\gamma}}$. Therefore, by Parseval's identity, we obtain that, for any $\gamma>\frac{1}{2}$,

$$
\begin{aligned}
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} & \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\left\|\left\lvert\, \partial_{x} \gamma^{\gamma} \tilde{V} \cdot\left(\left|\partial_{x}\right|^{\gamma-\frac{1}{2}-\varepsilon} \tilde{V}\right)^{2}\right.\right\|_{L^{2}} \\
& \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
\end{aligned}
$$

Case 2: $|k| \lesssim\left|k_{2}\right|$. In this case, we have

$$
\left|\hat{\mathcal{R}}_{3, k}(v)\right| \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon} \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\ \mid k_{2}+k_{3} \leq N}}\left|k_{1}\right|^{\frac{\gamma}{2}}\left|k_{2}\right|^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon}\left|k_{3}\right|^{\gamma-\frac{1}{2}-\varepsilon} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}}
$$

Therefore, we use Parseval's identity to obtain

$$
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\left\|\left|\partial_{x}\right|^{\frac{1}{2} \gamma} \tilde{V} \cdot\left|\partial_{x}\right|^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon} \tilde{V} \cdot\left|\partial_{x}\right|^{\gamma-\frac{1}{2}-\varepsilon} \tilde{V}\right\|_{L^{2}} .
$$

If $\gamma \neq 1$, we employ the Hölder and Sobolev inequalities to get

$$
\begin{aligned}
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} & \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\left\|\left|\partial_{x}\right|^{\frac{1}{2} \gamma} \tilde{V}\right\|_{L^{\frac{1}{1-\gamma}}} \cdot\left\|\left|\partial_{x}\right|^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon} \tilde{V}\right\|_{L^{\frac{2}{\gamma}}} \cdot\left\|\left|\partial_{x}\right|^{\gamma-\frac{1}{2}-\varepsilon} \tilde{V}\right\|_{L^{\infty}} \\
& \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
\end{aligned}
$$

If $\gamma=1$, we write

$$
\begin{aligned}
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} & \lesssim \tau^{2-\varepsilon}\left\|\left|\partial_{x}^{\frac{1}{2}}\right| \tilde{V}\right\|_{L^{\frac{1}{\varepsilon}}} \cdot\left\|\left|\partial_{x}^{1-\varepsilon}\right| \tilde{V}\right\|_{L^{\frac{2}{1-2 \varepsilon}}} \cdot\left\|\left|\partial_{x}^{\gamma-\frac{1}{2}-\varepsilon}\right| \tilde{V}\right\|_{L^{\infty}} \\
& \lesssim \tau^{2-\varepsilon}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
\end{aligned}
$$

Hence, for any $\frac{1}{2}<\gamma \leq 1$ we conclude that

$$
\left\|\mathcal{R}_{3}(v)\right\|_{L^{2}} \lesssim \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}\|v\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}^{3} .
$$

Case 3: $|k| \lesssim\left|k_{3}\right|$. It is same as case 2.
(ii) We have the following inequality

$$
\left|\left(\mathrm{e}^{2 i s k k_{1}}-1\right)\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right)\right| \lesssim 1 .
$$

Plugging the above inequality into (3.8), we obtain

$$
\begin{equation*}
\left|\hat{\mathcal{R}}_{3, k}(v)\right| \lesssim \tau \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right| \leq N}} \hat{V}_{k_{1}} \hat{V}_{k_{2}} \hat{V}_{k_{3}} . \tag{3.19}
\end{equation*}
$$

Then, from the definition of $T_{m}(M ; v)$ in 2.7), we observe that

$$
\mathcal{R}_{3}(v) \in \tau T_{3}(1 ; v)
$$

Finally, from Lemma 3.2 (iii), we get the result (ii).

## 4. Proof of Theorem 2.1

Note that the $L^{2}$ stability estimate depends on bounds of the numerical solution in $H^{\gamma}, \frac{1}{2}<\gamma \leq 1$. Therefore we first show the $H^{\gamma}$ bound before we give the proof of Theorem 2.1. This strategy was first used in [9]. In particular, the bound of the $H^{\gamma}$ norm of the numerical solution is independent of the degrees of freedom in the spatial discretisation, $N$. It only depends on $T$ and the bound $\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$. We are now in a position to show the bound in $H^{\gamma}$.

Proposition 4.1. Let $u_{\tau, N}^{n}$ be the numerical solution given in 2.4. Under the assumption that $u^{0} \in H^{\gamma}(\mathbb{T})$ for some $\frac{1}{2}<\gamma \leq 1$, there exists a positive constant $C$, such that for any integer $N \geq 1$

$$
\begin{equation*}
\left\|u_{\tau, N}^{n}\right\|_{H^{\gamma}} \leq C, \quad \text { for all } \quad 0 \leq n \tau \leq T \tag{4.1}
\end{equation*}
$$

where the constant $C$ only depends on $T$ and the bound $\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$.
Proof. Let $v^{n}=u_{\tau, N}^{n}$. From (3.13), we have

$$
\begin{align*}
v\left(t_{n+1}\right)-v^{n+1} & =v\left(t_{n+1}\right)-\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)+\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)-\Phi_{\tau, N}^{n}\left(v^{n}\right)  \tag{4.2}\\
& =\mathcal{L}^{n}+\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)-\Phi_{\tau, N}^{n}\left(v^{n}\right), \tag{4.3}
\end{align*}
$$

where $\mathcal{L}^{n}=v\left(t_{n+1}\right)-\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)$.
Furthermore, from (3.11) we get

$$
\mathcal{L}^{n}=\mathcal{R}_{1}(v)+\mathcal{R}_{2}(v)+\mathcal{R}_{3}(v)+\mathcal{R}_{4}(v) .
$$

Then, from (3.16), (3.17), (3.18) and Lemma 3.3 (ii), we have

$$
\begin{equation*}
\left\|\mathcal{L}^{n}\right\|_{H^{\gamma}} \leq C \tau \tag{4.4}
\end{equation*}
$$

Note that the constant $C$ only depends on $\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$.
Recall that $\Phi_{\tau, N}^{n}(f)$ defined in (3.12) can be written the following integral form:

$$
\begin{equation*}
\widehat{\Phi}_{\tau, N}^{n}(f)(k)=\hat{f}_{k}-i \sum_{\substack{k=k_{1}+k_{2}+k_{3} \\\left|k_{2}+k_{3}\right| \leq N \\|k| \leq N}} \int_{0}^{\tau} \mathrm{e}^{i t_{n} \phi}\left[\mathrm{e}^{2 i s k k_{1}}+\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right)\right] d s \widehat{\bar{f}}_{k_{1}} \hat{f}_{k_{2}} \hat{f}_{k_{3}} . \tag{4.5}
\end{equation*}
$$

Further, note that

$$
\left|\mathrm{e}^{i t_{n} \phi}\left[\mathrm{e}^{2 i s k k_{1}}+\left(\mathrm{e}^{2 i s k_{2} k_{3}}-1\right)\right]\right| \lesssim 1
$$

Then, by Lemma 3.1(i), we obtain

$$
\begin{equation*}
\left\|\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)-\Phi_{\tau, N}^{n}\left(v^{n}\right)\right\|_{H^{\gamma}} \leq(1+C \tau)\left\|v^{n}-v\left(t_{n}\right)\right\|_{H^{\gamma}}+C \tau\left\|v^{n}-v\left(t_{n}\right)\right\|_{H^{\gamma}}^{3} . \tag{4.6}
\end{equation*}
$$

A combination of the above estimates allows that

$$
\left\|v\left(t_{n+1}\right)-v^{n+1}\right\|_{H^{\gamma}} \leq(1+C \tau)\left\|v^{n}-v\left(t_{n}\right)\right\|_{H^{\gamma}}+C \tau\left\|v^{n}-v\left(t_{n}\right)\right\|_{H^{\gamma}}^{3}+C \tau .
$$

By iteration and Gronwall's lemma, we get

$$
\left\|v\left(t_{n+1}\right)-v^{n+1}\right\|_{H^{\gamma}} \leq C .
$$

That means

$$
\left\|v^{n+1}\right\|_{H^{\gamma}} \leq C
$$

This finishes the proof on the boundness in $H^{\gamma}$.
Now we start to prove Theorem 2.1.
From (3.16), (3.17), (3.18) and Lemma 3.3 (i), we have

$$
\begin{equation*}
\left\|\mathcal{L}^{n}\right\|_{L^{2}} \leq C \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}+C \tau N^{-\gamma} \tag{4.7}
\end{equation*}
$$

where the constant $C$ only depends on the bound $\|u\|_{L^{\infty}\left((0, T) ; H^{\gamma}\right)}$.
From (4.5), we have

$$
\begin{aligned}
& \left\|\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)-\Phi_{\tau, N}^{n}\left(v^{n}\right)\right\|_{L^{2}} \\
& \quad \leq\left\|v^{n}-v\left(t_{n}\right)\right\|_{L^{2}}+C \tau\left\|v^{n}-v\left(t_{n}\right)\right\|_{L^{2}}\left(\left\|v^{n}\right\|_{H^{\gamma}}^{2}+\left\|v\left(t_{n}\right)\right\|_{H^{\gamma}}^{2}\right)
\end{aligned}
$$

By Proposition 4.1, we have

$$
\left\|\Phi_{\tau, N}^{n}\left(v\left(t_{n}\right)\right)-\Phi_{\tau, N}^{n}\left(v^{n}\right)\right\|_{L^{2}} \leq(1+C \tau)\left\|v^{n}-v\left(t_{n}\right)\right\|_{L^{2}} .
$$

Combining (4.2) with (4.7) and the above estimate, we get

$$
\left\|v\left(t_{n+1}\right)-v^{n+1}\right\|_{H^{\gamma}} \leq(1+C \tau)\left\|v^{n}-v\left(t_{n}\right)\right\|_{H^{\gamma}}+C \tau^{\frac{3}{2} \gamma+\frac{1}{2}-\varepsilon}+C \tau N^{-\gamma}
$$

By iteration and Gronwall's lemma, we finally get

$$
\left\|v\left(t_{n+1}\right)-v^{n+1}\right\|_{H^{\gamma}} \leq C \tau^{\frac{3}{2} \gamma-\frac{1}{2}-\varepsilon}+C N^{-\gamma} .
$$

This concludes the proof.

## 5. Numerical experiments

In this section we carry out numerical experiments to support our theoretical analysis. We consider the nonlinear Schrödinger equation (1.1) with initial data

$$
\begin{equation*}
u^{0}(x)=\sum_{k \in \mathbb{Z}}(1+|k|)^{-\frac{1}{2}-\gamma} g_{k} \mathrm{e}^{i k x} \tag{5.1}
\end{equation*}
$$

where $\gamma$ and $\left(g_{k}\right)_{k \in \mathbb{Z}}$ are used to set the regularity of the data. The complex coefficients $g_{k}$ are chosen as uniformly distributed random variables in $[-1,1]+i[-1,1]$. They are generated with the matlab routine rand. This choice guarantees that $u^{0} \in H^{\gamma}$. however, as a consequence of the Paley-Zygmund Theorem (see, e.g., [5]), the initial data (5.1) satisfies the stronger regularity condition $\partial_{x}^{\gamma} u^{0} \in L^{p}, 2 \leq p<\infty$. This will be a slight issue when we study the temporal discretisation error. See the discussion below.

We start our numerical experiments with the spatial discretisation errors. On the left-hand side of Fig. 1, we present our results for a sufficiently small time step size $\tau$. This allows us to ignore the errors caused by temporal discretisation. We choose three different values of $\gamma \in\left[\frac{1}{2}, 1\right]$ to illustrate the spatial convergence rate of our scheme. In order to measure the spatial discretization error $u\left(t_{m}, \cdot\right)-u_{\tau, N}^{m}$ for fixed time $t_{m}$, we use the discrete $L^{2}$ norm

$$
\begin{equation*}
\|w\|_{L_{N}^{2}}^{2}=\frac{N}{2 \pi} \sum_{j=0}^{N-1}\left|w\left(x_{j}\right)\right|^{2}, \quad x_{j}=\frac{2 j}{N} \pi . \tag{5.2}
\end{equation*}
$$

The results of our numerical experiments agree well with the corresponding results of the theoretical analysis, see Theorem 2.1.

The temporal discretisation errors are displayed on the right-hand side of Fig. 1 for a sufficiently large $N$. This allows us to ignore the errors caused by spatial discretisation. In order to illustrate the time convergence rate, we present results for three different values of $\gamma \in\left[\frac{1}{2}, 1\right]$. We use again the norm specified in (5.2). Note that the observed rates of convergence are slightly better than predicted by Theorem 2.1. The reasons for this is the additional regularity, guaranteed by the Paley-Zygmund Theorem [5].


Figure 1. Spatial discretisation error at $\tau=2^{-15}, T=1$ for various values of $N$ and $\gamma$ (left) and temporal discretisation error for $N=2^{14}$ at $T=1$ for various values of $\tau$ and $\gamma$ (right).

## 6. Conclusion

We have constructed a fully discrete low-regularity integrator for the cubic NLS equation with nonsmooth initial data in one space dimension. The scheme can be computed with FFT with $\mathcal{O}(N \log N)$ operations per time step. We have proved convergence in $L^{2}(\mathbb{T})$ for initial data in $H^{\gamma}(\mathbb{T}), \frac{1}{2}<\gamma \leq 1$. Numerical results illustrate our convergence result.

## ACKNOWLEDGEMENTS

This research is supported by the NSFC key project under the grant number 11831003, and NSFC under the grant numbers 11971356. The second author also acknowledges financial support by the China Scholarship Council.

## References

[1] C. Besse, B. Bidégaray, and S. Descombes, Order estimates in time of splitting methods for the nonlinear Schrödinger equation, SIAM J. Numer. Anal. 40 (2002), pp. 26-40.
[2] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. I. Schrödinger equations, Geom. Funct. Anal. 3 (1993), pp. 107-156.
[3] E. Faou, Geometric Numerical Integration and Schrödinger Equations, European Mathematical Society Publishing House, Zürich, 2012.
[4] T. Jahnke and C. Lubich, Error bounds for exponential operator splittings, BIT 40 (2000), pp. 735-744.
[5] J. P. Kahane, Some random series of functions, Cambridge University Press, Cambridge, 1993.
[6] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Commun. Pure Appl. Math. 41 (1988), pp. 891-907.
[7] M. Knöller, A. Ostermann, and K. Schratz, A Fourier integrator for the cubic nonlinear Schrödinger equation with rough initial data, SIAM J. Numer. Anal. 57 (2019), pp. 1967-1986.
[8] B. Li and Y. Wu, A full discrete low-regularity integrator for the 1D period cubic nonlinear Schrödinger equation, arXiv:2101.03728, 2021.
[9] C. Lubich, On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations, Math. Comp. 77 (2008), pp. 2141-2153.
[10] A. Ostermann and K. Schratz, Low regularity exponential-type integrators for semilinear Schrödinger equations, Found. Comput. Math. 18 (2018), pp. 731-755.
[11] A. Ostermann, F. Rousset, and K. Schratz, Error estimates of a Fourier integrator for the cubic Schrödinger equation at low regularity, Found. Comput. Math. 21 (2021), pp. 725-765.
[12] A. Ostermann, F. Rousset, and K. Schratz, Fourier integrator for periodic NLS: low regularity estimates via discrete Bourgain spaces, $\operatorname{arXiv}: 2006.12785,2020$.
[13] J. M. Sanz-Serna, Methods for the numerical solution of the nonlinear Schrödinger equation, Math. Comp. 43 (1984), pp. 21-27.
[14] Y. Wu and F. Yao, A first-order Fourier integrator for the nonlinear Schrödinger equation on $\mathbb{T}$ without loss of regularity, arXiv:2010.02672, 2020.
A. Ostermann: Department of Mathematics, University of Innsbruck, Technikerstrasse 13, 6020 Innsbruck, Austria

Email address: alexander.ostermann@uibk.ac.at
F. Yao: School of Mathematical Sciences, South China University of Technology, Guangzhou, Guangdong 510640, P. R. China

Email address: fangyanyao@outlook.com


[^0]:    2020 Mathematics Subject Classification. Primary 65M12, 65M15, 35Q55.
    Key words and phrases. Low regularity, nonlinear Schrödinger equation, fully discrete, fast Fourier transform.

