# Mortar coupling of $h p$-discontinuous Galerkin and boundary element methods for the Helmholtz equation 

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#### Abstract

We design and analyze a coupling of a discontinuous Galerkin finite element method with a boundary element method to solve the Helmholtz equation with variable coefficients in three dimensions. The coupling is realized with a mortar variable that is related to an impedance trace on a smooth interface. The method obtained has a block structure with nonsingular subblocks. We prove quasi-optimality of the $h$ - and $p$-versions of the scheme, under a threshold condition on the approximability properties of the discrete spaces. Amongst others, an essential tool in the analysis is a novel discontinuous-to-continuous reconstruction operator on tetrahedral meshes with curved faces.


Keywords: discontinuous Galerkin method; boundary element method; mortar coupling; Helmholtz equation; variable sound speed

## 1 Introduction

A natural habitat of wave propagation problems are unbounded domains. An important class of numerical methods for this setting is the coupling of a volume-based method such as the finite element method (FEM) or one of its variants for a finite, chosen computational domain $\Omega$ and a boundary element method (BEM) for its unbounded exterior $\Omega^{\text {ext }}:=\mathbb{R}^{3} \backslash \bar{\Omega}$. In this paper, we study such a coupling technique for a time-harmonic acoustic scattering problem modelled by the Helmholtz equation in $\mathbb{R}^{3}$ and given by

$$
\begin{equation*}
-\operatorname{div}(\nu \nabla u)-(k n)^{2} u=f \quad \text { in } \mathbb{R}^{3} \tag{1.1}
\end{equation*}
$$

where the coefficients $\nu$ and $n$ are constant outside a sufficiently large ball, $k$ denotes the wave number, and $f$ is the right-hand side. Our focus is on a strategy that couples a high order discontinuous Galerkin finite element method (DGFEM) in the computational domain $\Omega$ with a BEM on $\Gamma:=\partial \Omega$ to account for $\Omega^{\text {ext }}$. We consider approximation spaces made of piecewise polynomial functions.

The present work is a continuation of the recent work [37], where the coupling of a conforming, high order FEM with a BEM is presented and analyzed. The coupling there is reminiscent of the symmetric coupling proposed in 14 and 30 for Poisson-type problems but uses an additional mortar variable that has the physical meaning of a Robin trace for incoming waves. A feature of the mortar-based coupling is that the resulting system has a block structure where the two blocks corresponding to the volume and to the BEM unknowns, respectively, are individually invertible. This allows for the use of existing discretization techniques for these blocks. Other FEM-BEM

[^0]coupling strategies for Helmholtz problems are possible and are discussed in 37. In contrast to the conforming setting of [37, our focus here is on a DGFEM for the discretization in $\Omega$, since the DGFEM has proved to be a very versatile discretization technique that can accommodate very well, for example, high order discretizations. High order methods are particularly suited for wave propagation problems [4, 36, 41, 43, 44. Further well known advantages of DG discretizations include the ease to realize adaptivity and accommodate nonuniform polynomial degree distributions. Moreover, DG formulations for Helmholtz problems in bounded domains have the potential to be unconditionally well posed [22, 26]. We refer, e.g., to [17, 21, 23, 29, 41, 52] for polynomial-based DG methods for the Helmholtz problem, to [10, 15, 28 for hybridized DG (HDG) methods, and to 16, 27 for discontinuous Petrov-Galerkin (DPG) methods.

For Poisson-type problems, couplings of several variants of the DGFEM with the BEM have been proposed and analyzed. The first analysis appears to be that of the symmetric coupling of the local DG (LDG) method with BEM in [8, 24, 25]. Generalizations to nonsymmetric couplings have been proposed in [46] and analyzed in [31]. The closely related coupling of finite volume methods with BEM is analyzed in [18-20]. A fairly general framework that uses mortar variables for coupling the DGFEM and the BEM can be found in [11,12]. In the limit $k \rightarrow 0$, which is not the focus of the present work, our method for (1.1) has similarities with those of 19,25 for the Laplace equation.

On a technical side, a main difficulty of the analysis of couplings of DG with BEM arises from the mapping properties of BEM operators that do not easily accommodate the discontinuous traces of DG functions. This is one of the reasons for using mortar variables for the coupling both for Poisson-type problems discussed above and the present Helmholtz equation. In our analysis, we tackle this issue with a new discontinuous-to-continuous operator in Theorem 4.4. From the many possible DG variants, we opted for an interior penalty-like DG method to keep the presentation as simple as possible, although other DG discretizations could be analyzed with similar techniques. Following the lead in [37, we employ a form of symmetric coupling using all four BEM operators. The sesquilinear forms are carefully designed to ensure both consistency and adjoint consistency. In particular, as compared to 37, the discretization of the coupling condition required us to introduce an additional (consistent) term in order to prove a discrete Gårding inequality.

Notation. For bounded Lipschitz domains $D \subset \mathbb{R}^{d}, d \geq 1$, we introduce the following norms and spaces: For integers $s \in \mathbb{N}_{0}$ and complex-valued functions $v$, we define the norms $\|v\|_{s, D}^{2}:=$ $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_{0}^{3}:|\boldsymbol{\alpha}| \leq s}\left\|D^{\boldsymbol{\alpha}} v\right\|_{0, D}^{2}$ and the seminorms $|v|_{s, D}^{2}:=\sum_{\boldsymbol{\alpha} \in \mathbb{N}_{0}^{3}:|\boldsymbol{\alpha}|=s}\left\|D^{\boldsymbol{\alpha}} v\right\|_{0, D}^{2}$. The Hilbert spaces $H^{s}(D)$ and $H_{0}^{s}(D)$ are defined as the closure of $C^{\infty}(D)$ and $C_{0}^{\infty}(D)$ with respect to the norm $\|\cdot\|_{s, D}$. We further set $H^{0}(D):=L^{2}(D)$. For a noninteger $s>0$, the spaces $H^{s}(D)$ and $H_{0}^{s}(D)$ are defined by interpolation between $H^{\lfloor s\rfloor}(D)$ and $H^{\lceil s\rceil}(D)$ and between $H_{0}^{\lfloor s\rfloor}(D)$ and $H_{0}^{\lceil s\rceil}(D)$, respectively. For $s>0$, the space $H^{-s}(D)$ is defined as the dual of $H_{0}^{s}(D)$ with norm

$$
\|v\|_{-s, D}:=\sup _{w \in H_{0}^{s}(D)} \frac{|\langle v, w\rangle|}{\|w\|_{s, D}},
$$

where $\langle\cdot, \cdot\rangle$ denotes the duality pairing, which coincides with the $L^{2}$ inner product whenever both $v, w \in L^{2}(D)$. The inner product in $H^{s}(D)$, denoted by $(\cdot, \cdot)_{s, D}$, is linear in the first argument and antilinear in the second argument.

For closed, connected, smooth 2-dimensional surfaces $\Gamma \subset \mathbb{R}^{3}$ and $s \geq 0$, we define the Sobolev spaces $H^{s}(\Gamma)$ as follows. Let $\left\{\varphi_{n}, \lambda_{n}\right\}_{n \in \mathbb{N}_{0}}$ be a sequence of eigenpairs of the Laplace-Beltrami operator on $\Gamma$, so that $\left\{\varphi_{n}\right\}_{n \in \mathbb{N}_{0}}$ is an orthonormal basis of $L^{2}(\Gamma)$. For $s \geq 0$, we define the norm $\|v\|_{s, \Gamma}^{2}:=\sum_{n}\left|v_{n}\right|^{2}\left(1+\lambda_{n}\right)^{s}$, where $v=\sum_{n} v_{n} \varphi_{n}$ is expanded in the basis $\left\{\varphi_{n}\right\}_{n \in \mathbb{N}_{0}}$. Then, $H^{s}(\Gamma):=\left\{v \in L^{2}(\Gamma):\|v\|_{s, \Gamma}<+\infty\right\}$. The $\|\cdot\|_{s, \Gamma}$ norm is equivalent to the one obtained by using local charts as described in [38]; see also [45, Sec. 5.4]. The mapping $v \mapsto\left(v_{n}\right)_{n \in \mathbb{N}_{0}}$, with $v_{n}=\left(v, \varphi_{n}\right)_{L^{2}(\Gamma)}$, is an isometric isomorphism between $H^{s}(\Gamma)$ and the sequence space $\left\{\left.\left(v_{n}\right)_{n \in \mathbb{N}_{0}}\left|\sum_{n}\right| v_{n}\right|^{2}\left(1+\lambda_{n}\right)^{s}<\infty\right\}$. Negative order Sobolev spaces are defined by duality and equipped with the norm

$$
\|v\|_{-s, \Gamma}:=\sup _{w \in H^{s}(\Gamma)} \frac{|\langle v, w\rangle|}{\|w\|_{s, \Gamma}} .
$$

Moreover, $H^{-s}(\Gamma)$ is equivalent to the space $\left\{\left.\left(v_{n}\right)_{n \in \mathbb{N}_{0}}\left|\sum_{n}\right| v_{n}\right|^{2}\left(1+\lambda_{n}\right)^{-s}<\infty\right\}$, endowed with the norm $\|v\|_{-s, \Gamma}^{2}=\sum_{n}\left|v_{n}\right|^{2}\left(1+\lambda_{n}\right)^{-s}$. When identifying the spaces $H^{s}(\Gamma)$ and $H^{-s}(\Gamma)$ with sequence spaces as above, the duality pairing takes the form

$$
\langle v, w\rangle=\sum_{n} v_{n} \overline{w_{n}},
$$

and, for $s \in \mathbb{R}$, the inner product in $H^{s}(\Gamma)$ takes the form

$$
(v, w)_{s, \Gamma}=\sum_{n \in \mathbb{N}_{0}} v_{n} \overline{w_{n}}\left(1+\lambda_{n}\right)^{s} .
$$

The seminorm $|\cdot|_{\frac{1}{2}, \Gamma}$ in $H^{\frac{1}{2}}(\Gamma)$ is defined by $|v|_{\frac{1}{2}, \Gamma}=\inf _{c \in \mathbb{C}}\|v-c\|_{\frac{1}{2}, \Gamma}$.
Let $s$ and $s^{\prime} \in \mathbb{R}$. For a bounded linear operator $K: H^{s}(\Gamma) \rightarrow H^{s^{\prime}}(\Gamma)$, the adjoint operator $K^{*}$ : $H^{-s^{\prime}}(\Gamma) \rightarrow H^{-s}(\Gamma)$ is defined by $\left\langle v, K^{*} w\right\rangle=\langle K v, w\rangle$, where the duality pairings are between the appropriate spaces. This also implies $\langle w, K v\rangle=\left\langle K^{*} w, v\right\rangle$.

As we deal with the Helmholtz problem, we also introduce the following $k$-weighted Sobolev norms for integers $s \geq 1$ on domains $D \subset \mathbb{R}^{3}$ :

$$
\|v\|_{s, k, D}^{2}:=\sum_{\boldsymbol{\alpha} \in \mathbb{N}_{0}^{3}:|\boldsymbol{\alpha}| \leq s} k^{2(s-|\boldsymbol{\alpha}|)}\left\|D^{\alpha} v\right\|_{0, D}^{2} .
$$

Finally, given $a, b \geq 0$, we write $a \lesssim b$ and $a \gtrsim b$ to indicate the existence of a positive constant $c$, whose dependence is specified at each occurrence, such that $a \leq c b$ and $a \geq c b$, respectively.

Outline of the paper. The mortar formulation of the three dimensional Helmholtz problem is detailed in Section 2, here, we also recall several properties of boundary integral operators. We introduce the DGFEM-BEM mortar method in Section 3. Such a discretization is characterized by a sesquilinear form satisfying a Gårding inequality and continuity estimates, which we prove in Sections 4 and 5 respectively. In Section 6, we provide results for the adjoint problem. Then we cope with the well posedness of the method and the $h$ - and $p$-error analysis in Section 7 We present numerical results verifying the theoretical results in Section 8 and state some conclusions in Section 9 . Three appendices conclude the paper: in the first one, we show a consistency property of the proposed DGFEM-BEM mortar coupling; in the second one, we construct a discontinuous-to-continuous reconstruction operator on curvilinear meshes with optimal $h$ - and $p$-stability and approximation properties, which is of independent interest; in the third one, we prove quantitative error estimates that are explicit in $h$ and $p$.

## 2 Helmholtz model problem, boundary integral operators, and mortar coupling

In this section, we describe the model problem, see Section 2.1, and present its continuous mortar formulation in Section 2.3. The setting is the same as that of [37, Secs. 2 and 3]. In order to make this paper self-contained, we report here all the necessary elements, including the definitions and some properties of the boundary integral operators for the 3D Helmholtz problem; see Section 2.2.

### 2.1 Helmholtz model problem

Let $\Omega \subset \mathbb{R}^{3}$ be a bounded domain with a connected $C^{\infty}$-smooth boundary $\Gamma$ and $\mathbf{n}_{\Gamma}$ be the outward pointing unit vector normal to $\Gamma$. Denote $\Omega^{\mathrm{ext}}:=\mathbb{R}^{3} \backslash \bar{\Omega}$ and let $k \in \mathbb{R}^{+}, k \geq k_{0}>0$, denote the wave number.

We assume that $\Omega^{\text {ext }}$ is occupied by a homogeneous medium with both the refractive index $n \in L^{\infty}\left(\mathbb{R}^{3} ; \mathbb{C}\right)$ and a scalar-valued positive diffusion coefficient $\nu \in C^{\infty}\left(\mathbb{R}^{3} ; \mathbb{R}\right)$ normalized to 1 , while $n$ and $\nu$ may vary within $\Omega$. That is, the supports of $1-n$ and $1-\nu$ are contained in $\Omega$ and
$\nu$ satisfies $0<\nu_{\min } \leq \nu(\mathbf{x}) \leq \nu_{\max }<+\infty$. In addition, we assume that $|n(\mathbf{x})| \geq c_{0}>0$ a.e. in $\mathbb{R}^{3}$. Given $f \in L^{2}(\Omega)$ with support contained in $\Omega$, we set

$$
\tilde{f}= \begin{cases}f & \text { in } \Omega \\ 0 & \text { in } \Omega^{\mathrm{ext}}\end{cases}
$$

Under the above assumptions on $n, \nu$, and $\tilde{f}$, there exists an open neighborhood $\mathcal{N}(\Gamma)$ of $\Gamma$ such that $n \equiv 1, \nu \equiv 1$, and $\widetilde{f} \equiv 0$ in $\Omega^{\mathrm{ext}} \cup \mathcal{N}(\Gamma)$.

We consider the Helmholtz problem: Find $u: \mathbb{R}^{3} \rightarrow \mathbb{C}$ such that

$$
\left\{\begin{array}{l}
-\operatorname{div}(\nu \nabla u)-(k n)^{2} u=\widetilde{f} \quad \text { in } \mathbb{R}^{3},  \tag{2.1}\\
\lim _{|\mathbf{x}| \rightarrow+\infty}|\mathbf{x}|\left(\partial_{|\mathbf{x}|} u-\mathrm{i} k u\right)=0
\end{array}\right.
$$

We rewrite problem (2.1) as a transmission problem. To that end, we define the following jump operators. For $v \in H^{1}\left(\mathbb{R}^{3} \backslash \Gamma\right)$, we denote the Dirichlet traces of $v_{\mid \Omega}$ and $v_{\mid \Omega^{\text {ext }}}$ on $\Gamma$ by $\gamma_{0}^{\text {int }}(v)$ and $\gamma_{0}^{\text {ext }}(v)$, respectively. The two Neumann traces $\left(\partial_{\mathbf{n}_{\Gamma}}=\mathbf{n}_{\Gamma} \cdot \nabla\right)$ on $\Gamma$ of a piecewise smooth function $v$ are denoted by $\gamma_{1}^{\text {int }}(\varphi)$ and $\gamma_{1}^{e x t}(\varphi)$. For sufficiently smooth functions $v$ defined in $\mathbb{R}^{3} \backslash \Gamma$, we then define the jumps

$$
\llbracket v \rrbracket_{\Gamma}=\gamma_{0}^{i n t}(v)-\gamma_{0}^{\text {ext }}(v), \quad \llbracket \partial_{\mathbf{n}_{\Gamma}} u \rrbracket_{\Gamma}:=\gamma_{1}^{i n t}(v)-\gamma_{1}^{\text {ext }}(v) .
$$

With these jumps in hand, we reformulate (2.1) as looking for solutions $u: \mathbb{R}^{3} \rightarrow \mathbb{C}$ of the following transmission problem:

$$
\begin{cases}-\operatorname{div}(\nu \nabla u)-(k n)^{2} u=f & \text { in } \Omega  \tag{2.2}\\ -\Delta u-k^{2} u=0 & \text { in } \Omega^{\mathrm{ext}} \\ \llbracket u \rrbracket_{\Gamma}=0, \llbracket \partial_{\mathbf{n}_{\Gamma}} u \rrbracket_{\Gamma}=0, & \\ \lim _{|\mathbf{x}| \rightarrow+\infty}|\mathbf{x}|\left(\partial_{|\mathbf{x}|} u-\mathrm{i} k u\right)=0 . & \end{cases}
$$

Here, we required the boundary $\Gamma$ to be globally smooth, whereas in Section 3 below we allow for a piecewise smooth $\Gamma$. The global smoothness assumption is needed to promote the regularity of the solution to problem (2.2) below, while the piecewise smoothness assumption is enough for the design of the method.

### 2.2 Boundary integral operators

The fundamental solution to the 3D Helmholtz problem is

$$
G_{k}(\mathbf{x}, \mathbf{y})=\frac{e^{\mathrm{i} k|\mathbf{x}-\mathbf{y}|}}{4 \pi|\mathbf{x}-\mathbf{y}|}
$$

Based on that, we define the single and double layer potentials as follows:

$$
\begin{array}{ll}
\widetilde{\mathcal{V}}_{k} \varphi(\mathbf{x})=\int_{\Gamma} G_{k}(\mathbf{x}-\mathbf{y}) \varphi(\mathbf{y}) d s(\mathbf{y}) & \forall \mathbf{x} \in \mathbb{R}^{3} \backslash \Gamma, \\
\widetilde{\mathcal{K}}_{k} \varphi(\mathbf{x})=\int_{\Gamma} \partial_{\mathbf{n}_{\Gamma}(\mathbf{y})} G_{k}(\mathbf{x}-\mathbf{y}) \varphi(\mathbf{y}) d s(\mathbf{y}) & \forall \mathbf{x} \in \mathbb{R}^{3} \backslash \Gamma,
\end{array}
$$

Starting from the potentials $\widetilde{\mathcal{V}}_{k}$ and $\widetilde{\mathcal{K}}_{k}$, we introduce the four standard boundary integral operators for the Helmholtz operator. Their properties are widely studied in the literature; see, e.g., [13, 38, 47, 51 and the references therein. The properties mentioned below have also been summarized in 37.

Single layer operator. Define $\mathcal{V}_{k}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ as

$$
\begin{equation*}
\mathcal{V}_{k} \varphi:=\gamma_{0}^{i n t}\left(\widetilde{\mathcal{V}}_{k} \varphi\right) \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma) \tag{2.3}
\end{equation*}
$$

For $C^{\infty}$-smooth $\Gamma$, the operator $\mathcal{V}_{k}$ extends to $\mathcal{V}_{k}: H^{-1+s}(\Gamma) \rightarrow H^{s}(\Gamma)$ for all $s \in \mathbb{R}$.

Double layer operator. Define $\mathcal{K}_{k}: H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ as

$$
\left(-\frac{1}{2}+\mathcal{K}_{k}\right) \varphi:=\gamma_{0}^{i n t}\left(\widetilde{\mathcal{K}}_{k} \varphi\right) \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)
$$

For $C^{\infty}$-smooth $\Gamma$, the operator $\mathcal{K}_{k}$ extends to $\mathcal{K}_{k}: H^{s}(\Gamma) \rightarrow H^{s}(\Gamma)$ for all $s \in \mathbb{R}$.
Adjoint double layer operator. Define $\mathcal{K}_{k}^{\prime}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ as

$$
\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right) \varphi:=\gamma_{1}^{i n t}\left(\widetilde{\mathcal{V}}_{k} \varphi\right) \quad \forall \varphi \in H^{-\frac{1}{2}}(\Gamma)
$$

For $C^{\infty}{ }_{\text {-smooth }} \Gamma$, the operator $\mathcal{K}_{k}^{\prime}$ extends to $\mathcal{K}_{k}^{\prime}: H^{-s}(\Gamma) \rightarrow H^{-s}(\Gamma)$ for all $s \in \mathbb{R}$.
Hypersingular boundary integral operator. Define $\mathcal{W}_{k}: H^{\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ as

$$
-\mathcal{W}_{k} \varphi:=\gamma_{1}^{i n t}\left(\widetilde{\mathcal{K}}_{k} \varphi\right) \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma)
$$

For $C^{\infty}$-smooth $\Gamma$, the operator $\mathcal{W}_{k}$ extends to $\mathcal{W}_{k}: H^{s}(\Gamma) \rightarrow H^{-1+s}(\Gamma)$ for all $s \in \mathbb{R}$.
Let $\mathcal{V}_{0}, \mathcal{K}_{0}, \mathcal{K}_{0}^{\prime}$, and $\mathcal{W}_{0}$ be the corresponding integral operators for zero wave number $k=0$. Then, for all $s \geq 0$, the difference operators are linear bounded operators in the following spaces

$$
\begin{array}{lc}
\mathcal{V}_{k}-\mathcal{V}_{0}: H^{-\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{5}{2}+s}(\Gamma), & \mathcal{K}_{k}-\mathcal{K}_{0}: H^{\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{5}{2}+s}(\Gamma) \\
\mathcal{K}_{k}^{\prime}-\mathcal{K}_{0}^{\prime}: H^{-\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{3}{2}+s}(\Gamma), & \mathcal{W}_{k}-\mathcal{W}_{0}: H^{\frac{1}{2}+s}(\Gamma) \rightarrow H^{\frac{3}{2}+s}(\Gamma) \tag{2.4}
\end{array}
$$

In other words, the difference operators possess enhanced shift properties with respect to those of each term in the difference; see, e.g., [37, Prop. 2.2] and [38, Thm. 7.2]. Moreover, $\mathcal{V}_{0}$ and $\mathcal{W}_{0}$ satisfy the following properties: there exist positive constants $c_{\mathcal{V}_{0}}, C_{\mathcal{V}_{0}}, c_{\mathcal{W}_{0}}$, and $C_{\mathcal{W}_{0}}$ such that

$$
\begin{array}{rlrl}
c_{\mathcal{V}_{0}}\|\varphi\|_{-\frac{1}{2}, \Gamma}^{2} & \leq\left\langle\varphi, \mathcal{V}_{0} \varphi\right\rangle \leq C_{\mathcal{V}_{0}}\|\varphi\|_{-\frac{1}{2}, \Gamma}^{2} & & \forall \varphi \in H^{-\frac{1}{2}}(\Gamma),  \tag{2.5}\\
c_{\mathcal{W}_{0}}|\varphi|_{\frac{1}{2}, \Gamma}^{2} & \leq\left\langle\mathcal{W}_{0} \varphi, \varphi\right\rangle \leq C_{\mathcal{W}_{0}}|\varphi|_{\frac{1}{2}, \Gamma}^{2} & \forall \varphi \in H^{\frac{1}{2}}(\Gamma) / \mathbb{C} .
\end{array}
$$

We also have the following properties:

$$
\mathcal{V}_{0}^{*}=\mathcal{V}_{0}, \quad \mathcal{K}_{0}^{*}=\mathcal{K}_{0}^{\prime}, \quad \mathcal{W}_{0}^{*}=\mathcal{W}_{0}
$$

### 2.3 Mortar coupling

In this section, we recall the mortar coupling described in 37. Instead of looking for solutions to (2.2), we aim to solve the following three coupled problems for $u: \Omega \rightarrow \mathbb{C}$ and $u^{e x t}, m: \Gamma \rightarrow \mathbb{C}$ :

$$
\begin{align*}
& \begin{cases}-\operatorname{div}(\nu \nabla u)-(k n)^{2} u=f & \text { in } \Omega, \\
\partial_{\mathbf{n}_{\Gamma}} u+i k u-m=0 & \text { on } \Gamma,\end{cases}  \tag{2.6}\\
& \left\{u^{e x t}=\mathcal{P}_{\text {ItD }} m \quad \text { on } \Gamma\right. \text {, }  \tag{2.7}\\
& \left\{u-\left[\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-i k u^{e x t}\right)\right]=0 .\right. \tag{2.8}
\end{align*}
$$

The operator $\mathcal{P}_{\text {ItD }}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ appearing in (2.7) maps the impedance mortar variable $m$ to the Dirichlet trace $u^{e x t}$ of the solution to the exterior problem. This operator was defined, e.g., in 9. pp. 124-126]. In order to characterize it explicitly, we introduce the combined integral operators

$$
\begin{equation*}
\mathcal{B}_{k}:=-\mathcal{W}_{k}-i k\left(\frac{1}{2}-\mathcal{K}_{k}\right), \quad \mathcal{A}_{k}^{\prime}:=\frac{1}{2}+\mathcal{K}_{k}^{\prime}+i k \mathcal{V}_{k} \tag{2.9}
\end{equation*}
$$

and recall their mapping properties, see, e.g., 9, Thm. 2.27]:

$$
\mathcal{B}_{k}: H^{s+\frac{1}{2}}(\Gamma) \rightarrow H^{s-\frac{1}{2}}(\Gamma), \quad \mathcal{A}_{k}^{\prime}: H^{s-\frac{1}{2}}(\Gamma) \rightarrow H^{s-\frac{1}{2}}(\Gamma)
$$

are bounded. Then, equation (2.7) is equivalent to

$$
\begin{equation*}
\mathcal{B}_{k} u^{e x t}+i k \mathcal{A}_{k}^{\prime}\left(u^{e x t}\right)-\mathcal{A}_{k}^{\prime} m=0 ; \tag{2.10}
\end{equation*}
$$

see [37, Prop. 3.2] and the references therein.
The variational formulation of problem (2.6)-(2.8) reads as follows:

$$
\begin{cases}\text { Find }\left(u, m, u^{e x t}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \text { such that } &  \tag{2.11}\\ (\nu \nabla u, \nabla v)_{0, \Omega}-\left((k n)^{2} u, v\right)_{0, \Omega}+i(k u, v)_{0, \Gamma}-\langle m, v\rangle=(f, v)_{0, \Omega} & \forall v \in H^{1}(\Omega), \\ \left\langle\left(\mathcal{B}_{k}+i k \mathcal{A}_{k}^{\prime}\right) u^{e x t}-\mathcal{A}_{k}^{\prime} m, v^{e x t}\right\rangle=0 & \forall v^{e x t} \in H^{\frac{1}{2}}(\Gamma), \\ \langle u, \lambda\rangle-\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-i k u^{e x t}\right), \lambda\right\rangle=0 & \forall \lambda \in H^{-\frac{1}{2}}(\Gamma) .\end{cases}
$$

As in 37, we introduce

$$
\begin{align*}
\mathcal{T} & \left(\left(u, m, u^{e x t}\right),\left(v, \lambda, v^{e x t}\right)\right)=(\nu \nabla u, \nabla v)_{0, \Omega}-\left((k n)^{2} u, v\right)_{0, \Omega}+i k(u, v)_{0, \Gamma}-\langle m, v\rangle \\
& -\left\langle\left(-\mathcal{W}_{k}-i k\left(\frac{1}{2}-\mathcal{K}_{k}\right)+i k\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+i k \mathcal{V}_{k}\right)\right) u^{e x t}\right.  \tag{2.12}\\
& \left.-\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+i k \mathcal{V}_{k}\right) m, v^{e x t}\right\rangle+\langle u, \lambda\rangle-\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-i k u^{e x t}\right), \lambda\right\rangle
\end{align*}
$$

Then, we can rewrite problem (2.11) in compact form:

$$
\left\{\begin{array}{l}
\text { Find }\left(u, m, u^{e x t}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \text { such that }  \tag{2.13}\\
\mathcal{T}\left(\left(u, m, u^{e x t}\right),\left(v, \lambda, v^{e x t}\right)\right)=(f, v)_{0, \Omega} \quad \forall\left(v, \lambda, v^{e x t}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) .
\end{array}\right.
$$

In [37, Thm. 3.5], the well posedness of (2.13) was proven, under the assumption of smoothness of $\Gamma$ and uniqueness of the solution to problem (2.2), based on the following Gårding inequality:

Theorem 2.1. (37, Thm. 3.6]) Let $\mathcal{T}(\cdot, \cdot)$ be defined as in (2.12), and assume that the interface $\Gamma$ is smooth. Then, there exists $c>0$ depending only on $k_{0}$ and $\Omega$ and, for each $k \geq k_{0}$, there is a positive constant $c_{G}(k)$ depending on $k$ and $\Omega$, such that, for all $\left(v, \lambda, v^{\text {ext }}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times$ $H^{\frac{1}{2}}(\Gamma)$,

$$
\begin{aligned}
\mathbb{R} \mathbb{E}\left(\mathcal{T}\left(\left(v, \lambda, v^{e x t}\right),\left(v, \lambda, v^{e x t}\right)\right)\right) \geq & c\left\{\left\|\nu^{1 / 2} \nabla v\right\|_{0, \Omega}^{2}+\|\lambda\|_{-\frac{1}{2}, \Gamma}^{2}+\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma}^{2}\right\} \\
& -\left\{k^{2}\|n v\|_{0, \Omega}^{2}+c_{G}(k)\left(\|\lambda\|_{-\frac{5}{2}, \Gamma}^{2}+\left\|v^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right)\right\} .
\end{aligned}
$$

## 3 DGFEM-BEM mortar coupling

We introduce a discontinuous Galerkin finite element method-boundary element method (DGFEMBEM) for the discretization of problem (2.6) -(2.8). As a DGFEM discretization of (2.6) in the interior domain $\Omega$, we use the method introduced in 41, which is based on the same variational formulation as that of 32. For the sake of completeness, we recall the main steps of its derivation in Section 3.1 below, in case of a smooth coefficient $\nu$. Equation (2.7) is discretized as in 37, while the discretization of (2.8) is obtained by a suitable modification described in Section 3.2 of what is proposed in [37. The complete discrete formulation is summarized in Section 3.3 .

### 3.1 DGFEM discretization of (2.6)

We shall work with regular, shape regular meshes of the (curved) domain $\Omega$. That is, the meshes will have no hanging nodes and the parametrizations of common edges or faces induced by the element maps of neighboring elements match; see [35, Def. 2.2] for the precise statement.

As in 35, Def. 2.2], we define a curved d-simplex $K, d=2,3$, as the image of a reference straight $d$-simplex $\widehat{K}$ through a $C^{1}$-diffeomorphism $\Phi_{K}$ satisfying

$$
\begin{equation*}
\left\|D \Phi_{K}\right\|_{L^{\infty}(\widehat{K})} \leq \gamma_{S R} h_{K}, \quad\left\|\left(D \Phi_{K}\right)^{-1}\right\|_{L^{\infty}(\widehat{K})} \leq \gamma_{S R} h_{K}^{-1} \tag{3.1}
\end{equation*}
$$

where $D$. denotes the Jacobian and $\gamma_{S R}>0$ is the shape regularity constant.
Condition (3.1) implies that $\Phi_{K}$ can be decomposed as $\Phi_{K}=\Phi_{K}^{\Delta}+\Psi_{K}$, where $\Phi_{K}^{\Delta}$ is an affine bijection and $\Psi_{K}$ is a $C^{1}$ mapping such that

$$
\begin{equation*}
c_{K}:=\sup _{\widehat{\mathbf{x}} \in \widehat{K}}\left\|D \Psi_{K}(\widehat{\mathbf{x}}) \cdot\left(D \Phi_{K}^{\Delta}\right)^{-1}\right\|_{L^{\infty}(\widehat{K})} \lesssim 1 \tag{3.2}
\end{equation*}
$$

To see (3.2), it is enough to fix any $\widehat{\mathbf{x}}_{0} \in \widehat{K}$ and take $\Phi_{K}^{\Delta}:=\Phi_{K}\left(\widehat{\mathbf{x}}_{0}\right)+D \Phi_{K}\left(\widehat{\mathbf{x}}_{0}\right)\left(\widehat{\mathbf{x}}-\widehat{\mathbf{x}}_{0}\right)$. This gives (3.2) with $\gamma_{S R}^{2}+1$ on the right-hand side. A face $F$ of a curved 3 -simplex $K$ is the image through $\Phi_{K}$ of a face $\widehat{F}$ of $\widehat{K}$.

Let $\left\{\Omega_{h}\right\}_{h}$ be a sequence of conforming, i.e., regular in the sense described above, decompositions of $\Omega$ into curved 3 -simplices with mesh granularity $h$. For $h$ sufficiently small, $c_{K} \leq c<1$ for all $K \in \Omega_{h}$. The union of the (open) internal and boundary faces of $\Omega_{h}$ are denoted by $\mathcal{F}_{h}^{I}$ and $\mathcal{F}_{h}^{B}$, respectively. We assume that all the faces in $\mathcal{F}_{h}^{I}$ are flat. The faces in $\mathcal{F}_{h}^{B}$ are curved 2 -simplices.

Given an element $K \in \Omega_{h}$, denote its diameter by $h_{K}$ and the outward pointing unit vector normal to $\partial K$ by $\mathbf{n}_{K}$. We introduce the mesh size function $\mathfrak{h}: \bar{\Omega} \rightarrow \mathbb{R}$, where $\mathfrak{h}_{\left.\right|_{K}}=h_{K}$ for all $K \in \Omega_{h}, \mathfrak{h}=\min \left\{h_{K_{1}}, h_{K_{2}}\right\}$ on each face in $\mathcal{F}_{h}^{I}$ shared by $K_{1}$ and $K_{2}$, and $\mathfrak{h}=h_{K}$ on each face in $\mathcal{F}_{h}^{B}$ on $\partial \Omega$. We may fix $\mathfrak{h}$ arbitrarily at mesh vertices and on edges because we shall not need it there.

To derive the DG formulation, we write the first equation of (2.6) in mixed form:

$$
\left\{\begin{array}{l}
\mathrm{i} k \boldsymbol{\sigma}=\nu \nabla u \\
-\mathrm{i} k \operatorname{div}(\boldsymbol{\sigma})-(k n)^{2} u=f \quad \text { in } \Omega .
\end{array}\right.
$$

On each element $K \in \Omega_{h}$, we multiply the above two equations by smooth functions $\boldsymbol{\tau}$ and $v$, respectively, and integrate by parts:

$$
\left\{\begin{array}{l}
\int_{K} \mathrm{i} k \boldsymbol{\sigma} \cdot \overline{\boldsymbol{\tau}}+\int_{K} u \overline{\operatorname{div}(\nu \boldsymbol{\tau})}-\int_{\partial K} \nu u \overline{\boldsymbol{\tau} \cdot \mathbf{n}_{K}}=0  \tag{3.3}\\
\int_{K}^{\mathrm{i}} k \boldsymbol{\sigma} \cdot \overline{\nabla v}-\int_{\partial K} \mathrm{i} k \boldsymbol{\sigma} \cdot \mathbf{n}_{K} \bar{v}-\int_{K}(k n)^{2} u \bar{v}=\int_{K} f \bar{v} .
\end{array}\right.
$$

We replace the traces of $u$ and $\boldsymbol{\sigma} \cdot \mathbf{n}_{K}$ in the integral on $\partial K$ with suitable numerical fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}} \cdot \mathbf{n}_{K}$, respectively, which will be defined later on in (3.7). Thus, we replace $u_{\mid \partial K}$ with $\widehat{u}_{\mid \partial K}$ in the first equation of (3.3), apply one more integration by parts, select $\boldsymbol{\tau}=\nabla v$, and end up with

$$
\begin{equation*}
\int_{K} \mathrm{i} k \boldsymbol{\sigma} \cdot \overline{\nabla v}-\int_{K} \nu \nabla u \cdot \overline{\nabla v}+\int_{\partial K} \nu(u-\widehat{u}) \overline{\nabla v \cdot \mathbf{n}_{K}}=0 \tag{3.4}
\end{equation*}
$$

Next, we replace $\boldsymbol{\sigma} \cdot \mathbf{n}_{K \mid \partial K}$ with $\widehat{\boldsymbol{\sigma}} \cdot \mathbf{n}_{K \mid \partial K}$ in the second equation of (3.3), and obtain

$$
\begin{equation*}
\int_{K} \mathrm{i} k \boldsymbol{\sigma} \cdot \overline{\nabla v}-\int_{\partial K} \mathrm{i} k \widehat{\boldsymbol{\sigma}} \cdot \mathbf{n}_{K} \bar{v}-\int_{K}(k n)^{2} u \bar{v}=\int_{K} f \bar{v} . \tag{3.5}
\end{equation*}
$$

Subtracting (3.4) from (3.5) and adding over all $K \in \Omega_{h}$ lead to the following broken variational formulation:

$$
\left\{\begin{array}{l}
\text { Find } u \in H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \text { such that for all } v \in H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right)  \tag{3.6}\\
\sum_{K \in \Omega_{h}}\left(\int_{K} \nu \nabla u \cdot \overline{\nabla v}-\int_{\partial K} \nu(u-\widehat{u}) \overline{\nabla v \cdot \mathbf{n}_{K}}\right)-\int_{\Gamma} \mathrm{i} k \widehat{\boldsymbol{\sigma}} \cdot \mathbf{n}_{\Gamma} \bar{v}-\int_{\Omega}(k n)^{2} u \bar{v}=\int_{\Omega} f \bar{v}
\end{array}\right.
$$

where

$$
H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right):=\left\{v \in L^{2}(\Omega): v_{\mid K} \in H^{1}(K) \forall K \in \Omega_{h}\right\}
$$

In order to complete the definition of the DGFEM method, we need to choose finite dimensional subspaces of $H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right)$ and define the numerical fluxes.

To that end, we introduce the following notation for spaces of mapped, piecewise polynomial functions of finite degree. Let $D \subset \mathbb{R}^{3}$ be an open, bounded Lipschitz domain with piecewise
$C^{\infty}$-smooth boundary, and $D_{h}$ a partition of $D$ into curved simplices with flat internal faces. Let $\ell \in \mathbb{N}_{0}$, and denote by $\mathbb{P}_{\ell}(\cdot)$ the space of polynomials of degree at most $\ell$ on the domain within the brackets. For $\ell \geq 1$ and $r=0,1$, we set

$$
\mathcal{S}^{\ell, r}\left(D, D_{h}\right)=\left\{v \in H^{r}(D): v_{\left.\right|_{K}} \circ \Phi_{K} \in \mathbb{P}_{\ell}(\widehat{K}) \quad \forall K \in D_{h}\right\} .
$$

For later use, we also define mapped, piecewise polynomial spaces on surface meshes. To that end, we assume that $S \subset \mathbb{R}^{3}$ is a closed, piecewise $C^{\infty}$-smooth surface and let $S_{h}$ be a partition of $S$ into curved 2 -simplices, which is the trace of a partition $D_{h}$ of its interior $D$ as above. For $\ell \geq 1$ and $r=0,1$, we set

$$
\mathcal{S}^{\ell, r}\left(S, S_{h}\right)=\left\{v \in H^{r}(S): v_{\mid F} \circ \Phi_{K_{F}} \in \mathbb{P}_{\ell}(\widehat{F}) \quad \forall F \in S_{h}\right\}
$$

where $K_{F}$ is the element of $D_{h}$ with $F$ as a face.
As for the DGFEM discretization of (3.6), we choose discretization spaces made by discontinuous piecewise polynomial functions:

$$
V_{h}:=\mathcal{S}^{p, 0}\left(\Omega, \Omega_{h}\right),
$$

and the numerical fluxes introduced in [32,41. We recall their definition in the case of a smooth coefficient $\nu$. We first introduce the following notation for the jump and the average functionals on $\mathcal{F}_{h}^{I}$ for smooth, scalar functions $v$ and vector-valued functions $\boldsymbol{\tau}$. At any $\mathbf{x} \in \mathcal{F}_{h}^{I}$ shared by the two elements $K_{\mathbf{x}}^{1}$ and $K_{\mathbf{x}}^{2}$, the jumps $\llbracket v \rrbracket$ and $\llbracket \boldsymbol{\tau} \rrbracket$, and the averages $\left.\{v\}\right\}$ and $\left.\{\boldsymbol{\tau}\}\right\}$ are defined as

$$
\begin{array}{ll}
\llbracket v \rrbracket(\mathbf{x}):=v_{\mid K_{\mathbf{x}}^{1}}(\mathbf{x}) \mathbf{n}_{K_{\mathbf{x}}^{1}}+v_{\mid K_{\mathbf{x}}^{2}}(\mathbf{x}) \mathbf{n}_{K_{\mathbf{x}}^{2}}, \quad\{v\}(\mathbf{x})=\frac{1}{2}\left(v_{\mid K_{\mathbf{x}}^{1}}(\mathbf{x})+v_{\mid K_{\mathbf{x}}^{2}}(\mathbf{x})\right), \\
\llbracket \boldsymbol{\tau} \rrbracket(\mathbf{x}):=\boldsymbol{\tau}_{\mid K_{\mathbf{x}}^{1}}(\mathbf{x}) \cdot \mathbf{n}_{K_{\mathbf{x}}^{1}}+\boldsymbol{\tau}_{\mid K_{\mathbf{x}}^{2}}(\mathbf{x}) \cdot \mathbf{n}_{K_{\mathbf{x}}^{2}}, \quad\{\boldsymbol{\tau}\}(\mathbf{x}):=\frac{1}{2}\left(\boldsymbol{\tau}_{\mid K_{\mathbf{x}}^{1}}(\mathbf{x})+\boldsymbol{\tau}_{\mid K_{\mathbf{x}}^{2}}(\mathbf{x})\right) .
\end{array}
$$

Then, given functions $\alpha, \beta>0$ in $L^{\infty}\left(\mathcal{F}_{h}^{I}\right)$ and $\delta \in(0,1 / 2]$ in $L^{\infty}\left(\mathcal{F}_{h}^{B}\right)$, we define the following numerical fluxes:

$$
\begin{align*}
\mathrm{i} k \widehat{\boldsymbol{\sigma}}=\mathrm{i} k \widehat{\boldsymbol{\sigma}}(u) & := \begin{cases}\nu\left\{\nabla_{h} u\right\}-\mathrm{i} k \nu \alpha \llbracket \rrbracket & \text { on } \mathcal{F}_{h}^{I}, \\
\nabla_{h} u-(1-\delta)\left(\nabla_{h} u+\mathrm{i} k u \mathbf{n}_{\Gamma}-m \mathbf{n}_{\Gamma}\right) & \text { on } \mathcal{F}_{h}^{B},\end{cases} \\
\widehat{u}=\widehat{u}(u) & := \begin{cases}\{u\}-(\mathrm{i} k)^{-1} \beta \llbracket \nabla_{h} u \rrbracket & \text { on } \mathcal{F}_{h}^{I}, \\
u+\delta\left(-(\mathrm{i} k)^{-1} \nabla_{h} u \cdot \mathbf{n}_{\Gamma}-u+(\mathrm{i} k)^{-1} m\right) & \text { on } \mathcal{F}_{h}^{B},\end{cases} \tag{3.7}
\end{align*}
$$

where $\nabla_{h}$ denotes the elementwise application of the gradient operator, and we recall that $m$ denotes the impedance boundary datum in (2.6).

Given positive constants $\mathfrak{a}, \mathfrak{b}$, and $\mathfrak{d}$, with $\mathfrak{a}$ sufficiently large, see Remark 4.1 below, and $\mathfrak{b}$ and $\mathfrak{d}$ sufficiently small, see (4.12), the functions $\alpha, \beta$ and $\delta$ are chosen as

$$
\begin{equation*}
\alpha(\mathbf{x})=\mathfrak{a} \frac{p^{2}}{k \mathfrak{h}(\mathbf{x})}, \quad \beta(\mathbf{x})=\mathfrak{b} \frac{k \mathfrak{h}(\mathbf{x})}{p} \quad \forall \mathbf{x} \in \mathcal{F}_{h}^{I}, \quad \delta(\mathbf{x})=\mathfrak{d} \frac{k \mathfrak{h}(\mathbf{x})}{p^{2}} \quad \forall \mathbf{x} \in \mathcal{F}_{h}^{B} \tag{3.8}
\end{equation*}
$$

The assumption $\delta \in(0,1 / 2]$ implies $\mathfrak{d} \in\left(0, p^{2} /\left(2 k h_{K}\right)\right)$.
The fluxes defined in (3.7) are single-valued on interior mesh faces and consistent, which entails the consistency of the resulting DGFEM scheme (Lemma 3.1). Furthermore, they satisfy the following combined consistency property:

$$
\mathrm{i} k \widehat{\boldsymbol{\sigma}} \cdot \mathbf{n}_{\Gamma}+\mathrm{i} k \widehat{u}=m \quad \text { on } \mathcal{F}_{h}^{B}
$$

In the error analysis, we deal with the interior DGFEM-error $u-u_{h}$, which is locally smooth but globally only in $L^{2}(\Omega)$. Thus, for $r>0$, we introduce the broken Sobolev spaces on $\Omega_{h}$ as

$$
H_{\mathrm{pw}}^{r}\left(\Omega_{h}\right):=\left\{v \in L^{2}(\Omega): v_{\mid K} \in H^{r}(K) \forall K \in \Omega_{h}\right\} .
$$

We also define the following two DG norms, which will be used in the analysis: Given $v \in$ $H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right)$, with $t>0$ arbitrarily small, we define

$$
\begin{align*}
\|v\|_{\mathrm{DG}(\Omega)}^{2}:= & \left\|\nu^{1 / 2} \nabla_{h} v\right\|_{0, \Omega}^{2}+\|k n v\|_{0, \Omega}^{2}+k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}  \tag{3.9}\\
& +k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}+k\left\|(1-\delta)^{1 / 2} v\right\|_{0, \Gamma}^{2},
\end{align*}
$$

and

$$
\|v\|_{\mathrm{DG}^{+}(\Omega)}^{2}:=\|v\|_{\mathrm{DG}(\Omega)}^{2}+k^{-1}\left\|\nu^{1 / 2} \alpha^{-1 / 2}\left\{\left\{\nabla_{h} v\right\}\right\}\right\|_{0, \mathcal{F}_{h}^{I}}^{2} .
$$

In Section 2, we required the boundary $\Gamma$ to be globally smooth, whereas in this section we can allow for a piecewise smooth $\Gamma$. The global smoothness assumption is needed to promote the regularity of the solution to problem (2.2), while the piecewise smoothness assumption is enough for the design of the method.

### 3.2 BEM discretization of (2.7) and discretization of (2.8)

On $\Gamma$, we introduce the curved simplicial mesh $\Gamma_{h}$, whose elements are given by the intersection of the elements in $\Omega_{h}$ and $\Gamma$. As already mentioned, for (2.7), whose variational formulation is given by the second equation in (2.11), we use the same discretization as in [37, eqns. (3.8) and (4.1)], namely, a standard conforming BEM method with approximation spaces

$$
Z_{h}:=\mathcal{S}^{p, 1}\left(\Gamma, \Gamma_{h}\right) \quad \text { and } \quad W_{h}:=\mathcal{S}^{p-1,0}\left(\Gamma, \Gamma_{h}\right)
$$

for $u^{e x t}$ and $m$, respectively.
Next, we focus on the discretization of (2.8), whose variational formulation is given by the third equation in (2.11). Compared to what was done in [37, we add suitable terms that will allow us to prove a discrete Gårding inequality, see Theorem 4.7 below, and retain consistency and adjoint consistency, see Lemma 3.1 and Proposition 6.5 below. To that end, it is convenient to write the integral terms on $\Gamma$ appearing in the DGFEM discretization in the interior domain $\Omega$ explicitly. Using the definition of the numerical fluxes (3.7) on $\mathcal{F}_{h}^{B}$, we write

$$
\begin{aligned}
- & \int_{\Gamma}\left(u_{h}-\widehat{u}\left(u_{h}\right)\right) \overline{\nabla v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma} \mathrm{i} k \widehat{\boldsymbol{\sigma}}\left(u_{h}\right) \cdot \mathbf{n}_{\Gamma} \overline{v_{h}} \\
= & -\int_{\Gamma}-\delta\left(-(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}-u_{h}+(\mathrm{i} k)^{-1} m_{h}\right) \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}} \\
& -\int_{\Gamma}\left(\nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}-(1-\delta)\left(\nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}+\mathrm{i} k u_{h}-m_{h}\right)\right) \overline{v_{h}} \\
= & -\int_{\Gamma} \delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma} \delta u_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}+\int_{\Gamma} \delta(\mathrm{i} k)^{-1} m_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}} \\
& -\int_{\Gamma} \delta \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma} \overline{v_{h}}+\int_{\Gamma}(1-\delta) \mathrm{i} k u_{h} \overline{v_{h}}-\int_{\Gamma}(1-\delta) m_{h} \overline{v_{h}} .
\end{aligned}
$$

Therefore, the contribution from the interior discretization to the coupling, i.e., the terms involving $m_{h}$, is

$$
\begin{align*}
\int_{\Gamma} \delta(\mathrm{i} k)^{-1} m_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma}(1-\delta) m_{h} \overline{v_{h}} & =-\int_{\Gamma} m_{h} \overline{\left(\overline{\delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}}\right)}  \tag{3.10}\\
& =-\left(m_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}\right)_{0, \Gamma}
\end{align*}
$$

We have to discretize the third equation in (2.11) in such a way that we have terms that match some of the terms in (3.10) when proving a discrete Gårding inequality; see Proposition 4.3 below.

For $m_{h} \in W_{h}$ and $u_{h}^{\text {ext }} \in Z_{h}$, we abbreviate, for convenience,

$$
\begin{equation*}
X_{h}:=\left(\frac{1}{2}+\mathcal{K}_{k}\right) u_{h}^{e x t}-\mathcal{V}_{k}\left(m_{h}-\mathrm{i} k u_{h}^{e x t}\right) \tag{3.11}
\end{equation*}
$$

and introduce the following discretization of the third equation of (2.11):

$$
\left\langle-\delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) u_{h}+\delta(\mathrm{i} k)^{-1} m_{h}, \lambda_{h}\right\rangle-\left\langle X_{h}, \lambda_{h}\right\rangle=0 \quad \forall \lambda_{h} \in W_{h}
$$

The term $-\delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}-\delta u_{h}$ is added in order to be able to prove the Gårding inequality, while the term $\delta(\mathrm{i} k)^{-1} m_{h}$ is added in order to restore consistency. The signs of the terms are chosen in a way that gives a convenient structure to the adjoint problem; see Section 6 below.

### 3.3 Complete discrete formulation

On $V_{h} \times V_{h}$, we define local the sesquilinear forms $a_{h}^{K}(\cdot, \cdot)$ for all $K \in \Omega_{h}$ by

$$
\begin{aligned}
a_{h}^{K}\left(u_{h}, v_{h}\right):= & \int_{K} \nu \nabla_{h} u_{h} \cdot \overline{\nabla_{h} v_{h}}-\int_{K}(k n)^{2} u_{h} \overline{v_{h}} \\
& -\sum_{F \subset \partial K \cap \mathcal{F}_{h}^{I}}\left(\int_{F} \nu\left(u_{h}-\widehat{u}\left(u_{h}\right)\right) \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{K}}+\int_{F} \mathrm{i} k \widehat{\boldsymbol{\sigma}}\left(u_{h}\right) \cdot \mathbf{n}_{K} \overline{v_{h}}\right),
\end{aligned}
$$

with fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ as in (3.7), and the global boundary sesquilinear form $b_{h}^{\Gamma}(\cdot, \cdot)$ by

$$
\begin{aligned}
b_{h}^{\Gamma}\left(u_{h}, v_{h}\right):= & -\int_{\Gamma} \delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma} \delta u_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}} \\
& -\int_{\Gamma} \delta \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma} \overline{v_{h}}+\int_{\Gamma}(1-\delta) \mathrm{i} k u_{h} \overline{v_{h}} .
\end{aligned}
$$

With $V_{h}=\mathcal{S}^{p, 0}\left(\Omega, \Omega_{h}\right), W_{h}=\mathcal{S}^{p-1,0}\left(\Gamma, \Gamma_{h}\right)$, and $Z_{h}=\mathcal{S}^{p, 1}\left(\Gamma, \Gamma_{h}\right)$, the full DGFEM-BEM method reads as follows:

$$
\left\{\begin{array}{l}
\text { Find }\left(u_{h}, m_{h}, u_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h} \text { such that, for all }\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right) \in V_{h} \times W_{h} \times Z_{h},  \tag{3.12}\\
\sum_{K \in \Omega_{h}} a_{h}^{K}\left(u_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(u_{h}, v_{h}\right)-\left(m_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}\right)_{0, \Gamma}=\left(f, v_{h}\right)_{0, \Omega}, \\
\left\langle\left(\mathcal{B}_{k}+\mathrm{i} k \mathcal{A}_{k}^{\prime}\right) u_{h}^{e x t}-\mathcal{A}_{k}^{\prime} m_{h}, v_{h}^{\mathrm{ext}}\right\rangle=0, \\
\left\langle-\delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) u_{h}+\delta(\mathrm{i} k)^{-1} m_{h}, \lambda_{h}\right\rangle-\left\langle X_{h}, \lambda_{h}\right\rangle=0,
\end{array}\right.
$$

where the combined integral operators $\mathcal{B}_{k}$ and $\mathcal{A}_{k}^{\prime}$ are as in (2.9) and $X_{h}$ is as in (3.11).
The definition of $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ in (3.7) entails

$$
\begin{aligned}
& \sum_{K \in \Omega_{h}} a_{h}^{K}\left(u_{h}, v_{h}\right)=\sum_{K \in \Omega_{h}}\left(\int_{K} \nu \nabla u_{h} \cdot \overline{\nabla v_{h}}-\int_{K}(k n)^{2} u_{h} \overline{v_{h}}\right) \\
& -\int_{\mathcal{F}_{h}^{I}} \nu\left(\llbracket u_{h} \rrbracket \cdot\left\{\left\{\overline{\nabla_{h} v_{h}}\right\}+\left\{\left[\nabla_{h} u_{h}\right\}\right\} \cdot \llbracket \overline{v_{h} \rrbracket} \rrbracket\right)-\int_{\mathcal{F}_{h}^{I}} \nu \beta(i k)^{-1} \llbracket \nabla_{h} u_{h} \rrbracket \llbracket \overline{\nabla_{h} v_{h}} \rrbracket+\int_{\mathcal{F}_{h}^{I}} \nu \alpha i k \llbracket u_{h} \rrbracket \cdot \llbracket \overline{v_{h}} \rrbracket .\right.
\end{aligned}
$$

By introducing the sesquilinear form

$$
\begin{align*}
\mathcal{T}_{h} & \left(\left(u_{h}, m_{h}, u_{h}^{e x t}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right) \\
:= & \sum_{K \in \Omega_{h}} a_{h}^{K}\left(u_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(u_{h}, v_{h}\right)-\left(m_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}\right)_{0, \Gamma} \\
& -\left\langle\left(-\mathcal{W}_{k}-\mathrm{i} k\left(\frac{1}{2}-\mathcal{K}_{k}\right)+\mathrm{i} k\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right)\right) u_{h}^{e x t}-\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right) m_{h}, v_{h}^{\text {ext }}\right\rangle  \tag{3.13}\\
& +\left\langle-\delta(\mathrm{i} k)^{-1} \nabla_{h} u_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) u_{h}+\delta(\mathrm{i} k)^{-1} m_{h}, \lambda_{h}\right\rangle \\
& -\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}\right) u_{h}^{e x t}-\mathcal{V}_{k}\left(m_{h}-\mathrm{i} k u_{h}^{e x t}\right), \lambda_{h}\right\rangle,
\end{align*}
$$

method (3.12) can be written in compact form as follows:

$$
\left\{\begin{array}{l}
\text { Find }\left(u_{h}, m_{h}, u_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h} \text { such that }  \tag{3.14}\\
\mathcal{T}_{h}\left(\left(u_{h}, m_{h}, u_{h}^{e x t}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right)=\left(f, v_{h}\right)_{0, \Omega} \quad \forall\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right) \in V_{h} \times W_{h} \times Z_{h}
\end{array}\right.
$$

Lemma 3.1. Let the exact solution ( $u, m, u^{e x t}$ ) to (2.6) -(2.8) belong to $H^{\frac{3}{2}+t}(\Omega) \times L^{2}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, for some $t>0$. Then, the discrete DGFEM-BEM coupling (3.14), or equivalently (3.12), is consistent, i.e.,

$$
\begin{equation*}
\mathcal{T}_{h}\left(\left(u, m, u^{e x t}\right),\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right)\right)=\left(f, v_{h}\right)_{0, \Omega} \quad \forall\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h} \tag{3.15}
\end{equation*}
$$

## Proof. See Appendix A

An immediate consequence of (3.14) and (3.15) is the following Galerkin orthogonality property: For all $\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right) \in V_{h} \times W_{h} \times Z_{h}$,

$$
\begin{equation*}
\mathcal{T}_{h}\left(\left(u-u_{h}, m-m_{h}, u^{e x t}-u_{h}^{e x t}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right)=0 \tag{3.16}
\end{equation*}
$$

## 4 A Gårding inequality

In this section, we establish in Theorem 4.7 a Gårding inequality for the form $\mathcal{T}_{h}((\cdot, \cdot, \cdot),(\cdot, \cdot, \cdot))$ defined in (3.13). We start with a remark and some preliminary results.
Remark 4.1. For any $K \in \Omega_{h}$, introduce $C_{\text {trace }}(p, K)$ as the smallest constant such that

$$
\begin{equation*}
\left\|\nabla v_{h}\right\|_{0, \partial K} \leq C_{\text {trace }}(p, K)\left\|\nabla v_{h}\right\|_{0, K} \quad \forall v_{h} \in V_{h} \tag{4.1}
\end{equation*}
$$

For straight elements, it is well known that $C_{\text {trace }}(p, K) \lesssim p h_{K}^{-1 / 2}$; see, e.g., 49, Thm. 4.76]. Under the shape regularity assumption (3.1), this is valid also for curved elements. In fact, given $v_{h} \in V_{h}$, let $\widehat{v}_{h}$ be the pull-back of $v_{\left.h\right|_{K}}$ through the mapping $\Phi_{K}: \widehat{K} \rightarrow K$. Since $\widehat{v}_{h}$ is a polynomial and $\widehat{K}$ is a straight simplex, we have

$$
\left\|\nabla v_{h}\right\|_{0, \partial K} \lesssim\left\|\widehat{\nabla} \widehat{v}_{h}\right\|_{0, \partial \widehat{K}} \lesssim p\left\|\widehat{\nabla} \widehat{v}_{h}\right\|_{0, \widehat{K}} \lesssim p h_{K}^{-1 / 2}\left\|\nabla v_{h}\right\|_{0, K} .
$$

In the light of this, we demand the following assumptions: for $h_{K}$ sufficiently small and $p$ sufficiently large,

$$
\begin{equation*}
\alpha(\mathbf{x}) \geq \frac{\aleph}{k} \max _{K \in\left\{K_{\mathbf{x}}^{-}, K_{\mathbf{x}}^{+}\right\}} C_{\text {trace }}^{2}(p, K) \quad \forall \mathbf{x} \in \mathcal{F}_{h}^{I}, \tag{4.2}
\end{equation*}
$$

where $\aleph$ is a constant, which will be fixed in the proof of Proposition 4.3 below; see equation (4.9).

The following coercivity/continuity result is valid.
Proposition 4.2. Let $\alpha$ satisfy (4.2) and $0<\delta<1 / 2$. Then, there exists a positive constant $c_{\text {coer }}$ independent of $h, k, p, \alpha, \beta$, and $\delta$, such that

$$
\begin{equation*}
\left|\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right| \geq c_{c o e r}\left\|v_{h}\right\|_{D G(\Omega)}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2} \quad \forall v_{h} \in V_{h} . \tag{4.3}
\end{equation*}
$$

Moreover, for any $t>0$, there exists a positive constant $c_{c}$ independent of $h, k, p, \alpha, \beta$, and $\delta$, such that

$$
\begin{align*}
\left|\sum_{K \in \Omega_{h}} a_{h}^{K}(u, v)+b_{h}^{\Gamma}(u, v)\right| & \leq c_{c}\|u\|_{D G^{+}(\Omega)}\|v\|_{D G^{+}(\Omega)} & \forall u, v \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right),  \tag{4.4}\\
\left|\sum_{K \in \Omega_{h}} a_{h}^{K}\left(u, v_{h}\right)+b_{h}^{\Gamma}\left(u, v_{h}\right)\right| & \leq c_{c}\|u\|_{D G^{+}(\Omega)}\left\|v_{h}\right\|_{D G(\Omega)} & \forall u \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right), \forall v_{h} \in V_{h},  \tag{4.5}\\
\left|\sum_{K \in \Omega_{h}} a_{h}^{K}\left(u_{h}, v\right)+b_{h}^{\Gamma}\left(u_{h}, v\right)\right| & \leq c_{c}\left\|u_{h}\right\|_{D G(\Omega)}\|v\|_{D G^{+}(\Omega)} & \forall u_{h} \in V_{h}, \forall v \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right) . \tag{4.6}
\end{align*}
$$

Proof. The proof of [41, Prop. 3.1] applies also in our context. As for (4.3), we also refer to the proof of Proposition 4.3 below. As for (4.4)-(4.6), we use the trace inequality (4.1) and assumption (4.2).

The coercivity bound (4.3) can be refined, as described in the following result, which is instrumental in the proof of the Gårding inequality in Theorem 4.7 below.

Proposition 4.3. Given $\varepsilon>0$, there exist $\mathfrak{a}_{0}>0, \mathfrak{b}_{0}>0$, and $\mathfrak{d}_{0}>0$ independent of $k$ such that, for all $\mathfrak{a} \geq \mathfrak{a}_{0}, \mathfrak{b} \leq \mathfrak{b}_{0}$, and $\mathfrak{d} \leq \mathfrak{d}_{0}$ in (3.8), and for all $v_{h} \in V_{h}$, the following bound is valid:

$$
\begin{gather*}
(\mathbb{R E}+\varepsilon \mathbb{I M})\left(\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right) \geq \frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2} \\
+\frac{1}{2} \varepsilon\left(k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}\right.  \tag{4.7}\\
\left.+k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}+k\left\|(1-\delta) v_{h}\right\|_{0, \Gamma}^{2}\right) .
\end{gather*}
$$

Proof. The proof is a modification of that of [41, Prop. 3.1]. We begin by observing that

$$
\begin{align*}
& (\mathbb{R E}+\varepsilon \mathbb{M} \mathbb{M})\left(\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right) \\
& =\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2}-2 \mathbb{R} \mathbb{E}\left(\int_{\mathcal{F}_{h}^{I}} \nu \llbracket v_{h} \rrbracket \overline{\left.\left\{\nabla_{h} v_{h}\right\}\right\}}\right)-2 \mathbb{R} \mathbb{E}\left(\int_{\Gamma} \delta v_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}\right)  \tag{4.8}\\
& \quad+\varepsilon\left(k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \mathcal{F}_{h}^{B}}^{2}\right. \\
& \left.\quad+k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k\left\|(1-\delta)^{1 / 2} v_{h}\right\|_{0, \Gamma}^{2}\right) .
\end{align*}
$$

Using the Young inequality with weight $\varepsilon k / 2$ entails

$$
\begin{aligned}
\left|2 \mathbb{R} \mathbb{E}\left(\int_{\mathcal{F}_{h}^{I}} \nu \llbracket v_{h} \rrbracket \overline{\left\{\left\{\nabla_{h} v_{h}\right\}\right.}\right)\right| & \leq \frac{\varepsilon k}{2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\frac{2}{\varepsilon k}\left\|\nu^{1 / 2} \alpha^{-1 / 2}\left\{\nabla_{h} v_{h}\right\}\right\|_{0, \mathcal{F}_{h}^{I}}^{2} \\
& \leq \frac{\varepsilon k}{2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\sum_{K \in \Omega_{h}} \frac{1}{\varepsilon k}\left\|\nu^{1 / 2} \alpha^{-1 / 2} \nabla_{h} v_{h}\right\|_{0, \partial K \backslash \Gamma}^{2} .
\end{aligned}
$$

For the second summand, we use (4.1) and (4.2) to obtain

$$
\sum_{K \in \mathcal{T}_{h}} \frac{1}{\varepsilon k}\left\|\nu^{1 / 2} \alpha^{-1 / 2} \nabla_{h} v_{h}\right\|_{0, \partial K \backslash \Gamma}^{2} \leq \sum_{K \in \mathcal{T}_{h}} \frac{\nu_{\max }}{\varepsilon \aleph}\left\|\nabla_{h} v_{h}\right\|_{0, K}^{2} \leq \sum_{K \in \mathcal{T}_{h}} \frac{\nu_{\max }}{\varepsilon \aleph \nu_{\min }}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, K}^{2}
$$

Fix

$$
\begin{equation*}
\aleph=2 \nu_{\max } /\left(\varepsilon \nu_{\min }\right) \tag{4.9}
\end{equation*}
$$

and get

$$
\sum_{K \in \mathcal{T}_{h}} \frac{1}{\varepsilon k}\left\|\nu^{1 / 2} \alpha^{-1 / 2} \nabla_{h} v_{h}\right\|_{0, \partial K \backslash \Gamma}^{2} \leq \frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2} .
$$

We deduce

$$
\begin{equation*}
\left|2 \mathbb{R} \mathbb{E}\left(\int_{\mathcal{F}_{h}^{I}} \nu \llbracket v_{h} \rrbracket \overline{\left\{\left\{\nabla_{h} v_{h}\right\}\right.}\right)\right| \leq \frac{\varepsilon k}{2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2} . \tag{4.10}
\end{equation*}
$$

We deal with the fourth term on the right-hand side of (4.8) analogously: For any constant $t>0$, we have

$$
\left|2 \mathbb{R E}\left(\int_{\Gamma} \delta v_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}\right)\right| \leq t k\left\|\frac{\delta}{1-\delta}\right\|_{\infty, \Gamma}\left\|(1-\delta)^{1 / 2} v_{h}\right\|_{0, \Gamma}^{2}+\frac{1}{t k}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2} .
$$

Take $t=\frac{1}{2}\left\|\frac{\delta}{1-\delta}\right\|_{\infty, \Gamma}^{-1}$ and get

$$
\begin{equation*}
\left|2 \mathbb{R E}\left(\int_{\Gamma} \delta v_{h} \overline{\nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}}\right)\right| \leq \frac{k}{2}\left\|(1-\delta)^{1 / 2} v_{h}\right\|_{0, \Gamma}^{2}+\frac{2}{k}\left\|\frac{\delta}{1-\delta}\right\|_{\infty, \Gamma}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2} . \tag{4.11}
\end{equation*}
$$

Inserting (4.10) and (4.11) into (4.8), we get

$$
\begin{aligned}
(\mathbb{R E} & +\varepsilon \mathbb{I M})\left(\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right) \geq \frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2} \\
& +\varepsilon k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\left(\varepsilon-2\left\|\frac{\delta}{1-\delta}\right\|_{\infty, \Gamma}\right) k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2} \\
& +\frac{1}{2} \varepsilon k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\frac{1}{2} \varepsilon k\left\|(1-\delta)^{1 / 2} v_{h}\right\|_{0, \Gamma}^{2} .
\end{aligned}
$$

Take $\mathfrak{d}$ such that

$$
\begin{equation*}
2\left\|\frac{\delta}{1-\delta}\right\|_{\infty, \Gamma} \leq \frac{1}{2} \varepsilon \tag{4.12}
\end{equation*}
$$

and deduce

$$
\begin{aligned}
(\mathbb{R} \mathbb{E}+ & \varepsilon \mathbb{M} \mathbb{M})\left(\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right) \geq \frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2} \\
+ & \frac{1}{2} \varepsilon\left(k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}\right. \\
& \left.+k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+k\left\|(1-\delta)^{1 / 2} v_{h}\right\|_{0, \Gamma}^{2}\right)
\end{aligned}
$$

whence the assertion follows.
An explicit choice of $\varepsilon$ in the bound (4.7) is given in the proof of the Gårding inequality in Theorem 4.7.

Next, we present a discontinuous-to-continuous reconstruction operator for piecewise smooth functions on curvilinear simplicial meshes.

Theorem 4.4. Let $\Omega_{h}$ be a shape-regular mesh on $\Omega$ as defined in Section 3.1. Then, there exists $c>0$ depending only on $\Omega$ and $\gamma_{S R}$ in (3.1) such that, for each $\ell \in \mathbb{N}$, there exists a linear operator $\mathcal{P}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow H^{1}(\Omega)$ that satisfies, for all $v \in H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right)$,

$$
\begin{align*}
\|\nabla \mathcal{P} v\|_{0, \Omega} & \leq c\left(\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right)  \tag{4.13}\\
\|\mathcal{P} v\|_{0, \Omega} & \leq c\left(\left\|\mathfrak{h} \ell^{-2} \nabla_{h} v\right\|_{0, \Omega}+\|v\|_{0, \Omega}+\left\|\mathfrak{h}^{1 / 2} \ell^{-1} \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right),  \tag{4.14}\\
\left\|\mathfrak{h}^{-1 / 2} \ell(I-\mathcal{P}) v\right\|_{0, \Gamma} & \leq c\left(\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right) . \tag{4.15}
\end{align*}
$$

Proof. We postpone the proof to Appendix B below.
Remark 4.5. The parameter $\ell$ appearing in the statement of Theorem4.4 is not necessarily related to the polynomial degree of the DGFEM space under consideration. However, it will be apparent in Theorem 4.7 and Proposition 5.1 below that a natural choice in our framework is in fact $\ell=p$.

Remark 4.6. Theorem 4.4 relates to similar results in the literature; see, e.g., [7, Sec. 5.2] and 33, Prop. 5.2]. With respect to the first reference, we provide here optimal estimates also on curvilinear simplicial meshes; moreover, differently from the second reference, we also present stability estimates for the elemental $L^{2}$ norm. Furthermore, we define the reconstruction operator for piecewise sufficiently smooth functions, without restricting to piecewise polynomial functions. The price to pay is that the image of this operator is not an $H^{1}$-conforming piecewise polynomial space over the decomposition $\Omega_{h}$, but rather on a sufficiently fine shape regular refinement of $\Omega_{h}$; see Appendix B below for more details.

We are left to prove the main result of the section, namely the following discrete Gårding inequality for the form defined in (3.13).

Theorem 4.7. Let $\mathcal{T}_{h}((\cdot, \cdot, \cdot),(\cdot, \cdot, \cdot))$ be defined as in (3.13) and the interface $\Gamma$ be smooth. Then, the following Gärding inequality is valid: there exist a constant $\varepsilon>0$ only depending on $\Omega$ and $\gamma_{S R}$ in (3.1) (see (4.30)), three constants $\mathfrak{a}_{0}>0, \mathfrak{b}_{0}>0$, and $\mathfrak{d}_{0}>0$ depending additionally on $\nu$, and a positive constant $c_{G}(k)$ depending additionally on $k$ such that, for all $\mathfrak{a} \geq \mathfrak{a}_{0}, \mathfrak{b} \leq \mathfrak{b}_{0}$, and $\mathfrak{d} \leq \mathfrak{d}_{0}$ in (3.8),

$$
\begin{align*}
& (\mathbb{R E}+\varepsilon \mathbb{M}) \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right),\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right)\right) \\
& \gtrsim\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}+\varepsilon k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}+\varepsilon k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2} \\
& \quad+\varepsilon k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\varepsilon k\left\|(1-\delta) v_{h}\right\|_{0, \Gamma}^{2}+\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}+\left\|v_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma}^{2}  \tag{4.16}\\
& \quad-\left\|k n v_{h}\right\|_{0, \Omega}^{2}-c_{G}(k)\left(\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|v_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) \quad \forall\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h} .
\end{align*}
$$

The hidden constant depends on $\mathcal{V}_{0}$ and $c_{\mathcal{W}_{0}}$ in (2.5) but not on $k$.

Proof. Observe that

$$
\begin{aligned}
& \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right)\right) \\
& =\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)-\left(\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}\right)_{0, \Gamma} \\
& \quad-\left\langle\left(-\mathcal{W}_{k}-\mathrm{i} k\left(\frac{1}{2}-\mathcal{K}_{k}\right)+\mathrm{i} k\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right)\right) v_{h}^{\mathrm{ext}}-\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right) \lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle \\
& \quad-\overline{\left\langle\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}-(1-\delta) v_{h}-\delta(\mathrm{i} k)^{-1} \lambda_{h}\right\rangle}-\left\langle\lambda_{h},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v_{h}^{\mathrm{ext}}-\mathcal{V}_{k}\left(\lambda_{h}-\mathrm{i} k v_{h}^{\mathrm{ext}}\right)\right\rangle .
\end{aligned}
$$

Equivalently, we write

$$
\begin{aligned}
& \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right)=\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right) \\
& \quad+\left\langle\mathcal{W}_{k} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle+\mathrm{i} k\left\langle\left(\frac{1}{2}-\mathcal{K}_{k}\right) v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right) v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle \\
& \quad+\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right) \lambda_{h}, v_{h}^{\text {ext }}\right\rangle+\mathrm{i} k\left\langle\mathcal{V}_{k} \lambda_{h}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k^{-1}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma}^{2} \\
& \quad-\left\langle\lambda_{h},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v_{h}^{\text {ext }}\right\rangle+\overline{\left\langle\lambda_{h}, \mathcal{V}_{k} \lambda_{h}\right\rangle}-\mathrm{i} k \overline{\left\langle\lambda_{h}, \mathcal{V}_{k} v_{h}^{\text {ext }}\right\rangle} \\
& \quad-2 \mathbb{R E}\left(\left\langle\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\rangle\right)-2 \mathrm{i} \mathbb{M} \mathbb{M}\left(\left\langle(1-\delta) \lambda_{h}, v_{h}\right\rangle\right)
\end{aligned}
$$

and thus

$$
\begin{aligned}
& \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right)=\sum_{K \in \Omega_{h}} a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right) \\
& \quad+\overline{\left\langle\lambda_{h}, \mathcal{V}_{k} \lambda_{h}\right\rangle}+\left\langle\mathcal{W}_{k} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle+k^{2}\left\langle\mathcal{V}_{k} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k\left\langle\left(\mathcal{K}_{k}^{\prime}+\mathcal{K}_{k}\right) v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle \\
& \quad+\left[\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right) \lambda_{h}, v_{h}^{\text {ext }}\right\rangle-\overline{\left\langle\lambda_{h},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v_{h}^{\text {ext }}\right\rangle}\right]-\mathrm{i} k^{-1}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma}^{2} \\
& \quad+\mathrm{i} k\left\langle\mathcal{V}_{k} \lambda_{h}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k \overline{\left\langle\lambda_{h}, \mathcal{V}_{k} v_{h}^{\text {ext }}\right\rangle}-2 \mathbb{R} \mathbb{E}\left(\left\langle\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\rangle\right)-2 \mathrm{i} \mathbb{M} \mathbb{M}\left(\left\langle(1-\delta) \lambda_{h}, v_{h}\right\rangle\right) .
\end{aligned}
$$

For some $\varepsilon>0$ to be fixed sufficiently small below, we take the $\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M}$ part on both sides and
get

$$
\left.\begin{array}{l}
(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M}) \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right)=\sum_{K \in \Omega_{h}}(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left[a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right] \\
+(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M})\left[\overline{\left[\lambda_{h}, \mathcal{V}_{k} \lambda_{h}\right\rangle}+\left\langle\mathcal{W}_{k} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle+k^{2}\left\langle\mathcal{V}_{k} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k\left\langle\left(\mathcal{K}_{k}^{\prime}+\mathcal{K}_{k}\right) v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle\right] \\
+(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M})\left[\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right) \lambda_{h}, v_{h}^{\text {ext }}\right\rangle\right]-(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M})\left[\left\langle\lambda_{h},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v_{h}^{\text {ext }}\right\rangle\right.
\end{array}\right] \begin{aligned}
& -\varepsilon k^{-1}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma}^{2}+(\mathbb{R E}+\varepsilon \mathbb{M})\left[+\mathrm{i} k\left\langle\mathcal{V}_{k} \lambda_{h}, v_{h}^{\text {ext }}\right\rangle-\mathrm{i} k \overline{\left\langle\lambda_{h}, \mathcal{V}_{k} v_{h}^{\text {ext }}\right\rangle}\right] \\
& -2 \mathbb{R} \mathbb{E}\left(\left\langle\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\rangle\right)-2 \varepsilon \mathbb{M} \mathbb{M}\left(\left\langle(1-\delta) \lambda_{h}, v_{h}\right\rangle\right)  \tag{4.17}\\
& =: \sum_{K \in \Omega_{h}}(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M})\left[a_{h}^{K}\left(v_{h}, v_{h}\right)+b_{h}^{\Gamma}\left(v_{h}, v_{h}\right)\right]+\sum_{i=1}^{10} T_{i} .
\end{aligned}
$$

We deal with the terms $T_{i}$, for $i=1, \ldots, 10$, separately.
The continuity of $\mathcal{V}_{k}-\mathcal{V}_{0}: H^{-\frac{3}{2}}(\Gamma) \rightarrow H^{\frac{3}{2}}(\Gamma)$, see (2.4), the fact that $\left\langle\lambda_{h}, \mathcal{V}_{0} \lambda_{h}\right\rangle$ is real, and $\varepsilon \lesssim 1$ imply

$$
\begin{align*}
& T_{1}:=(\mathbb{R E}+\varepsilon \mathbb{M} \mathbb{M})\left(\overline{\left\langle\lambda_{h}, \mathcal{V}_{k} \lambda_{h}\right\rangle}\right) \\
& \quad=\left\langle\lambda_{h}, \mathcal{V}_{0} \lambda_{h}\right\rangle+(\mathbb{R E}+\varepsilon \mathbb{M})\left(\overline{\left\langle\lambda_{h},\left(\mathcal{V}_{k}-\mathcal{V}_{0}\right) \lambda_{h}\right\rangle}\right)  \tag{4.18}\\
& \stackrel{(2.5)}{\geq} c_{\mathcal{V}_{0}}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-(1+\varepsilon)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}\left\|\left(\mathcal{V}_{k}-\mathcal{V}_{0}\right) \lambda_{h}\right\|_{\frac{3}{2}, \Gamma} \geq c_{\mathcal{V}_{0}}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-c_{1}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2} .
\end{align*}
$$

Analogously, the continuity of $\mathcal{W}_{k}-\mathcal{W}_{0}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$, see (2.4), and the fact that $\left\langle\mathcal{W}_{0} v_{h}^{\text {ext }}, v_{h}^{\text {ext }}\right\rangle$ is real and $\varepsilon \lesssim 1$ imply

$$
\begin{align*}
& T_{2}:=(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left(\left\langle\mathcal{W}_{k} v_{h}^{\text {ext }}, v_{h}^{\mathrm{ext}}\right\rangle\right) \\
& =\left\langle\mathcal{W}_{0} v_{h}^{\text {ext }}, v_{h}^{\mathrm{ext}}\right\rangle+(\mathbb{R E}+\varepsilon \mathbb{M})\left(\left\langle\left(\mathcal{W}_{k}-\mathcal{W}_{0}\right) v_{h}^{\mathrm{ext}}, v_{h}^{\mathrm{ext}}\right\rangle\right) \\
& \stackrel{(2.5)}{\geq} c_{\mathcal{W}_{0}}\left|v_{h}^{\mathrm{ext}}\right|_{\frac{1}{2}, \Gamma}^{2}-(1+\varepsilon)\left\|\left(\mathcal{W}_{k}-\mathcal{W}_{0}\right) v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}\left\|v_{h}^{\text {ext }}\right\|_{-\frac{1}{2}, \Gamma} \geq c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{2}(k)\left\|v_{h}^{\text {ext }}\right\|_{-\frac{1}{2}, \Gamma}^{2} . \tag{4.19}
\end{align*}
$$

By the discussion after (2.3) the operator $\mathcal{V}_{k}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{1}{2}}(\Gamma)$ is continuous so that we get for $\varepsilon \in(0,1]$

$$
\begin{equation*}
T_{3}:=k^{2}(\mathbb{R E}+\varepsilon \mathbb{M} \mathbb{M})\left(\left\langle\mathcal{V}_{k} v_{h}^{\mathrm{ext}}, v_{h}^{\mathrm{ext}}\right\rangle\right) \geq-c_{3}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2} \tag{4.20}
\end{equation*}
$$

Owing to the continuity of $\mathcal{K}_{k}^{\prime}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ and $\mathcal{K}_{k}: H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$, and $\varepsilon \lesssim 1$, we note that

$$
\begin{align*}
T_{4} & :=-(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left(\mathrm{i} k\left\langle\mathcal{K}_{k}^{\prime} v_{h}^{\mathrm{ext}}, v_{h}^{\mathrm{ext}}\right\rangle+\mathrm{i} k\left\langle\mathcal{K}_{k} v_{h}^{\mathrm{ext}}, v_{h}^{\mathrm{ext}}\right\rangle\right) \\
& \geq-k(1+\varepsilon)\left\|\mathcal{K}_{k}^{\prime} v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}-k(1+\varepsilon)\left\|\mathcal{K}_{k} v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma} \\
& \geq-\frac{c_{\mathcal{W}_{0}}}{5}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c k^{2}\left\|\mathcal{K}_{k}^{\prime} v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\frac{c_{\mathcal{W}_{0}}}{5}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c k^{2}\left\|\mathcal{K}_{k} v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}  \tag{4.21}\\
& \geq-\frac{2}{5} c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{4}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2} .
\end{align*}
$$

Next, we focus on the term $T_{5}$. We observe that

$$
\begin{aligned}
T_{5} & :=(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left(\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right) \lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle-\overline{\left\langle\lambda_{h},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v_{h}^{\mathrm{ext}}\right\rangle}\right) \\
& =\frac{1}{2}(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left(\left\langle\lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle-\overline{\left\langle\lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle}\right)+(\mathbb{R} \mathbb{E}+\varepsilon \mathbb{M} \mathbb{M})\left(\left\langle\mathcal{K}_{k}^{\prime} \lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle-\overline{\left\langle\lambda_{h}, \mathcal{K}_{k} v_{h}^{\mathrm{ext}}\right\rangle}\right) \\
& =: T_{5,1}+T_{5,2} .
\end{aligned}
$$

First, we focus on the term $T_{5,1}$ :

$$
\begin{equation*}
T_{5,1}=\varepsilon \mathbb{M}\left(\left\langle\lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle\right) \geq-\varepsilon\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma} \geq-\frac{1}{2} \varepsilon\left(\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}+\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}\right) \tag{4.22}
\end{equation*}
$$

To show a bound on the term $T_{5,2}$, we use [40, eqn. (1.2)], (2.4), and $\varepsilon \lesssim 1$ :

$$
\begin{align*}
T_{5,2} & =(\mathbb{R E}+\varepsilon \mathbb{I M})\left(\left\langle\left(\mathcal{K}_{k}^{\prime}-\mathcal{K}_{0}^{\prime}\right) \lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle-\overline{\left\langle\lambda_{h},\left(\mathcal{K}_{k}-\mathcal{K}_{0}\right) v_{h}^{\mathrm{ext}}\right\rangle}\right) \\
& \geq-(1+\varepsilon)\left\|\left(\mathcal{K}_{k}^{\prime}-\mathcal{K}_{0}^{\prime}\right) \lambda_{h}\right\|_{\frac{1}{2}, \Gamma}\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}-\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}\left\|\left(\mathcal{K}_{k}-\mathcal{K}_{0}\right) v_{h}^{\mathrm{ext}}\right\|_{\frac{3}{2}, \Gamma}  \tag{4.23}\\
& \gtrsim-c_{5,2}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}-c_{5,2}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2} .
\end{align*}
$$

We show a bound on the term $T_{6}$ using the polynomial inverse inequality of [5, Lemma A.1] with constant $c_{\text {inv }}^{G}$ and (3.8):

$$
\begin{equation*}
T_{6}:=-\varepsilon k^{-1}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma}^{2} \geq-\varepsilon c_{\text {inv }}^{G} \mathfrak{d}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2} \tag{4.24}
\end{equation*}
$$

Using the continuity of $\mathcal{V}_{k}: H^{-\frac{3}{2}}(\Gamma) \rightarrow H^{-\frac{1}{2}}(\Gamma)$ and $\varepsilon \lesssim 1$, we get

$$
\begin{align*}
T_{7} & :=k(\mathbb{R E}+\varepsilon \mathbb{M} \mathbb{M})\left(\mathrm{i}\left\langle\mathcal{V}_{k} \lambda_{h}, v_{h}^{\mathrm{ext}}\right\rangle\right) \geq-(1+\varepsilon) k\left\|\mathcal{V}_{k} \lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma} \\
& \geq-\frac{1}{10} c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{7}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2} . \tag{4.25}
\end{align*}
$$

Besides, using the continuity of $\mathcal{V}_{k}: H^{\frac{1}{2}}(\Gamma) \rightarrow H^{\frac{3}{2}}(\Gamma)$, we prove that

$$
\begin{align*}
T_{8} & :=-k(\mathbb{R E}+\varepsilon \mathbb{M} \mathbb{M})\left(\mathrm{i}\left\langle\lambda_{h}, \mathcal{V}_{k} v_{h}^{\mathrm{ext}}\right\rangle\right) \geq-(1+\varepsilon) k\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}\left\|\mathcal{V}_{k} v_{h}^{\mathrm{ext}}\right\|_{\frac{3}{2}, \Gamma} \\
& \geq-\frac{c_{\mathcal{W}_{0}}}{5}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{8}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2} . \tag{4.26}
\end{align*}
$$

Next, we focus on the term $T_{9}$. Using again the polynomial inverse inequality of [5, Lemma A.1], the Young inequality with weight $\varepsilon / 4$, and (3.8), we arrive at

$$
\begin{align*}
T_{9} & :=-2 \mathbb{R} \mathbb{E}\left(\left\langle\lambda_{h}, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\rangle\right) \geq-2 k^{-1 / 2}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma} k^{-1 / 2}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma} \\
& \geq-\frac{4}{\varepsilon} k^{-1}\left\|\delta^{1 / 2} \lambda_{h}\right\|_{0, \Gamma}^{2}-\frac{\varepsilon}{4} k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}  \tag{4.27}\\
& \stackrel{(3.8)}{\geq}-\frac{4 c_{\mathrm{inv}}^{G} \mathfrak{v}}{\varepsilon}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\frac{\varepsilon k^{-1}}{4}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}
\end{align*}
$$

As for the term $T_{10}$, we proceed as follows. Recall that

$$
T_{10}:=-2 \varepsilon \mathbb{M} \mathbb{M}\left\langle(1-\delta) \lambda_{h}, v_{h}\right\rangle .
$$

Let $\mathcal{P}$ be the operator introduced in Theorem4.4, with $\ell=p$. Then, we use a trace inequality and again the polynomial inverse inequality of [5, Lemma A.1] to deduce

$$
\begin{aligned}
\left\langle\lambda_{h}, v_{h}\right\rangle & =\left\langle\lambda_{h}, \mathcal{P} v_{h}\right\rangle+\left\langle\lambda_{h},(1-\mathcal{P}) v_{h}\right\rangle \\
& \leq\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}\left\|\mathcal{P} v_{h}\right\|_{\frac{1}{2}, \Gamma}+\left\|\mathfrak{h}^{1 / 2} p^{-1} \lambda_{h}\right\|_{0, \Gamma}\left\|\mathfrak{h}^{-1 / 2} p(I-\mathcal{P}) v_{h}\right\|_{0, \Gamma} \\
\text { (4.13) } & \lesssim \| 4.15) \\
& \lesssim \lambda_{h} \|_{-\frac{1}{2}, \Gamma}\left(\left\|\nabla_{h} v_{h}\right\|_{0, \Omega}+\left\|v_{h}\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} p \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right) \\
& \lesssim\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}\left(\nu_{\min }^{-1 / 2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}+\left(k_{0} c_{0}\right)^{-1}\left\|k n v_{h}\right\|_{0, \Omega}+\nu_{\min }^{-1 / 2} k^{1 / 2} \mathfrak{a}^{-1 / 2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right),
\end{aligned}
$$

where the last inequality follows from the bounds $\nu \geq \nu_{\text {min }}$ and $|k n| \geq k_{0} c_{0}$, and from the definition of $\alpha$ in (3.8).

Let $\widetilde{\varepsilon}>0$ be a positive constant, which will be fixed below; see (4.29). The Young inequality gives

$$
\begin{align*}
-2 \varepsilon \mathbb{M} \mathbb{M} & \left(\left\langle(1-\delta) \lambda_{h}, v_{h}\right\rangle\right) \geq-\widetilde{\varepsilon}^{-1} \varepsilon\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}  \tag{4.28}\\
& \quad-c_{10} \widetilde{\varepsilon} \varepsilon\left(\nu_{\min }^{-1}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}+\left(k_{0} c_{0}\right)^{-2}\left\|k n v_{h}\right\|_{0, \Omega}^{2}+\nu_{\min }^{-1} k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}\right),
\end{align*}
$$

where $c_{10}$ depends on $\mathfrak{a}^{-1 / 2}$. Provided that $\mathfrak{a}$ is sufficiently large and $\mathfrak{d}, \mathfrak{b}$ are sufficiently small, depending on $\varepsilon$, we insert (4.7), (4.18), (4.19), (4.20), (4.21), (4.22), (4.23), (4.25), (4.26), (4.27),
and (4.28) into (4.17), and arrive at

$$
\begin{aligned}
& (\mathbb{R E}+\varepsilon \mathbb{M}) \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\mathrm{ext}}\right)\right) \\
& \geq c_{\mathcal{V}_{0}}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-c_{1}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{2}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}-c_{3}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2} \\
& \quad-\frac{2}{5} c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{4}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\frac{\varepsilon}{2}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\frac{\varepsilon}{2}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2} \\
& \quad-c_{5,2}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}-c_{5,2}(k)\left\|v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\varepsilon c_{\mathrm{inv}}^{G} \mathfrak{v}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2} \\
& \quad-\frac{1}{10} c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{7}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}-\frac{1}{5} c_{\mathcal{W}_{0}}\left\|v_{h}^{\mathrm{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-c_{8}(k)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2} \\
& \quad-c_{\mathrm{inv}}^{G} \mathfrak{v} 4 \varepsilon^{-1}\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\widetilde{\varepsilon}^{-1} \varepsilon\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2} \\
& \quad-c_{10} \widetilde{\varepsilon} \varepsilon\left(\nu_{\min }^{-1}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}+\left(k_{0} c_{0}\right)^{-2}\left\|k n v_{h}\right\|_{0, \Omega}^{2}+\nu_{\min }^{-1} k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}\right) \\
& \quad+\frac{1}{2}\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left\|k n v_{h}\right\|_{0, \Omega}^{2}+\frac{1}{2}\left(\varepsilon k^{-1}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\varepsilon k\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}\right) \\
& \quad+\frac{1}{2}\left(\varepsilon k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2}+\varepsilon k\left\|(1-\delta) v_{h}\right\|_{0, \Gamma}^{2}\right)-\frac{\varepsilon}{4} k^{-1}\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2} .
\end{aligned}
$$

Simple computations yield

$$
\begin{aligned}
& (\mathbb{R E}+\varepsilon \mathbb{M}) \mathcal{T}_{h}\left(\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right),\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)\right) \\
& \geq\left(1 / 2-c_{10} \widetilde{\varepsilon} \varepsilon \nu_{\min }^{-1}\right)\left\|\nu^{1 / 2} \nabla_{h} v_{h}\right\|_{0, \Omega}^{2}-\left(c_{10} \widetilde{\varepsilon} \varepsilon\left(k_{0} c_{0}\right)^{-2}+1\right)\left\|k n v_{h}\right\|_{0, \Omega}^{2}+k^{-1} \varepsilon / 4\left\|\delta^{1 / 2} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}^{2} \\
& \quad+\varepsilon k\left(1 / 2-c_{10} \widetilde{\varepsilon} \nu_{\min }^{-1}\right)\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\frac{\varepsilon}{2 k}\left\|\nu^{1 / 2} \beta^{1 / 2} \llbracket \nabla_{h} v_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}^{2}+\frac{1}{2} \varepsilon k\left\|(1-\delta) v_{h}\right\|_{0, \Gamma}^{2} \\
& \quad+\left(c_{\mathcal{V}_{0}}-\frac{\varepsilon}{2}-\varepsilon c_{\mathrm{inv}}^{G} \mathfrak{d}-4 c_{\mathrm{inv}}^{G} \mathfrak{d} \varepsilon^{-1}-\widetilde{\varepsilon}^{-1} \varepsilon\right)\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}^{2}-\left(c_{1}(k)+c_{5,2}(k)+c_{7}(k)+c_{8}(k)\right)\left\|\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2} \\
& \quad+\left(\frac{3}{10} c_{\mathcal{W}_{0}}-\frac{\varepsilon}{2}\right)\left\|v_{h}^{\operatorname{ext}}\right\|_{\frac{1}{2}, \Gamma}^{2}-\left(c_{2}(k)+c_{3}(k)+c_{4}(k)+c_{5,2}(k)\right)\left\|v_{h}^{\operatorname{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2} .
\end{aligned}
$$

We select

$$
\begin{equation*}
\widetilde{\varepsilon}:=\frac{\nu_{\min }}{4 c_{10}}, \tag{4.29}
\end{equation*}
$$

and fix $\varepsilon$ as

$$
\begin{equation*}
\varepsilon:=\min \left\{\frac{c_{\mathcal{V}_{0}}}{3\left(1 / 2+c_{\mathrm{inv}}^{G}+4 c_{10} \nu_{\min }^{-1}\right)}, \frac{c_{\mathcal{W}_{0}}}{10}, 1\right\} \tag{4.30}
\end{equation*}
$$

where we recall that the constants $c_{\mathcal{V}_{0}}$ and $c_{\mathcal{W}_{0}}$ are from (2.5), $\nu_{\text {min }}$ is a lower bound of the coefficient $\nu$ (see Section(2.1), $c_{\mathrm{inv}}^{G}$ is the inverse inequality constant in (4.24), and $c_{10}$ is from (4.28).

Using (4.29) and (4.30), we investigate the constants of the terms appearing in the DG norm:

* $\left(1 / 2-c_{10} \widetilde{\varepsilon} \varepsilon \nu_{\min }^{-1}\right) \geq 1 / 2-\varepsilon / 4>1 / 4$;
* $-\left(c_{10} \widetilde{\varepsilon} \varepsilon\left(k_{0} c_{0}\right)^{-2}+1\right) \geq-\left(\frac{1}{4} \nu_{\min } \varepsilon\left(k_{0} c_{0}\right)^{-2}+1\right)$;
* $\varepsilon k\left(1 / 2-c_{10} \widetilde{\varepsilon} \nu_{\min }^{-1}\right)=\varepsilon k / 4$;
* by taking $\mathfrak{d}$ in (3.8) also fulfilling

$$
\mathfrak{d} \leq \mathfrak{d}_{0} \leq \frac{c \mathcal{V}_{0}}{12 c_{\mathrm{inv}}^{G}} \varepsilon,
$$

we also have

$$
\begin{aligned}
\left(c_{\mathcal{V}_{0}}-\frac{\varepsilon}{2}-4 c_{\mathrm{inv}}^{G} \mathfrak{d}-c_{\mathrm{inv}}^{G} \mathfrak{d} \varepsilon^{-1}-\widetilde{\varepsilon}^{-1} \varepsilon\right) & =c_{\mathcal{V}_{0}}-\varepsilon\left(\frac{1}{2}+c_{\mathrm{inv}}^{G} \mathfrak{d}+4 c_{10} \nu_{\min }^{-1}\right)-4 c_{\mathrm{inv}}^{G} \mathfrak{d} \varepsilon^{-1} \\
& \geq 2 c_{\mathcal{V}_{0}} / 3-4 c_{\mathrm{inv}}^{G} \mathfrak{d} \varepsilon^{-1}=c_{\mathcal{V}_{0}} / 3
\end{aligned}
$$

this term is positive as well;

* $\left(\frac{3}{10} c_{\mathcal{W}_{0}}-\varepsilon\right)>c_{\mathcal{W}_{0}} / 5$.

The assertion follows.

## 5 Continuity of $\mathcal{T}_{h}((\cdot, \cdot, \cdot),(\cdot, \cdot, \cdot))$

In this section, we prove the continuity of $\mathcal{T}_{h}((\cdot, \cdot, \cdot),(\cdot, \cdot, \cdot))$. To that end, we introduce the two following energy norms, which extend the $\operatorname{DG}(\Omega)$ and $\mathrm{DG}^{+}(\Omega)$ norms to the DGFEM-BEM coupling:

$$
\begin{aligned}
\left\|\left(u, m, u^{e x t}\right)\right\|_{\mathrm{DG}(\Omega)}^{2} & :=\|u\|_{\mathrm{DG}(\Omega)}^{2}+\|m\|_{-\frac{1}{2}, \Gamma}^{2}+\left\|u^{e x t}\right\|_{\frac{1}{2}, \Gamma}^{2}, \\
\left\|\left(u, m, u^{e x t}\right)\right\|_{\mathrm{DG}^{+}(\Omega)}^{2} & :=\|u\|_{\mathrm{DG}^{+}(\Omega)}^{2}+\|m\|_{-\frac{1}{2}, \Gamma}^{2}+\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma}^{2}+\left\|u^{e x t}\right\|_{\frac{1}{2}, \Gamma}^{2} .
\end{aligned}
$$

Proposition 5.1. For all $\left(u, m, u^{e x t}\right),\left(v, \lambda, v^{e x t}\right) \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right) \times L^{2}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ for some regularity parameter $t>0$, the following continuity bound is valid:

$$
\begin{equation*}
\left|\mathcal{T}_{h}\left(\left(u, m, u^{e x t}\right),\left(v, \lambda, v^{e x t}\right)\right)\right| \lesssim\left\|\left(u, m, u^{e x t}\right)\right\|_{D G^{+}(\Omega)}\left\|\left(v, \lambda, v^{e x t}\right)\right\|_{D G^{+}(\Omega)}, \tag{5.1}
\end{equation*}
$$

where the hidden constant depends on $k$. If $\left(u, m, u^{e x t}\right)$ or $\left(v, \lambda, v^{e x t}\right)$ is in $V_{h} \times W_{h} \times Z_{h}$, then we can replace the corresponding $\|\cdot\|_{D G^{+}(\Omega)}$ norm in (5.1) with $\|\cdot\|_{D G(\Omega)}$.

Proof. We present the estimates of the terms in the sesquilinear form $\mathcal{T}_{h}\left(\left(u, m, u^{e x t}\right),\left(v, \lambda, v^{e x t}\right)\right)$ defined in (3.13) separately.

First, to estimate the term $\sum_{K \in \Omega_{h}} a_{h}^{K}(u, v)+b_{h}^{\Gamma}(u, v)$, we use (4.4).
For the terms involving the integral operators, we use the definitions of the combined integral operators in (2.9), and the mapping properties described in Section 2.2. More precisely, we write

$$
\begin{aligned}
\mid-\left\langle\left(-\mathcal{W}_{k}-\mathrm{i} k( \right.\right. & \left.\left.\left.\frac{1}{2}-\mathcal{K}_{k}\right)+\mathrm{i} k\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right)\right) u^{e x t}-\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right) m, v^{e x t}\right\rangle \mid \\
& \leq\left\|\mathcal{B}_{k} u^{e x t}\right\|_{-\frac{1}{2}, \Gamma}\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma}+k\left\|\mathcal{A}_{k}^{\prime} u^{e x t}\right\|_{-\frac{1}{2}, \Gamma}\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma}+\left\|\mathcal{A}_{k}^{\prime} m\right\|_{-\frac{1}{2}, \Gamma}\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma} \\
& \lesssim\left\|u^{e x t}\right\|_{\frac{1}{2}, \Gamma}\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma}+\|m\|_{-\frac{1}{2}, \Gamma}\left\|v^{e x t}\right\|_{\frac{1}{2}, \Gamma},
\end{aligned}
$$

where we have used $\left\|u^{e x t}\right\|_{-\frac{1}{2}, \Gamma} \leq\left\|u^{e x t}\right\|_{\frac{1}{2}, \Gamma}$ and

$$
\begin{aligned}
\left|-\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-\mathrm{i} k u^{e x t}\right), \lambda\right\rangle\right| & \leq\left\|\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-\mathrm{i} k u^{e x t}\right)\right\|_{\frac{1}{2}, \Gamma}\|\lambda\|_{-\frac{1}{2}, \Gamma} \\
& \lesssim\left(\left\|u^{e x t}\right\|_{\frac{1}{2}, \Gamma}+\|m\|_{-\frac{1}{2}, \Gamma}\right)\|\lambda\|_{-\frac{1}{2}, \Gamma}
\end{aligned}
$$

Next, we focus on the coupling terms. Several of the following estimates are already established in the proof of Theorem 4.7 However, we cannot use the polynomial inverse inequality here. With the Cauchy-Schwarz inequality and the definition of $\delta$ in (3.8), we get

$$
\begin{aligned}
\left|\left(m, \delta(\mathrm{i} k)^{-1} \nabla_{h} v \cdot \mathbf{n}_{\Gamma}\right)_{0, \Gamma}\right| & \leq k^{-1}\left\|\delta^{1 / 2} m\right\|_{0, \Gamma}\left\|\delta^{1 / 2} \nabla_{h} v \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma} \\
& \leq \mathfrak{d}^{1 / 2}\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma} k^{-1 / 2}\left\|\delta^{1 / 2} \nabla_{h} v \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}
\end{aligned}
$$

The next coupling term is dealt with as follows:

$$
\begin{aligned}
\left|\left\langle-\delta(\mathrm{i} k)^{-1} \nabla_{h} u \cdot \mathbf{n}_{\Gamma}, \lambda\right\rangle\right| & \leq k^{-1}\left\|\delta^{1 / 2} \nabla_{h} u \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma}\left\|\delta^{1 / 2} \lambda\right\|_{0, \Gamma} \\
& \leq k^{-1 / 2}\left\|\delta^{1 / 2} \nabla_{h} u \cdot \mathbf{n}_{\Gamma}\right\|_{0, \Gamma} \mathfrak{d}^{1 / 2}\left\|\mathfrak{h}^{1 / 2} p^{-1} \lambda\right\|_{0, \Gamma} .
\end{aligned}
$$

Furthermore, we get

$$
\left|\left\langle\delta(\mathrm{i} k)^{-1} m, \lambda\right\rangle\right| \leq k^{-1}\left\|\delta^{1 / 2} m\right\|_{0, \Gamma}\left\|\delta^{1 / 2} \lambda\right\|_{0, \Gamma} \leq \mathfrak{d}^{1 / 2}\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma} \mathfrak{d}^{1 / 2}\left\|\mathfrak{h}^{1 / 2} p^{-1} \lambda\right\|_{0, \Gamma} .
$$

As for the two remaining coupling terms, we employ the reconstruction operator $\mathcal{P}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow$ $H^{1}(\Omega)$ introduced in Theorem 4.4 and write

$$
\begin{aligned}
|-\langle m,(1-\delta) v\rangle| & \lesssim|\langle m, \mathcal{P} v\rangle|+|\langle m,(1-\mathcal{P}) v\rangle| \\
& \leq\|m\|_{-\frac{1}{2}, \Gamma}\|\mathcal{P} v\|_{\frac{1}{2}, \Gamma}+\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma}\left\|\mathfrak{h}^{-1 / 2} p(I-\mathcal{P}) v\right\|_{0, \Gamma} .
\end{aligned}
$$

Properties (4.13)-(4.15) with $\ell=p, \mathfrak{h} p^{-2} \leq 1$, and the definition of $\alpha$ in (3.8) lead to

$$
\begin{aligned}
& \mid\langle m,(1-\delta) v\rangle \left\lvert\, \lesssim\|m\|_{-\frac{1}{2}, \Gamma}\left(\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} p \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}+\|v\|_{0, \Omega}\right)\right. \\
& \quad+\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma}\left(\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} p \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right) \\
& \lesssim\|m\|_{-\frac{1}{2}, \Gamma}\left(\nu_{\min }^{-1 / 2}\left\|\nu^{1 / 2} \nabla_{h} v\right\|_{0, \Omega}+\nu_{\min }^{-1 / 2} k^{1 / 2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}+\left(k_{0} c_{0}\right)^{-1}\|k n v\|_{0, \Omega}\right) \\
& \quad+\left\|\mathfrak{h}^{1 / 2} p^{-1} m\right\|_{0, \Gamma}\left(\nu_{\min }^{-1 / 2}\left\|\nu^{1 / 2} \nabla_{h} v\right\|_{0, \Omega}+\nu_{\min }^{-1 / 2} k^{1 / 2}\left\|\nu^{1 / 2} \alpha^{1 / 2} \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}\right),
\end{aligned}
$$

where the last inequality follows from the bounds $\nu \geq \nu_{\min }$ and $|k n| \geq k_{0} c_{0}$. The hidden constant depends additionally on $\mathfrak{a}^{-\frac{1}{2}}$.

We proceed in the same way to estimate the term $|\langle(1-\delta) u, \lambda\rangle|$, and the assertion follows combining the above bounds.

When dealing with discrete functions, estimate (5.1) can be improved using the polynomial inverse inequality of [5, Lemma A.1]:

$$
\left\|\mathfrak{h}^{1 / 2} p^{-1} \lambda_{h}\right\|_{0, \Gamma} \lesssim\left\|\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma} \quad \forall \lambda_{h} \in W_{h} .
$$

Thus, we can replace $\|\cdot \cdot\|_{\mathrm{DG}^{+}(\Omega)}$ by $\|\cdot\|_{\mathrm{DG}(\Omega)}$ for discrete functions.

## 6 Adjoint problem

In this section, we introduce and analyze the adjoint problem of (3.12).
The dual problem to (2.11) is: given $\left(r, r_{m}, r^{e x t}\right) \in L^{2}(\Omega) \times H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$,

$$
\left\{\begin{array}{l}
\text { find }\left(\psi, \psi_{m}, \psi^{e x t}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) \text { such that }  \tag{6.1}\\
(\nu \nabla v, \nabla \psi)_{0, \Omega}-\left((k n)^{2} v, \psi\right)_{0, \Omega}+\mathrm{i} k(v, \psi)_{0, \Gamma}-\langle\lambda, \psi\rangle \\
\quad-\left\langle\left(\mathcal{B}_{k}+i k \mathcal{A}_{k}^{\prime}\right) v^{e x t}-\mathcal{A}_{k}^{\prime} \lambda, \psi^{e x t}\right\rangle+\overline{\left\langle\psi_{m}, v\right\rangle}-\overline{\left\langle\psi_{m},\left(\frac{1}{2}+\mathcal{K}_{k}\right) v^{e x t}-\mathcal{V}_{k}\left(\lambda-i k v^{e x t}\right)\right\rangle} \\
=\left((w, r)_{0, \Omega}+\left(\xi, r_{m}\right)_{-\frac{3}{2}, \Gamma}+\left(w^{e x t}, r^{e x t}\right)_{-\frac{1}{2}, \Gamma}\right) \\
\quad \forall\left(v, \lambda, v^{e x t}\right) \in H^{1}(\Omega) \times H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma) .
\end{array}\right.
$$

We recall some technical results from [37].
Lemma 6.1. ( [37, Lemma 3.6]) The following identities are valid: For all $\varphi \in H^{-\frac{1}{2}}(\Gamma)$ and for all $\psi \in H^{\frac{1}{2}}(\Gamma)$,

$$
\begin{align*}
& \mathcal{V}_{k}^{*} \varphi=\overline{\mathcal{V}_{k} \bar{\varphi}}, \quad \mathcal{K}_{k}^{*} \psi=\overline{\mathcal{K}_{k}^{\prime} \bar{\psi}}  \tag{6.2}\\
& \left(\mathcal{K}_{k}^{\prime}\right)^{*} \varphi=\overline{\mathcal{K}_{k} \bar{\varphi}}, \quad \mathcal{W}_{k}^{*} \psi=\overline{\mathcal{W}_{k} \bar{\psi}} \tag{6.3}
\end{align*}
$$

where we recall that $\cdot *$ denotes the adjoint operator.
Lemma 6.2. ( [37, Lemma 3.7]) Let $s \in \mathbb{R}^{+}$. Given $r_{m} \in H^{s-\frac{3}{2}}(\Gamma)$ and $r^{e x t} \in H^{s-\frac{1}{2}}(\Gamma)$, there exist $R_{m} \in H^{s+\frac{3}{2}}(\Gamma)$ and $R^{e x t} \in H^{s+\frac{1}{2}}(\Gamma)$ such that

$$
\left\|R_{m}\right\|_{s+\frac{3}{2}, \Gamma}=\left\|r_{m}\right\|_{s-\frac{3}{2}, \Gamma}, \quad\left\|R^{e x t}\right\|_{s+\frac{1}{2}, \Gamma}=\left\|r^{e x t}\right\|_{s-\frac{1}{2}, \Gamma}
$$

and

$$
\begin{align*}
\left\langle\xi, R_{m}\right\rangle & =\left(\xi, r_{m}\right)_{-\frac{3}{2}, \Gamma} & & \forall \xi \in H^{-\frac{1}{2}}(\Gamma), \\
\left(w^{e x t}, R^{e x t}\right)_{0, \Gamma} & =\left(w^{e x t}, r^{e x t}\right)_{-\frac{1}{2}, \Gamma} & & \forall w^{e x t} \in L^{2}(\Gamma) . \tag{6.4}
\end{align*}
$$

Indeed, the global problem (6.1) can be split into three problems as detailed in the following result.

Lemma 6.3. ( $\sqrt[37]{ }$ Lemma 3.8]) Let $\left(r, r_{m}, r^{e x t}\right) \in L^{2}(\Omega) \times H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$ and $R_{m}$ and $R^{e x t}$ be the representers of $r_{m}$ and $r^{e x t}$ constructed in Lemma 6.2. Then, problem (6.1) is equivalent to the variational formulation of the following three coupled problems: Find $\left(\psi, \psi_{m}, \psi^{e x t}\right) \in H^{1}(\Omega) \times$ $H^{-\frac{1}{2}}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ such that

$$
\left.\begin{array}{l} 
\begin{cases}-\operatorname{div}(\nu \nabla \bar{\psi})-(k n)^{2} \bar{\psi}=\bar{r} & \text { in } \Omega, \\
\nabla \bar{\psi} \cdot \mathbf{n}_{\Gamma}+i k \bar{\psi}+\overline{\psi_{m}}=0 & \text { on } \Gamma,\end{cases} \\
\left\{-\bar{\psi}+\left(\frac{1}{2}+\mathcal{K}_{k}+i k \mathcal{V}_{k}\right) \overline{\psi^{e x t}}+\mathcal{V}_{k} \overline{\psi_{m}}=\overline{R_{m}} \quad \text { on } \Gamma,\right.
\end{array}\right\} \begin{aligned}
& \left\{\left(\mathcal{W}_{k}+i k\left(\frac{1}{2}-\mathcal{K}_{k}^{\prime}\right)-i k\left(\frac{1}{2}+\mathcal{K}_{k}+i k \mathcal{V}_{k}\right)\right) \overline{\psi^{e x t}}-\left(\left(\frac{1}{2}+\mathcal{K}_{k}^{\prime}\right)+i k \mathcal{V}_{k}\right) \overline{\psi_{m}}=\overline{R^{e x t}} \quad \text { on } \Gamma .\right.
\end{aligned}
$$

Well posedness as well as regularity results for problem (6.1) are given in the following theorem.
Theorem 6.4. ([37, Thm. 3.12]) Given $s \in \mathbb{R}_{0}^{+}$and

$$
r \in H^{s}(\Omega), \quad r_{m} \in H^{s-\frac{3}{2}}(\Gamma), \quad r^{e x t} \in H^{s-\frac{1}{2}}(\Gamma)
$$

let $\left(\psi, \psi_{m}, \psi^{e x t}\right)$ be the solution to (6.5)-(6.7). Then, $\left(\psi, \psi_{m}, \psi^{\text {ext }}\right)$ satisfies

$$
\psi \in H^{s+2}(\Omega), \quad \psi_{m} \in H^{s+\frac{1}{2}}(\Gamma), \quad \psi^{e x t} \in H^{s+\frac{3}{2}}(\Gamma)
$$

together with the a priori estimates

$$
\begin{equation*}
\|\psi\|_{s+2, \Omega}+\left\|\psi_{m}\right\|_{s+\frac{1}{2}, \Gamma}+\left\|\psi^{e x t}\right\|_{s+\frac{3}{2}, \Gamma} \lesssim_{k}\left(\|r\|_{s, \Omega}+\left\|r_{m}\right\|_{s-\frac{3}{2}, \Gamma}+\left\|r^{e x t}\right\|_{s-\frac{1}{2}, \Gamma}\right) \tag{6.8}
\end{equation*}
$$

In the next proposition, we prove that the adjoint formulation of (3.12) is in fact an approximation of the adjoint problem (6.1), i.e., of the coupled problems (6.5)-6.7).
Proposition 6.5 (adjoint consistency). Let the right-hand side ( $r, r_{m}, r^{e x t}$ ) of (6.1) belong to $L^{2}(\Omega) \times H^{-\frac{3}{2}}(\Gamma) \times H^{-\frac{1}{2}}(\Gamma)$. Then, the solution $\left(\psi, \psi_{m}, \psi^{e x t}\right)$ of (6.1) belongs to $H^{2}(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times$ $H^{\frac{3}{2}}(\Gamma)$ and satisfies

$$
\begin{equation*}
\mathcal{T}_{h}\left(\left(w, \xi, w^{e x t}\right),\left(\psi, \psi_{m}, \psi^{e x t}\right)\right)=(w, r)+\left(\xi, r_{m}\right)_{-\frac{3}{2}, \Gamma}+\left(w^{e x t}, r^{e x t}\right)_{-\frac{1}{2}, \Gamma} \tag{6.9}
\end{equation*}
$$

for all $\left(w, \xi, w^{e x t}\right) \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right) \times L^{2}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$, for some $t>0$.
Proof. According to Theorem 6.4, $\left(\psi, \psi_{m}, \psi^{e x t}\right)$ belongs to $H^{2}(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)$.
STEP 1: $\left(\psi, \psi_{m}, \psi^{\text {ext }}\right)$ satisfies $\mathcal{T}_{h}\left((w, 0,0),\left(\psi, \psi_{m}, \psi^{\text {ext }}\right)\right)=(w, r)_{0, \Omega}$ for all $w \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right)$.
Since $\psi \in H^{2}(\Omega)$, on each internal face we have

$$
\begin{equation*}
\llbracket \psi \rrbracket=0, \quad \llbracket \nabla \psi \rrbracket=0, \quad\{\nabla \nabla \psi\}=\nabla \psi . \tag{6.10}
\end{equation*}
$$

We multiply the first equation in (6.5) by $w \in H_{\mathrm{pw}}^{\frac{3}{2}+t}\left(\Omega_{h}\right)$ and integrate by parts elementwise to get

$$
\begin{equation*}
\sum_{K \in \Omega_{h}}\left(-\int_{\partial K} \nu w \overline{\nabla \psi \cdot \mathbf{n}_{\Gamma}}+\int_{K} \nu \nabla w \cdot \overline{\nabla \psi}\right)-\int_{\Omega}(k n)^{2} w \bar{\psi}=\int_{\Omega} w \bar{r} \tag{6.11}
\end{equation*}
$$

With the aid of the boundary condition in (6.5), the definition of the parameter $\delta$ in (2.4), and the fact that $\nu=1$ on $\Gamma$, we manipulate the boundary term in (6.11) as follows:

$$
\begin{aligned}
&-\sum_{K \in \Omega_{h}} \int_{\partial K} \nu w \overline{\nabla \psi \cdot \mathbf{n}_{\Gamma}}=-\int_{\mathcal{F}_{h}^{I}} \nu \llbracket w \rrbracket \cdot \overline{\nabla \psi}-\int_{\mathcal{F}_{h}^{B}} w \overline{\nabla \psi \cdot \mathbf{n}_{\Gamma}} \\
&=-\int_{\mathcal{F}_{h}^{I}} \nu \llbracket w \rrbracket \cdot \overline{\nabla \psi}+\int_{\mathcal{F}_{h}^{B}} \mathrm{i} k w \bar{\psi}+\int_{\mathcal{F}_{h}^{B}} w \overline{\psi_{m}}-\int_{\mathcal{F}_{h}^{B}} \delta w \overline{\nabla \psi \cdot \mathbf{n}_{\Gamma}} \\
&-\int_{\mathcal{F}_{h}^{B}} \delta \mathrm{i} k w \bar{\psi}-\int_{\mathcal{F}_{h}^{B}} \delta w \overline{\psi_{m}}-\int_{\mathcal{F}_{h}^{B}} \delta(\mathrm{i} k)^{-1} \nabla w \cdot \mathbf{n}_{\Gamma} \overline{\nabla \psi \cdot \mathbf{n}_{\Gamma}} \\
&-\int_{\mathcal{F}_{h}^{B}} \delta \nabla w \cdot \mathbf{n}_{\Gamma} \bar{\psi}-\int_{\mathcal{F}_{h}^{B}} \delta(\mathrm{i} k)^{-1} \nabla w \cdot \mathbf{n}_{\Gamma} \overline{\psi_{m}}
\end{aligned}
$$

Inserting the above identity into (6.11) and adding some terms with property (6.10), we see that STEP 1 is valid.

STEP 2: $\left(\psi, \psi_{m}, \psi^{e x t}\right)$ satisfies $\mathcal{T}_{h}\left((0, \xi, 0),\left(\psi, \psi_{m}, \psi^{e x t}\right)\right)=\left(\xi, r_{m}\right)_{-\frac{3}{2}, \Gamma}$ for all $\xi \in L^{2}(\Gamma)$. First, we multiply (6.6) by $\xi \in L^{2}(\Gamma)$ :

$$
\begin{equation*}
-\int_{\Gamma} \xi \bar{\psi}+\int_{\Gamma} \xi\left(1 / 2+\mathcal{K}_{k}+\mathrm{i} k \mathcal{V}_{k}\right) \overline{\psi^{e x t}}+\int_{\Gamma} \xi \mathcal{V}_{k} \overline{\psi_{m}}=\int_{\Gamma} \xi \overline{R_{m}} \tag{6.12}
\end{equation*}
$$

Identities (6.2) and (6.3) lead to

$$
\begin{aligned}
& \int_{\Gamma} \xi\left(1 / 2+\mathcal{K}_{k}+\mathrm{i} k \mathcal{V}_{k}\right) \overline{\psi^{e x t}}=\left\langle\xi, \overline{\left(1 / 2+\mathcal{K}_{k} \overline{\psi^{e x t}}\right.}\right\rangle+\mathrm{i} k\left\langle\xi, \overline{\mathcal{V}_{k} \overline{\psi^{e x t}}}\right\rangle \\
& \quad=\left\langle\xi,\left(1 / 2+\mathcal{K}_{k}\right)^{*} \psi^{e x t}\right\rangle+\mathrm{i} k\left\langle\xi, \mathcal{V}_{k}^{*} \psi^{e x t}\right\rangle=\left\langle\left(1 / 2+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right) \xi, \psi^{e x t}\right\rangle
\end{aligned}
$$

and

$$
\int_{\Gamma} \xi \mathcal{V}_{k} \overline{\psi_{m}}=\left\langle\xi, \overline{\mathcal{V}_{k} \overline{\psi_{m}}}\right\rangle=\left\langle\xi, \mathcal{V}_{k}^{*} \psi_{m}\right\rangle=\left\langle\mathcal{V}_{k} \xi_{h}, \psi_{m}\right\rangle
$$

Inserting these two terms into (6.12) and adding the boundary condition in (6.5) with the parameter $\delta$, yields STEP 2. To deal with the right-hand side of (6.12), we have used (6.4).

STEP 3: $\left(\psi, \psi_{m}, \psi^{e x t}\right)$ satisfies $\mathcal{T}_{h}\left(\left(0,0, w^{e x t}\right),\left(\psi, \psi_{m}, \psi^{e x t}\right)\right)=\left(w^{e x t}, r^{e x t}\right)_{-\frac{1}{2}, \Gamma}$ for all $w^{e x t} \in$ $H^{\frac{1}{2}}(\Gamma)$.
We multiply (6.7) by $w^{e x t} \in H^{\frac{1}{2}}(\Gamma)$ :

$$
\begin{aligned}
& \int_{\Gamma} w^{e x t}\left(\mathcal{W}_{k}+\mathrm{i} k\left(1 / 2-\mathcal{K}_{k}^{\prime}\right)-\mathrm{i} k\left(1 / 2+\mathcal{K}_{k}+\mathrm{i} k \mathcal{V}_{k}\right)\right) \overline{\psi^{e x t}}+\int_{\Gamma} w^{e x t}\left(\left(1 / 2+\mathcal{K}_{k}^{\prime}\right)+\mathrm{i} k \mathcal{V}_{k}\right) \overline{\psi_{m}} \\
& =\int_{\Gamma} w^{e x t} \overline{R^{e x t}}
\end{aligned}
$$

We use again identities (6.2) and (6.3) and write

$$
\begin{aligned}
& \int_{\Gamma} w^{e x t}\left(\mathcal{W}_{k}+\mathrm{i} k\left(1 / 2-\mathcal{K}_{k}^{\prime}\right)-\mathrm{i} k\left(1 / 2+\mathcal{K}_{k}+\mathrm{i} k \mathcal{V}_{k}\right)\right) \overline{\psi^{e x t}} \\
& =\left\langle w^{e x t}, \overline{\mathcal{W}_{k} \overline{\psi^{e x t}}}\right\rangle+\mathrm{i} k\left\langle w^{e x t}, \overline{\left.\left(1 / 2-\mathcal{K}_{k}^{\prime}\right) \overline{\psi^{e x t}}\right\rangle-\mathrm{i} k\left\langle w^{e x t}, \overline{\left(1 / 2+\mathcal{K}_{k}\right) \overline{\psi^{e x t}}}\right\rangle-(\mathrm{i} k)^{2}\left\langle w^{e x t}, \overline{\mathcal{V}_{k} \overline{\psi^{e x t}}}\right\rangle} \begin{array}{l}
=\left\langle w^{e x t}, \mathcal{W}_{k}^{*} \psi^{e x t}\right\rangle+\mathrm{i} k\left\langle w^{e x t},\left(1 / 2-\mathcal{K}_{k}\right)^{*} \psi^{e x t}\right\rangle-\mathrm{i} k\left\langle w^{e x t},\left(1 / 2+\mathcal{K}_{k}^{\prime}\right)^{*} \psi^{e x t}\right\rangle-(\mathrm{i} k)^{2}\left\langle w^{e x t}, \mathcal{V}_{k}^{*} \psi^{e x t}\right\rangle \\
=\left\langle\left(\mathcal{W}_{k}+\mathrm{i} k\left(1 / 2-\mathcal{K}_{k}\right)-\mathrm{i} k\left(1 / 2+\mathcal{K}_{k}^{\prime}+\mathrm{i} k \mathcal{V}_{k}\right)\right) w^{e x t}, \psi^{e x t}\right\rangle,
\end{array}\right.
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\Gamma} w^{e x t}\left(\left(1 / 2+\mathcal{K}_{k}^{\prime}\right)+\mathrm{i} k \mathcal{V}_{k}\right) \overline{\psi_{m}}=\left\langle w^{e x t}, \overline{\left(1 / 2+\mathcal{K}_{k}^{\prime}\right) \overline{\psi_{m}}}\right\rangle+\mathrm{i} k\left\langle w^{e x t}, \overline{\mathcal{V}_{k} \overline{\psi_{m}}}\right\rangle \\
& =\left\langle w^{e x t},\left(1 / 2+\mathcal{K}_{k}\right)^{*} \psi_{m}\right\rangle+\mathrm{i} k\left\langle w^{e x t}, \mathcal{V}_{k}^{*} \psi_{m}\right\rangle=\left\langle\left(1 / 2+\mathcal{K}_{k}+\mathrm{i} k \mathcal{V}_{k}\right) w^{e x t}, \psi_{m}\right\rangle .
\end{aligned}
$$

Thus, using the above terms and (6.4) for the right-hand side of (6.12) shows STEP 3. Combining STEPS $1-3$ gives the assertion.

## 7 Error analysis

In this section, we prove the well posedness of scheme (3.14) as well as the convergence rate of the $h$ - and $p$-versions of the method. We require the following approximability property.
Assumption 7.1. Let $\left(\psi, \psi_{m}, \psi^{e x t}\right) \in H^{2}(\Omega) \times H^{\frac{1}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma)$ satisfy $\|\psi\|_{2, \Omega}+\left\|\psi_{m}\right\|_{\frac{1}{2}, \Gamma}+\left\|\psi^{e x t}\right\|_{\frac{3}{2}, \Gamma} \leq$ 1. Then, for every $\varepsilon>0$, there exists $\eta_{0}(\varepsilon)>0$ such that for $h$ and $p$ satisfying $h p^{-1} \in\left(0, \eta_{0}(\varepsilon)\right]$ there exists $\left(\psi_{h}, \psi_{m h}, \psi_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h}$ such that

$$
\left(\left\|\psi-\psi_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}+\left\|\psi_{m}-\psi_{m h}\right\|_{-\frac{1}{2}, \Gamma}+\left\|\psi^{e x t}-\psi_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma}+\left\|\frac{\mathfrak{h}^{1 / 2}}{p}\left(\psi_{m}-\psi_{m h}\right)\right\|_{0, \Gamma}\right) \leq \varepsilon
$$

Theorem 7.2. Let the solution ( $u, m, u^{e x t}$ ) to (2.13) be in $H^{\frac{3}{2}+t}(\Omega) \times L^{2}(\Gamma) \times H^{\frac{1}{2}}(\Gamma)$ for some $t>$ 0 , and $\left(u_{h}, m_{h}, u_{h}^{\text {ext }}\right) \in V_{h} \times W_{h} \times Z_{h}$ be the discrete solution of method (3.14) with flux parameters defined in (3.8) and satisfying the assumptions of Theorem 4.7. Furthermore, let Assumption 7.1 be valid. Then, there exists $\eta_{0}>0$ such that for $h$, $p$ satisfying $h p^{-1} \in\left(0, \eta_{0}\right]$ and for all $\left(v_{h}, \lambda_{h}, v_{h}^{e x t}\right)$ in $V_{h} \times W_{h} \times Z_{h}$,

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{D G(\Omega)} & +\left\|m-m_{h}\right\|_{-\frac{1}{2}, \Gamma}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma} \\
& \lesssim\left\|u-v_{h}\right\|_{D G^{+}(\Omega)}+\left\|m-\lambda_{h}\right\|_{-\frac{1}{2}, \Gamma}+\left\|u^{e x t}-v_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma}+\left\|\mathfrak{h}^{1 / 2} p^{-1}\left(m-\lambda_{h}\right)\right\|_{0, \Gamma} .
\end{aligned}
$$

The hidden constant depends on $k$.
Proof. We use Schatz' argument 48]; see also 263741. For convenience, we write $x:=\left(u, m, u^{e x t}\right)$ and $x_{h}:=\left(u_{h}, m_{h}, u_{h}^{e x t}\right)$. For all $y_{h}:=\left(v_{h}, \lambda_{h}, v_{h}^{\text {ext }}\right)$ in $V_{h} \times W_{h} \times Z_{h}$ we get

$$
\begin{equation*}
\left\|x-x_{h}\right\|_{\mathrm{DG}(\Omega)} \leq\left\|x-y_{h}\right\|_{\mathrm{DG}(\Omega)}+\left\|y_{h}-x_{h}\right\|_{\mathrm{DG}(\Omega)} \tag{7.1}
\end{equation*}
$$

We use the discrete Gårding inequality (4.16) to estimate

$$
\begin{align*}
\left\|y_{h}-x_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} \lesssim & \mathcal{T}_{h}\left(y_{h}-x_{h}, y_{h}-x_{h}\right) \\
& +2\left\|k n\left(v_{h}-u_{h}\right)\right\|_{0, \Omega}^{2}+c_{G}(k)\left(\left\|\lambda_{h}-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|v_{h}^{e x t}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) . \tag{7.2}
\end{align*}
$$

We estimate the first term on the right-hand side of (7.2). Using (3.16) to replace $x_{h}$ by $x$ in the first argument, applying Proposition 5.1. where the second argument is discrete, and using the Young inequality lead to

$$
\begin{align*}
\mathcal{T}_{h}\left(y_{h}-x_{h}, y_{h}-x_{h}\right) & =\mathcal{T}_{h}\left(y_{h}-x, y_{h}-x_{h}\right) \\
& \lesssim \varepsilon_{1}^{-1}\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}^{2}+\varepsilon_{1}\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} \tag{7.3}
\end{align*}
$$

where $\varepsilon_{1}>0$ will be fixed later on. Next, we estimate the compact perturbation term appearing in (7.2). The triangle inequality yields

$$
\begin{align*}
2\left\|k n\left(v_{h}-u_{h}\right)\right\|_{0, \Omega}^{2}+ & c_{G}(k)\left(\left\|\lambda_{h}-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|v_{h}^{\operatorname{ext}}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) \\
\leq & 2\left\|k n\left(u-v_{h}\right)\right\|_{0, \Omega}^{2}+c_{G}(k)\left(\left\|m-\lambda_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|u^{e x t}-v_{h}^{\mathrm{ext}}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right)  \tag{7.4}\\
& +2\left\|k n\left(u-u_{h}\right)\right\|_{0, \Omega}^{2}+c_{G}(k)\left(\left\|m-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) .
\end{align*}
$$

We apply a standard duality argument for the last two terms. More precisely, we consider (6.9) with $r=2(k n)^{2}\left(u-u_{h}\right), r_{m}=c_{G}(k)\left(m-m_{h}\right), r^{e x t}=c_{G}(k)\left(u^{e x t}-u_{h}^{e x t}\right)$, and $x-x_{h}$ for the test function. We collect the solution to the adjoint problem into the vector $\Psi:=\left(\psi, \psi_{m}, \psi^{e x t}\right)$ and we get

$$
2\left\|k n\left(u-u_{h}\right)\right\|_{0, \Omega}^{2}+c_{G}(k)\left(\left\|m-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right)=\mathcal{T}_{h}\left(x-x_{h}, \Psi\right)
$$

Next, we use the Galerkin orthogonality (3.16) to subtract an arbitrary $\Psi_{h}:=\left(\psi_{h}, \psi_{m h}, \psi_{h}^{e x t}\right) \in$ $V_{h} \times W_{h} \times Z_{h}$ to the right-hand side, and apply the continuity estimate (5.1) (the first argument in the second term is discrete):

$$
\begin{align*}
2\left\|k n\left(u-u_{h}\right)\right\|_{0, \Omega}^{2}+ & c_{G}(k)\left(\left\|m-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) \\
& =\mathcal{T}_{h}\left(x-x_{h}, \Psi-\Psi_{h}\right)=\mathcal{T}_{h}\left(x-y_{h}, \Psi-\Psi_{h}\right)+\mathcal{T}_{h}\left(y_{h}-x_{h}, \Psi-\Psi_{h}\right)  \tag{7.5}\\
& \lesssim\left(\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}+\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}\right)\left\|\Psi-\Psi_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}
\end{align*}
$$

From (6.8), we see that

$$
\|\psi\|_{2, \Omega}+\left\|\psi_{m}\right\|_{\frac{1}{2}, \Gamma}+\left\|\psi^{e x t}\right\|_{\frac{3}{2}, \Gamma} \lesssim\left\|u-u_{h}\right\|_{0, \Omega}+\left\|m-m_{h}\right\|_{-\frac{1}{2}, \Gamma}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma} .
$$

This, together with Assumption 7.1, yields

$$
\left\|\Psi-\Psi_{h}\right\|_{\mathrm{DG}^{+}(\Omega)} \lesssim \varepsilon\left(\|\psi\|_{2, \Omega}+\left\|\psi_{m}\right\|_{\frac{1}{2}, \Gamma}+\left\|\psi^{e x t}\right\|_{\frac{3}{2}, \Gamma}\right) \lesssim \varepsilon\left\|x-x_{h}\right\|_{\mathrm{DG}(\Omega)} .
$$

We insert this bound into (7.5) and merge the resulting bound with (7.4):

$$
\begin{align*}
2\left\|k n\left(v_{h}-u_{h}\right)\right\|_{0, \Omega}^{2} & +c_{G}(k)\left(\left\|\lambda_{h}-m_{h}\right\|_{-\frac{3}{2}, \Gamma}^{2}+\left\|v_{h}^{e x t}-u_{h}^{e x t}\right\|_{-\frac{1}{2}, \Gamma}^{2}\right) \\
& \lesssim\left\|x-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2}+\left(\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}+\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}\right) \varepsilon\left\|x-x_{h}\right\|_{\mathrm{DG}(\Omega)}  \tag{7.6}\\
& \lesssim(1+\varepsilon)\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}^{2}+\varepsilon\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} .
\end{align*}
$$

Eventually, we insert (7.3) and (7.6) in (7.2) and, writing $c$ for the constant implied in all the previous estimates, we get

$$
\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} \leq c\left(\varepsilon_{1}^{-1}+1+\varepsilon\right)\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}^{2}+c\left(\varepsilon_{1}+\varepsilon\right)\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} .
$$

Assuming that $\varepsilon$ in Assumption 7.1 is sufficiently small and taking $\varepsilon_{1}$ small enough, we shift the second term to the left-hand side:

$$
\begin{equation*}
\left(1-c\left(\varepsilon+\varepsilon_{1}\right)\right)\left\|x_{h}-y_{h}\right\|_{\mathrm{DG}(\Omega)}^{2} \lesssim\left\|x-y_{h}\right\|_{\mathrm{DG}^{+}(\Omega)}^{2} . \tag{7.7}
\end{equation*}
$$

Inserting (7.7) in (7.1) concludes the proof.

The quasi-optimality result Theorem 7.2 can lead to quantitative error estimates that are explicit in the mesh size $h$ and the polynomial degree $p$. To obtain higher order rates of convergence, the element maps $\Phi_{K}$ need to have more regularity than what has been assumed so far at the outset of Section 3.1. To be concrete, one can make the following assumption as in 6.

Assumption 7.3. Given $s \in \mathbb{N}$, there is a constant $\widetilde{c}_{B}>0$ such that

$$
\left\|D^{l} \Phi_{K}\right\|_{L^{\infty}(\widehat{K})} \leq \widetilde{c}_{B} h_{K}^{l}, \quad 2 \leq l \leq s+1
$$

Remark 7.4. Scenarios for the constructions of triangulations and element maps that ensure the validity of Assumption 7.3 are provided in [6].

Corollary 7.5. Let $s \in \mathbb{N}$ and Assumption 7.3 be valid. Set $h:=\max _{K}\left(h_{K}\right)$. Let the solution ( $u, m, u^{e x t}$ ) to (2.13) belong to $H^{s+1}(\Omega) \times H^{s-\frac{1}{2}}(\Gamma) \times H^{s+\frac{1}{2}}(\Gamma)$ and $\left(u_{h}, m_{h}, u_{h}^{e x t}\right) \in V_{h} \times W_{h} \times Z_{h}$ be the discrete solution of method (3.14) with flux parameters defined in (3.8) and satisfying the assumptions of Theorem 4.7. Then, there are constants $\eta_{0}=\eta_{0}(k)$ and $c(k)>0$ such that, under the scale resolution condition $h p^{-1} \in\left(0, \eta_{0}\right]$, the following bound is valid

$$
\begin{aligned}
& \left\|u-u_{h}\right\|_{D G(\Omega)}+\left\|m-m_{h}\right\|_{-\frac{1}{2}, \Gamma}+\left\|u^{e x t}-u_{h}^{e x t}\right\|_{\frac{1}{2}, \Gamma} \\
& \quad \leq c(k) h^{\min (p, s)} p^{-s+\frac{1}{2}}\left(\|u\|_{s+1, \Omega}+\|m\|_{s-\frac{1}{2}, \Gamma}+\left\|u^{e x t}\right\|_{s+\frac{1}{2}, \Gamma}\right)
\end{aligned}
$$

Proof. See Appendix C

Remark 7.6 (suboptimality in $p$ ). The suboptimality by half an order in the polynomial degree $p$ is due to the $p$-scaling of the parameter $\alpha$ in the definition of the DGFEM norm in (3.9). Under further assumptions on the mesh, it is possible to argue as in [41, Sec. 4.2.2] to obtain p-optimal estimates.

Remark 7.7 (exponential convergence). Exponential convergence that is explicit in $h$ and $p$ for analytic solutions can be proved if the element maps $\Phi_{K}$ are assumed to be of the form $\Phi_{K}=$ $R_{K} \circ A_{K}$ with an analytic map $R_{K}$ and an affine map $A_{K}$. We refer to [41, Assump. 4.1] for details. See also [39, Sec. 3.3.2] for the concept of "patchwise structured meshes".

## 8 Numerical results

In this section, we present numerical results validating the convergence rate detailed in Corollary 7.5

We implemented method (3.12) by combining the NGSolve package 3 with the BEM ++ library [1,50]. In particular, we proceeded as in [37], yet replacing the interior discretization with the novel discontinuous Galerkin part. In order to solve the resulting algebraic linear system, we used a GMRES iteration with a preconditioner based on $\mathcal{H}$-matrix $L U$-decomposition provided by the H2Lib library [2].

We considered sequences of quasiuniform tetrahedral meshes $\Omega_{h}$ in $\Omega$ and used the trace of the corresponding interior finite element mesh as a partition $\Gamma_{h}$ of $\Gamma$. As for the choice of the discretization spaces, we picked $V_{h}=S^{p, 0}\left(\Omega, \Omega_{h}\right)$ as the space of discontinuous piecewise polynomials of order $p$ over the tetrahedral meshes $\Omega_{h}$, whereas we picked $Z_{h}=S^{p, 1}\left(\Gamma, \Gamma_{h}\right)$ and $W_{h}=S^{p-1,0}\left(\Gamma, \Gamma_{h}\right)$ as the spaces of continuous and discontinuous piecewise polynomials of orders $p$ and $p-1$ over the triangulation $\Gamma_{h}$ of $\Gamma$, respectively.

We are interested in studying the convergence of the following relative errors:

$$
\frac{\left\|u-u_{h}\right\|_{0, \Omega}}{\|u\|_{0, \Omega}}, \quad \frac{\left\|\nabla_{h}\left(u-u_{h}\right)\right\|_{0, \Omega}}{\|\nabla u\|_{0, \Omega}}, \quad h^{\frac{1}{2}} \frac{\left\|m-m_{h}\right\|_{0, \Gamma}}{\|m\|_{0, \Gamma}}, \quad h^{-\frac{1}{2}} \frac{\left\|u^{e x t}-u_{h}^{e x t}\right\|_{0, \Gamma}}{\left\|u^{e x t}\right\|_{0, \Gamma}}
$$

For the $h$-version of the method, the last two error measures scale like the relative errors in the $H^{-\frac{1}{2}}(\Gamma)$ and the $H^{\frac{1}{2}}(\Gamma)$, respectively. The stabilization parameters of the DG method (3.8) are taken to be $\mathfrak{a}_{0}:=10$, and $\mathfrak{b}_{0}:=\mathfrak{d}_{0}:=0.1$.

We investigated the performance of method (3.12) for the domain $\Omega:=(-1,1)^{3}$ and the coefficients $\nu=1$ and $n=1$ in (2.1), and prescribe the exact smooth solution

$$
u(x, y, z):= \begin{cases}\sin (k x) \cos (k y) & (x, y, z) \in \Omega  \tag{8.1}\\ \frac{e^{i k \sqrt{x^{2}+y^{2}+z^{2}}}}{\sqrt{x^{2}+y^{2}+z^{2}}} & \text { otherwise }\end{cases}
$$

The function $u$ solves the Helmholtz equation in $\mathbb{R}^{3}$ but has nonzero Dirichlet and Neumann jumps. This case is not covered by the theory in Sections 2 , but can be incorporated into method (3.12) via a suitable modification of the right-hand sides.

The coupling strategy based on the mortar variable $m$ aims at solvability for all wave numbers $k$. To underline this feature, we select the wave numbers $k$ as $k:=n \sqrt{3} \pi$ for $n=1$, 2 , which are the first two nonzero eigenvalues of the Dirichlet and Neumann Laplacian on the unit cube. Figures 1 and 2 show that the method (3.12) delivers optimal convergence rates of the errors after some pre-asymptotic phase, which is expected due to dispersion errors ("pollution" effect) typical of wave propagation problems. These rates partly surpass those predicted by Corollary 7.5, which only considers a convergence of the combined error, i.e., the rate of all contributions would be dominated by the lowest order contribution, namely, the $H^{1}(\Omega)$ seminorm. A similar superconvergence phenomenon is well known for the simpler Poisson problem and analyzed in details in [42].

We also considered the $p$-version of the method with wave numbers $k:=4 \sqrt{3} \pi \sim 21.7$ and $k:=$ $2 \sqrt{3} \pi \sim 10.88$. We fixed an underlying uniform mesh of size $h \approx 1 / 4$ and considered the exact solution as in (8.1). For both wave numbers, we observe exponential convergence after a small preasymptotic regime; see Figure 3,

## 9 Conclusions

We introduced a DGFEM-BEM mortar coupling for three dimensional Helmholtz problems with variable coefficients. Upon showing that the discrete sesquilinear form satisfies a Gårding inequality and continuity bounds, we showed quasi-optimality of the $h$ - and $p$-versions of the scheme. The theoretical results are validated by numerical examples. Notably, theoretical and numerical results are valid regardless of whether the wave number is a Dirichlet or Neumann Laplace eigenvalue.

As a pivot result of independent interest, we constructed a discontinuous-to-continuous reconstruction operator on tetrahedral meshes, with optimal $h$ - and $p$-stability properties in the $H^{1}$ seminorm and in the $L^{2}$ norm, covering the case of curvilinear meshes.


Figure 1: $h$-version. Wave number $k=2 \sqrt{3} \pi$.

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Figure 2: $h$-version. Wave number $k=\sqrt{3} \pi$.
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(a) $k:=4 \sqrt{3} \pi$.

(b) $k:=2 \sqrt{3} \pi$.

Figure 3: $p$-version.
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## A Consistency of method (3.12)

Proof of Lemma 3.1. Proving assertion (3.15) is equivalent to proving that the continuous solution $\left(u, m, u^{e x t}\right)$ solves also the three equations in (3.12). Since $u \in H^{\frac{3}{2}+t}(\Omega)$ we have that

$$
\begin{equation*}
\llbracket u \rrbracket=0, \quad \llbracket \nabla u \rrbracket=0, \quad\{\nabla u u\}=\nabla u \quad \text { on } \mathcal{F}_{h}^{I} . \tag{A.1}
\end{equation*}
$$

We multiply (2.6) by $\overline{v_{h}} \in V_{h}$ and integrate elementwise by parts to get

$$
\sum_{K \in \Omega_{h}}\left(-\int_{\partial K} \nu \nabla u \cdot \mathbf{n}_{\Gamma} \overline{v_{h}}+\int_{K} \nu \nabla u \cdot \overline{\nabla v_{h}}\right)-\int_{\Omega}(k n)^{2} u \overline{v_{h}}=\int_{\Omega} f \overline{v_{h}} .
$$

With the aid of the boundary condition in (2.6), inserting the parameter $\delta$, and using the fact that $\nu=1$ on $\Gamma$, we manipulate the boundary term as follows:

$$
\begin{aligned}
- & \sum_{K \in \Omega_{h}} \int_{\partial K} \nu \nabla u \cdot \mathbf{n}_{\Gamma} \overline{v_{h}} \\
= & -\int_{\Gamma} \delta \nabla u \cdot \mathbf{n}_{\Gamma} \overline{v_{h}}-\int_{\Gamma}(1-\delta) m \overline{v_{h}}+\int_{\Gamma} \mathrm{i} k(1-\delta) u \overline{v_{h}}-\int_{\mathcal{F}_{h}^{I}} \nu \nabla u \cdot \llbracket \overline{v_{h}} \rrbracket \\
& +\int_{\Gamma}(\mathrm{i} k)^{-1} \delta m \overline{\nabla v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma}(\mathrm{i} k)^{-1} \delta \nabla u \cdot \mathbf{n}_{\Gamma} \overline{\nabla v_{h} \cdot \mathbf{n}_{\Gamma}}-\int_{\Gamma} \delta u \overline{\nabla v_{h} \cdot \mathbf{n}_{\Gamma}}
\end{aligned}
$$

Properties (A.1) and the above identity lead to the consistency of the first equation of (3.12), i.e.,

$$
\sum_{K \in \Omega_{h}} a_{h}^{K}\left(u, v_{h}\right)+b_{h}^{\Gamma}\left(u, v_{h}\right)-\left(m, \delta(\mathrm{i} k)^{-1} \nabla_{h} v_{h} \cdot \mathbf{n}_{\Gamma}+(1-\delta) v_{h}\right)_{0, \Gamma}=\left(f, v_{h}\right)_{0, \Omega} \quad \forall v_{h} \in V_{h}
$$

To show the consistency of the second equation of (3.12), we multiply (2.10), which is an equivalent formulation of (2.7), by $\overline{v_{h}^{\text {ext }}} \in Z_{h}$ and integrate over $\Gamma$ :

$$
\left\langle\left(\mathcal{B}_{k}+\mathrm{i} k \mathcal{A}_{k}^{\prime}\right) u^{e x t}-\mathcal{A}_{k}^{\prime} m, v_{h}^{\mathrm{ext}}\right\rangle=0 \quad \forall v_{h}^{\mathrm{ext}} \in Z_{h} .
$$

Eventually, multiplying (2.8) by $\overline{\lambda_{h}} \in W_{h}$ and integrating over $\Gamma$, we get

$$
\left\langle u, \lambda_{h}\right\rangle-\left\langle\left(\frac{1}{2}+\mathcal{K}_{k}\right) u^{e x t}-\mathcal{V}_{k}\left(m-i k u^{e x t}\right), \lambda_{h}\right\rangle=0
$$

Similarly as above, the boundary condition in (2.6) leads to

$$
\left\langle-\delta(\mathrm{i} k)^{-1} \nabla u \cdot \mathbf{n}_{\Gamma}, \lambda_{h}\right\rangle+\left\langle-\delta u, \lambda_{h}\right\rangle+\left\langle\delta(\mathrm{i} k)^{-1} m, \lambda_{h}\right\rangle=0 .
$$

Summing up the last two equations shows the consistency of the third equation in (3.12).

## B An $h p$-stable, discontinuous-to-continuous reconstruction operator on curvilinear simplicial meshes

Here, we prove Theorem 4.4 .
Let the mesh $\Omega_{h}$ satisfy the shape regularity assumption (3.1) and $v \in H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right)$. We construct the operator $\mathcal{P}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow H^{1}(\Omega)$ as the composition $\mathcal{P}:=\mathcal{P}_{2} \circ \mathcal{P}_{1}$ of two operators $\mathcal{P}_{2}, \mathcal{P}_{1}$ that we define below. Preliminarily, for each $K \in \Omega_{h}$, we construct a quasi-uniform, shape regular simplicial decomposition $\widetilde{\Omega}_{h}^{K}$ of $K$, such that the size of each element $\widetilde{K}$ of $\widetilde{\Omega}_{h}^{K}$ is comparable to $\widetilde{h}_{K}:=h_{K} / \ell^{2}$. Denote the union of all $\widetilde{\Omega}_{h}^{K}$ by $\widetilde{\Omega}_{h}$. By using a standard refinement strategy on the original mesh, we can additionally ensure that $\widetilde{\Omega}_{h}$ does not contain hanging nodes. We also introduce

$$
\begin{equation*}
\widetilde{V}_{h}:=\left\{v \in S^{1,0}\left(\Omega, \widetilde{\Omega}_{h}\right) \mid v_{\left.\right|_{K}} \in S^{1,1}\left(K, \widetilde{\Omega}_{h}^{K}\right) \quad \forall K \in \Omega_{h}\right\} \tag{B.1}
\end{equation*}
$$

the space of the mapped, piecewise linear polynomials over $\widetilde{\Omega}_{h}$, which are continuous in each $K \in$ $\Omega_{h}$ but possibly discontinuous at the interfaces of $\Omega_{h}$.

We define $\mathcal{P}_{1}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow \widetilde{V}_{h}$ as follows. For each $K \in \Omega_{h}, \mathcal{P}_{1}\left(v_{h \mid K}\right) \in \mathcal{S}^{1,1}\left(K, \widetilde{\Omega}_{h}^{K}\right)$ is the quasi-interpolant of $v$ defined in [6] Sec. 4]. As for $\mathcal{P}_{2}: \widetilde{V}_{h} \rightarrow \mathcal{S}^{1,1}\left(\Omega, \widetilde{\Omega}_{h}\right) \subset H^{1}(\Omega)$, we choose the lowest-order, Oswald-type operator introduced by Karakashian and Pascal in [34]. This operator interpolates the arithmetical averages of the degrees of freedom at each vertex of the mesh $\widetilde{\Omega}_{h}$. Thus, we are actually going to prove Theorem 4.4 with $\mathcal{P}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow \mathcal{S}^{1,1}\left(\Omega, \widetilde{\Omega}_{h}\right) \subset H^{1}(\Omega)$. For simplicity, throughout this section we assume that $h / \ell^{2} \lesssim 1$ and $\ell \in \mathbb{N}$. The other cases follow similarly but would incur some cumbersome notation/case distinctions.

Before proving (4.13)-(4.15), we recall two propositions, which summarize the properties of the operators $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$.
Proposition B.1. For any element $K \in \Omega_{h}$, the quasi-interpolant $\mathcal{P}_{1}: H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right) \rightarrow \widetilde{V}_{h}$ satisfies the following estimates:

$$
\begin{align*}
&\left\|\nabla \mathcal{P}_{1} v\right\|_{0, K}  \tag{B.2}\\
&\left\|\|\nabla v\|_{0, K},\right.  \tag{B.3}\\
&\left\|v-\mathcal{P}_{1} v\right\|_{0, K} \lesssim\left\|\mathfrak{h} \ell^{-2} \nabla v\right\|_{0, K},  \tag{B.4}\\
& \|\left[\mathcal{P}_{1} v \rrbracket \|_{0, \partial K \backslash \Gamma}\right. \lesssim\|v\|\left\|_{0, \partial K \backslash \Gamma}+\right\| \mathfrak{h}^{1 / 2} \ell^{-1} \nabla_{h} v \|_{0, \omega_{K}},
\end{align*}
$$

where $\omega_{K}$ in (B.4) denotes the set of elements sharing a face with $K$.
Proof. Bounds (B.2) and (B.3) follow from [6, Thm. 4.1] locally on $K$ as the domain to obtain a function on the subtriangulation $\widetilde{\Omega}_{h}^{K}$. We can apply [6, Thm. 4.1] since $\widetilde{\Omega}_{h}^{K}$ fulfills (3.1) and thus (3.2), which is the condition required there.

To show ( (B.4), we fix a facet $F$ shared by the elements $K$ and $K^{\prime}$. We get

$$
\begin{aligned}
\left\|\llbracket \mathcal{P}_{1} v \rrbracket\right\|_{0, F} & \leq\|\llbracket v \rrbracket\|_{0, F}+\left\|\llbracket v-\mathcal{P}_{1} v \rrbracket\right\|_{0, F} \\
& \leq\|\llbracket v \rrbracket\|_{0, F}+\left\|\left(v-\mathcal{P}_{1} v\right)_{\mid K}\right\|_{0, F}+\left\|\left(v-\mathcal{P}_{1} v\right)_{\mid K^{\prime}}\right\|_{0, F} .
\end{aligned}
$$

For brevity, we only consider the third term on the right-hand side. Transforming to the reference element, applying a multiplicative trace estimate and transforming back gives

$$
\left\|\left(v-\mathcal{P}_{1} v\right)_{\mid K^{\prime}}\right\|_{0, F} \lesssim\left\|\mathfrak{h}^{-1 / 2}\left(v-\mathcal{P}_{1} v\right)\right\|_{0, K^{\prime}}+\left\|v-\mathcal{P}_{1} v\right\|_{0, K^{\prime}}^{1 / 2}\left\|\nabla\left(v-\mathcal{P}_{1} v\right)\right\|_{0, K^{\prime}}^{1 / 2} .
$$

Inserting ( $\overline{\mathrm{B} .2)}$ ) and ( $\overline{\mathrm{B} .3}$ ) yields ( (B.4).
Proposition B.2. The Oswald-type operator $\mathcal{P}_{2}: \widetilde{V}_{h} \rightarrow \mathcal{S}^{1,1}\left(\Omega, \widetilde{\Omega}_{h}\right)$ satisfies the following properties:

$$
\begin{equation*}
\left\|\widetilde{v}_{h}-\mathcal{P}_{2} \widetilde{v}_{h}\right\|_{0, \Omega} \lesssim\left\|\mathfrak{h}^{1 / 2} \ell^{-1} \llbracket \widetilde{v}_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}, \quad\left\|\nabla_{h}\left(\widetilde{v}_{h}-\mathcal{P}_{2} \widetilde{v}_{h}\right)\right\|_{0, \Omega} \lesssim\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket \widetilde{v}_{h} \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}} . \tag{B.5}
\end{equation*}
$$

Proof. We claim that

$$
\left\|\widetilde{v}_{h}-\mathcal{P}_{2} \widetilde{v}_{h}\right\|_{0, \Omega}^{2} \lesssim \sum_{K \in \Omega_{h}}\left\|\widetilde{h}_{K}^{1 / 2} \llbracket \widetilde{v}_{h} \rrbracket\right\|_{0, \partial K \backslash \Gamma}^{2}, \quad\left\|\nabla_{h}\left(\widetilde{v}_{h}-\mathcal{P}_{2} \widetilde{v}_{h}\right)\right\|_{0, \Omega}^{2} \lesssim \sum_{K \in \Omega_{h}}\left\|\widetilde{h}_{K}^{-1 / 2} \llbracket \widetilde{v}_{h} \rrbracket\right\|_{0, \partial K \backslash \Gamma}^{2}
$$

This follows as in the proof of [34, Thm. 2.2], which only makes use of the definition of the Lagrangian degrees of freedom of $\mathcal{P}_{2} \widetilde{v}_{h}$ as arithmetical averages of the degrees of freedom of $\widetilde{v}_{h}$ and of the scaling properties of the basis functions. We remark that [34, Thm. 2.2] states the estimate in the $H^{1}$ seminorm; the estimate in the $L^{2}$ norm follows along the same lines; see also [7, Lemma 5.3]. Then, the estimates in (B.5) follow from the definition of $\widetilde{h}_{K}=h_{K} / \ell^{2}$ and the fact that function $\widetilde{v}_{h}$ is continuous within each element $K \in \Omega_{h}$, i.e., no extra jumps are introduced along the edges of the refined triangulation $\widetilde{\Omega}_{h}$.

As an immediate consequence of the shape regularity of $\Omega_{h}$ and the locality of the operator $\mathcal{P}_{1}$, we get

$$
\begin{equation*}
\left\|\mathfrak{h}^{-1} \ell^{2}\left(\mathcal{P}_{1} v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, \Omega} \stackrel{\sqrt{\text { B. } 5)}}{\lesssim}\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket \mathcal{P}_{1} v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}} \stackrel{\sqrt{\text { B.4 }}}{\lesssim}\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}}+\left\|\nabla_{h} v\right\|_{0, \Omega} . \tag{B.6}
\end{equation*}
$$

We prove further properties of the operator $\mathcal{P}_{2}$. First, proceeding as in Remark 4.1, we have the following inverse estimate for mapped, affine functions:

$$
\begin{equation*}
\|\nabla q\|_{0, \widetilde{K}} \lesssim \widetilde{h}_{K}^{-1}\|q\|_{0, \widetilde{K}}=\left\|\mathfrak{h}^{-1} \ell^{2} q\right\|_{0, \widetilde{K}} \quad \forall \widetilde{K} \in \widetilde{\Omega}_{h}^{K}, \forall q \in \mathcal{S}^{1,1}\left(K, \widetilde{\Omega}_{h}^{K}\right) \tag{B.7}
\end{equation*}
$$

Next, we observe that

$$
\begin{align*}
& \left\|\nabla_{h} \mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right\|_{0, \Omega} \leq\left\|\nabla_{h}\left(\mathcal{P}_{1} v\right)\right\|_{0, \Omega}+\left\|\nabla_{h}\left(\mathcal{P}_{1} v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, \Omega} \\
& \stackrel{(\overline{B .2]}}{\lesssim}\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\nabla_{h}\left(\mathcal{P}_{1} v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, \Omega} \stackrel{\overline{\text { B. } 7 /}}{\lesssim}\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1} \ell^{2}\left(\mathcal{P}_{1} v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, \Omega}  \tag{B.8}\\
& \stackrel{\left(\frac{B .6]}{}\right.}{\lesssim}\left\|\nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{-1 / 2} \ell \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}} .
\end{align*}
$$

From this and the triangle inequality, we get (4.13).
In order to prove (4.14), we observe that the following approximation property of the operator $\mathcal{P}_{2}$ is valid:

$$
\begin{align*}
&\left\|v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right\|_{0, \Omega} \leq\left\|v-\mathcal{P}_{1} v\right\|_{0, \Omega}+\left\|\mathcal{P}_{1} v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right\|_{0, \Omega} \\
& \stackrel{\text { B.3], (B.6) }}{\lesssim}\left\|\mathfrak{h} \ell^{-2} \nabla_{h} v\right\|_{0, \Omega}+\left\|\mathfrak{h}^{1 / 2} \ell^{-1} \llbracket v \rrbracket\right\|_{0, \mathcal{F}_{h}^{I}} . \tag{B.9}
\end{align*}
$$

Then, (4.14) follows by the triangle inequality.
We are left to prove (4.15). To that end, we use a scaling argument. Given $v \in H_{\mathrm{pw}}^{1}\left(\Omega_{h}\right)$, for any $K \in \Omega_{h}$, let $\widehat{v}$ be the polynomial pull-back of $v_{\left.\right|_{K}}$ through the mapping $\Phi_{K}: \widehat{K} \rightarrow K$. We denote the counterparts of $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$ acting on the polynomials on $\widehat{K}$ by $\widehat{\mathcal{P}}_{1}$ and $\widehat{\mathcal{P}}_{2}$, respectively. For any boundary face $F \in \mathcal{F}_{h}^{B}$, we denote the pull-back of $F$ through $\Phi_{K}$ by $\widehat{F}$, where $K$ is the only element such that $F \subset \partial K$. For all $F \in \mathcal{F}_{h}^{B}$, we apply a scaling argument, the multiplicative trace inequality, and the Young inequality to get

$$
\begin{aligned}
\left\|v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right\|_{0, F}^{2} & \lesssim\left\|\mathfrak{h}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{F}}^{2} \\
& \lesssim\left\|\mathfrak{h}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}}^{2}+\left\|\mathfrak{h}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}}\left\|\mathfrak{h} \widehat{\nabla}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}} \\
& \ell \geq 1 \\
& \lesssim\left\|\mathfrak{h} \ell\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}}^{2}+\left\|\mathfrak{h} \ell^{-1} \widehat{\nabla}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}(\widehat{\mathcal{P}} \widehat{v})\right)\right\|_{0, \widehat{K}}^{2} .
\end{aligned}
$$

Scaling back to $K$, summing over all the elements, and using the locality of the operators $\mathcal{P}_{1}$ and $\mathcal{P}_{2}$, as well the shape regularity of the meshes to insert the factor $\mathfrak{h}^{-1 / 2} \ell$, we deduce

$$
\begin{aligned}
& \left\|\mathfrak{h}^{-1 / 2} \ell\left(v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, \Gamma}^{2} \\
& \quad \lesssim \sum_{K \in \Omega_{h} \text { with } \bar{K} \cap \partial \Omega \in \mathcal{F}_{h}^{B}}\left\|\mathfrak{h}^{1 / 2} \ell^{2}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}}^{2}+\left\|\mathfrak{h}^{1 / 2} \widehat{\nabla}\left(\widehat{v}-\widehat{\mathcal{P}}_{2}\left(\widehat{\mathcal{P}}_{1} \widehat{v}\right)\right)\right\|_{0, \widehat{K}}^{2} \\
& \quad \lesssim \sum_{K \in \Omega_{h} \text { with } \bar{K} \cap \partial \Omega \in \mathcal{F}_{h}^{B}}\left\|\mathfrak{h}^{-1} \ell^{2}\left(v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, K}^{2}+\left\|\nabla\left(v-\mathcal{P}_{2}\left(\mathcal{P}_{1} v\right)\right)\right\|_{0, K}^{2} \\
& \quad \begin{array}{l}
\text { B.9), (B.8] }
\end{array}\left\|\nabla_{h} v\right\|_{0, \Omega}^{2}+\| \mathfrak{h}^{-1 / 2} \ell\left[v \rrbracket \|_{0, \mathcal{F}_{h}^{I}}^{2},\right.
\end{aligned}
$$

whence the assertion follows.

## C Explicit error estimates

Proof of Corollary 7.5. We start by noting that, for the special case $s=1$, the arguments below show that Assumption 7.1 is valid with $\varepsilon=O(h / p)$. By Theorem 7.2, this fixes $\eta_{0}$.

To simplify the exposition, we restrict our attention to the case $p \geq s$. The case $p<s$ is a pure $h$-version that is shown along similar lines. We shall nevertheless write $\min (p, s)=s$ at the appropriate places.

By [6, Lemma 2.3], for any $v \in H^{s+1}(\Omega)$, Assumption 7.3 implies that the following estimate for the pull-back $\widehat{v}:=\left.v\right|_{K} \circ \Phi_{K}$ is valid for all $K \in \Omega_{h}$ :

$$
\begin{equation*}
\|\widehat{v}\|_{s+1, \widehat{K}} \leq c h_{K}^{s+1-3 / 2}\|v\|_{s+1, K} . \tag{C.1}
\end{equation*}
$$

We also note that, for $j \in\{0,1\}$ and for each face $F$ of element $K$ with corresponding pull-back $\widehat{F}:=\Phi_{K}^{-1}(F)$, bounds (3.1) imply

$$
\begin{equation*}
|\widehat{v}|_{j, \widehat{K}} \sim h_{K}^{j-3 / 2}|v|_{j, K}, \quad|\widehat{v}|_{0, \widehat{F}} \sim h_{K}^{-1}|v|_{0, F}, \quad|\widehat{\nabla} \widehat{v}|_{0, \widehat{F}} \sim|\nabla v|_{0, F} \tag{C.2}
\end{equation*}
$$

Properties (C.2) allow for transferring approximation results on the reference element $\widehat{K}$ to the physical elements $K$ ("scaling argument"). The last preliminary ingredient are $p$-explicit approximation results on the reference element for which we refer, e.g., to [43, Lemma B.3, Thm. B.4]. As in, e.g., 41, combining the polynomial approximation results on $\widehat{K}$ with (C.2) and (C.1) allows for showing that

$$
\begin{equation*}
\inf _{v_{h} \in S^{p, 0}\left(\Omega, \Omega_{h}\right)}\left\|u-v_{h}\right\|_{\mathrm{DG}^{+}(\Omega)} \leq c h^{\min (p, s)} p^{-s+\frac{1}{2}}\|u\|_{s+1, \Omega} . \tag{C.3}
\end{equation*}
$$

For the approximation of $u^{e x t}$ and $m$, we obviate the discussion of changes of variables in fractional Sobolev norms by resorting to appropriate liftings. For the approximation of $u^{e x t}$, let $U^{\text {ext }} \in$ $H^{s+1}(\Omega)$ be a lifting of $u^{e x t}$ with $\left\|U^{e x t}\right\|_{s+1, \Omega} \lesssim\left\|u^{e x t}\right\|_{s+\frac{1}{2}, \Gamma}$. Since the mesh $\Omega_{h}$ is a regular mesh (see the discussion at the outset of Section 3.1), 43, Thm. B.4] provides an $H^{1}(\Omega)$-conforming approximation with optimal convergence properties:

$$
\inf _{v_{h} \in S^{p, 1}\left(\Omega, \Omega_{h}\right)}\left\|U^{e x t}-v_{h}\right\|_{1, \Omega} \leq c h^{\min (p, s)} p^{-s}\left\|U^{e x t}\right\|_{s+1, \Omega} \leq c h^{\min (p, s)} p^{-s}\left\|u^{e x t}\right\|_{s+\frac{1}{2}, \Gamma} .
$$

By taking the trace of $v_{h}$ on $\Gamma$, we obtain the desired approximation of $u^{e x t}$. Finally, for $m$, let $M \in H^{s}(\Omega)$ be a lifting of $m \in H^{s-\frac{1}{2}}(\Gamma)$ with $\|M\|_{H^{s}(\Omega)} \lesssim\|m\|_{H^{s-\frac{1}{2}}(\Gamma)}$. Let $m_{h} \in S^{p-1,0}\left(\Gamma, \Gamma_{h}\right)$ be the $L^{2}(\Gamma)$-projection of $m$ into $S^{p-1,0}\left(\Gamma, \Gamma_{h}\right)$. For each face $F \in \mathcal{F}_{h}^{B}$, denote by $K_{F} \in \Omega_{h}$ the element that has $F$ as a face. Using approximation results on the reference element $\widehat{K}$ and the "scaling arguments" (C.2) we get

$$
\begin{equation*}
\left\|m-m_{h}\right\|_{0, F} \leq c h_{K}^{\min (p, s)-1 / 2} p^{-s+1 / 2}\|M\|_{s, K_{F}} . \tag{C.4}
\end{equation*}
$$

By summation over all faces $F \in \mathcal{F}_{h}^{B}$, we arrive at

$$
\left\|\mathfrak{h}^{1 / 2} p^{-1}\left(m-m_{h}\right)\right\|_{0, \Gamma} \lesssim h^{\min (p, s)} p^{-s-1 / 2}\left\|\left(m-m_{h}\right)\right\|_{s-\frac{1}{2}, \Gamma} .
$$

The $H^{-\frac{1}{2}}(\Gamma)$-estimate is obtained by a standard duality argument using the orthogonality provided by the $L^{2}(\Gamma)$-projection:

$$
\begin{equation*}
\left\|m-m_{h}\right\|_{-\frac{1}{2}, \Gamma}=\sup _{v \in H^{\frac{1}{2}}(\Gamma)} \frac{\left|\left\langle m-m_{h}, v\right\rangle\right|}{\|v\|_{\frac{1}{2}, \Gamma}}=\sup _{v \in H^{\frac{1}{2}}(\Gamma)} \inf _{v_{h} \in S^{p-1,0}\left(\Gamma, \Gamma_{h}\right)} \frac{\left|\left\langle m-m_{h}, v-v_{h}\right\rangle\right|}{\|v\|_{\frac{1}{2}, \Gamma}} . \tag{C.5}
\end{equation*}
$$

The infimum is estimated by taking $v_{h}$ as the $L^{2}(\Gamma)$-projection of $v$ into $S^{p-1,0}\left(\Gamma, \Gamma_{h}\right)$. To estimate $v-v_{h}$, let $V \in H^{1}(\Omega)$ be a lifting of $v \in H^{\frac{1}{2}}(\Gamma)$ with $\|V\|_{1, \Omega} \lesssim\|v\|_{\frac{1}{2}, \Gamma}$. By the same arguments as in (C.4) (taking $s=1$ ), we have

$$
\left\|v-v_{h}\right\|_{0, F} \leq c h_{K}^{\min (p, 1)-1 / 2} p^{-1+1 / 2}\|V\|_{1, K_{F}} .
$$

Inserting this in C.5 yields

$$
\begin{aligned}
\left\|m-m_{h}\right\|_{-\frac{1}{2}, \Gamma} & \lesssim \sup _{v \in H^{\frac{1}{2}}(\Gamma)} \frac{\sum_{F \in \mathcal{F}_{h}^{B}}\left\|m-m_{h}\right\|_{0, F}\left\|v-v_{h}\right\|_{0, F}}{\|v\|_{\frac{1}{2}, \Gamma}} \\
& \lesssim \sup _{v \in H^{\frac{1}{2}}(\Gamma)} \frac{1}{\|v\|_{\frac{1}{2}, \Gamma}} \sum_{F \in \mathcal{F}_{h}^{B}} p^{-s} h_{K}^{\min (p, s)-1 / 2+1-1 / 2}\|M\|_{s, K_{F}}\|V\|_{1, K_{F}} \\
& \lesssim h^{\min (p, s)} p^{-s}\|m\|_{s-\frac{1}{2}, \Gamma},
\end{aligned}
$$

which completes the proof.


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