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Abstract

This paper discusses a novel three field formulation for the Darcy-Forchheimer flow with a nonlinear viscosity depending on the temperature coupled with the heat equation. We show unique solvability of the variational problem by using; Galerkin method, Brouwer's fixed point and some compactness properties. We propose and study in detail a finite element approximation. A priori error estimate is then derived and convergence is obtained. A solution technique is formulated to solve the non-linear problem and numerical experiments that validate the theoretical findings are presented.

Keywords: Darcy-Forchheimer equations, heat equation, finite element.

AMS subject classification: 65N30, 76M10, 35J85

Contents

1 Introduction

In this work we study numerically the distribution of temperature in a fluid modeled by the incompressible Darcy-Forchheimer equations:

$$\nu(\theta)\boldsymbol{u} + \beta|\boldsymbol{u}|\boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \Omega\ , \tag{1.2}$$

$$-\kappa\Delta\theta + (\boldsymbol{u}\cdot\nabla)\theta = g \quad \text{in } \Omega, \tag{1.3}$$

with $|\cdot|$ is the Euclidean vector norm $|\boldsymbol{u}|^2 = \boldsymbol{u}\cdot\boldsymbol{u}$ and Ω is a bounded open set in $\mathbb{R}^d$ (d=2,3) with a Lipschitz-continuous boundary $\partial\Omega$ with an outer normal $\boldsymbol{n}$ of length one. $g : \Omega \longrightarrow \mathbb{R}$ is the external heat source, while $\boldsymbol{f} : \Omega \longrightarrow \mathbb{R}^d$ is the external body force per unit volume acting on the fluid. In (1.1),...,(1.3), $\boldsymbol{u}$ is the velocity and θ the temperature, while p is the pressure. The thermal conductivity κ is positive and ν is the viscosity and depend on the temperature. β represent the Forchheimer number of the porous media. Note that when $\beta = 0$, the first equation is reduced to Darcy's equation. The reader interested in the derivation of the model (1.1),...,(1.3) can consult [1, 2]. The coupling in (1.1),...,(1.3) are represented through the convective term $(\boldsymbol{u}\cdot\nabla)\theta$ and the expression $\nu(\theta)\boldsymbol{u}$. The system of equations is supplemented by the boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega \qquad \text{and} \qquad \theta = \theta_0 \quad \text{on } \partial\Omega\ , \tag{1.4}$$

where $\theta_0 : \partial\Omega \longrightarrow \mathbb{R}^d$ is the given temperature distribution on the boundary $\partial\Omega$. It should be noted that the first term in the left hand side of equation (1.1) is sometimes replaced (considering the coupling) by $\nu(\theta)K^{-1}\boldsymbol{u}$ where K is the permeability tensor assumed to be uniformly positive definite and bounded. Thus from the mathematical viewpoint, this omission will not changed the results we obtain in this work. The mathematical analysis of Darcy-Forchheimer's equation (1.1) and (1.2) has been considered in [3], while mixed finite element methods are examined in [4, 5, 6]. A compressible Darcy-Forchheimer's model is thoroughly analysed numerically in [7] making use of Crouzeix-Raviart's element, while its implementation is discussed in [8]. Time dependent problem is analysed in [9]. The literature on numerical analysis dealing with heat convection in a liquid medium whose motion is described by the Stokes/Navier Stokes is rich and among others we mentioned the papers [10, 11, 12, 13, 14]. In [15, 16], numerical analysis of Darcy's model coupled with heat equation is examined with specific interest of presenting; the existence theory, convergence of the numerical scheme, a priori error estimates, and numerical simulations.

Hence it is a natural to consider the numerical analysis of the heat equation with Darcy-Forchheimer equation in a porous media. The analysis we present here borrow from [16], but differ from it because the presence of the nonlinear term $\beta\,|\boldsymbol{u}|\,\boldsymbol{u}$ introduces more computations. The aim of this work is to present a numerical investigation of (1.1),...,(1.4) using the mixed method in which the boundary condition on the velocity is treated as a natural one. We start this investigation by presenting two equivalent weak formulations associated to (1.1),...,(1.4), follow by a discussion of the existence theory by making use of Galerkin's approximation, Brouwer's fixed point, a priori estimates and passage to the limit. Next, unique solvability is derived when the data are suitably restricted. The continuous formulation is approximated using conforming finite element scheme where the compatibility condition between the velocity and pressure is observed. We study the existence of the finite element solution, and derive uniqueness of solution with a more tighten condition than the one obtained in the continuous analysis. Convergence and a priori error estimates for the finite element solution are derived by making use of Babuska-Brezzi's conditions for mixed problems. It is interesting to note at this juncture that in [7], the authors are interested in L^2 a priori error estimate for the velocity with $W^{1,4}(\Omega)$ regularity while in our work we derived L^3 a priori error estimate for the velocity with $W^{1,3}(\Omega)$ regularity. The third contribution of this work is the implementation of the finite element solution and the resulting numerical simulations. The strategy we adopt to compute the finite solution find its roots in the works R. Glowinski (see particularly [26]). Indeed, we proceed in two steps as follows:
(i) the finite element problem we have can be seen as a limit of an evolution problem,
(ii) we discretize in time the evolution problem with Marchuk-Yanenko's method.
The above algorithm has been tested in many problems and the results one obtains validate the theoretical findings of this work. The rest of the work is organised as follows

- Section 2 is concerned with the weak formulations, the construction of weak solution.
- Section 3 is devoted to the finite element approximation, and its a priori analysis.
- Section 4 is devoted to the formulation of the iterative scheme, numerical experiments, and conclusions.

2 Analysis of the continuous problem

2.1 Preliminaries

To write the system (1.1),...,(1.4) in a variational form, we need some preliminaries. We shall use the standard notations (see [23]). Thus for $\alpha = (\alpha_1, \cdots, \alpha_d)$ denoting a set of non-negative integers. Let $|\alpha| = \alpha_1 + \cdots + \alpha_d$ and define the partial derivative ∂^k by

$$\partial^k v = \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}\,.$$

Then for any non-negative integer m and number $r \geq 1$, recall the Sobolev space

$$W^{m,r}(\Omega) = \{v \in L^r(\Omega);\ \partial^\alpha v \in L^r(\Omega),\ \forall\ |\alpha| \leq m\}\ ,$$

equipped with the seminorm

$$|v|_{W^{m,r}(\Omega)} = \left[\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha v|^r dx \right]^{1/r} ,$$

and the norm (for which it is a Banach space)

$$\|v\|_{W^{m,r}(\Omega)} = \left[\sum_{0\le k\le m} |v|^r_{W^{k,r}(\Omega)} \right]$$

with the usual extension when $r = \infty$. When $r = 2$, this space is the Hilbert space $H^m(\Omega)$. The definitions of these spaces are extended in the usual way to vectors and tensors, with the same notation, but with the following modification for the norms in the non-Hilbert case. For a vector or a tensor $\boldsymbol{u}$, we set

$$\|\boldsymbol{u}\|_{L^p(\Omega)} = \left[\int_\Omega |\boldsymbol{u}(\boldsymbol{x})|^p dx \right]^{1/p} .$$

We recall that for vanishing boundary conditions, we set

$$H^1_0(\Omega) = \left\{ v \in H^1(\Omega) : \ v|_{\partial\Omega} = 0 \right\} .$$

The following Sobolev imbedding will be used: for any real number $p \ge 1$ when $d = 2$, or $1 \le p \le \frac{2d}{d-2}$ when $d \ge 3$, there exists constants c_p and c^0_p such that

$$\text{for all } v \in H^1(\Omega) \ , \ \|v\|_{L^p(\Omega)} \le c_p \|v\|_{H^1(\Omega)} \, , \tag{2.1}$$

and

$$\text{for all } v \in H^1_0(\Omega) \ , \ \ \|v\|_{L^p(\Omega)} \le c^0_p \|\nabla v\| \, , \tag{2.2}$$

noting that (2.2) is Poincaré's inequality when $p = 2$. Now, in order to describe the spaces where the unknowns lies, we observe that Darcy's equations has two variational formulations (see [13, 17]), and the spaces involved in those formulations differ. We present in this text the two possibilities. Firstly, for the formulation which enables to treat the boundary condition on $\boldsymbol{u}$ as natural one, we observe by multiplying the equation (1.1) by $\boldsymbol{u}$ and then integrate the resulting equation that $\boldsymbol{u}$ should be an element of $L^3(\Omega)^d$. Hence the velocity is studied in the space $L^3(\Omega)^d$. With the velocity space in mind, the gradient of the pressure should be in $L^{3/2}(\Omega)^d$. Therefore the space of pressure is

$$M = \left\{ q(\boldsymbol{x}) \in W^{1,3/2}(\Omega); \quad \int_\Omega q(\boldsymbol{x}) dx = 0 \right\} .$$

It is worth mentioning that the zero mean value is added on the pressure to avoid the pressure given in (1.1), (1.2) to be determined up to a constant. We recall that there exists c such that

$$\text{for all } q \in M, \ \ c \int_\Omega |q(\boldsymbol{x})|^{3/2} \, dx \le \int_\Omega |\nabla q(\boldsymbol{x})|^{3/2} \, dx \, .$$

Thus on M, $\|\nabla\cdot\|_{L^{3/2}(\Omega)}$ is a norm equivalent to $W^{1,3/2}(\Omega)$ norm. Finally, the temperature is an element of $H^1(\Omega)$. For the second formulation, we define the spaces

$$H(\mathrm{div},\Omega) = \left\{ \boldsymbol{v} \in L^3(\Omega)^d : \ \mathrm{div}\, \boldsymbol{v} \in L^3(\Omega) \right\} ,$$
$$H_0(\mathrm{div},\Omega) = \{ \boldsymbol{v} \in H(\mathrm{div},\Omega) : \ \boldsymbol{v}\cdot\boldsymbol{n}|_\Gamma = 0 \} ,$$

equipped with the norm

$$\|\boldsymbol{v}\|^2_{H(\mathrm{div},\Omega)} = \|\boldsymbol{v}\|^2_{L^3(\Omega)^d} + \|\,\mathrm{div}\,\boldsymbol{v}\|^2_{L^3(\Omega)} .$$

The pressure in the second formulation is then defined in the space

$$L_0^{3/2}(\Omega) = \left\{ q \in L^{3/2}(\Omega) : \ \int_\Omega q(x)dx = 0 \right\},$$

again here the temperature is an element of $H^1(\Omega)$.

For the mathematical analysis of (1.1),...,(1.4), some structural conditions are needed on the function ν. We assume that $\nu(\cdot)$ is a bounded continuous function defined on $\mathbb{R}^+$ satisfying for some ν_0, ν_1, ν_2 in $\mathbb{R}^+$,

$$\nu \in \mathcal{C}^1(\mathbb{R}^+) \ \text{ and for any } s \in \mathbb{R}^+, \ \ 0 < \nu_0 \le \nu(s) \le \nu_1 \text{ and } \ |\nu'(s)| \le \nu_2 . \qquad (2.3)$$

It is important to note that if the function $\nu(\theta)$ is unbounded, the analysis of the problem will be very hard. But on the other hand since $\nu(\theta)$ is neither infinite nor zero, the conditions (2.3) are reasonable and simplifies the analysis. In the following, we assume that

$$g \in L^2(\Omega) \ \ , \quad \theta_0 \in H^{1/2}(\partial\Omega) \quad \text{and} \quad \boldsymbol{f} \in L^2(\Omega)^d . \qquad (2.4)$$

2.2 Variational formulations

We introduce the following functionals that will be used to write down the weak form of the problem in abstract setting.

$$\begin{array}{rccl}
a: & L^3(\Omega)^d \times L^3(\Omega)^d & \longrightarrow & \mathbb{R} \\
 & (\boldsymbol{u},\boldsymbol{v}) & \longrightarrow & a(\theta:\boldsymbol{u},\boldsymbol{v}) = \displaystyle\int_\Omega \nu(\theta)\boldsymbol{u}\cdot\boldsymbol{v}dx , \\
c: & H^1(\Omega)\times H^1(\Omega) & \longrightarrow & \mathbb{R} \\
 & (\theta,\rho) & \longrightarrow & c(\theta,\rho) = \kappa \displaystyle\int_\Omega \nabla\theta\cdot\nabla\rho dx , \\
b_1: & L^3(\Omega)^d \times M & \longrightarrow & \mathbb{R} \\
 & (\boldsymbol{v},q) & \longrightarrow & b_1(\boldsymbol{v},q) = \displaystyle\int_\Omega \nabla q\cdot\boldsymbol{v}dx \ , \\
d: & L^3(\Omega)^d \times H^1(\Omega)\times H^1(\Omega) & \longrightarrow & \mathbb{R} \\
 & (\boldsymbol{v},\theta,\rho) & \longrightarrow & d(\boldsymbol{v},\theta,\rho) = \displaystyle\int_\Omega (\boldsymbol{v}\cdot\nabla)\theta\rho dx .
\end{array}$$

We consider the variational problem:

$$\begin{cases} \text{Find} \quad (\boldsymbol{u}, p, \theta) \in L^3(\Omega)^d \times M \times H^1(\Omega), \quad \text{such that} \\ \theta = \theta_0 \quad \text{on} \quad \partial\Omega, \\ \text{and for all} \quad (\boldsymbol{v}, q, \rho) \in L^3(\Omega)^d \times M \times H_0^1(\Omega), \\ a(\theta; \boldsymbol{u}, \boldsymbol{v}) + \beta \displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u} \cdot \boldsymbol{v} dx + b_1(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx\,, \\ b_1(\boldsymbol{u}, q) = 0\,, \\ c(\theta, \rho) + d(\boldsymbol{u}, \theta, \rho) = \displaystyle\int_\Omega g\rho\, dx\,. \end{cases} \tag{2.5}$$

We note from [18] that

$$\text{for all } (\boldsymbol{v}, q) \in N \times M, \quad \int_\Omega \boldsymbol{v} \cdot \nabla q dx + \int_\Omega q \operatorname{div} \boldsymbol{v} dx = \langle q, \boldsymbol{v} \cdot \boldsymbol{n} \rangle_{\partial\Omega}\,,$$

with

$$N = \Big\{ \boldsymbol{v} \in L^3(\Omega)^d : \ \operatorname{div} \boldsymbol{v} \in L^{3d/(d+3)}(\Omega) \Big\}\ .$$

We then deduce that $\boldsymbol{v} \cdot \boldsymbol{n}$ belongs to the dual space of $W^{1/3,3/2}(\partial\Omega)$. We claim that

Lemma 2.1 *Any triplet $(\boldsymbol{u}, p, \theta)$ in $L^3(\Omega)^d \times M \times H^1(\Omega)$ that solves (1.1),...,(1.3) in the sense of distributions in Ω, the first equation in (1.4) in the sense of traces in the dual space of $W^{1/3,3/2}(\partial\Omega)$, and the second equation in (1.4) in the sense of traces in $H^{1/2}(\partial\Omega)$, is a solution of (2.5). Conversely, any solution $(\boldsymbol{u}, p, \theta)$ of (2.5) solves (1.1),...,(1.4) in the above sense.*

One of the crucial point in the analysis of (2.5) is the inf-sup condition

$$\forall q \in M, \qquad \sup_{0 \neq \boldsymbol{v} \in L^3(\Omega)^d} \frac{b_1(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{L^3(\Omega)}} \geq \|\nabla q\|_{L^{3/2}(\Omega)} \tag{2.6}$$

obtained by the following dual representation of the norm

$$\|\nabla q\|_{L^{3/2}(\Omega)} = \sup_{0 \neq \boldsymbol{v} \in L^3(\Omega)^d} \frac{\displaystyle\int_\Omega \nabla q \cdot \boldsymbol{v} dx}{\|\boldsymbol{v}\|_{L^3(\Omega)^d}}\,.$$

The kernel of $b_1(\cdot, \cdot)$ is defined as follows

$$K(\Omega) = \Big\{ \boldsymbol{v} \in L^3(\Omega)^d, \quad \forall q \in M, \quad b_1(\boldsymbol{v}, q) = 0 \Big\}\,,$$

which is

$$K(\Omega) = \Big\{ \boldsymbol{v} \in L^3(\Omega)^d, \quad \operatorname{div} \boldsymbol{v}|_\Omega = 0 \ \text{ and } \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial\Omega} = 0 \Big\}\ .$$

With the space $K(\Omega)$, we define the following problem

$$
\begin{cases}
\text{Find } \ (\boldsymbol{u},\theta) \in K(\Omega)\times H^1(\Omega), \ \text{ such that} \\
\theta=\theta_0 \ \text{ on } \ \partial\Omega, \\
\text{and for all } \ (\boldsymbol{v},\rho)\in K(\Omega)\times H_0^1(\Omega), \\
a(\theta;\boldsymbol{u},\boldsymbol{v})+\beta\displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\,, \\
c(\theta,\rho)+d(\boldsymbol{u},\theta,\rho)=\displaystyle\int_\Omega g\rho\,dx\,.
\end{cases}
\tag{2.7}
$$

We claim that

Proposition 2.1 *the variational problem (2.5) is equivalent to the variational problem (2.7).*

Proof. Let $(\boldsymbol{u},\theta,p)\in L^3(\Omega)^d\times H_0^1(\Omega)\times M$ be the solution of (2.5), then $\boldsymbol{u}$ is an element of $K(\Omega)$ and $(\boldsymbol{u},\theta)$ solves (2.7).
Conversely, let $(\boldsymbol{u},\theta)\in K(\Omega)\times H_0^1(\Omega)$ be the solution of (2.7). For $\boldsymbol{v}\in L^3(\Omega)^d$, we define

$$
L(\boldsymbol{v})=a(\theta,\boldsymbol{u},\boldsymbol{v})+\beta\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx-\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx\,.
$$

It is linear and continuous on $L^3(\Omega)^d$ and since $(\boldsymbol{u},\theta)$ is a solution of (2.5), it vanishes on $K(\Omega)$. Hence there exists a unique function p in $W^{1,3/2}(\Omega)$ such that

$$
\begin{aligned}
&\text{for all } \boldsymbol{v}\in L^3(\Omega)^d\ ,\ L(\boldsymbol{v})=b_1(\boldsymbol{v},p),\\
&\|\nabla p\|_{L^{3/2}(\Omega)}\le \sup_{0\neq\boldsymbol{v}\in L^3(\Omega)^d}\frac{L(\boldsymbol{v})}{\|\boldsymbol{v}\|_{L^3(\Omega)^d}}\,,
\end{aligned}
$$

which is the end of the proof. □

Having in mind proposition 2.1, we can restrict the analysis to the variational problem (2.7). We note that $c(\cdot,\cdot)$ is continuous and elliptic on $H^1(\Omega)$; this means that for (θ,ρ) element of $H^1(\Omega)\times H^1(\Omega)$

$$
c(\theta,\rho)\le\kappa\|\theta\|_{H^1(\Omega)}\|\rho\|_{H^1(\Omega)}\ ,\quad c(\rho,\rho)=\kappa\|\nabla\rho\|^2\ge\kappa c\|\rho\|^2_{H^1(\Omega)}\,. \tag{2.8}
$$

The trilinear form $d(\cdot,\cdot,\cdot)$ enjoys the following properties (see R. Temam [24]): for all $(\boldsymbol{v},\theta,\rho)\in H^1(\Omega)^d\times H_0^1(\Omega)\times H_0^1(\Omega)$ and $\boldsymbol{v}$ is such that $\operatorname{div}\boldsymbol{v}|_\Omega=0$, then

$$
\begin{aligned}
&d(\boldsymbol{v},\theta,\rho)=-d(\boldsymbol{v},\rho,\theta)\ ,\\
&d(\boldsymbol{v},\rho,\rho)=0\,.
\end{aligned}
\tag{2.9}
$$

The second variational formulation we present in this work reads;

$$\begin{cases} \text{Find } \ (\boldsymbol{u},p,\theta) \in H_0(\text{div},\Omega) \times L_0^{3/2}(\Omega) \times H^1(\Omega), \ \text{ such that} \\ \theta = \theta_0 \ \text{ on } \ \partial\Omega, \\ \text{and for all } \ (\boldsymbol{v},q,\rho) \in H_0(\text{div},\Omega) \times L_0^{3/2}(\Omega) \times H_0^1(\Omega), \\ a(\theta;\boldsymbol{u},\boldsymbol{v}) + \beta \displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx + b_2(\boldsymbol{v},p) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\,, \\ b_2(\boldsymbol{u},q) = 0\,, \\ c(\theta,\rho) + d(\boldsymbol{u},\theta,\rho) = \displaystyle\int_\Omega g\rho\,dx\,, \end{cases} \tag{2.10}$$

with

$$b_2(\boldsymbol{v},q) = -\int_\Omega q\,\text{div}\,\boldsymbol{v}dx\,.$$

One observes that any triplet $(\boldsymbol{u},p,\theta)$ in $H_0(\text{div},\Omega)\times L_0^{3/2}(\Omega)\times H^1(\Omega)$ that solves (1.1),...,(1.3) in the sense of distributions in Ω, and the first equation in (1.4) is understood in the sense of traces in the dual space of the trace space of $L^{3/2}(\Omega)$, and the second equation in (1.4) is understood in the sense of traces in $H^{1/2}(\partial\Omega)$, is a solution of (2.10). Conversely, any solution $(\boldsymbol{u},p,\theta)$ of (2.10) solves (1.1),...,(1.4) in the above sense.
We recall the inf-sup condition between $H_0(\text{div},\Omega)$ and $L_0^{3/2}(\Omega)$, there exists $\beta > 0$ such that

$$\text{for all } q \in L_0^{3/2}(\Omega), \qquad \sup_{0\neq\boldsymbol{v}\in H_0(\text{div},\Omega)} \frac{-\displaystyle\int_\Omega q\,\text{div}\,\boldsymbol{v}dx}{\|\boldsymbol{v}\|_{H(\text{div},\Omega)}} \geq \beta\|q\|_{L^{3/2}(\Omega)}\,, \tag{2.11}$$

obtained by solving the equation

$$\begin{cases} -\Delta\psi = |q|^{-1/2}\,q - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |q|^{-1/2}\,qdx \text{ in } \Omega\,, \\ \nabla\psi\cdot\boldsymbol{n} = 0 \text{ on } \Gamma = \partial\Omega\,, \end{cases}$$

and setting $\boldsymbol{v} = \nabla\psi$. Indeed $\widetilde{q} = |q|^{-1/2}\,q - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega |q|^{-1/2}\,qdx$ is an element of $L_0^3(\Omega) = \{q \in L^3(\Omega):\ (q,1) = 0\}$. Thus $\psi \in W^{2,3}(\Omega)$ and the usual regularity of elliptic equation implies that

$$\|\psi\|_{2,3} \leq c\|\widetilde{q}\|_{L^3(\Omega)} \leq c\|q\|^{1/2}_{L^{3/2}(\Omega)}\,.$$

Moreover,

$$\begin{aligned} &-(\text{div}\,\boldsymbol{v},q) = -(\Delta\psi,q) = (\widetilde{q},q) = \|q\|^{3/2}_{L^{3/2}(\Omega)}\,, \text{and} \\ &\|\boldsymbol{v}\|^2_{H(\text{div},\Omega)} = \|\boldsymbol{v}\|^2_{L^3(\Omega)^d} + \|\,\text{div}\,\boldsymbol{v}\|^2_{L^3(\Omega)} \leq c\|\Delta\psi\|^2_{L^3(\Omega)} \leq c\|q\|_{L^{3/2}(\Omega)}\,. \end{aligned}$$

We conclude this paragraph with the following result obtained by application of Green's formula

Lemma 2.2 *The variational formulations (2.5) and (2.10) are equivalent.*

With the equivalence between the variational problems (2.5) and (2.10), we focus next on the analysis of (2.5).

2.3 A priori estimates and Existence of solution

In what follows, c is a positive constant that may vary from one line to the next one. The aim of this paragraph is to construct the weak solution of (2.5). The solution is constructed by using Galerkin's method, Brouwer's fixed theorem and compactness results.
We first claim that

Proposition 2.2 *There exist positive constants c_1, c_2 such that if $(\boldsymbol{u}, \theta, p)$ is given by (2.5), then*

$$\frac{\nu_0}{2}\|\boldsymbol{u}\|^2_{L^2(\Omega)^d} + \beta\|\boldsymbol{u}\|^3_{L^3(\Omega)^d} \le \frac{1}{2\nu_0}\|\boldsymbol{f}\|^2_{L^2(\Omega)^d} \ ,$$
$$\|\theta\|_{H^1(\Omega)} \le c_1\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_1}{\kappa}\|g\| \ ,$$
$$\|\nabla p\|_{L^{3/2}(\Omega)^d} \le c_2\|\boldsymbol{f}\|_{L^2(\Omega)^d} + c_2\|\boldsymbol{u}\|_{L^3(\Omega)^d} \,.$$

Proof. We first recall the Young's inequality

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{q/p}}b^q \ \text{ with } \frac{1}{p}+\frac{1}{q} = 1 \ , \ a \text{ and } b \text{ are positive}\,. \tag{2.12}$$

We take in (1.2) $\boldsymbol{v} = \boldsymbol{u}$ and using (2.3) yields

$$\nu_0\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}\,dx + \beta\int_\Omega |\boldsymbol{u}|^3\,dx \le \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx \ . \tag{2.13}$$

Finally using (2.12) one deduces the estimate on the velocity.
Next, we state the following result [18] (see Chap 4, Lemma 2.3): For any $\delta > 0$, there exists a lifting $\widetilde{\theta}_0$ of θ_0 which satisfies

$$\|\widetilde{\theta}_0\|_{L^6(\Omega)} \le \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)} \quad \text{and} \quad \|\widetilde{\theta}_0\|_{H^1(\Omega)} \le c\|\theta_0\|_{H^{1/2}(\partial\Omega)} \ , \tag{2.14}$$

where c is a positive constant independent of δ. We set $\widetilde{\theta} = \theta - \widetilde{\theta}_0$, note that $\widetilde{\theta}|_{\partial\Omega} = 0$ and take $\rho = \widetilde{\theta}$ in (2.5). Noticing that $((\boldsymbol{u}\cdot\nabla)\widetilde{\theta}, \widetilde{\theta}) = 0$, we obtain

$$\begin{aligned} c(\widetilde{\theta},\widetilde{\theta}) &= -c(\widetilde{\theta}_0,\widetilde{\theta}) - ((\boldsymbol{u}\cdot\nabla)\widetilde{\theta}_0,\widetilde{\theta}) + \int_\Omega g\widetilde{\theta}\,dx \\ &= -c(\widetilde{\theta}_0,\widetilde{\theta}) + ((\boldsymbol{u}\cdot\nabla)\widetilde{\theta},\widetilde{\theta}_0) + \int_\Omega g\widetilde{\theta}\,dx\,. \end{aligned}$$

Using Holder's inequality on the right hand side together with Poincaré Friedrichs's inequality and (2.14), yields

$$\begin{aligned} \kappa\|\nabla\widetilde{\theta}\|^2 &\le \kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|\boldsymbol{u}\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|g\|\|\widetilde{\theta}\| \\ &\le \kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\widetilde{\theta}\| + \|\boldsymbol{u}\|_{L^3(\Omega)}\|\widetilde{\theta}_0\|_{L^6(\Omega)}\|\nabla\widetilde{\theta}\| + \|g\|\|\widetilde{\theta}\| \\ &\le c\kappa\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\widetilde{\theta}\| + c\delta\|\boldsymbol{u}\|_{L^3(\Omega)}\|\nabla\widetilde{\theta}\|\|\theta_0\|_{H^{1/2}(\partial\Omega)} + c\|g\|\|\nabla\widetilde{\theta}\|\,. \end{aligned}$$

For $\delta = 1/\|\boldsymbol{u}\|_{L^3(\Omega)}$, we obtain

$$\|\nabla\widetilde{\theta}\| \;\leq\; c\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c}{\kappa}\|g\|\,. \tag{2.15}$$

We deduce the estimate on θ using the decomposition $\theta = \widetilde{\theta} + \widetilde{\theta}_0$, the triangle's inequality, (2.14) and (2.15).
As for the pressure, one has

$$b_1(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx - \beta\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx - \int_\Omega \nu(\theta)\boldsymbol{u}\cdot\boldsymbol{v}dx\ ,$$

which with the inf-sup condition gives

$$\|\nabla p\|_{L^{3/2}(\Omega)^d} \leq \sup_{\boldsymbol{v}\in L^3(\Omega)^d}\frac{b_1(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_{L^3(\Omega)^d}} \leq c\|\boldsymbol{f}\|_{L^2(\Omega)^d} + \beta\|\boldsymbol{u}\|_{L^{3/2}(\Omega)^d} + \nu_1\|\boldsymbol{u}\|_{L^{3/2}(\Omega)^d}$$

Hence the proof is complete after utilization of Holder's inequality. □

As far as the existence and uniqueness are concerned, from the equivalence property between the variational problems (2.5) and (2.7) (see proposition 2.1), it is enough to establish the existence and uniqueness of a solution $(\boldsymbol{u}, \theta)$ of (2.7). The variational problem (2.7) is a mixed variational problem with monotone operator and we refer in general to [22] for a systematic manner of mathematical analysis of such problems. For the implementation of the method mentioned above, we recall that with the lifting $\widetilde{\theta}_0$ introduced earlier, (2.7) is rewritten as follows;

$$\begin{cases}\text{Find}\ \ (\boldsymbol{u},\theta)\in K(\Omega)\times H^1_0(\Omega),\ \ \text{such that}\\ \text{and for all}\ \ (\boldsymbol{v},\rho)\in K(\Omega)\times H^1_0(\Omega),\\ a(\theta+\widetilde{\theta}_0;\boldsymbol{u},\boldsymbol{v}) + \beta\displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\,,\\ c(\theta+\widetilde{\theta}_0,\rho) + d(\boldsymbol{u},\theta+\widetilde{\theta}_0,\rho) = \displaystyle\int_\Omega g\rho\,dx\,.\end{cases} \tag{2.16}$$

We claim that

Proposition 2.3 *The problem (2.16) admits at least one solution* $(\boldsymbol{u},\theta)\in K(\Omega)\times H^1_0(\Omega)$.

Proof. It is done in several steps.

Step 1: Galerkin approximation. First, Since $K(\Omega)$ is separable, there are $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, ...$ elements of $K(\Omega)$, linear independent to each other such that

$$\bigcup_{n=1}^{\infty}\{\boldsymbol{\psi}_n\}\subset K(\Omega), \qquad \overline{\{\boldsymbol{\psi}_1,\boldsymbol{\psi}_2,...,\boldsymbol{\psi}_n,...\}} = K(\Omega)\,.$$

Let $K_n(\Omega) = \{\boldsymbol{\psi}_1, \boldsymbol{\psi}_2, ..., \boldsymbol{\psi}_n\}$. Next since $H_0^1(\Omega)$ is separable then there are $\phi_1, ..., \phi_n$ elements of $H_0^1(\Omega)$, linear independent to each other such that

$$\bigcup_{n=1}^{\infty} \{\phi_n\} \subset H_0^1(\Omega), \qquad \overline{\{\phi_1, \phi_2, ..., \phi_n, ...\}} = H_0^1(\Omega) .$$

Let $W^n = \{\phi_1, \phi_2, ..., \phi_n\}$. The Galerkin problem associated to (2.16) reads;

$$\begin{cases} \text{Find } \ (\boldsymbol{u}^n, \theta^n) \in K_n(\Omega) \times W^n, \ \text{ such that} \\ \text{and for all } \ (\boldsymbol{v}, \rho) \in K_n(\Omega) \times W^n, \\ a(\theta^n + \widetilde{\theta}_0; \boldsymbol{u}^n, \boldsymbol{v}) + \beta \displaystyle\int_\Omega |\boldsymbol{u}^n| \boldsymbol{u}^n \cdot \boldsymbol{v} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx \, , \\ c(\theta^n + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}^n, \theta^n + \widetilde{\theta}_0, \rho) = \displaystyle\int_\Omega g\rho \, dx \, . \end{cases} \tag{2.17}$$

To prove the existence of $(\boldsymbol{u}^n, \theta^n)$, we will apply the fixed point of Brouwer.
Step 2: Brouwer's fixed point. Let $(\boldsymbol{u}, \theta) \in K(\Omega) \times H_0^1(\Omega)$ and $(\boldsymbol{v}, \rho) \in K(\Omega) \times H_0^1(\Omega)$, we define the mapping $\mathcal{F}$ as follows

$$\begin{aligned} \mathcal{F}(\boldsymbol{u}, \theta)(\boldsymbol{v}, \rho) =& a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + \beta \int_\Omega |\boldsymbol{u}| \boldsymbol{u} \cdot \boldsymbol{v} dx + c(\theta + \widetilde{\theta}_0, \rho) + d(\boldsymbol{u}, \theta + \widetilde{\theta}_0, \rho) \\ & - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, dx - \int_\Omega g\rho \, dx \, . \end{aligned}$$

We need to prove that $\mathcal{F}$ is continuous and non-negative.

(A) $\mathcal{F}$ is continuous. Indeed let $(\boldsymbol{u}^n, \theta^n)_{n \geq 1}$ be a sequence of functions in $K(\Omega) \times H_0^1(\Omega)$ such that

$$\begin{aligned} & \boldsymbol{u}^n \to \boldsymbol{u} \ \text{ strongly in } \ L^3(\Omega)^d, \\ & \theta^n \to \theta \ \text{ strongly in } H^1(\Omega) \ . \end{aligned} \tag{2.18}$$

We would like to show that $\mathcal{F}(\boldsymbol{u}^n, \theta^n)(\boldsymbol{v}, \rho) \longrightarrow \mathcal{F}(\boldsymbol{u}, \theta)(\boldsymbol{v}, \rho)$.
From the convergence (2.18), the property of $\nu(\cdot)$ (see (2.3)), we deduce that

$$\begin{aligned} & \text{for any } \boldsymbol{v} \in K(\Omega) \\ & \nu(\theta^n + \widetilde{\theta}_0)\boldsymbol{v} \to \nu(\theta + \widetilde{\theta}_0)\boldsymbol{v} \text{ almost everywhere in } \Omega \\ & \text{and} \\ & \|\nu(\theta^n + \widetilde{\theta}_0)\boldsymbol{v}\|_{L^{3/2}(\Omega)^d} \leq \nu_1 ||\boldsymbol{v}||_{L^{3/2}(\Omega)^d}. \end{aligned} \tag{2.19}$$

Thus from the Lebesgue dominated convergence theorem one deduces that

$$\text{for all } \boldsymbol{v} \in K(\Omega), \quad \lim_{n \to \infty} \nu(\theta^n + \widetilde{\theta}_0)\boldsymbol{v} = \nu(\theta + \widetilde{\theta}_0)\boldsymbol{v} \ \text{ strongly in } L^{3/2}(\Omega) \ . \tag{2.20}$$

Thus

$$\text{for all } \boldsymbol{v} \in K(\Omega), \quad \lim_{n \to \infty} a(\theta^n + \widetilde{\theta}_0; \boldsymbol{u}^n, \boldsymbol{v}) = a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) \ . \tag{2.21}$$

Passing to the limit in $c(\cdot,\cdot)$ is direct and follows from the strong convergence of θ^n in $H^1(\Omega)$. The strong convergence properties (2.18) allows us to pass to the limit in the trilinear form $d(\cdot,\cdot,\cdot)$. Finally since $|\boldsymbol{u}|\boldsymbol{u}$ is monotone, we have (see [22])

$$\lim_{n\to\infty}\int_\Omega |\boldsymbol{u}^n|\boldsymbol{u}^n\boldsymbol{v}dx = \int_\Omega |\boldsymbol{u}|\boldsymbol{u}\boldsymbol{v}dx\ .$$

We then conclude that $\mathcal{F}$ is continuous.

(B) there is a constant r for which $\mathcal{F}(\boldsymbol{v},\rho)(\boldsymbol{v},\rho)$ is positive outside the ball $B(0,r)$.

Having in mind (2.3) we have

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) =& \int_\Omega \nu(\theta+\widetilde{\theta}_0)|\boldsymbol{v}|^2dx + \beta\,\|\boldsymbol{v}\|^3_{L^3(\Omega)^d} + \kappa\|\nabla\theta\|^2 + \kappa\int_\Omega \nabla\widetilde{\theta}_0\cdot\nabla\theta dx - d(\boldsymbol{v},\theta,\widetilde{\theta}_0)\\ &- \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx - \int_\Omega g\theta\,dx\\ \geq&\nu_0\,\|\boldsymbol{v}\|^2 + \beta\,\|\boldsymbol{v}\|^3_{L^3(\Omega)^d} + \kappa\|\nabla\theta\|^2 - \kappa\|\nabla\widetilde{\theta}_0\|\|\nabla\theta\| - \beta\|\boldsymbol{v}\|_{L^3(\Omega)^d}\|\widetilde{\theta}_0\|_{L^6(\Omega)}\|\nabla\theta\|\\ &- \|\boldsymbol{f}\|\|\boldsymbol{v}\| - c\|g\|\|\nabla\theta\|\,,\end{aligned}$$

where we have used Hölder's inequality, Poincaré's inequality. Inserting (2.14) and (2.12) in the previous inequality yields

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) \geq&\nu_0\,\|\boldsymbol{v}\|^2 + \beta\,\|\boldsymbol{v}\|^3_{L^3(\Omega)^d} + \kappa\|\nabla\theta\|^2 - \kappa c\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\nabla\theta\|\\ &- \delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{v}\|_{L^3(\Omega)^d}\|\nabla\theta\| - \|\boldsymbol{f}\|\|\boldsymbol{v}\| - c\|g\|\|\nabla\theta\|\\ \geq&\frac{\nu_0}{2}\,\|\boldsymbol{v}\|^2 + \|\boldsymbol{v}\|^2_{L^3(\Omega)^d}\left(\beta\|\boldsymbol{v}\|_{L^3(\Omega)} - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right) + \left(\frac{\kappa}{4} - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\theta\|^2\\ &- \frac{1}{2\nu_0\lambda}\|\boldsymbol{f}\|^2 - \frac{c}{\kappa}\|g\|^2 - c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.\end{aligned} \tag{2.22}$$

We distinguish two cases.

- If $\|\boldsymbol{v}\|_{L^3(\Omega)^d}\geq 1$, then $\|\boldsymbol{v}\|_{L^3(\Omega)} - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)} \geq 1 - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}$, and (2.22) implies

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) \geq&\frac{\nu_0}{2}\,\|\boldsymbol{v}\|^2 + \left(\beta - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\boldsymbol{v}\|^2_{L^3(\Omega)^d} + \left(\frac{\kappa}{4} - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\theta\|^2\\ &- \frac{1}{2\nu_0}\|\boldsymbol{f}\|^2 - \frac{c}{\kappa}\|g\|^2 - c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

We take $\delta>0$ such that

$$\delta\|\theta_0\|_{H^{1/2}(\partial\Omega)} \leq \min\left(2\beta,\frac{\kappa}{2}\right)\,,$$

and we deduce that

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) \geq&\frac{\nu_0}{2}\,\|\boldsymbol{v}\|^2 + \min\left(\beta - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}, \frac{\kappa}{4} - \frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\left(\|\nabla\theta\|^2 + \|\boldsymbol{v}\|^2_{L^3(\Omega)^d}\right)\\ &- \frac{1}{2\nu_0}\|\boldsymbol{f}\|^2 - \frac{c}{\kappa}\|g\|^2 - c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

We take r such that

$$\min\left(\beta-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}, \frac{\kappa}{4}-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right) r^2 > \frac{1}{2\nu_0}\|\boldsymbol{f}\|^2+\frac{c}{\kappa}\|g\|^2+c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.$$

Hence for any $(\boldsymbol{v},\theta)$ element of $K(\Omega)\times H^1_0(\Omega)$, with $\|\boldsymbol{v}\|^2_{L^3(\Omega)^d}+||\theta||_1^2=r^2$, $\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta)$ is non-negative.
We recall that $\bigcup_n K_n(\Omega)\times W^n$ is dense in $K(\Omega)\times H^1_0(\Omega)$, and the properties established for $\mathcal{F}$ are valid for $K(\Omega)\times H^1_0(\Omega)$ replaced by $K_n(\Omega)\times W^n$. Thus the Brouwer's fixed point is applicable. Hence there is $(\boldsymbol{u}^n,\theta^n)$ element of $K_n(\Omega)\times W^n$ such that $\mathcal{F}(\boldsymbol{u}^n,\theta^n)(\boldsymbol{v},\rho)=0$ for all $(\boldsymbol{v},\rho)\in K_n(\Omega)\times W^n$.

- If $\|\boldsymbol{v}\|_{L^3(\Omega)^d}<1$. Then (2.22) gives

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) \;\geq\; & \frac{\nu_0}{2}\|\boldsymbol{v}\|^2+\beta\|\boldsymbol{v}\|^3_{L^3(\Omega)^d}-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\left(\frac{\kappa}{4}-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\theta\|^2\\ & -\frac{1}{2\nu_0}\|\boldsymbol{f}\|^2-\frac{c}{\kappa}\|g\|^2-c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,. \end{aligned}\tag{2.23}$$

We first assume that $2\delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}\leq\kappa$. Now, let $0<\varepsilon<1$, $\beta>\varepsilon$ and δ such that $\delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}\leq 2\left(\beta\|\boldsymbol{v}\|^3_{L^3(\Omega)}-\varepsilon\|\boldsymbol{v}\|^2_{L^3(\Omega)}\right)=2\|\boldsymbol{v}\|^2_{L^3(\Omega)}\left(\beta\|\boldsymbol{v}\|_{L^3(\Omega)}-\varepsilon\right)\leq 2(\beta-\varepsilon)$. Thus for

$$\delta\|\theta_0\|_{H^{1/2}(\partial\Omega)}\leq\min\left(\frac{\kappa}{2},2(\beta-\varepsilon)\right)\,,$$

the inequality (2.23) gives

$$\begin{aligned}\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta) \;\geq\; & \frac{\nu_0}{2}\|\boldsymbol{v}\|^2+\varepsilon\|\boldsymbol{v}\|^2_{L^3(\Omega)^d}+\left(\frac{\kappa}{4}-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)\|\nabla\theta\|^2\\ & -\frac{1}{2\nu_0}\|\boldsymbol{f}\|^2-\frac{c}{\kappa}\|g\|^2-c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)}\,.\end{aligned}$$

Let $r^2=\|\boldsymbol{v}\|^2_{L^3(\Omega)^d}+\|\nabla\theta\|^2$. So by taking r such that

$$\min\left(\varepsilon,\frac{\kappa}{4}-\frac{\delta}{2}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\right)r^2>\frac{1}{2\nu_0}\|\boldsymbol{f}\|^2+\frac{c}{\kappa}\|g\|^2+c\kappa\|\theta_0\|^2_{H^{1/2}(\partial\Omega)},$$

we deduce that for any $(\boldsymbol{v},\theta)$ element of $K(\Omega)\times H^1_0(\Omega)$, $\mathcal{F}(\boldsymbol{v},\theta)(\boldsymbol{v},\theta)$ is non-negative.
We recall that $\bigcup_n K_n(\Omega)\times W^n$ is dense in $K(\Omega)\times H^1_0(\Omega)$, and the properties established for $\mathcal{F}$ are valid for $K(\Omega)\times H^1_0(\Omega)$ replaced by $K_n(\Omega)\times W^n$. Thus the Brouwer's fixed point is applicable. Hence there is $(\boldsymbol{u}^n,\theta^n)$ element of $K_n(\Omega)\times W^n$ such that $\mathcal{F}(\boldsymbol{u}^n,\theta^n)(\boldsymbol{v},\rho)=0$ for all $(\boldsymbol{v},\rho)\in K_n(\Omega)\times W^n$.
step 3: a priori estimates and passage to the limit. The a priori estimates obtained in proposition 2.2 will also hold in the discrete setting $K_n(\Omega)\times W^n$. Hence

$$\begin{aligned}&\frac{\nu_0}{2}\|\boldsymbol{u}^n\|^2_{L^2(\Omega)^d}+\beta\|\boldsymbol{u}^n\|^3_{L^3(\Omega)^d}\leq\frac{1}{2\nu_0}\|\boldsymbol{f}\|^2_{L^2(\Omega)^d}\,,\\ &\|\theta^n\|_{H^1(\Omega)}\leq c_1\|\theta_0\|_{H^{1/2}(\partial\Omega)}+c_1\|g\|\,.\end{aligned}$$

Then we can find a subsequence, denoted also $(\boldsymbol{u}^n, \theta^n)$, such that

$$\begin{aligned} \boldsymbol{u}^n &\to \boldsymbol{u} \quad \text{in } L^3(\Omega)^d \text{ weakly} \\ \theta^n &\to \theta \quad \text{in } H_0^1(\Omega) \text{ weakly} . \end{aligned}$$

Now owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exits a subsequence, still denoted by $(\boldsymbol{u}^n, \theta^n)$, such that

$$\begin{aligned} &(\boldsymbol{u}^n, \theta^n) \to (\boldsymbol{u}, \theta) \quad \text{weakly in } L^3(\Omega)^d \times H^1(\Omega) \\ &\text{and} \\ &\theta^n \to \theta \quad \text{strongly in } L^4(\Omega) . \end{aligned}$$

Hence one can pass to the limit in (2.17). We note that passing to the limit for linear term is direct and only necessitate the weak convergence, but for the nonlinear terms, we need the arguments used for the continuity of $\mathcal{F}$. Hence the proof of proposition 2.3 is now complete. □

To study the unique solvability of (2.7) it is convenient to recall the following monotonicity and continuity properties (see [28, 29])

$$\begin{aligned} &\text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \\ &\text{for } s > 2, \quad c|\boldsymbol{y} - \boldsymbol{x}|^s \le \left(|\boldsymbol{x}|^{s-2}\boldsymbol{x} - |\boldsymbol{y}|^{s-2}\boldsymbol{y}, \boldsymbol{y} - \boldsymbol{x}\right) \\ &\text{for } s \ge 2, \quad \left| \, |\boldsymbol{x}|^{s-2}\boldsymbol{x} - |\boldsymbol{y}|^{s-2}\boldsymbol{y}\right| \le c|\boldsymbol{y} - \boldsymbol{x}| \left(|\boldsymbol{x}| + |\boldsymbol{y}|\right)^{s-2} \end{aligned} \tag{2.24}$$

with c independent of $\boldsymbol{x}, \boldsymbol{y}$.

We conclude on the solvability of (2.7) by reporting on the situation where the solution of (2.7) is unique. We claim that

Proposition 2.4 *Let $(\boldsymbol{u}, \theta) \in K(\Omega) \times H^1(\Omega)$ be the solution of (2.7). There exists a positive constant c depending only on Ω such that if for κ, $\boldsymbol{f}$, g, θ_0, ν_0 and ν_2 the relation*

$$\frac{c\,\nu_2}{\kappa \nu_0 \beta^2} \|\boldsymbol{f}\|^{1/3} \left(\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\| \right) \le \frac{1}{(\beta\nu_0)^{1/3}}$$

is satisfied, then the solution of (2.7) is unique.

Proof of Proposition 2.4. Let $(\boldsymbol{u}_1, \theta_1)$, and $(\boldsymbol{u}_2, \theta_2)$ solutions of (2.7). Then from the velocity equation of (2.7), we obtain

$$\begin{aligned} &\beta \int_\Omega (|\boldsymbol{u}_2|\boldsymbol{u}_2 - |\boldsymbol{u}_1|\boldsymbol{u}_1) \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) dx \\ &= \int_\Omega \nu(\theta_1)\boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) - \int_\Omega \nu(\theta_2)\boldsymbol{u}_2 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) dx \\ &= \int_\Omega \left(\nu(\theta_1) - \nu(\theta_2)\right) \boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) - \int_\Omega \nu(\theta_2) \left(\boldsymbol{u}_2 - \boldsymbol{u}_1\right) \cdot \left(\boldsymbol{u}_2 - \boldsymbol{u}_1\right) dx , \end{aligned}$$

which is re written as follows

$$\beta \int_\Omega (|\boldsymbol{u}_2|\boldsymbol{u}_2 - |\boldsymbol{u}_1|\boldsymbol{u}_1) \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) dx + \int_\Omega \nu(\theta_2)\, (\boldsymbol{u}_2 - \boldsymbol{u}_1) \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1)\, dx$$
$$= \int_\Omega (\nu(\theta_1) - \nu(\theta_2))\, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) dx\,.$$

Using (2.3), (2.24), (2.1) and Proposition 2.2 one has

$$\begin{aligned} c\beta \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^3_{L^3(\Omega)^d} + \nu_0 \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2 &\le \int_\Omega (\nu(\theta_1) - \nu(\theta_2))\, \boldsymbol{u}_1 \cdot (\boldsymbol{u}_2 - \boldsymbol{u}_1) dx \\ &\le \nu_2 \int_\Omega |\theta_1 - \theta_2|\ |\boldsymbol{u}_1|\ |\boldsymbol{u}_2 - \boldsymbol{u}_1|\, dx \\ &\le \nu_2 \|\theta_1 - \theta_2\|_{L^6(\Omega)}\ \|\boldsymbol{u}_1\|\ \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \\ &\le \frac{\nu_2}{\nu_0} \|\boldsymbol{f}\|\, \|\theta_1 - \theta_2\|_{L^6(\Omega)}\ \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \\ &\le c\frac{\nu_2}{\nu_0} \|\boldsymbol{f}\|\, \|\theta_1 - \theta_2\|_{H^1(\Omega)}\ \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \end{aligned} \tag{2.25}$$

Clearly (2.25) together with Young's inequality (2.12) implies that

$$c\beta \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|^2_{L^3(\Omega)^d} \le c\frac{\nu_2}{\beta\nu_0} \|\boldsymbol{f}\|\, \|\nabla(\theta_1 - \theta_2)\|\ . \tag{2.26}$$

Next, from the temperature equation in (2.7), one deduces that

$$c(\theta_1 - \theta_2, \theta_1 - \theta_2) = d(\boldsymbol{u}_2, \theta_2, \theta_1 - \theta_2) - d(\boldsymbol{u}_1, \theta_1, \theta_1 - \theta_2)\,,$$

which with the definition of $c(\cdot,\cdot)$, the property of the trilinear form $d(\cdot,\cdot,\cdot)$, Proposition 2.2, (2.1) gives

$$\begin{aligned} \kappa \|\nabla(\theta_2 - \theta_1)\|^2 &= d(\boldsymbol{u}_2 - \boldsymbol{u}_1, \theta_2, \theta_2 - \theta_1) \\ &\le \int_\Omega |\boldsymbol{u}_2 - \boldsymbol{u}_1|\ |\nabla\theta_2|\, |\theta_1 - \theta_2| dx \\ &\le c\, \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \|\nabla\theta_2\|\, \|\theta_1 - \theta_2\|_{L^6(\Omega)} \\ &\le c\left(\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right) \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \|\nabla(\theta_1 - \theta_2)\|\,, \end{aligned}$$

which implies that

$$\|\nabla(\theta_2 - \theta_1)\| \le \frac{c}{\kappa}\left(\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right) \|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d}\ . \tag{2.27}$$

Putting together (2.27) and (2.26), one obtains

$$\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} \left[\|\boldsymbol{u}_2 - \boldsymbol{u}_1\|_{L^3(\Omega)^d} - \frac{c\,\nu_2}{\kappa\nu_0\beta^2} \|\boldsymbol{f}\| \left(\left(1 + \frac{1}{\kappa}\right) \|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\|\right)\right] \le 0\,.$$

Thus for unique solvability we require that

$$\frac{c\,\nu_2}{\kappa\nu_0\beta^2}\|\boldsymbol{f}\|\left(\left(1+\frac{1}{\kappa}\right)\|\theta_0\|_{H^{1/2}(\partial\Omega)}+\frac{1}{\kappa}\|g\|\right) \quad\leq\quad \|\boldsymbol{u}_2-\boldsymbol{u}_1\|_{L^3(\Omega)^d}\leq\left(\frac{c}{\beta\nu_0}\right)^{1/3}\|\boldsymbol{f}\|^{2/3},$$

which ends the proof. □

Remark 2.1 *The restriction condition noted for the uniqueness of solution is not restrictive given that we have a nonlinear problem. It is also noted that we do not need extra regularity of the solution for uniqueness.*

We discuss next the finite element approximation associated to the problem (2.5). The finite element associated to (2.10) will not be addressed in this work as it differ enormously to the formulation associated to (2.5).

3 Finite element approximation

3.1 Discrete problem

We start by recalling some preliminaries and later we formulate the finite element problem. From now on, we assume that Ω is a polygon or polyhedron. In order to approximate the problem (2.5) and (2.10), we introduce a regular family $(\mathcal{T}_h)_h$ of triangulations of Ω by closed triangles ($d=2$) or tetrahedra ($d=3$), in the usual sense;

(a) For each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_h$; $\overline{\Omega}=\bigcup_{K\in\mathcal{T}_h} K$

(b) For each h, the intersection of two different elements of $\mathcal{T}_h$ if not empty, is a corner, a whole edge or a whole face of both elements.

(c) The ratio of the diameter h_K of an element K in $\mathcal{T}_h$ to the diameter ρ_K of its inscribed circle or sphere is bounded by a constant independent of K and h, that is

$$\frac{h_K}{\rho_K}\leq\sigma\ , \quad \text{for all}\ \ K\in\mathcal{T}_h\,.$$

As standard, h stands for the maximum of the diameters of the elements of $\mathcal{T}_h$. For each non-negative integer n and any K in $\mathcal{T}_h$, let $\mathbb{P}_l(K)$ denote the space of restrictions to K of polynomials with d variables and total degree less than or equal to l.
In what follows, c stand for generic constant which may vary from line to line but is always independent of h. We discretize the temperature θ in a finite dimensional space $H^1_{0h}(\Omega)\subset H^1_0(\Omega)$ given as follows

$$H^1_{0h}(\Omega)=\left\{\theta_h\in\mathcal{C}^0(\overline{\Omega})\cap H^1_0(\Omega):\ \text{for all}\ K\in\mathcal{T}_h,\ \theta_h|_K\in\mathbb{P}_1(K)\right\}\,.$$

The space $L^3(\Omega)^d$ is approximated by $L^3_h(\Omega)^d$ given as follows

$$L^3_h(\Omega)^d = \left\{ \boldsymbol{v}_h \in L^3(\Omega)^d \cap \mathcal{C}(\overline{\Omega})^d, \text{ for all } K \in \mathcal{T}_h, \quad \boldsymbol{v}_h|_K \in (\mathbb{P}_1(K) + \text{bubble})^d \right\},$$

where $\mathbb{P}_1(K)$+bubble is the sum of a polynomial of $\mathbb{P}_1(K)$ and a bubble function $b_K(\boldsymbol{x}) = \alpha_1(\boldsymbol{x})...\alpha_{d+1}(\boldsymbol{x})$, for any $K \in \mathcal{T}_h$, and denoting the vertices of $K \in \mathcal{T}_h$ by a_i, $1 \leq i \leq d+1$, and its corresponding barycentric coordinates by α_i. Note that $b_K(\boldsymbol{x}) = 0$ on ∂K and that $b_K(\boldsymbol{x}) > 0$ on K.
The pressure is approximated in the space M_h given as follows

$$M_h = \{q_h \in M \cap \mathcal{C}(\overline{\Omega}), \quad \text{for all } K \in \mathcal{T}_h, \quad q_h|_K \in \mathbb{P}_1(K)\},$$

With these preliminaries in place, we approximate (2.7) by the following finite element scheme

$$\begin{cases} \text{Find } \ (\boldsymbol{u}_h, \theta_h, p_h) \in L^3_h(\Omega)^d \times H^1_{0h}(\Omega) \times M_h, \quad \text{such that} \\ \text{for all } \ (\boldsymbol{v}_h, s_h, q_h) \in L^3_h(\Omega)^d \times H^1_{0h}(\Omega) \times M_h, \\ a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega |\boldsymbol{u}_h| \boldsymbol{u}_h \cdot \boldsymbol{v}_h dx + b_1(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx \,, \\ b_1(\boldsymbol{u}_h, q_h) = 0 \,, \\ c(\theta_h + \widetilde{\theta}_0, s_h) + \widetilde{d}(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, s_h) = \displaystyle\int_\Omega g s_h \, dx \,, \end{cases} \tag{3.1}$$

with the trilinear form $\widetilde{d}(\cdot,\cdot,\cdot)$ given by R.Temam [24]

$$\widetilde{d}(\boldsymbol{v}_h, \theta_h, \rho_h) = d(\boldsymbol{v}_h, \theta_h, \rho_h) + \frac{1}{2}\left((\operatorname{div} \boldsymbol{v}_h)\theta_h, \rho_h\right) = \frac{1}{2}\left(d(\boldsymbol{v}_h, \theta_h, \rho_h) - d(\boldsymbol{v}_h, \rho_h, \theta_h)\right).$$

It is noted that $\widetilde{d}(\cdot,\cdot,\cdot)$ is consistent with $d(\cdot,\cdot,\cdot)$ in the sense that

$$\text{for all } (\boldsymbol{v}, \theta, \rho) \in K(\Omega) \times H^1(\Omega) \times H^1(\Omega) \quad , \ \widetilde{d}(\boldsymbol{v}, \theta, \rho_h) = d(\boldsymbol{v}, \theta, \rho) \,.$$

Furthermore $\widetilde{d}(\cdot,\cdot,\cdot)$ is anti-symmetry meaning that

$$\begin{aligned} &\text{for all } (\boldsymbol{v}_h, \theta_h, \rho_h) \in L^3_h(\Omega)^d \times H^1_{0h}(\Omega) \times H^1_{0h}(\Omega) \,, \\ &\widetilde{d}(\boldsymbol{v}_h, \theta_h, \rho_h) = -\widetilde{d}(\boldsymbol{v}_h, \rho_h, \theta_h) \,. \end{aligned} \tag{3.2}$$

We recall that the discrete version of inf-sup condition (2.6) holds: there exists γ (independent of h) such that

$$\gamma \|\nabla q_h\|_{L^{3/2}(\Omega)} \leq \sup_{\boldsymbol{v}_h \in L^3_h(\Omega)^d} \frac{b_1(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_{L^3(\Omega)^d}} \quad \text{for all} \quad q_h \in M_h \,. \tag{3.3}$$

3.2 Existence

We recall that the kernel of $b_1(\cdot,\cdot)$ in $L^3_h(\Omega)^d$ is

$$K_h(\Omega) = \left\{ \boldsymbol{v}_h \in L^3_h(\Omega)^d : \text{ for all } q_h \in M_h, \quad \int_\Omega q_h \operatorname{div} \boldsymbol{v}_h = 0 \right\},$$

and the reduced problem reads;

$$\begin{cases} \text{Find } \ (\boldsymbol{u}_h,\theta_h) \in K_h(\Omega) \times H^1_{0h}(\Omega), \quad \text{such that} \\ \text{for all } \ (\boldsymbol{v}_h, s_h) \in K_h(\Omega) \times H^1_{0h}(\Omega), \\ a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega |\boldsymbol{u}_h| \boldsymbol{u}_h \cdot \boldsymbol{v}_h dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx\,, \\ c(\theta_h + \widetilde{\theta}_0, s_h) + \widetilde{d}(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, s_h) = \displaystyle\int_\Omega g s_h \, dx\,. \end{cases} \tag{3.4}$$

This is a finite dimensional, square system of nonlinear equations. We address next the solvability of (3.4) by Brouwer's Fixed point arguments. To this end, for fixed $(\boldsymbol{u}_h,\theta_h) \in K_h(\Omega) \times H^1_{0h}(\Omega)$, one introduces $F(\boldsymbol{u}_h,\theta_h)$ in $K_h(\Omega) \times H^1_{0h}(\Omega)$ by

$$\begin{aligned}(F(\boldsymbol{u}_h,\theta_h);(\boldsymbol{v}_h,s_h)) =& a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) + \beta \int_\Omega |\boldsymbol{u}_h| \boldsymbol{u}_h \cdot \boldsymbol{v}_h dx + c(\theta_h + \widetilde{\theta}_0, s_h) + \widetilde{d}(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, s_h) \\ & - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx - \int_\Omega g s_h \, dx\,.\end{aligned}$$

From the proof of proposition 2.3, we claim that the mapping $F : K_h(\Omega) \times H^1_{0h}(\Omega) \longrightarrow K_h(\Omega) \times H^1_{0h}(\Omega)$ is continuous.
Next, using the anti-symmetry property (3.2) of $\widetilde{d}(\cdot,\cdot,\cdot)$, (2.14), Poincaré's inequality (2.2) and Young's inequality one obtains

$$\begin{aligned} & (F(\boldsymbol{u}_h,\theta_h);(\boldsymbol{u}_h,\theta_h)) \\ =& \int_\Omega \nu(\theta_h + \widetilde{\theta}_0) \, |\boldsymbol{u}_h|^2 \, dx + \beta \int_\Omega |\boldsymbol{u}_h|^3 dx + \kappa \int_\Omega |\nabla \theta_h|^2 \, dx \\ & + \frac{1}{2} \Big(d(\boldsymbol{u}_h, \widetilde{\theta}_0, \theta_h) - d(\boldsymbol{u}_h, \theta_h, \widetilde{\theta}_0) \Big) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_h \, dx - \int_\Omega g \theta_h \, dx + c(\widetilde{\theta}_0, \theta_h) \\ \geq& \nu_0 \, \|\boldsymbol{u}_h\|^2 + \beta \, \|\boldsymbol{u}_h\|^3_{L^3(\Omega)^d} + \kappa \, \|\nabla \theta_h\|^2 - c \|\boldsymbol{u}_h\|_{L^3(\Omega)^d} \|\nabla \theta_h\| \left(\|\nabla \widetilde{\theta}_0\| + \|\widetilde{\theta}_0\|_{L^6(\Omega)} \right) \\ & - c\|\boldsymbol{f}\| \|\boldsymbol{u}_h\|_{L^3(\Omega)^d} - \|g\| \|\theta_h\| - \kappa \|\nabla \widetilde{\theta}_0\| \|\nabla \theta_h\| \\ \geq& \|\boldsymbol{u}_h\|_{L^3(\Omega)^d} \left[\beta \, \|\boldsymbol{u}_h\|^2_{L^3(\Omega)^d} - c \left(\|\widetilde{\theta}_0\|_{H^{1/2}(\partial\Omega)} + \delta \|\widetilde{\theta}_0\|_{H^{1/2}(\partial\Omega)} \right) \|\boldsymbol{u}_h\|_{L^3(\Omega)^d} - c\|\boldsymbol{f}\| \right] + \nu_0 \, \|\boldsymbol{u}_h\|^2 \\ & + \|\nabla \theta_h\| \left[-c \|\nabla \theta_h\| \left(\|\widetilde{\theta}_0\|_{H^{1/2}(\partial\Omega)} + \delta \|\widetilde{\theta}_0\|_{H^{1/2}(\partial\Omega)} \right) + \left(\kappa \, \|\nabla \theta_h\| - c\|g\| - \kappa c \|\widetilde{\theta}_0\|_{H^{1/2}(\partial\Omega)} \right) \right]. \end{aligned}$$

Hence to show that $(F(\boldsymbol{u}_h,\theta_h);(\boldsymbol{u}_h,\theta_h))$ is non-negative for all $(\boldsymbol{u}_h,\theta_h) \in K_h(\Omega) \times H^1_{0h}(\Omega)$, we follow the proof of proposition 2.3. By Brouwer's Fixed point, this proves existence of at least one solution of (3.4). Using the equivalence between (3.1) and (3.4), we construct the pressure and claim that

Proposition 3.1 *The finite element problem (3.1) admits at least one solution* $(\boldsymbol{u}_h, \theta_h, p_h) \in L^3_h(\Omega)^d \times H^1_{0h}(\Omega) \times M_h$*, and there exist positive constants* c_1, c_2 *independent of* h *such that*

$$\begin{aligned}
&\frac{\nu_0}{2}\|\boldsymbol{u}_h\|^2_{L^2(\Omega)^d} + \beta\|\boldsymbol{u}_h\|^3_{L^3(\Omega)^d} \le \frac{1}{2\nu_0}\|\boldsymbol{f}\|^2 \ ,\\
&\|\theta_h\|_{H^1(\Omega)} \le c_1\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{c_1}{\kappa}\|g\| + \frac{c_1}{\kappa(\beta\nu_0)^{1/3}}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{f}\|^{2/3} \ ,\\
&\|\nabla p_h\|_{L^{3/2}(\Omega)^d} \le c_2\|\boldsymbol{f}\| + c_2\|\boldsymbol{u}_h\|_{L^3(\Omega)^d} \,.
\end{aligned}$$

The estimate above are obtained as in proposition 2.2 combined with (3.2). One can observes the small difference between the estimate on the temperature in proposition 2.2 and proposition 3.1. This difference being the fact that in the discrete setting $\operatorname{div} \boldsymbol{u}_h$ is non zero.
We end this analysis with this result

Proposition 3.2 *Let* $(\boldsymbol{u}_h, \theta_h) \in K_h(\Omega) \times H^1_{0h}(\Omega)$ *be the solution of (3.4). There exists a positive constant* c *depending only on* Ω *such that if for* κ, $\boldsymbol{f}$, g, θ_0, ν_0 *and* ν_2 *the relation*

$$c\frac{\nu_2}{\kappa\nu_0}\|\boldsymbol{f}\|^{1/3}\left(\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\| + \frac{1}{\kappa(\beta\nu_0)^{1/3}}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{f}\|^{2/3}\right) \le \frac{1}{(\beta\nu_0)^{1/3}}$$

is satisfied, then the solution of (3.4) is unique.

Proof. Let $(\boldsymbol{u}_{1h}, \theta_{1h})$, and $(\boldsymbol{u}_{2h}, \theta_{2h})$ solutions of (3.4). We follow to the line the proof of proposition 2.4 and obtain

$$\|\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}\|^2_{L^3(\Omega)^d} \le c\frac{\nu_2}{\beta^2\nu_0}\|\boldsymbol{f}\|\,\|\nabla(\theta_{1h} - \theta_{2h})\| \,. \tag{3.5}$$

Next, from the temperature equation in (3.4), and the anti-symmetry property of $\widetilde{d}(\cdot,\cdot,\cdot)$ one deduces that

$$c(\theta_{1h} - \theta_{2h}, \theta_{1h} - \theta_{2h}) = \widetilde{d}(\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}, \theta_{2h} + \widetilde{\theta}_0, \theta_{1h} - \theta_{2h})$$

which with the definition of $c(\cdot,\cdot)$, Hölder's inequality, the bound obtained in Proposition 3.1, (2.1), (2.14) one obtains

$$\begin{aligned}
\kappa\|\nabla(\theta_{2h} - \theta_{1h})\| &\le c\|\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}\|_{L^3(\Omega)^d}\left(\|\nabla\theta_{2h}\| + \|\nabla\widetilde{\theta}_0\|\right)\\
&\le c\|\boldsymbol{u}_{2h} - \boldsymbol{u}_{1h}\|_{L^3(\Omega)^d}\left(\|\theta_0\|_{H^{1/2}(\partial\Omega)} + \frac{1}{\kappa}\|g\| + \frac{1}{\kappa(\beta\nu_0)^{1/3}}\|\theta_0\|_{H^{1/2}(\partial\Omega)}\|\boldsymbol{f}\|^{2/3}\right) ,
\end{aligned}$$

which together with (3.5) ends the proof. □

3.3 Convergence

With the estimates in Proposition 3.1, we can extract a subsequence of h, still denoted by h, and functions $\boldsymbol{u} \in L^3(\Omega)^d$, $F \in L^{3/2}(\Omega)^d$ and $\theta \in H_0^1(\Omega)$ such that

$$\begin{aligned} &\lim_{h\to 0} \boldsymbol{u}_h = \boldsymbol{u} \text{ weakly in } L^3(\Omega)^d \\ &\lim_{h\to 0} \nabla p_h = F \text{ weakly in } L^{3/2}(\Omega)^d \\ &\lim_{h\to 0} \theta_h = \theta \text{ weakly in } H_0^1(\Omega). \end{aligned} \tag{3.6}$$

The compactness of the imbedding of $H^1(\Omega)$ into $L^p(\Omega)$ for any real number $p \geq 2$ implies that

$$\text{for all } p \in [2, \infty), \quad \lim_{h\to 0} \theta_h = \theta \text{ strongly in } L^p(\Omega)\,.$$

To pass to the limit in (3.1), we need the following approximation properties of the discrete spaces.

Assumption 3.1 *(a) There exists an operator $\Pi_h : L^3(\Omega)^d \to L_h^3(\Omega)^d$ continuous such that*

$$\begin{aligned} &\text{for all } (\boldsymbol{v}, q_h) \in L^3(\Omega)^d \times M_h, \\ &\int_\Omega \Pi_h \boldsymbol{v} \cdot \nabla q_h dx = \int_\Omega \boldsymbol{v} \cdot \nabla q_h dx\,, \\ &\lim_{h\to 0} \Pi_h \boldsymbol{v} = \boldsymbol{v} \quad \text{strongly in } L^3(\Omega)^d\,. \end{aligned} \tag{3.7}$$

(b) There exists an operator $r_h : M \to M_h$ continuous such that

$$\text{for all } q \in M\ ,\ \lim_{h\to 0} \|r_h q - q\|_{W^{1,3/2}(\Omega)} = 0\ . \tag{3.8}$$

(c) There exists $R_h : H_0^1(\Omega) \to H_{0h}^1(\Omega)$ continuous such that

$$\text{for all } \rho \in H_0^1(\Omega)\ ,\ \lim_{h\to 0} \|R_h \rho - \rho\|_1 = 0\ . \tag{3.9}$$

It is noted that the operators mentioned in Assumption 3.1 can be constructed (in fact this will be done in the next paragraph).
We first note that

Proposition 3.3 *Let $(\boldsymbol{u}_h, \theta_h, p_h)$ be the solution of (3.1) and $(\boldsymbol{u}, \theta, F)$ be the limit functions given in (3.6). Assume that the conditions mentioned in Assumption 3.1 are valid. Then $\boldsymbol{u}$ belong to $L^3(\Omega)^d$, and $(\boldsymbol{u}, \theta)$ satisfies the first equation of (2.16).*

Proof. (A). First, we show that $\boldsymbol{u}$ is in $L^3(\Omega)^d$.
Let $q \in M$ and choose $q_h = r_h(q)$ in (3.1) this gives $(\nabla r_h(q), \boldsymbol{u}_h) = 0$. The weak convergence of $\boldsymbol{u}_h$ and the strong convergence of $r_h(q)$ implies that $(\nabla q, \boldsymbol{u}) = 0$. Having in mind that $q \in M$, we have $\nabla q \in L^{3/2}(\Omega)^d$ and by duality $\boldsymbol{u}$ is in $L^3(\Omega)^d$.

(B). The strong convergence of θ_h and the continuity of $\nu(\cdot)$ (see (2.3)) implies that $\nu(\theta_h + \widetilde{\theta}_0) \to \nu(\theta + \widetilde{\theta}_0)$ from which we deduce that $\nu(\theta_h + \widetilde{\theta}_0)\boldsymbol{u}_h \to \nu(\theta + \widetilde{\theta}_0)\boldsymbol{u}$ almost everywhere in Ω. Finally with Lebesgue dominated convergence we have

$$\nu(\theta_h + \widetilde{\theta}_0)\boldsymbol{u}_h \to \nu(\theta + \widetilde{\theta}_0)\boldsymbol{u} \ \text{ weakly in } L^3(\Omega)^d\,. \tag{3.10}$$

We recall that

$$\begin{cases} \text{for all } \ \boldsymbol{v}_h \in K_h(\Omega) \\ a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega |\boldsymbol{u}_h|\boldsymbol{u}_h \cdot \boldsymbol{v}_h dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h \, dx\,. \end{cases}$$

Let $\boldsymbol{v} \in K(\Omega)$ and take $\boldsymbol{v}_h = \Pi_h \boldsymbol{v}$. Then

$$a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \Pi_h\boldsymbol{v}) + \beta \int_\Omega |\boldsymbol{u}_h|\boldsymbol{u}_h \cdot \Pi_h\boldsymbol{v} dx = \int_\Omega \boldsymbol{f} \cdot \Pi_h\boldsymbol{v}\, dx\,. \tag{3.11}$$

The weak convergence property (3.10) together with the strong convergence of Π_h imply that

$$a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \Pi_h\boldsymbol{v}) \longrightarrow a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v})\ ,$$
$$\int_\Omega \boldsymbol{f} \cdot \Pi_h\boldsymbol{v}\, dx \longrightarrow \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx\,.$$

For the nonlinear term $(|\boldsymbol{u}_h|\boldsymbol{u}_h, \Pi_h\boldsymbol{v})$, we use the fact that the mapping $\boldsymbol{v} \to |\boldsymbol{v}|\boldsymbol{v}$ is monotone, hence continuous and $\boldsymbol{u}_h$ converges weakly to $\boldsymbol{u}$ in $L^3(\Omega)^d$. Hence $|\boldsymbol{u}_h|\boldsymbol{u}_h$ converges weakly to $|\boldsymbol{u}|\boldsymbol{u}$, which combined with the strong convergence of Π_h gives $(|\boldsymbol{u}_h|\boldsymbol{u}_h, \Pi_h\boldsymbol{v}) \longrightarrow (|\boldsymbol{u}|\boldsymbol{u}, \boldsymbol{v})$. Thus $(\boldsymbol{u}, \theta)$ satisfies

$$\text{for all } \boldsymbol{v} \in K(\Omega)\quad,\quad a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + \beta \int_\Omega |\boldsymbol{u}|\boldsymbol{u} \cdot \boldsymbol{v} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx\,.$$

□

Secondly

Proposition 3.4 *Under the assumption of Proposition 7, $\nu(\theta_h+\widetilde{\theta}_0)^{1/2}\boldsymbol{u}_h$ converges strongly to $\nu(\theta + \widetilde{\theta}_0)^{1/2}\boldsymbol{u}$ in $L^2(\Omega)^d$.*

Proof. We have that

$$\begin{cases} \text{for all } \ (\boldsymbol{v}_h, q_h) \in L^3_h(\Omega)^d \times M_h \\ a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega |\boldsymbol{u}_h|\boldsymbol{u}_h \cdot \boldsymbol{v}_h dx + b_1(\boldsymbol{v}_h, p_h) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, dx \\ a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u} \cdot \boldsymbol{v}_h dx + b_1(\boldsymbol{v}_h, p) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_h\, dx\,, \\ b_1(\boldsymbol{u}, q_h) = 0 \text{ and } b_1(\boldsymbol{u}_h, q_h) = 0\,, \end{cases}$$

from which we deduce that

$$\begin{cases} \text{for all } \ (\boldsymbol{v}_h, q_h) \in L^3_h(\Omega)^d \times M_h \\ b_1(\boldsymbol{u} - \boldsymbol{u}_h, q_h) = 0\ , \\ a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) - a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) \\ +\beta \displaystyle\int_\Omega (|\boldsymbol{u}|\boldsymbol{u} - |\boldsymbol{u}_h|\boldsymbol{u}_h) \cdot \boldsymbol{v}_h dx + b_1(\boldsymbol{v}_h, p - r_h(p)) + b_1(\boldsymbol{v}_h, r_h(p) - p_h) = 0\,. \end{cases} \tag{3.12}$$

We take $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h - \Pi_h\boldsymbol{u}, p_h - r_h(p))$ in (3.12) and obtain that

$$
\begin{aligned}
&a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h - \Pi_h\boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) + \beta \int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\Pi_h\boldsymbol{u}|\Pi_h\boldsymbol{u}) \cdot (\boldsymbol{u}_h - \Pi_h\boldsymbol{u}) dx \\
&= -a(\theta_h + \widetilde{\theta}_0; \Pi_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) - a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) + a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) \\
&- a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) + \beta \int_\Omega (|\boldsymbol{u}|\boldsymbol{u} - |\Pi_h\boldsymbol{u}|\Pi_h\boldsymbol{u}) \cdot (\boldsymbol{u}_h - \Pi_h\boldsymbol{u}) dx \\
&+ b_1(\boldsymbol{u}_h - \Pi_h\boldsymbol{u}, p - r_h(p)) + b_1(\boldsymbol{u}_h - \Pi_h\boldsymbol{u}, r_h(p) - p_h) \\
&\text{and} \\
&b_1(\boldsymbol{u}_h - \Pi_h\boldsymbol{u}, r_h(p) - p_h) = b_1(\boldsymbol{u} - \Pi_h\boldsymbol{u}, r_h(p) - p_h)\,.
\end{aligned}
$$

Now, $|\boldsymbol{v}|\boldsymbol{v}$ is monotone, hence $\int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\Pi_h\boldsymbol{u}|\Pi_h\boldsymbol{u}) \cdot (\boldsymbol{u}_h - \Pi_h\boldsymbol{u}) dx$ is non-negative, and one gets

$$
\begin{aligned}
&a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h - \Pi_h\boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) \\
&\leq -a(\theta_h + \widetilde{\theta}_0; \Pi_h\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) - a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) + a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) \\
&- a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) + \beta \int_\Omega (|\boldsymbol{u}|\boldsymbol{u} - |\Pi_h\boldsymbol{u}|\Pi_h\boldsymbol{u}) \cdot (\boldsymbol{u}_h - \Pi_h\boldsymbol{u}) dx \\
&+ b_1(\boldsymbol{u}_h - \Pi_h\boldsymbol{u}, p - r_h(p)) + b_1(\boldsymbol{u} - \Pi_h\boldsymbol{u}, r_h(p) - p_h)\,.
\end{aligned}
\tag{3.13}
$$

Owing to the weak convergence of $\boldsymbol{u}_h$ in $L^3(\Omega)^2$, the strong convergence of $\Pi_h\boldsymbol{u}$ in $L^3(\Omega)^2$, the strong convergence of θ_h in $H^1(\Omega)$, the continuity of both $\nu(\cdot)$ and $|\boldsymbol{u}|\boldsymbol{u}$, the first term in the right hand side of (3.13) tends to zero. Similarly, the strong convergence of $r_h(p)$ in M and the weak convergence of $\Pi_h\boldsymbol{u}$ and $\boldsymbol{u}_h$, both in $L^3(\Omega)^d$ show that the sixth term in the right hand side of (3.13) tends to zero. Finally, the strong convergence of $\Pi_h\boldsymbol{u}$ in $L^3(\Omega)^d$, the strong convergence of both p_h and $r_h(p)$ in M imply that the last term in the right hand side of (3.13) tends to zero. Consequently,

$$
\lim_{h\to 0} a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h - \Pi_h\boldsymbol{u}, \boldsymbol{u}_h - \Pi_h\boldsymbol{u}) = 0,
$$

thus yielding the asserted result by making use of the triangle inequality. □

We now turn to the pressure and note that ∇p_h is bounded in $L^{3/2}(\Omega)^d$ and having in mind that $p_h \in M$, Poincaré's inequality is applicable, so p_h is also bounded. Hence

$$
\lim_{h\to 0} \nabla p_h = F \text{ weakly in } L^{3/2}(\Omega)^d \text{ and } \lim_{h\to 0} p_h = p \text{ weakly in } L^{3/2}(\Omega)\,.
$$

We then need to show that $F = \nabla p$.

Proposition 3.5 *Let $(\boldsymbol{u}_h, \theta_h, p_h)$ be the solution of (3.1) and $(\boldsymbol{u}, \theta, F)$ be the limit functions given in (3.6). Assume that the conditions mentioned in Assumption 3.1 are valid.*

Then $F = \nabla p$ *and moreover* $(\boldsymbol{u}, \theta, p)$ *satisfies*

$$\begin{cases} \text{for all } \ (\boldsymbol{v}, q) \in L^3(\Omega)^d \times M \\ a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}) + \beta \displaystyle\int_\Omega |\boldsymbol{u}|\boldsymbol{u} \cdot \boldsymbol{v} dx + b_1(\boldsymbol{v}, p) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \\ b_1(\boldsymbol{u}, q) = 0\,. \end{cases}$$

The proof follows the lines of the proof of Lemma 2 in [7]. The only difference here is the presence of the temperature.
Thirdly, we claim that

Proposition 3.6 *Let* $(\boldsymbol{u}_h, \theta_h, p_h)$ *be the solution of (3.1), and* $(\boldsymbol{u}, \theta, p)$ *given by (3.6) and satisfying the first two equations of (2.5). Assume that the conditions mentioned in Assumption 3.1 are valid. Then*

$$\lim_{h \to 0} \boldsymbol{u}_h = \boldsymbol{u} \quad \text{strongly in } L^3(\Omega)^d$$
$$\lim_{h \to 0} \nabla p_h = \nabla p \quad \text{strongly in } L^{3/2}(\Omega)^d\,.$$

The proof follows the lines of the proof of Theorem 6 and Theorem 7 in [7]. The only difference here is the temperature.
We now turn to the temperature and let $\rho \in H_0^1(\Omega)$ and take $s_h = R_h \rho$ in (3.4) to obtain

$$c(\theta_h + \widetilde{\theta}_0, R_h\rho) + \widetilde{d}(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, R_h\rho) = \int_\Omega g R_h \rho\, dx\,.$$

Owing to the weak convergence of θ_h in $H^1(\Omega)$ and the strong convergence of $R_h\rho$ in $H^1(\Omega)$ we have

$$\lim_{h \to 0} c(\theta_h + \widetilde{\theta}_0, R_h\rho) = c(\theta + \widetilde{\theta}_0, \rho)$$
$$\lim_{h \to 0} \int_\Omega g R_h \rho\, dx = \int_\Omega g \rho\, dx\,.$$

As for the trilinear term $\widetilde{d}(\cdot, \cdot, \cdot)$, we first consider the equivalent relation

$$\widetilde{d}(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, R_h\rho) = \frac{1}{2}\left(d(\boldsymbol{u}_h, \theta_h + \widetilde{\theta}_0, R_h\rho) - d(\boldsymbol{u}_h, R_h\rho, \theta_h + \widetilde{\theta}_0) \right)\,,$$

and follows the lines of the proof of Theorem 3.9 in [16].

Remark 3.1 *It should be noted that since the solution* $(\boldsymbol{u}, p, \theta)$ *is unique, the entire sequences* $(\boldsymbol{u}_h, p_h, \theta_h)$ *converge and not just subsequences.*

Remark 3.2 *The convergence result obtained can be interpreted in two different ways:*
(a) We actually obtained the convergence of the finite element solution $(\boldsymbol{u}_h, p_h, \theta_h)$ *towards to the solution of the continuous problem* $(\boldsymbol{u}, p, \theta)$*. Thus ensuring a sort of reliability/guarantee of the approximation.*
(b) We have indeed constructed the solution $(\boldsymbol{u}, p, \theta)$ *of the continuous problem via the finite element solution* $(\boldsymbol{u}_h, p_h, \theta_h)$*.*

3.4 A priori error estimates

We tackle the convergence here by computing a priori error estimates. These errors are obtained with the assumption that the problems (2.7) and (3.1) are uniquely solvable. It is worth noting that without uniqueness no convergence result can be formulated.
We make precise and sharpen the approximation properties in the statement of Assumption 3.1 .
The interpolation operator for the temperature is $R_h : W^{1,p}(\Omega) \cap H_0^1(\Omega) \longrightarrow H_{0h}^1(\Omega)$, is the regularized operator constructed by Bernardi and Girault in [19] (for d=2), or by Scott and Zhang [20] (when d=3) satisfying the following a priori error estimates: for all $K \in \mathcal{T}_h$, $m = 0, 1$, $l = 0, 1$, and all $p \geq 2$,

$$\text{for all } \rho \in W^{l+1,p}(\Omega),\ |\rho - R_h\rho|_{W^{m,p}(K)} \leq c(p,m,l)h^{l+1-m}|\rho|_{W^{l+1,p}(\triangle_K)}\,, \tag{3.14}$$

where $\triangle_K$ is the macro element containing the values of ρ used to define $R_h(\rho)$.
The velocity will be interpolated by a variant of the Clement's type operator in Scott and Zhang [20], i.e $\Pi_h : L^3(\Omega)^d \to \{\boldsymbol{v}_h \in \mathcal{C}(\overline{\Omega})^d,\ \text{for all } K \in \mathcal{T}_h,\ \ \boldsymbol{v}_h|_K \in \mathbb{P}_1(K)^d\}$ which is locally stable, meaning that

$$\text{for all } \boldsymbol{v} \in L^3(\Omega)^d,\ \ \|\Pi_h \boldsymbol{v}\|_{L^3(K)^d} \leq c\|\boldsymbol{v}\|_{L^3(\triangle_K)^d}\,, \tag{3.15}$$

and has the following a priori error estimates: for $m = 0, 1$ and $1 \leq l \leq 2$,

$$\text{for all } \ \boldsymbol{v} \in H^l(\Omega)^d,\ \ |\boldsymbol{v} - \Pi_h\boldsymbol{v}|_{H^m(K)^d} \leq ch^{l-m}\,|\boldsymbol{v}|_{H^l(\triangle_K)^d}\ . \tag{3.16}$$

The pressure is interpolated by $r_h : W^{1,p}(\Omega) \cap M \longrightarrow M_h$, a modification of R_h constructed in [21], and satisfying; for all $K \in \mathcal{T}_h$, $m = 0, 1$, $l = 0, 1$, and all $p \geq 2$,

$$\text{for all } \rho \in W^{l+1,p}(\Omega),\ |\rho - r_h\rho|_{W^{m,p}(K)} \leq c(p,m,l)h^{l+1-m}|\rho|_{W^{l+1,p}(\triangle_K)}\,, \tag{3.17}$$

where $\triangle_K$ is the macro element containing the values of ρ used to define $r_h(\rho)$.

We claim that

Theorem 3.1 *Let $(\boldsymbol{u}, p, \theta)$ the solution of (2.7). Let $(\boldsymbol{u}_h, p_h, \theta_h)$ the solution of (3.1). We suppose that $\nabla\theta$ and $\nabla\widetilde{\theta}_0$ are elements of $L^3(\Omega)^d$ and we take $\kappa, \nu_0, \beta, \nu_2, \boldsymbol{f}$ such that*

$$\|\nabla(\theta + \widetilde{\theta}_0)\|_{L^3(\Omega)^d} \leq c\nu_0 \left(1 + \frac{\nu_2}{\kappa(\beta\nu_0)^{1/3}}\|\boldsymbol{f}\|^{2/3}\right)^{-1}\,. \tag{3.18}$$

Then the following a priori errors hold for all $(\boldsymbol{w}_h, q_h, t_h) \in L_h^3(\Omega)^d \times M_h \times H_{0h}^1(\Omega)$

$$\begin{aligned} \|\nabla(p - p_h)\|_{L^{3/2}(\Omega)^d} \ &\leq\ c\|\nabla(q_h - p)\|_{L^{3/2}(\Omega)^d} + c\|\nabla(\theta_h - \theta)\| \\ &\quad +c\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^3(\Omega)^d}\,, \end{aligned} \tag{3.19}$$

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^3(\Omega)^d} \ &\leq\ c\|\boldsymbol{w}_h - \boldsymbol{u}\|^{2/3} + c\|\boldsymbol{w}_h - \boldsymbol{u}\|_{L^3(\Omega)^d} + c\|\nabla(q_h - p)\|^{1/2}_{L^{3/2}(\Omega)^d} \\ &\quad +c\|\nabla(t_h - \theta)\|^{1/2} + c\|\boldsymbol{w}_h - \boldsymbol{u}\|^{1/2}_{L^3(\Omega)^d} + c\|\boldsymbol{u} - \boldsymbol{w}_h\|^{2/3}_{L^3(\Omega)^d} \end{aligned} \tag{3.20}$$

$$\|\nabla(\theta_h - \theta)\| \leq \quad c\|\nabla(\theta - t_h)\| + c\|\boldsymbol{u} - \boldsymbol{w}_h\|_{L^3(\Omega)^d} + c\|\boldsymbol{u}_h - \boldsymbol{u}_h\|_{L^3(\Omega)^d}\,. \tag{3.21}$$

Proof. Let $(\boldsymbol{u}, \theta, p)$ and $(\boldsymbol{u}_h, \theta_h, p_h)$ solve respectively (2.7) and (3.1). The proof is conducted in three steps.

<u>error analysis on the pressure</u>. Since we have conforming finite elements spaces, we deduce from velocity equations of (2.7) and (3.1) that for all $\boldsymbol{v}_h \in L_h^3(\Omega)^d$

$$\begin{aligned} b_1(\boldsymbol{v}_h, p - p_h) =& a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) \\ &+ \beta \int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{u}|\boldsymbol{u})\, \boldsymbol{v}_h dx\,. \end{aligned} \tag{3.22}$$

From the definition of $a(\cdot,\cdot)$, the mean value-theorem, Hölder's inequality, (2.3) and (2.1) one has

$$\begin{aligned} & a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) \\ &= \int_\Omega \nu(\theta_h + \widetilde{\theta}_0)\boldsymbol{u}_h \cdot \boldsymbol{v}_h dx - \int_\Omega \nu(\theta + \widetilde{\theta}_0)\boldsymbol{u} \cdot \boldsymbol{v}_h dx \\ &= \int_\Omega \nu(\theta_h + \widetilde{\theta}_0)(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{v}_h dx - \int_\Omega \left(\nu(\theta + \widetilde{\theta}_0) - \nu(\theta_h + \widetilde{\theta}_0)\right) \boldsymbol{u} \cdot \boldsymbol{v}_h dx \\ &= \int_\Omega \nu(\theta_h + \widetilde{\theta}_0)(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{v}_h dx + \int_\Omega \nu'(\theta^*)(\theta_h - \theta)\boldsymbol{u} \cdot \boldsymbol{v}_h dx \\ &\leq \nu_1 \int_\Omega |\boldsymbol{u}_h - \boldsymbol{u}|\ |\boldsymbol{v}_h| dx + \nu_2 \int_\Omega |\theta_h - \theta|\ |\boldsymbol{u}|\, |\boldsymbol{v}_h| dx \\ &\leq \nu_1 \|\boldsymbol{u}_h - \boldsymbol{u}\| \|\boldsymbol{v}_h\| + \nu_2 \|\theta_h - \theta\|_{L^6(\Omega)} \|\boldsymbol{u}\| \|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \\ &\leq \nu_1 \|\boldsymbol{u}_h - \boldsymbol{u}\| \|\boldsymbol{v}_h\| + c\nu_2 \|\nabla(\theta_h - \theta)\| \|\boldsymbol{u}\| \|\boldsymbol{v}_h\|_{L^3(\Omega)^d}\,. \end{aligned} \tag{3.23}$$

Next using Hölder's inequality and (2.24)

$$\begin{aligned} & \int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{u}|\boldsymbol{u}) \cdot \boldsymbol{v}_h dx \\ \leq\ & \|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \left(\int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{u}|\boldsymbol{u})^{3/2} dx \right)^{2/3} \\ \leq\ & c\|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \left(\int_\Omega |\boldsymbol{u}_h - \boldsymbol{u}|^{3/2} \left(|\boldsymbol{u}_h|^{3/2} + |\boldsymbol{u}|^{3/2} \right) dx \right)^{2/3} \\ \leq\ & c\|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^3(\Omega)^d} \left[\|\boldsymbol{u}_h\|_{L^3(\Omega)^d} + \|\boldsymbol{u}\|_{L^3(\Omega)^d} \right]. \end{aligned} \tag{3.24}$$

Returning to (3.22) with (3.23) and (3.24), and using the bounds on the velocity in proposition 2.3 and proposition 3.1 one obtains that

$$\begin{cases} \text{for all } \boldsymbol{v}_h \in L_h^3(\Omega)^d\,, \\ b_1(\boldsymbol{v}_h, p - p_h) \leq\ \nu_1 \|\boldsymbol{u}_h - \boldsymbol{u}\| \|\boldsymbol{v}_h\| + \dfrac{c\nu_2}{\nu_0} \|\boldsymbol{f}\| \|\nabla(\theta_h - \theta)\| \|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \\ + \dfrac{c}{(\nu_0\beta)^{1/3}} \|\boldsymbol{f}\|^{2/3} \|\boldsymbol{v}_h\|_{L^3(\Omega)^d} \|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^3(\Omega)^d}\,. \end{cases} \tag{3.25}$$

Let $q_h \in M_h$, it follows from the inf-sup condition (3.3) and (3.25) that

$$\begin{aligned}
\gamma\|\nabla(q_h - p_h)\|_{L^{3/2}(\Omega)} &\leq \sup_{\boldsymbol{v}_h \in L_h^3(\Omega)^d} \frac{b_1(\boldsymbol{v}_h, q_h - p_h)}{\|\boldsymbol{v}_h\|_{L^3(\Omega)^d}} \\
&\leq \sup_{\boldsymbol{v}_h \in L_h^3(\Omega)^d} \frac{b_1(\boldsymbol{v}_h, q_h - p) + b_1(\boldsymbol{v}_h, p - p_h)}{\|\boldsymbol{v}_h\|_{L^3(\Omega)^d}} \\
&\leq \|\nabla(q_h - p)\|_{L^{3/2}(\Omega)} + \nu_1\|\boldsymbol{u}_h - \boldsymbol{u}\| + \frac{c\nu_2}{\nu_0}\|\boldsymbol{f}\|\|\nabla(\theta_h - \theta)\| \\
&\quad + \frac{c}{(\nu_0\beta)^{1/3}}\|\boldsymbol{f}\|^{2/3}\|\boldsymbol{u}_h - \boldsymbol{u}\|_{L^3(\Omega)^d}\,. \qquad (3.26)
\end{aligned}$$

Using the triangle inequality, we recover the estimate on the pressure.
error analysis on the velocity. Since we have conforming finite elements spaces, we deduce from (2.7) and (3.1) that

$$\begin{cases}
\text{for all } \ (\boldsymbol{v}_h, q_h) \in L_h^3(\Omega)^d \times M_h, \\
a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) + \beta \displaystyle\int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{u}|\boldsymbol{u}) \cdot \boldsymbol{v}_h dx = b_1(\boldsymbol{v}_h, p - p_h)\,, \\
b_1(\boldsymbol{u}_h - \boldsymbol{u}, q_h) = 0\,.
\end{cases} \qquad (3.27)$$

Using the mean value theorem, one has

$$a(\theta_h + \widetilde{\theta}_0; \boldsymbol{u}_h, \boldsymbol{v}_h) - a(\theta + \widetilde{\theta}_0; \boldsymbol{u}, \boldsymbol{v}_h) = \int_\Omega \nu'(\theta^*)(\theta_h - \theta)\boldsymbol{u} \cdot \boldsymbol{v}_h dx + \int_\Omega \nu(\theta + \widetilde{\theta}_0)(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{v}_h dx\,.$$

Thus we can re-write (3.27) as follows

$$\begin{cases}
\text{for all } \ (\boldsymbol{v}_h, q_h) \in L_h^3(\Omega)^d \times M_h, \\
\displaystyle\int_\Omega \nu(\theta + \widetilde{\theta}_0)(\boldsymbol{u}_h - \boldsymbol{u}) \cdot \boldsymbol{v}_h dx + \beta \int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{u}|\boldsymbol{u}) \cdot \boldsymbol{v}_h dx \\
= b_1(\boldsymbol{v}_h, p - p_h) - \displaystyle\int_\Omega \nu'(\theta^*)(\theta_h - \theta)\boldsymbol{u} \cdot \boldsymbol{v}_h dx\,, \\
b_1(\boldsymbol{u}_h - \boldsymbol{u}, q_h) = 0\,.
\end{cases} \qquad (3.28)$$

Let $\boldsymbol{w}_h \in L_h^3(\Omega)^d$, inserting it in (3.28) by using the linearity of operators involved, one gets

$$\begin{cases}
\text{for all } \ (\boldsymbol{v}_h, q_h) \in L_h^3(\Omega)^d \times M_h, \\
\displaystyle\int_\Omega \nu(\theta + \widetilde{\theta}_0)(\boldsymbol{u}_h - \boldsymbol{w}_h) \cdot \boldsymbol{v}_h dx + \int_\Omega \nu(\theta + \widetilde{\theta}_0)(\boldsymbol{w}_h - \boldsymbol{u}) \cdot \boldsymbol{v}_h dx \\
+\beta \displaystyle\int_\Omega (|\boldsymbol{u}_h|\boldsymbol{u}_h - |\boldsymbol{w}_h|\boldsymbol{w}_h) \cdot \boldsymbol{v}_h dx + \beta \int_\Omega (|\boldsymbol{w}_h|\boldsymbol{w}_h - |\boldsymbol{u}|\boldsymbol{u}) \cdot \boldsymbol{v}_h dx \\
= b_1(\boldsymbol{v}_h, p - p_h) - \displaystyle\int_\Omega \nu'(\theta^*)(\theta_h - \theta)\boldsymbol{u} \cdot \boldsymbol{v}_h dx\,, \\
b_1(\boldsymbol{u} - \boldsymbol{w}_h, q_h - p_h) + b_1(\boldsymbol{w}_h - \boldsymbol{u}_h, q_h - p_h) = 0\,.
\end{cases} \qquad (3.29)$$

We take successively $\boldsymbol{v}_h = \boldsymbol{u}_h$ and $\boldsymbol{v}_h = \boldsymbol{w}_h$ in (3.29) and take the difference in the resulting equations. This gives

$$\begin{aligned}
& \int_\Omega \nu(\theta+\widetilde{\theta}_0)|\boldsymbol{w}_h-\boldsymbol{u}_h|^2 dx + \beta\int_\Omega (|\boldsymbol{w}_h|\boldsymbol{w}_h - |\boldsymbol{u}_h|\boldsymbol{u}_h)\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx \\
= \;& \int_\Omega \nu(\theta+\widetilde{\theta}_0)(\boldsymbol{w}_h-\boldsymbol{u})\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx + \beta\int_\Omega (|\boldsymbol{w}_h|\boldsymbol{w}_h - |\boldsymbol{u}|\boldsymbol{u})\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx \quad (3.30)\\
+ \;& b_1(\boldsymbol{w}_h-\boldsymbol{u}, p_h-q_h) + b_1(\boldsymbol{w}_h-\boldsymbol{u}_h, q_h-p) + \int_\Omega \nu'(\theta^*)(\theta_h-\theta)\boldsymbol{u}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)dx\,.
\end{aligned}$$

We need to bound from below the left hand side of (3.30) and from above the right hand of (3.30). First from (2.3) and (2.24)

$$\begin{aligned}
& \int_\Omega \nu(\theta+\widetilde{\theta}_0)|\boldsymbol{w}_h-\boldsymbol{u}_h|^2 dx + \beta\int_\Omega (|\boldsymbol{w}_h|\boldsymbol{w}_h - |\boldsymbol{u}_h|\boldsymbol{u}_h)\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx \\
& \geq \nu_0\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^2 + c\beta\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^3_{L^3(\Omega)^d}\,.
\end{aligned}$$

Secondly from (2.3), following the way (3.24) is derived, using the bound on the velocity in proposition 2.3

$$\begin{aligned}
& \int_\Omega \nu(\theta+\widetilde{\theta}_0)(\boldsymbol{w}_h-\boldsymbol{u})\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx + \beta\int_\Omega (|\boldsymbol{w}_h|\boldsymbol{w}_h - |\boldsymbol{u}|\boldsymbol{u})\cdot(\boldsymbol{w}_h-\boldsymbol{u}_h)dx \\
& \leq \nu_1\|\boldsymbol{w}_h-\boldsymbol{u}\|\|\boldsymbol{w}_h-\boldsymbol{u}_h\| + \beta c\|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\left(\|\boldsymbol{w}_h\|_{L^3(\Omega)^d} + \|\boldsymbol{u}\|_{L^3(\Omega)^d}\right) \\
& \leq \nu_1\|\boldsymbol{w}_h-\boldsymbol{u}\|\|\boldsymbol{w}_h-\boldsymbol{u}_h\| + \beta c\|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\left(\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d} + 2\|\boldsymbol{u}\|_{L^3(\Omega)^d}\right) \\
& \leq \nu_1\|\boldsymbol{w}_h-\boldsymbol{u}\|\|\boldsymbol{w}_h-\boldsymbol{u}_h\| + \beta c\|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\left(\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d} + c\frac{\|\boldsymbol{f}\|^{2/3}}{(\nu_0\beta)^{1/3}}\right)\,.
\end{aligned}$$

Finally from (2.3), (2.1), the bound on the velocity in proposition 2.3

$$\begin{aligned}
& b_1(\boldsymbol{w}_h-\boldsymbol{u}, p_h-q_h) + b_1(\boldsymbol{w}_h-\boldsymbol{u}_h, q_h-p) + \int_\Omega \nu'(\theta^*)(\theta_h-\theta)\boldsymbol{u}\cdot(\boldsymbol{u}_h-\boldsymbol{w}_h)dx \\
& \leq \|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\|\nabla(p_h-q_h)\|_{L^{3/2}(\Omega)} + \|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\nabla(q_h-p)\|_{L^{3/2}(\Omega)} \\
& + \frac{c\nu_2}{(\nu_0\beta)^{1/3}}\|\boldsymbol{f}\|^{2/3}\|\nabla(\theta_h-\theta)\|\|\boldsymbol{u}_h-\boldsymbol{w}_h\|\,.
\end{aligned}$$

Returning to (3.30) with the above inequalities, one has

$$\begin{aligned}
& \nu_0\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^2 + c\beta\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^3_{L^3(\Omega)^d} \\
\leq & \nu_1\|\boldsymbol{w}_h-\boldsymbol{u}\|\|\boldsymbol{w}_h-\boldsymbol{u}_h\| + \beta c\|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\boldsymbol{w}_h-\boldsymbol{u}\|^2_{L^3(\Omega)^d} \\
& + \frac{c\beta}{(\nu_0\beta)^{1/3}}\|\boldsymbol{f}\|^{2/3}\|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d} \\
& + \|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\|\nabla(p_h-q_h)\|_{L^{3/2}(\Omega)} + \|\boldsymbol{w}_h-\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\nabla(q_h-p)\|_{L^{3/2}(\Omega)} \\
& + \frac{c\nu_2}{(\nu_0\beta)^{1/3}}\|\boldsymbol{f}\|^{2/3}\|\nabla(\theta_h-\theta)\|\|\boldsymbol{u}_h-\boldsymbol{w}_h\|,
\end{aligned}$$

and by (2.12), (3.26) and the triangle inequality one obtains

$$\frac{\nu_0}{2}\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^2+c\beta\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^3_{L^3(\Omega)^d}\leq\frac{\nu_1^2}{2\nu_0}\|\boldsymbol{w}_h-\boldsymbol{u}\|^2+c(\beta)\left(\|\boldsymbol{w}_h-\boldsymbol{u}\|^3_{L^3(\Omega)^d}+\|\nabla(q_h-p)\|^{3/2}_{L^{3/2}(\Omega)}\right)$$
$$+c(\beta,\nu_2,\nu_0,\|\boldsymbol{f}\|)\|\nabla(t_h-\theta)\|^{3/2}+c(\nu_1,\nu_0,\beta,\|\boldsymbol{f}\|)\|\boldsymbol{w}_h-\boldsymbol{u}\|^{3/2}_{L^3(\Omega)^d}$$
$$+c\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\left(\|\nabla(q_h-p)\|_{L^{3/2}(\Omega)}+\nu_1\|\boldsymbol{w}_h-\boldsymbol{u}\|+\frac{\nu_2\|\boldsymbol{f}\|}{\nu_0}\|\nabla(t_h-\theta)\|+\frac{\|\boldsymbol{f}\|^{2/3}}{(\nu_0\beta)^{1/3}}\|\boldsymbol{w}_h-\boldsymbol{u}\|_{L^3(\Omega)^d}\right)$$
$$+\frac{c\|\boldsymbol{f}\|\nu_2}{\nu_0}\|\nabla(\theta_h-t_h)\|\|\boldsymbol{w}_h-\boldsymbol{u}\|+\frac{c\nu_2\|\boldsymbol{f}\|^{2/3}}{(\beta\nu_0)^{1/3}}\|\nabla(\theta_h-t_h)\|\|\boldsymbol{u}_h-\boldsymbol{w}_h\|\,. \tag{3.31}$$

Hence one needs to estimate the temperature to close that inequality.

<u>error analysis on the temperature</u>. Since $H^1_{0h}(\Omega)\subset H^1_0(\Omega)$, from the temperature equations in (2.7) and (3.1), we deduce that

$$\text{for all } s_h\in H^1_{0h}(\Omega)^d\quad,\quad c(\theta-\theta_h,s_h)=\widetilde{d}(\boldsymbol{u}_h,\theta_h+\widetilde{\theta}_0,s_h)-\widetilde{d}(\boldsymbol{u},\theta+\widetilde{\theta}_0,s_h)\,.$$

For any $t_h\in H^1_{0h}(\Omega)$, we let $s_h=\theta_h-t_h$. Using the anti-property of $\widetilde{d}(\cdot,\cdot,\cdot)$ (see (3.2)) one gets

$$\begin{aligned}\kappa\|\nabla(\theta_h-t_h)\|^2=&c(\theta-t_h,\theta_h-t_h)+\widetilde{d}(\boldsymbol{u}_h,\theta_h-\theta,\theta_h-t_h)-\widetilde{d}(\boldsymbol{u}-\boldsymbol{w}_h,\theta+\widetilde{\theta}_0,\theta_h-t_h)\\&-\widetilde{d}(\boldsymbol{w}_h-\boldsymbol{u}_h,\theta+\widetilde{\theta}_0,\theta_h-t_h)\\=&c(\theta-t_h,\theta_h-t_h)+\widetilde{d}(\boldsymbol{u}_h,t_h-\theta,\theta_h-t_h)+\widetilde{d}(\boldsymbol{u}-\boldsymbol{w}_h,\theta_h-t_h,\theta+\widetilde{\theta}_0)\\&+\widetilde{d}(\boldsymbol{w}_h-\boldsymbol{u}_h,\theta_h-t_h,\theta+\widetilde{\theta}_0)\,.\end{aligned}$$

From Hölder's inequality, (2.1) one deduces that

$$\begin{aligned}\kappa\|\nabla(\theta_h-t_h)\|\leq&\kappa\|\nabla(\theta-t_h)\|+c\|\boldsymbol{u}_h\|_{L^3(\Omega)^d}\|\nabla(t_h-\theta)\|+c\|\boldsymbol{u}-\boldsymbol{w}_h\|_{L^3(\Omega)^d}\|\nabla(\theta+\widetilde{\theta}_0)\|\\&+c\|\boldsymbol{w}_h-\boldsymbol{u}_h\|\|\nabla(\theta+\widetilde{\theta}_0)\|_{L^3(\Omega)}\,.\end{aligned}\tag{3.32}$$

We replace (3.32) in (3.31) and use (2.12) to obtain

$$\left(-\|\nabla(\theta+\widetilde{\theta}_0)\|_{L^3(\Omega)}-\frac{c\nu_2\|\boldsymbol{f}\|^{2/3}}{\kappa(\beta\nu_0)^{1/3}}\|\nabla(\theta+\widetilde{\theta}_0)\|_{L^3(\Omega)}+\frac{\nu_0}{2}\right)\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^2+c\beta\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^3_{L^3(\Omega)^d}$$
$$\leq\frac{\nu_1^2}{2\nu_0}\|\boldsymbol{w}_h-\boldsymbol{u}\|^2+c(\beta)\left(\|\boldsymbol{w}_h-\boldsymbol{u}\|^3_{L^3(\Omega)^d}+\|\nabla(q_h-p)\|^{3/2}_{L^{3/2}(\Omega)}\right)+c(\beta,\nu_2,\nu_0,\|\boldsymbol{f}\|)\|\nabla(t_h-\theta)\|^{3/2}$$
$$+c(\nu_1,\nu_0,\beta,\|\boldsymbol{f}\|)\|\boldsymbol{w}_h-\boldsymbol{u}\|^{3/2}_{L^3(\Omega)^d}+c\|\boldsymbol{u}-\boldsymbol{w}_h\|^2_{L^3(\Omega)^d}+c\|\nabla(\theta+\widetilde{\theta}_0)\|_{L^3(\Omega)}\|\boldsymbol{w}_h-\boldsymbol{u}\|^2\,. \tag{3.33}$$

Finally applying the assumption (3.18), one gets

$$\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^2+\|\boldsymbol{w}_h-\boldsymbol{u}_h\|^3_{L^3(\Omega)^d}\leq c_1\|\boldsymbol{w}_h-\boldsymbol{u}\|^2+c_2\|\boldsymbol{w}_h-\boldsymbol{u}\|^3_{L^3(\Omega)^d}+c_3\|\nabla(q_h-p)\|^{3/2}_{L^{3/2}(\Omega)}$$
$$+c_4\|\nabla(t_h-\theta)\|^{3/2}+c_5\|\boldsymbol{w}_h-\boldsymbol{u}\|^{3/2}_{L^3(\Omega)^d}+c_6\|\boldsymbol{u}-\boldsymbol{w}_h\|^2_{L^3(\Omega)^d}+c_7\|\boldsymbol{w}_h-\boldsymbol{u}\|^2\,. \tag{3.34}$$

The estimate on the velocity is obtained by application of the triangle inequality. The inequality on the temperature is obtained from (3.32) and application of the triangle inequality. □

Remark 3.3 *Considering theorem 3.1, together with the operators R_h, Π_h, r_h introduced earlier, and taken $(\boldsymbol{u}, p, \theta)$ in $H^1(\Omega)^d \times W^{2,3/2}(\Omega) \times W^{2,3}(\Omega)$, we deduce that*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^3(\Omega)^d}+\|p-p_h\|_{W^{1,3/2}(\Omega)}+\|\theta-\theta_h\|_{H^1(\Omega)} \le ch^{1/2}\left(\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d} + \|p\|_{W^{2,3/2}(\Omega)} + \|\theta\|_{W^{2,3}(\Omega)}\right).$$

We believe that the sup-optimality is due amongst other to the nonlinear term in the equations.

4 Numerical experiments

4.1 Iterative scheme

The finite element problem (3.1) is nonlinear hence iterative/incremental scheme need to be formulated for its resolution. A direct strategy to make the above system less nonlinear and weaken the coupling between its various equations is to linearize the nonlinear terms, but the main drawback of this approach is that the stiffness matrix of the resulting system of equations is not fixed, thus will require more computational time. We formulate in the lines that follows a strategy based on linearization and operator splitting. This methodology allow us to decouple the computation of the velocity and pressure from the temperature. The starting procedure is the Laplace equation for the temperature and a linear Darcy's equations for the velocity and pressure. This initial step reads:
Find $(\theta_h^0, \boldsymbol{u}_h^0, p_h^0) \in H^1_{0h}(\Omega) \times L^3_h(\Omega)^d \times M_h$ solution of

$$\text{for all } \; s_h \in H^1_{0h}(\Omega), \qquad c(\theta_h^0, s_h) = \int_\Omega g s_h \, dx - c(\widetilde{\theta}_0, s_h)\,, \tag{4.1}$$

and

$$\begin{cases} \text{for all } \; (\boldsymbol{v}_h, q_h) \in L^3_h(\Omega)^d \times M_h, \\ \displaystyle\int_\Omega \nu(\theta_h^0 + \widetilde{\theta}_0)\nabla \boldsymbol{u}_h^0 : \nabla \boldsymbol{v}_h dx + b_1(\boldsymbol{v}_h, p_h^0) = \int_\Omega \boldsymbol{f}\cdot \boldsymbol{v}_h \, dx\,, \\ b_1(\boldsymbol{u}_h^0, q_h) = 0\,. \end{cases} \tag{4.2}$$

Then knowing $(\theta_h^0, \boldsymbol{u}_h^0, p_h^0)$, compute the sequence $(\theta_h^n, \boldsymbol{u}^n, p_h^n)$ for $n \ge 1$, until an adequate stopping condition is satisfied, by decoupling the temperature from the velocity and pressure as follows:
- Knowing $(\boldsymbol{u}_h^n, p_h^n, \theta_h^n)$ compute $\boldsymbol{u}_h^{n+1/3}$, $\boldsymbol{u}_h^{n+2/3}$ in $L^3_h(\Omega)^d$, $(\boldsymbol{u}_h^{n+1}, p_h^{n+2/3}) \in L^3_h(\Omega)^d \times M_h$ such that for all $(\boldsymbol{v}_h, q_h) \in L^3_h(\Omega)^d \times M_h$

$$\frac{1}{\alpha}\left(\nabla(\boldsymbol{u}_h^{n+1/3} - \boldsymbol{u}_h^n), \nabla \boldsymbol{v}_h\right) + \frac{1}{2}a(\theta_h^n + \widetilde{\theta}_0; \boldsymbol{u}_h^{n+1/3}, \boldsymbol{v}_h) = 0\,, \tag{4.3}$$

$$\begin{cases} \dfrac{1}{\alpha}\left(\nabla(\boldsymbol{u}_h^{n+2/3}-\boldsymbol{u}_h^{n+1/3}),\nabla\boldsymbol{v}_h\right)+\dfrac{1}{2}a(\theta_h^n+\widetilde{\theta}_0;\boldsymbol{u}_h^{n+2/3},\boldsymbol{v}_h)+b_1(\boldsymbol{v}_h,p_h^{n+2/3})=\displaystyle\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}_h\,dx\,,\\ b_1(\boldsymbol{u}_h^{n+2/3},q_h)=0\,. \end{cases} \tag{4.4}$$

$$\frac{1}{\alpha}\left(\nabla(\boldsymbol{u}_h^{n+1}-\boldsymbol{u}_h^{n+2/3}),\nabla\boldsymbol{v}_h\right)+\beta\left(\left|\boldsymbol{u}_h^{n+2/3}\right|\boldsymbol{u}_h^{n+1},\boldsymbol{v}_h\right)=0\,. \tag{4.5}$$

Finally we compute $\theta_h^{n+1}\in H^1_{0h}(\Omega)$ solution of

$$\begin{cases} \text{for all}\quad s_h\in H^1_{0h}(\Omega)\\ c(\theta_h^{n+1},s_h)+\widetilde{d}(\boldsymbol{u}_h^{n+1},\theta_h^{n+1},s_h)=\displaystyle\int_\Omega g s_h\,dx-c(\widetilde{\theta}_0,s_h)-\widetilde{d}(\boldsymbol{u}_h^{n+1},\widetilde{\theta}_0,s_h)\,. \end{cases} \tag{4.6}$$

The equations (4.3),(4.4), (4.5) are obtained by adding the artificial derivative in time, with α being the positive parameter chosen to enhance convergence.
The algorithm (4.3), (4.4), (4.5) and (4.6) has been used in [26, 30], and justify in this context by realising that when t approaches ∞

$$\boldsymbol{u}\longrightarrow\boldsymbol{u}_\infty\quad\text{in } L^2 \tag{4.7}$$

with $\boldsymbol{u}_\infty$ the velocity from (2.7) and $\boldsymbol{u}$ the velocity from the evolution problem

$$\begin{cases} \boldsymbol{u}(0)=\boldsymbol{u}_0\,,\\ \text{for all}\quad (\boldsymbol{v},\rho)\in K(\Omega)\times H^1_0(\Omega),\\ (\partial_t\nabla\boldsymbol{u},\nabla\boldsymbol{v})+a(\theta;\boldsymbol{u},\boldsymbol{v})+\beta\displaystyle\int_\Omega|\boldsymbol{u}|\boldsymbol{u}\cdot\boldsymbol{v}dx=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,dx\,,\\ c(\theta,\rho)+d(\boldsymbol{u},\theta,\rho)=\displaystyle\int_\Omega g\rho\,dx\,. \end{cases} \tag{4.8}$$

Thus computing the solution of (2.7) is almost equivalent to compute the solution of (4.8) for big enough time.

4.2 Simulations

We now study the numerical behavior of the algorithm (4.3), (4.4), (4.5) and (4.6). The test problems used are designed to illustrate the numerical behavior of the algorithm rather than the actual Darcy-Forchheimer model coupled with the heat. We have implemented algorithm (4.3), (4.4), (4.5) and (4.6) by taking $\nu(\theta)=1+e^{-\theta}$.

4.2.1 Flow past a circular cylinder

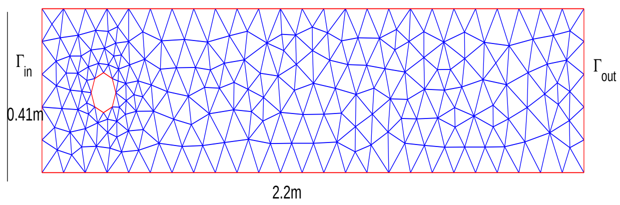
This is classical problem in computational fluid mechanic and has been studied by many authors [31, 32, 33]. The geometry together with the boundary conditions are given in Figure 1 and (4.9)

$$\begin{cases} u_1=0.3/0.41^2*4y(0.41-y),\ u_2=0\quad\text{on}\quad\Gamma_{in}=\{0\}\times(0,0.41),\\ u_1=0.3/0.41^2*4y(0.41-y),\ u_2=0\quad\text{on}\quad\Gamma_{out}=\{2.2\}\times(0,0.41)\,. \end{cases} \tag{4.9}$$

Of course it is assumed that on the other part of the boundary of Ω, homogeneous boundary conditions is prescribed. We also use $\kappa = 1$, while the value of β is indicated on the plots. For the temperature θ and heat source g, we take

$$\begin{aligned}
&\theta(x,y) = xy(2.2-x)(0.41-y)((x-0.2)^2+(y-0.2)^2-0.5^2)\\
&g(x,y) = -\kappa\Delta\theta(x,y)\\
&\theta_0(x,y) = 0, \quad f(x,y) = 0.
\end{aligned}$$

Figure 1: Geometry and boundary conditions



In this example, we take $h = 1/10$, $\alpha = 0.01$ and the convergence of the evolutionary problem is observed when $T = 1$. In the results plotted in Figure 2, it is observed that the flow is unidirectional and uniform far from the circle, while the temperature is uniformly distributed. It is worth mentioning that the same pattern were obtained for $\beta = 0.001$ and $\beta = 0.01$.

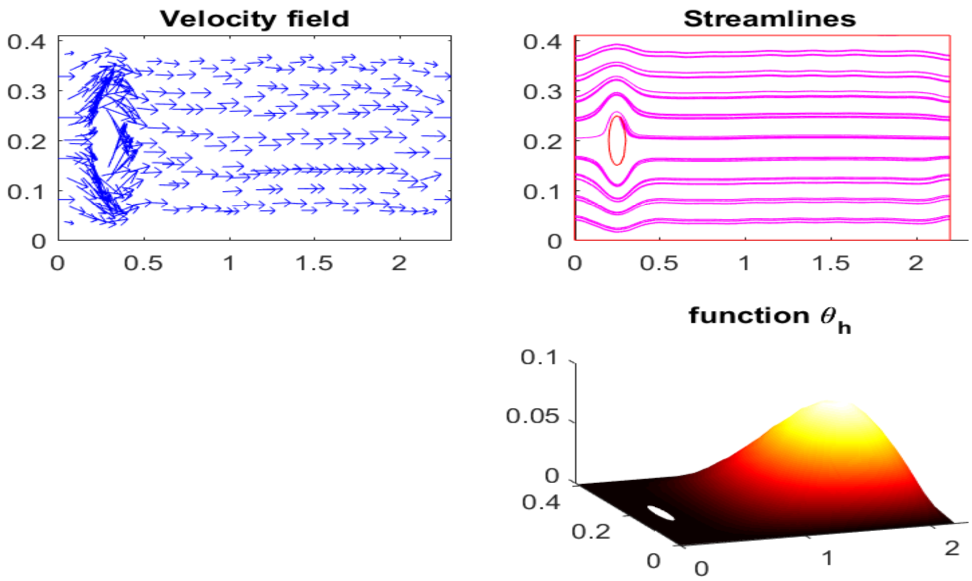


Figure 2: velocity-streamlines and temperature for $\beta = 1/2$

4.2.2 Driven cavity flow

This is a standard benchmark for assessing the performance of algorithms for many flow problems and it has been studied by many authors in different context(see [30, 31, 32, 34]). The configuration is as depicted in Figure 3. It corresponds to a flow in a box $\Omega = (0,1)^2$, with the boundary $\partial\Omega = \Gamma \cup S$ with

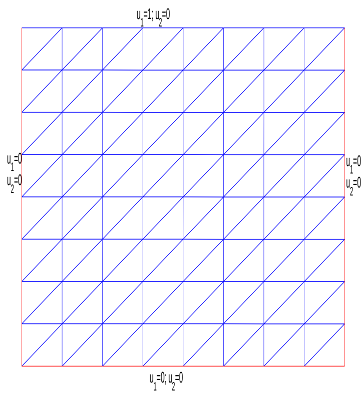
$$\begin{aligned} \Gamma &= \{(0,y)/0 < y < 1\} \cup \{(x,0)/0 < x < 1\} \cup \{(1,y)/0 < y < 1\} \\ S &= \{(x,1)/0 < x < 1\}. \end{aligned}$$

For this example, we also use $\kappa = 1$ and the following functions

$$\begin{aligned} &\theta(x,y) = xy(1-x)(1-y) \\ &g(x,y) = -\kappa\Delta\theta(x,y) \\ &\theta_0(x,y) = 0, \quad f(x,y) = 0. \end{aligned}$$

For the simulations we have used $h = 1/20$, $\alpha = 0.01$ and the convergence of the evolutionary problem is observed when $T = 1$. The stream function, the velocity and temperature distribution are represented in Figure 4, Figure 5 and Figure 6 below. It is noted when β is decreasing we have less iterations.

Figure 3: Driven Cavity description



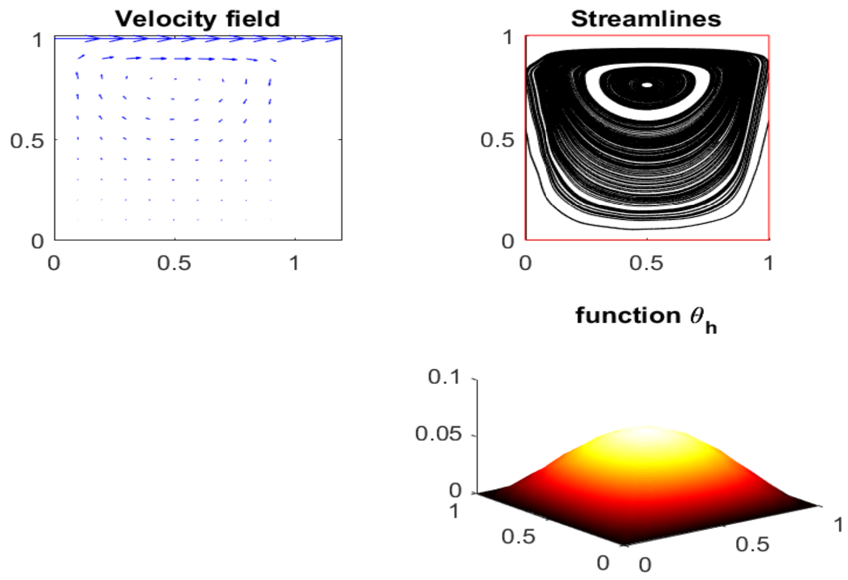


Figure 4: velocity-streamlines-temperature for $\beta = 0.25$, CPU=32.25s, iter=97

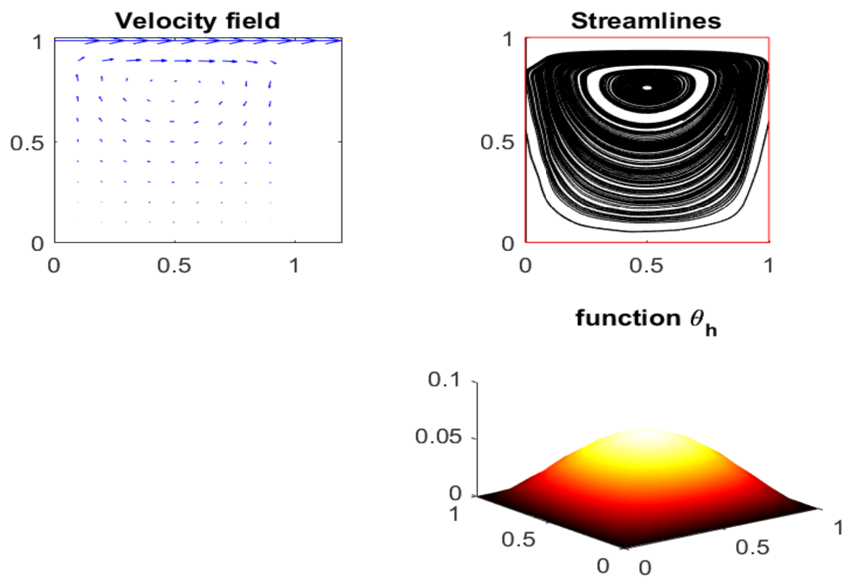


Figure 5: velocity-streamlines-temperature for $\beta = 0.1$, CPU=32.1, iter=92

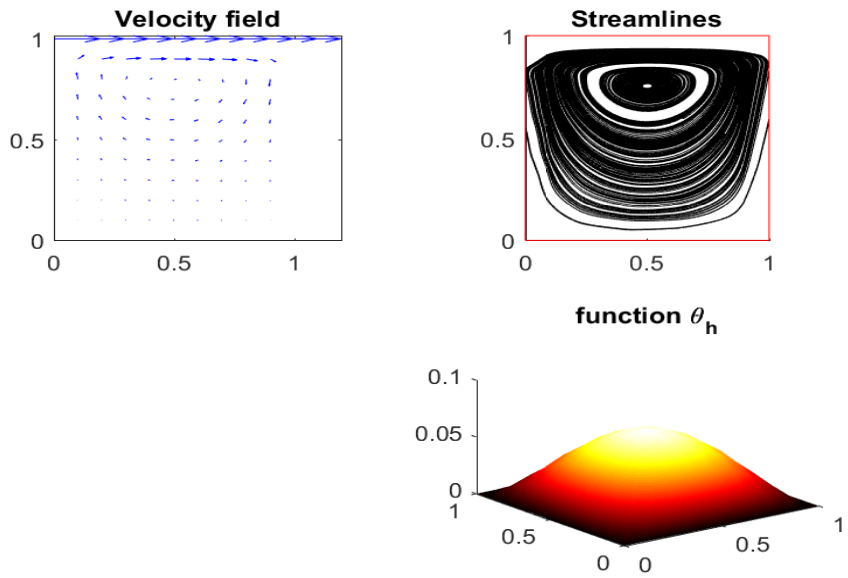


Figure 6: velocity-streamlines-temperature for $\beta = 0.01$, CPU=31.51, iter=89

4.2.3 Rate of convergence

In this test, we are interested in the rate of convergence of the finite element solution $(\boldsymbol{u}_h, p_h, \theta_h)$. We recall that (see remark 3.3)

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{L^3(\Omega)^d}+\|p-p_h\|_{W^{1,3/2}(\Omega)}+\|\theta-\theta_h\|_{H^1(\Omega)} \leq ch^{1/2}\left(\|\boldsymbol{u}\|_{W^{1,3}(\Omega)^d}+\|p\|_{W^{2,3/2}(\Omega)}+\|\theta\|_{W^{2,3}(\Omega)}\right).$$

We consider a flow region $\Omega = (0,1)^2$ with analytical solution $(\boldsymbol{u}, p, \theta)$ given as follows

$$\begin{cases} u_1(x,y) = -\sin(\pi x)\cos(\pi y) \\ u_2(x,y) = \cos(\pi x)\sin(\pi y) \\ p(x,y) = -\frac{1}{\pi}\sin(\pi x)\cos(\pi y) \\ \theta(x,y) = 2\sin^2(\pi x)\sin(\pi y). \end{cases} \tag{4.10}$$

The right-hand side $\boldsymbol{f}$ and g are taken for (4.10) to be the exact solution given β and $\kappa = 1$. The stream function, the velocity and temperature distribution are represented in Figure 7, and Figure 8 below, and the rate of convergence is calculated in table 1 and table 2, which is agreement with the theoretical findings.

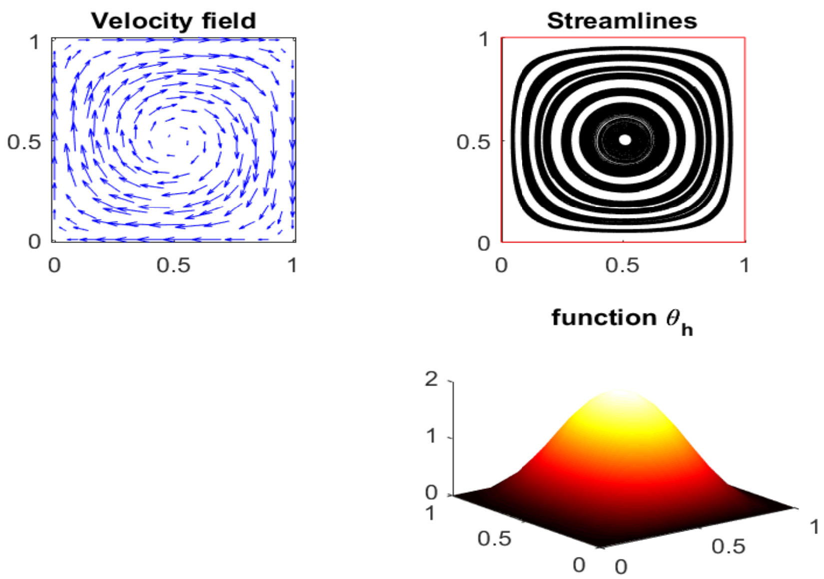


Figure 7: velocity-streamlines-temperature for $\beta = 1/2$

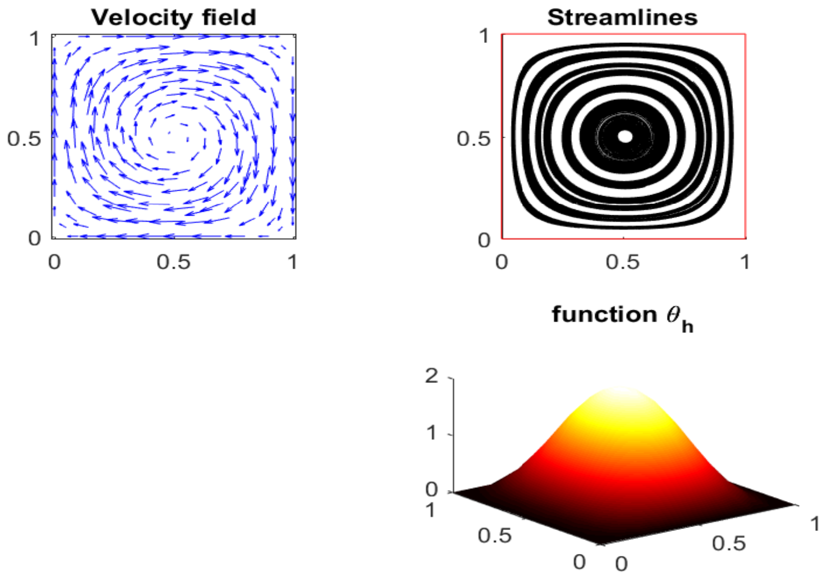


Figure 8: velocity-streamlines-temperature for $\beta = 1$

h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|$	Rate	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^3}$	Rate	$\|p - p_h\|_{W^{1,3/2}}$	Rate
1/4	2.571e-2		1.051e-4		5.057e-1	
1/8	1.451e-2	0.82	1.703e-4	0.70	3.737e-1	0.44
1/16	6.921e-3	1.06	1.033e-4	0.69	2.345e-1	0.67
1/32	3.512e-3	0.97	6.987e-5	0.57	3.407e-1	0.54

Table 1: Convergence rates with $\beta = 0.25$

h	$\|\theta_{ref} - \theta_h\|$	Rate	$\|\theta_{ref} - \theta_h\|_1$	Rate
1/4	9.844e-2		1.009e+0	
1/8	2.873e-2	1.77	5.067e-1	0.97
1/16	7.124e-3	2.01	2.478e-1	1.03
1/32	1.885e-3	1.91	1.246e-1	0.95

Table 2: Convergence rates with $\beta = 0.25$

4.3 Conclusion

We have presented a Darcy-Forchheimer's equation coupled with the heat equation in its continuous and finite element version. Conditions for the unique solvability of the continuous and finite element equations are investigated. Convergence of the finite element solution is obtained by making used of Babuska-Brezzi's theory for mixed formulations. Finally, numerical algorithm for the actual computation of the finite element problem is formulated and implemented. Numerical simulations exhibited validate the predictions of the theory developed.

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