

A Posteriori Estimates of Taylor-Hood Element for Stokes Problem Using Auxiliary Subspace Techniques

Jiachuan Zhang · Ran Zhang · Xiaoshen Wang

Abstract Based on the auxiliary subspace techniques, a hierarchical basis *a posteriori* error estimator is proposed for the Stokes problem in two and three dimensions. For the error estimator, we need to solve only two global diagonal linear systems corresponding to the degree of freedom of velocity and pressure respectively, which reduces the computational cost sharply. The upper and lower bounds up to an oscillation term of the error estimator are also shown to address the reliability of the adaptive method without saturation assumption. Numerical simulations are performed to demonstrate the effectiveness and robustness of our algorithm.

Keywords Adaptive method · Taylor-Hood element · Auxiliary subspace techniques · *A posteriori* error estimate · Stokes problem

Mathematics Subject Classification (2010) 65N15 · 65N30 · 65M12 · 76D07

1 Introduction

In this paper, we propose an *a posteriori* error estimator based on the auxiliary subspace techniques for the Taylor-Hood finite element method (FEM) [8,9] to solve Stokes equations [18,25] with Dirichlet boundary condition

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \quad (1.1)$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \quad (1.2)$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma, \quad (1.3)$$

where $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a bounded polygonal or polyhedral domain with the boundary Γ . The function \mathbf{u} is a vector velocity field and p is the pressure. The functions \mathbf{f} and \mathbf{g} are given Lebesgue square-integrable functions on Ω and Γ , respectively. The problem (1.1)-(1.3) has a unique solution in the sense that p is only determined up to an additive constant. In the later sections, we will analyze the case of $\mathbf{g} = \mathbf{0}$, and the case $\mathbf{g} \neq \mathbf{0}$ is similar.

A posteriori error estimators and adaptive FEM can be used to solve the problems with local singularities effectively. Hierarchical basis *a posteriori* estimator is a popular approach and has been proven to be robust and efficient, whose origins can be traced back to [26,27]. In this approach, let V_k and W_{k+d} be the approximation space and auxiliary space, respectively, where $V_k \cap W_{k+d} = \{0\}$ (to be specified in Section 2). The solution of approximation problem (2.8) is denoted by $(\hat{\mathbf{u}}, \hat{p}) \in V_k$. Then the approximation error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ can be estimated in auxiliary space W_{k+d} with the help of the error problem

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(2.15). Traditionally, the upper and lower bounds of error estimations need to make use of a saturation assumption, i.e. the best approximation of (\mathbf{u}, p) in $V_k \cup W_{k+d}$ is strictly better than its best approximation in V_k . Although saturation assumption is widely accepted in *a posteriori* error analysis [16, 1] and satisfied in the case of small data oscillation [13], it is not difficult to construct counter-examples for particular problems on particular meshes [11]. To remove the saturation assumption, Araya et al. presented an adaptive stabilized FEM combined with a hierarchical basis *a posteriori* error estimator in a special auxiliary bubble function spaces for generalized Stokes problem and Navier-Stokes equations. The error analysis of upper and lower bounds avoids the use of saturation assumption. Although the construction of auxiliary space needs a transformation operator in the reference element, it provides a novel idea for removing saturation assumption in reliability analysis [2, 3, 4, 5]. Hakula et al. constructed the auxiliary space directly on each element for the second order elliptic problem and elliptic eigenvalue problem and proved that the error is bounded by the error estimator up to oscillation terms without the saturation assumption [17, 15].

The contribution of this paper is twofold. Firstly, we extend the auxiliary subspace techniques in [17] to the Stokes problem in two and three dimensions. More specifically, we construct auxiliary spaces for velocity and pressure, respectively and prove that these auxiliary spaces satisfy the inf-sup condition shown in Lemma 2.2. The error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ can be bounded by the solution of the error problem (2.15), the term $\|\nabla \cdot \hat{\mathbf{u}}\|$ and the oscillation term $osc(\mathbf{f})$ (Theorem 3.1). We emphasize that the error analysis does not use the saturation assumption. The other contribution of the present work is the diagonalization of the error problem to reduce the computational cost. Considering that the Stokes problem is a saddle point problem, we replace part of the matrix, which is related to velocity only, with a diagonal matrix in (4.4) to construct the second error problem shown in (4.8). Then the solution of (4.8) combined with the term $\|\nabla \cdot \hat{\mathbf{u}}\|$ and the oscillation term $osc(\mathbf{f})$ can be used to bound the error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ (Theorem 4.2). Here, obtaining the pressure and velocity requires solving a non-diagonal and diagonal linear system, respectively. To further reduce the computation, the diagonal matrix is obtained by multiplying the diagonal matrix of pressure correlation matrix by a constant c_s related to the number of the bases of pressure in each element. Now, the linear systems of pressure and velocity are both diagonal, which is the third error problem shown in (4.25) whose solution combined with the term $\|\nabla \cdot \hat{\mathbf{u}}\|$ and the oscillation term $osc(\mathbf{f})$ can be used to bound the error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ (Theorem 4.4).

The rest of the work is organized as follows. In Section 2, the FEM spaces, the approximation problem, and the first error problem are introduced. Section 3 presents a quasi-interpolant based on moment conditions and develops *a posteriori* error estimation related to the first error problem for the Stokes equation. In Section 4, to reduce the computational cost, the system diagonalization techniques are developed for velocity (the second error problem) and pressure (the third error problem), respectively. The *a posteriori* error estimates of the corresponding error problems are shown. In Section 5, we obtained the local and global *a posteriori* error estimators, and an adaptive FEM is proposed based on the solution of the third error problem and term $\|\nabla \cdot \hat{\mathbf{u}}\|$. In Section 6, numerical experiment results are presented to verify the effectiveness of our adaptive algorithm. The last section is devoted to some concluding remarks.

2 Approximation Problem and Error Problem

The following notations are used in this paper

$$\begin{aligned} a(\mathbf{w}, \mathbf{v}) &= \int_{\Omega} \nabla \mathbf{w} : \nabla \mathbf{v}, \\ b(\mathbf{v}, q) &= \int_{\Omega} q \nabla \cdot \mathbf{v}, \\ a_1((\mathbf{w}, r), (\mathbf{v}, q)) &= a(\mathbf{w}, \mathbf{v}) - b(\mathbf{v}, r) + b(\mathbf{w}, q), \\ f(\mathbf{v}) &= \int_{\Omega} \mathbf{f} \cdot \mathbf{v}, \end{aligned}$$

for all $(\mathbf{v}, q), (\mathbf{w}, r) \in V := [H_0^1(\Omega)]^d \times L_0^2(\Omega)$, where

$$\begin{aligned} H_0^1(\Omega) &= \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma\}, \\ L_0^2(\Omega) &= \{q \in L^2(\Omega) \mid \int_{\Omega} q = 0\}. \end{aligned}$$

The variational formulation of (1.1)-(1.3) is: Find $(\mathbf{u}, p) \in V$ such that

$$a_1((\mathbf{u}, p), (\mathbf{v}, q)) = f(\mathbf{v}), \quad (2.1)$$

for all $(\mathbf{v}, q) \in V$.

We denote by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ the standard norm and semi-norm of Sobolev space with $m \geq 0$, respectively. For the sake of convenience, we will use $\|\cdot\|$ and $|\cdot|$ for $\|\cdot\|_{0,\Omega}$ and $|\cdot|_{0,\Omega}$, respectively. For the coupling space V , we define

$$\|(\mathbf{v}, q)\|_V = \sqrt{\|\nabla \mathbf{v}\|^2 + \|q\|^2}. \quad (2.2)$$

From Cauchy-Schwarz inequality, $a_1(\cdot, \cdot)$ is continuous, i.e.

$$|a_1((\mathbf{w}, r), (\mathbf{v}, q))| \leq \mathfrak{C}_1 \|(\mathbf{w}, r)\|_V \|(\mathbf{v}, q)\|_V. \quad (2.3)$$

From Proposition 4.69 in [25], $a_1(\cdot, \cdot)$ satisfies the estimates

$$\inf_{(\mathbf{v}, q) \in V \setminus \{0\}} \sup_{(\mathbf{w}, r) \in V \setminus \{0\}} \frac{a_1((\mathbf{v}, q), (\mathbf{w}, r))}{\|(\mathbf{v}, q)\|_V \|(\mathbf{w}, r)\|_V} \geq \mathfrak{c}_1. \quad (2.4)$$

We refer to \mathfrak{C}_1 and \mathfrak{c}_1 as the continuity and inf-sup constant, respectively.

2.1 Approximation Problem

Let \mathcal{T} be a family of conforming, shape-regular simplicial partition of Ω . Let \mathcal{F} denote the set of $(d-1)$ -dimensional sub-simplices, the “faces” of \mathcal{T} , and further decompose it as $\mathcal{F} = \mathcal{F}_I \cup \mathcal{F}_D$, where \mathcal{F}_I comprises those faces in the interior of Ω , and \mathcal{F}_D comprises those faces in Γ . To ensure that the Taylor-Hood element satisfies the stability condition (inf-sup condition), we make the following assumptions for \mathcal{T} :

Assumption 1. \mathcal{T} contains at least three triangles in the case of $d = 2$.

Assumption 2. Every element $T \in \mathcal{T}$ has at least one vertex in the interior of Ω in the case of $d = 3$.

In our scheme, in order to have a conforming approximation we shall choose the finite-dimensional spaces VV_{k+1} and VP_k with $k \geq 1$ (called Hood-Taylor or Taylor-Hood element)

$$VV_{k+1} = \{\hat{\mathbf{v}} \in [H_0^1(\Omega)]^d \mid \hat{\mathbf{v}}|_T \in [P_{k+1}]^d, \forall T \in \mathcal{T}\} \subset [H_0^1(\Omega)]^d, \quad (2.5)$$

$$VP_k = \{\hat{q} \in H^1(\Omega) \mid \hat{q}|_T \in P_k(K), \forall T \in \mathcal{T}, \int_{\Omega} \hat{q} = 0\} \subset L_0^2(\Omega), \quad (2.6)$$

$$V_k = VV_{k+1} \times VP_k. \quad (2.7)$$

A mixed finite element method to approximate (2.1) is called an **approximation problem**: Find $(\hat{\mathbf{u}}, \hat{p}) \in V_k$ such that

$$a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q})) = f(\hat{\mathbf{v}}), \quad (2.8)$$

for all $(\hat{\mathbf{v}}, \hat{q}) \in V_k$.

Remark 2.1 The solvability of the approximation problem (2.8) can be found in [8, 9, 10].

2.2 Error Problem

Given a simplex $T \subset \mathbb{R}^d$ of diameter h_T , we define $\mathcal{S}_j(T)$, $0 \leq j \leq d$ to be the set of sub-simplices of T of dimension j . The cardinality is $|\mathcal{S}_j(T)| = \binom{d+1}{j+1}$. We denote by \mathcal{S}_j the set of sub-simplices of the triangulation of dimension j , in particular, $\mathcal{S}_{d-1} = \mathcal{F}_I \cup \mathcal{F}_D$ and $\mathcal{S}_d = \mathcal{T}$. Recall that $P_m(S)$ is the set of polynomials of total degree $\leq m$ with domain S , and note that $\dim P_m(S) = \binom{m+j}{j}$ for $S \in \mathcal{S}_j(T)$. Denoting the vertices of T by $\{z_0, \dots, z_d\}$, we let $\lambda_i \in P_1(T)$, $0 \leq i \leq d$, be the corresponding barycentric coordinates, uniquely defined by the relation $\lambda_i(z_i) = \delta_{ij}$. We denote by $F_j \in \mathcal{S}_{d-1}(T)$ the sub-simplex not containing z_j .

The fundamental element and face bubbles for T are given by $(j = 0, 1, \dots, d)$

$$b_T = \prod_{k=0}^d \lambda_k \in P_{d+1}(T), \quad b_{F_j} = \prod_{\substack{k=0 \\ k \neq j}}^d \lambda_k \in P_d(T).$$

We also define general element and face bubbles of degree m ,

$$Q_m(T) = \{\hat{v} = b_T \hat{w} \in P_m(T) \mid \hat{w} \in P_{m-d-1}(T)\}, \quad (2.9)$$

$$Q_m(F_j) = \{\hat{v} = b_{F_j} \hat{w} \in P_m(T) \mid \hat{w} \in P_{m-d}(T)\} \ominus Q_m(T). \quad (2.10)$$

From now on, we use the shorthand $W_1 \ominus W_2 = \text{span}\{W_1 \setminus W_2\}$ for vector spaces W_1 and W_2 . So $W_1 \ominus W_2$ is the largest subspace of W_1 such that $W_1 \cap W_2 = \{0\}$. The functions in $Q_m(T)$ are precisely those in $P_m(T)$ that vanish on ∂T , and the functions in $Q_m(F_j)$ are precisely those in $P_m(T)$ that vanish on $\partial T \setminus F_j$. It is clear that $Q_m(T) \cap Q_m(F_j) = \{0\}$ and $Q_m(F_i) \cap Q_m(F_j) = \{0\}$ for $i \neq j$. The collection of face bubbles of degree m can be denoted by

$$Q_m(\partial T) = \bigoplus_{j=0}^d Q_m(F_j).$$

Then we define the local space

$$R_m(T) = Q_m(T) \oplus Q_m(\partial T),$$

which contains all element and face bubbles of degree m related to T defined in (2.9) and (2.10), and the corresponding global finite element spaces

$$R_m = \{\hat{v} \in H_0^1(\Omega) \mid \hat{v}|_T \in R_m(T) \text{ for each } T \in \mathcal{T}\}.$$

Lemma 2.1 *A function $\hat{v} \in R_m(T)$ is uniquely determined by the moments*

$$\int_S \hat{v} \kappa, \quad \forall \kappa \in P_{m-\ell-1}(S), \quad \forall S \in S_\ell(T), \quad d-1 \leq \ell \leq d. \quad (2.11)$$

Proof As is shown in [6], a function $v \in P_m(T)$ is uniquely determined by the moments

$$\int_S \hat{v} \kappa, \quad \forall \kappa \in P_{m-\ell-1}(S), \quad \forall S \in S_\ell(T), \quad 0 \leq \ell \leq d,$$

where $\int_S \hat{v} \kappa$ with $S \in S_0(T)$ is understood to be the evaluation of \hat{v} at the vertex S . Since $\hat{v} \in R_m(T)$ is uniquely determined by its moments on T and $F_j, j = 0, \dots, d$, the result is clear. \square

Given $k \in \mathbb{N}$, we define the local error space for velocity by element and face bubbles

$$WV_{k+d+1}(T) = [R_{k+d+1}(T) \ominus R_{k+1}(T)]^d,$$

and for pressure by element bubbles

$$WP_{k+d}(T) = Q_{k+d}(T) \ominus Q_k(T).$$

The velocity and pressure error spaces are constructed this way to satisfy the inf-sup condition shown in Lemma 2.3.

The corresponding global finite element spaces, defined by the degrees of freedom and local spaces, are given by

$$WV_{k+d+1} = \{\hat{\mathbf{w}} \in [H_0^1(\Omega)]^d \mid \hat{\mathbf{w}}|_T \in WV_{k+d+1}(T) \text{ for each } T \in \mathcal{T}\}, \quad (2.12)$$

$$WP_{k+d} = \{\hat{r} \in L_0^2(\Omega) \cap H^1(\Omega) \mid \hat{r}|_T \in WP_{k+d}(T) \text{ for each } T \in \mathcal{T}\}, \quad (2.13)$$

$$W_{k+d} = WV_{k+d+1} \times WP_{k+d}, \quad (2.14)$$

where $V_k \cap W_{k+d} = \{0\}$. Then the **error problem** is: Find $(\hat{e}_u, \hat{e}_p) \in W_{k+d}$ such that

$$a_1((\hat{e}_u, \hat{e}_p), (\hat{\mathbf{v}}, \hat{q})) = f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q})), \quad (2.15)$$

for any $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}$.

2.3 Solvability of Error Problem

The error problem is stable (in the sense of inf-sup condition) in W_{k+d} from the following Lemma 2.2 and Lemma 2.3. Let \overline{P}_k be the set of homogeneous polynomials of degree $k \geq 1$. Then we define

$$\overline{WV}_{k+j+1} = WV_{k+d+1} \cap [\overline{P}_{k+j+1}]^d, \quad \overline{WP}_{k+j} = WP_{k+d} \cap \overline{P}_{k+j},$$

where $1 \leq j \leq d$.

Lemma 2.2 *Under Assumptions 1 and 2, there exist positive constants μ_j ($1 \leq j \leq d$) independent of h such that*

$$\sup_{\hat{\mathbf{v}} \in \overline{WV}_{k+j+1}} \frac{b(\hat{\mathbf{v}}, \hat{q})}{\|\hat{\mathbf{v}}\|_{1,\Omega}} \geq \mu_j \|\hat{q}\|, \quad \forall \hat{q} \in \overline{P}_{k+j},$$

where $k \geq 1$.

Proof The proof can be found in Appendix A. □

Lemma 2.3 *There exists a positive constant μ independent of h such that*

$$\sup_{\hat{\mathbf{v}} \in WV_{k+d+1}} \frac{b(\hat{\mathbf{v}}, \hat{q})}{\|\hat{\mathbf{v}}\|_{1,\Omega}} \geq \mu \|\hat{q}\|, \quad \forall \hat{q} \in WP_{k+d}, \quad (2.16)$$

where $k \geq 1$.

Proof It follows from $\overline{WP}_{k+j} \subset \overline{P}_{k+j}$ and Lemma 2.2 that

$$\sup_{\hat{\mathbf{v}} \in \overline{WV}_{k+j+1}} \frac{b(\hat{\mathbf{v}}, \hat{q})}{\|\hat{\mathbf{v}}\|_{1,\Omega}} \geq \mu_j \|\hat{q}\|, \quad \forall \hat{q} \in \overline{WP}_{k+j},$$

for $k \geq 1$ and $1 \leq j \leq d$. Then set $\mu = \min_{1 \leq j \leq d} \mu_j$ and complete the proof from the facts

$$WV_{k+d+1} = \bigoplus_{j=1}^d \overline{WV}_{k+j+1}, \quad WP_{k+d} = \bigoplus_{j=1}^d \overline{WP}_{k+j}.$$

□

From Lemma 2.3, the proof of the following lemma is similar to that of Proposition 4.69 in [25]. For the sake of completeness, we give the proof here.

Lemma 2.4 *The bilinear form $a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))$ satisfies the estimate*

$$\inf_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_V \|(\hat{\mathbf{w}}, \hat{r})\|_V} \geq \frac{\mu^2}{(1 + \mu)^2} \quad (2.17)$$

where μ is a constant defined in Lemma 2.3.

Proof Let $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}$ be an arbitrary but fixed function. The definition of $a_1(\cdot, \cdot)$ immediately implies that

$$a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}}, \hat{q})) = \|\nabla \hat{\mathbf{v}}\|.$$

Due to Lemma 2.3, there is a velocity field $\hat{\mathbf{w}}_{\hat{q}} \in WV_{k+d+1}$ with $\|\nabla \hat{\mathbf{w}}_{\hat{q}}\| = 1$ such that

$$\int_{\Omega} \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} \geq \mu \|\hat{q}\|.$$

We therefore obtain for every $\delta > 0$

$$\begin{aligned} a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) &= a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}}, \hat{q})) - \delta \|\hat{q}\| a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}_{\hat{q}}, 0)) \\ &= \|\nabla \hat{\mathbf{v}}\|^2 - \delta \|\hat{q}\| \int_{\Omega} \nabla \hat{\mathbf{v}} : \nabla \hat{\mathbf{w}}_{\hat{q}} + \delta \|\hat{q}\| \int_{\Omega} \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} \\ &\geq \|\nabla \hat{\mathbf{v}}\|^2 - \delta \|\nabla \hat{\mathbf{v}}\| \|\hat{q}\| + \delta \mu \|\hat{q}\|^2 \\ &\geq (1 - \frac{\delta}{2\mu}) \|\nabla \hat{\mathbf{v}}\|^2 + \frac{1}{2} \delta \mu \|\hat{q}\|^2. \end{aligned}$$

The choice of $\delta = \frac{2\mu}{1+\mu^2}$ yields

$$a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) \geq \frac{\mu^2}{1+\mu^2} \|(\hat{\mathbf{v}}, \hat{q})\|_V^2.$$

On the other hand, we have

$$\begin{aligned} \|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_V &\leq \|(\hat{\mathbf{v}}, \hat{q})\|_V + \|(\delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, 0)\|_V \\ &= \|(\hat{\mathbf{v}}, \hat{q})\|_V + \delta \|\hat{q}\| \|\nabla \hat{\mathbf{w}}_{\hat{q}}\| \\ &= \|(\hat{\mathbf{v}}, \hat{q})\|_V + \delta \|\hat{q}\| \\ &\leq (1 + \delta) \|(\hat{\mathbf{v}}, \hat{q})\|_V \\ &= \frac{1 + \mu^2 + 2\mu}{1 + \mu^2} \|(\hat{\mathbf{v}}, \hat{q})\|_V. \end{aligned}$$

Combining these estimates we arrive at

$$\begin{aligned} \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_V \|(\hat{\mathbf{w}}, \hat{r})\|_V} &\geq \frac{a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_V \|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_V} \\ &\geq \frac{\mu^2}{1 + \mu^2}. \end{aligned}$$

Since $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}$ was arbitrary, this completes the proof. \square

Theorem 2.1 *The error problem (2.15) has a unique solution.*

Proof For the system (2.15), one can easily check that $a_1(\cdot, \cdot)$ is a continuous bilinear form on $W_{k+d} \times W_{k+d} \subset V \times V$ by (2.3) and satisfies the inf-sup condition by Lemma 2.4. In addition, $f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q}))$ is a continuous linear functional on W_{k+d} and the bilinear form $a_1(\cdot, \cdot)$ satisfies

$$a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}}, \hat{q})) = \|\nabla \hat{\mathbf{v}}\|^2 \geq C \|\hat{\mathbf{v}}\|^2 > 0, \quad \hat{\mathbf{v}} \neq 0,$$

by Poincaré's inequalities. So by Theorem 5.2.1 in [7], the scheme (2.15) has a unique solution. \square

3 A Posteriori Error Estimation

In this section, a quasi-interpolant based on moment conditions will be shown in Lemma 3.1, which is used to get the *a posteriori* error estimate shown in Theorem 3.1.

Lemma 3.1 *Given $\mathbf{v} \in [H^1(\Omega)]^d$, there exists a $\hat{\mathbf{v}} \in VV_{k+1}$ and $\hat{\mathbf{w}} \in WW_{k+d+1}$ such that*

- (1) $\int_T (\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}) \cdot \boldsymbol{\kappa} = 0$ for all $\boldsymbol{\kappa} \in [P_k(T)]^d$ and $T \in \mathcal{T}$.
- (2) $\int_F (\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}) \cdot \boldsymbol{\kappa} = 0$ for all $\boldsymbol{\kappa} \in [P_{k+1}(F)]^d$ and $F \in \mathcal{F}_I$.
- (3) $|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}|_{m,T} \leq C_{\mathcal{T}} h_T^{1-m} |\mathbf{v}|_{1,\Omega_T}$ for $m = 0, 1$, where Ω_T is a local patch of elements containing T .
- (4) $|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}|_{0,F} \leq C_{\mathcal{T}} h_F^{1/2} |\mathbf{v}|_{1,\Omega_F}$, where h_F is the diameter of $F \in \mathcal{F}$, and $\Omega_F = \Omega_T$ for some $T \in \mathcal{T}$ with $F \subset \partial T$.
- (5) $|\hat{\mathbf{w}}|_{1,T} \leq C_{\mathcal{T}} |\mathbf{v}|_{1,\Omega_T}$ for each $T \in \mathcal{T}$.

where $C_{\mathcal{T}}$ depends only on the dimension d , polynomial degree k , and the shape-regularity of \mathcal{T} .

Proof Since functions in $R_{k+d+1}(T)$ are uniquely determined by the moments (2.11), for $m = 0, 1$ the function $\langle\langle \cdot \rangle\rangle_{m,T} : [R_{k+d+1}(T)]^d \rightarrow \mathbb{R}^+$ defined by

$$\langle\langle \phi \rangle\rangle_{m,T} = \max_{\substack{S \in \mathcal{S}_{\ell}^{\ell}(T) \\ d-1 \leq \ell \leq d}} \sup_{\boldsymbol{\kappa} \in [P_{k+d-\ell}(S)]^d} \frac{h_T^{d/2-\ell/2-m}}{\|\boldsymbol{\kappa}\|_{0,S}} \int_S \phi \cdot \boldsymbol{\kappa}$$

is a norm on $[R_{k+d+1}(T)]^d$.

Let $\tilde{T} = \{y = h_T^{-1}x : x \in T\}$, and for each $\psi : T \rightarrow \mathbb{R}$, define $\tilde{\psi} : \tilde{T} \rightarrow \mathbb{R}$ by $\tilde{\psi}(y) = \psi(h_T y)$. Analogous definitions are given for the sub-simplices of T and \tilde{T} and functions defined on them. It is clear that $|\phi|_{m,T} = h_T^{d/2-m} |\tilde{\phi}|_{m,\tilde{T}}$, where $|\cdot|_{0,T} = \|\cdot\|_{0,T}$. We also have for any $S \in \mathcal{S}_{\ell}(T)$

$$\frac{h_T^{d/2-\ell/2-m}}{\|\kappa\|_{0,S}} \int_S \phi \cdot \kappa = \frac{h_T^{d/2-\ell/2-m}}{h_T^{\ell/2} \|\tilde{\kappa}\|_{0,\tilde{S}}} \int_{\tilde{S}} \tilde{\phi} \cdot \tilde{\kappa} h_T^\ell = \frac{h_T^{d/2-m}}{\|\tilde{\kappa}\|_{0,\tilde{S}}} \int_{\tilde{S}} \tilde{\phi} \cdot \tilde{\kappa}.$$

Since $h_{\tilde{T}} = 1$, we set that $\langle\langle \phi \rangle\rangle_{m,T} = h_T^{d/2-m} \langle\langle \tilde{\phi} \rangle\rangle_{m,\tilde{T}}$. Therefore there exists a scale-invariant constant $C_{\mathcal{T}} > 0$ that depends solely on k, d , and m such that

$$|\phi|_{m,T} = h_T^{d/2-m} |\tilde{\phi}|_{m,\tilde{T}} \leq C_{\mathcal{T}} h_T^{d/2-m} \langle\langle \tilde{\phi} \rangle\rangle_{m,\tilde{T}} = C_{\mathcal{T}} \langle\langle \phi \rangle\rangle_{m,T}. \quad (3.1)$$

At this stage, we see that the local constant $C_{\mathcal{T}}$ in (3.1) may depend on the shape of T , but not its diameter. For the rest of the argument, we make a shape-regularity assumption on \mathcal{T} .

Next, denote by $\hat{\mathbf{v}}_1 \in VV_{k+1}$ the Scott-Zhang interpolant of \mathbf{v} satisfying [21]

$$\|\mathbf{v} - \hat{\mathbf{v}}_1\|_{m,T} \leq C_{\mathcal{T}} h_T^{1-m} |\mathbf{v}|_{1,\Omega_T}, \quad m = 0, 1, \quad (3.2)$$

$$\|\mathbf{v} - \hat{\mathbf{v}}_1\|_{0,\partial T} \leq C_{\mathcal{T}} h_T^{1/2} |\mathbf{v}|_{1,\Omega_T}, \quad (3.3)$$

on each $T \in \mathcal{T}$. Set $\hat{\mathbf{v}}_2 \in [R_{k+d+1}]^d$ such that

$$\int_S \hat{\mathbf{v}}_2 \cdot \kappa = \int_S (\mathbf{v} - \hat{\mathbf{v}}_1) \cdot \kappa, \quad \forall \kappa \in [P_{k+d-\ell}(S)]^d, \quad \forall S \in S_\ell, \quad d-1 \leq \ell \leq d.$$

By (3.1)-(3.3) we get

$$\begin{aligned} |\hat{\mathbf{v}}_2|_{m,T} &\leq C_{\mathcal{T}} \max_{\substack{S \in S_\ell(T) \\ d-1 \leq \ell \leq d}} \sup_{\kappa \in [P_{k+d-\ell}(S)]^d} \frac{h_T^{d/2-\ell/2-m}}{\|\kappa\|_{0,S}} \int_S \hat{\mathbf{v}}_2 \cdot \kappa \\ &= C_{\mathcal{T}} \max_{\substack{S \in S_\ell(T) \\ d-1 \leq \ell \leq d}} \sup_{\kappa \in [P_{k+d-\ell}(S)]^d} \frac{h_T^{d/2-\ell/2-m}}{\|\kappa\|_{0,S}} \int_S (\mathbf{v} - \hat{\mathbf{v}}_1) \cdot \kappa \\ &\leq C_{\mathcal{T}} (h_T^{1/2-m} \|\mathbf{v} - \hat{\mathbf{v}}_1\|_{0,\partial T} + h_T^{-m} \|\mathbf{v} - \hat{\mathbf{v}}_1\|_{0,T}) \leq C h_T^{1-m} |\mathbf{v}|_{1,\Omega_T}. \end{aligned}$$

Uniquely decomposing $\hat{\mathbf{v}}_2$ as $\hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_3 + \hat{\mathbf{w}}$ with $\hat{\mathbf{v}}_3 \in VV_{k+1}$ and $\hat{\mathbf{w}} \in WV_{k+d+1}$, and setting $\hat{\mathbf{v}} = \hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_3$ so that $\hat{\mathbf{v}} + \hat{\mathbf{w}} = \hat{\mathbf{v}}_1 + \hat{\mathbf{v}}_2$, we see that properties (1)-(2) clearly hold, and

$$\|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{m,T} \leq \|\mathbf{v} - \hat{\mathbf{v}}_1\|_{m,T} + \|\hat{\mathbf{v}}_2\|_{m,T} \leq C_{\mathcal{T}} h_T^{1-m} |\mathbf{v}|_{1,\Omega_T}.$$

Therefore by the standard trace inequalities and the shape regularity of the mesh, we also have on $F \subset \partial T$

$$\|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{0,F} \leq C_{\mathcal{T}} (h_F^{-1/2} \|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{0,T} + h_F^{1/2} \|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{1,T}) \leq C_{\mathcal{T}} h_F^{1/2} |\mathbf{v}|_{1,\Omega_F}.$$

Hence, properties (3)-(4) are satisfied.

Finally, since $VV_{k+1}(T) \cap WV_{k+d+1}(T) = \{0\}$, the strengthened Cauchy-Schwarz inequality [14] gives the existence of a constant $\gamma \in [0, 1)$ such that

$$\int_T \nabla \hat{\mathbf{w}} \cdot \nabla \hat{\mathbf{v}}_3 \leq \gamma |\hat{\mathbf{w}}|_{1,T} |\hat{\mathbf{v}}_3|_{1,T}.$$

Consequently, we have

$$\begin{aligned} |\hat{\mathbf{v}}_2|_{1,T}^2 &= |\hat{\mathbf{w}}|_{1,T}^2 + |\hat{\mathbf{v}}_3|_{1,T}^2 + 2 \int_T \nabla \hat{\mathbf{w}} : \nabla \hat{\mathbf{v}}_3 \\ &\geq |\hat{\mathbf{w}}|_{1,T}^2 + |\hat{\mathbf{v}}_3|_{1,T}^2 - 2\gamma |\hat{\mathbf{w}}|_{1,T} |\hat{\mathbf{v}}_3|_{1,T} \geq (1 - \gamma^2) |\hat{\mathbf{w}}|_{1,T}^2. \end{aligned}$$

Therefore we find $|\hat{\mathbf{w}}|_{1,T} \leq \sqrt{(1 - \gamma^2)^{-1}} |\hat{\mathbf{v}}_2|_{1,T} \leq C_{\mathcal{T}} |\mathbf{v}|_{1,\Omega_T}$. \square

For $(\mathbf{v}, q) \in V$, we have

$$a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{v}, q)) = f(\mathbf{v}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\mathbf{v}, q)), \quad (3.4)$$

where (\mathbf{u}, p) and $(\hat{\mathbf{u}}, \hat{p})$ are the solutions of (2.1) and (2.8), respectively. So,

$$\begin{aligned} a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{v}, q)) &= \sum_{T \in \mathcal{T}} \int_T (\mathbf{f} \cdot \mathbf{v} - \nabla \hat{\mathbf{u}} : \nabla \mathbf{v} + \nabla \cdot \mathbf{v} \hat{p} - \nabla \cdot \hat{\mathbf{u}} q) \\ &= \sum_{T \in \mathcal{T}} \int_T (\mathbf{f} \cdot \mathbf{v} - (-\Delta \hat{\mathbf{u}} \cdot \mathbf{v} + \nabla \hat{p} \cdot \mathbf{v} - \nabla \cdot \hat{\mathbf{u}} q)) \\ &\quad + \sum_{T \in \mathcal{T}} \int_{\partial T} (-\nabla \hat{\mathbf{u}} \cdot \mathbf{n}_T \cdot \mathbf{v} + \hat{p} \mathbf{v} \cdot \mathbf{n}_T) \end{aligned}$$

Lemma 3.2 For any $(\mathbf{v}, q) \in V$, $(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d}$, and $(\hat{\mathbf{v}}, \hat{q}) \in V_k$, it holds that

$$a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{v}, q)) = a_1((\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p), (\hat{\mathbf{w}}, \hat{r})) + \mathcal{R}(\mathbf{v} - \hat{\mathbf{w}} - \hat{\mathbf{v}}, q - \hat{r} - \hat{q}) \quad (3.5)$$

where (\mathbf{u}, p) and $(\hat{\mathbf{u}}, \hat{p})$ are the solutions of (2.1) and (2.8), respectively, and

$$\begin{aligned} \mathcal{R}(\mathbf{w}, r) &= f(\mathbf{w}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\mathbf{w}, r)) \\ &= \sum_{T \in \mathcal{T}} \int_T ((\mathbf{f} - \mathbf{R}_T) \cdot \mathbf{w} + \nabla \cdot \hat{\mathbf{u}} r) + \sum_{F \in \mathcal{F}_I} \int_F \mathbf{r}_F \cdot \mathbf{w}, \end{aligned}$$

for any $(\mathbf{w}, r) \in [H_0^1(\Omega)]^d \times L_0^2(\Omega)$ and

$$\begin{aligned} \mathbf{R}_T &= (-\Delta \hat{\mathbf{u}} + \nabla \hat{p})|_T, \\ \mathbf{r}_F &= (-\nabla \hat{\mathbf{u}} \cdot \mathbf{n}_T + \hat{p} \mathbf{n}_T)|_T - (-\nabla \hat{\mathbf{u}} \cdot \mathbf{n}_{T'} + \hat{p} \mathbf{n}_{T'})|_{T'}. \end{aligned}$$

Here, T and T' are the simplices sharing the face F , and \mathbf{n}_T and $\mathbf{n}_{T'}$ are their outward unit normals.

Proof From (2.8), (2.15), and (3.4), we obtain

$$\begin{aligned} &\mathcal{R}(\mathbf{v} - \hat{\mathbf{w}} - \hat{\mathbf{v}}, q - \hat{r} - \hat{q}) \\ &= \mathcal{R}(\mathbf{v}, q) - \mathcal{R}(\hat{\mathbf{w}}, \hat{r}) - \mathcal{R}(\hat{\mathbf{v}}, \hat{q}) \\ &= f(\mathbf{v}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\mathbf{v}, q)) - (f(\hat{\mathbf{w}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{w}}, \hat{r}))) - (f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q}))) \\ &= a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{v}, q)) - a_1((\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p), (\hat{\mathbf{w}}, \hat{r})), \end{aligned}$$

which completes the proof. \square

We define the local oscillation for each $T \in \mathcal{T}$ by

$$\text{osc}(\mathbf{f}, T)^2 = h_T^2 \inf_{\boldsymbol{\kappa} \in [P_k(T)]^d} \|\mathbf{f} - \boldsymbol{\kappa}\|_{0,T}^2.$$

Then define

$$\text{osc}(\mathbf{f})^2 = \sum_{T \in \mathcal{T}} \text{osc}(\mathbf{f}, T)^2. \quad (3.6)$$

Theorem 3.1 Let (\mathbf{u}, p) , $(\hat{\mathbf{u}}, \hat{p})$, and $(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)$ be the solutions of (2.1), (2.8), and (2.15), respectively. There are constants $\hat{\mathfrak{C}}_* = \frac{\mu^2}{2\mathfrak{C}_1(1+\mu)^2}$ and $\hat{\mathfrak{C}}^* = \frac{\mathfrak{C}_1}{\mathfrak{c}_1}$ such that

$$\hat{\mathfrak{C}}_* \|(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)\|_V + \frac{1}{2\sqrt{d}} \|\nabla \cdot \hat{\mathbf{u}}\| \leq \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V \leq \hat{\mathfrak{C}}^* \|(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)\|_V + \frac{1}{\mathfrak{c}_1} \|\nabla \cdot \hat{\mathbf{u}}\| + \frac{C_{\mathcal{T}}}{\mathfrak{c}_1} \text{osc}(\mathbf{f}), \quad (3.7)$$

where constants $\mathfrak{C}_1, \mathfrak{c}_1, \mu$, and $C_{\mathcal{T}}$ are defined in (2.3), (2.4), (2.16), and Lemma 3.1.

Proof Given $q \in L_0^2(\Omega)$, there exists $\hat{r} \in WP_{k+d}$ such that $\|q - \hat{r}\| \leq \|q\|$ since $WP_{k+d} \subset L_0^2(\Omega)$. Then combining Lemma 3.1, Lemma 3.2, and noting $\mathbf{R}_T \in [P_k(T)]^2, \mathbf{r}_F \in [P_k(F)]^2$, we determine that

$$\begin{aligned} |a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{v}, q))| &\leq |a_1((\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p), (\hat{\mathbf{w}}, \hat{r}))| + \sum_{T \in \mathcal{T}} \|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{0,T} \inf_{\boldsymbol{\kappa} \in [P_k(T)]^d} \|\mathbf{f} - \boldsymbol{\kappa}\|_{0,T} \\ &\quad + \sum_{T \in \mathcal{T}} \|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{0,T} \inf_{\boldsymbol{\kappa} \in [P_k(T)]^d} \|\mathbf{R}_T - \boldsymbol{\kappa}\|_{0,T} \\ &\quad + \left| \sum_{T \in \mathcal{T}} \int_T (q - \hat{q} - \hat{r}) \nabla \cdot \hat{\mathbf{u}} \right| + \sum_{F \in \mathcal{F}_I} \|\mathbf{v} - \hat{\mathbf{v}} - \hat{\mathbf{w}}\|_{0,F} \inf_{\boldsymbol{\kappa} \in [P_{k+1}(F)]^d} \|\mathbf{r}_F - \boldsymbol{\kappa}\|_{0,F} \\ &\leq \mathfrak{C}_1 \|(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)\|_V \|(\hat{\mathbf{w}}, \hat{r})\|_V + C_{\mathcal{T}} \sum_{T \in \mathcal{T}} h_T \|\mathbf{v}\|_{1,\Omega_T} \inf_{\boldsymbol{\kappa} \in [P_k(T)]^d} \|\mathbf{f} - \boldsymbol{\kappa}\|_{0,T} \\ &\quad + \sum_{T \in \mathcal{T}} \|q\| \|\nabla \cdot \hat{\mathbf{u}}\| \\ &\leq \mathfrak{C}_1 \|(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)\|_V \|(\mathbf{v}, q)\|_V + C_{\mathcal{T}} \text{osc}(\mathbf{f}) \|(\mathbf{v}, q)\|_V + \|\nabla \cdot \hat{\mathbf{u}}\| \|(\mathbf{v}, q)\|_V, \end{aligned}$$

for any $\hat{\mathbf{w}} \in WV_{k+d+1}$ and $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}$. Then the right inequality of (3.7) follows from the inf-sup condition (2.4) of continuous problem:

$$\mathfrak{c}_1 \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V \leq \sup_{(\mathbf{w}, r) \in V \setminus \{0\}} \frac{a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\mathbf{w}, r))}{\|(\mathbf{w}, r)\|_V}.$$

From (2.17), (2.15), and (2.1),

$$\begin{aligned} \frac{\mu^2}{(1 + \mu)^2} \|(\hat{\mathbf{e}}_u, \hat{e}_p)\|_V &\leq \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\hat{\mathbf{e}}_u, \hat{e}_p), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{w}}, \hat{r})\|_V} \\ &= \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{w}}, \hat{r})\|_V} \\ &= \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{w}}, \hat{r})\|_V} \\ &\leq \mathfrak{C}_1 \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V. \end{aligned}$$

Since $\nabla \cdot \mathbf{u} = 0$, we have

$$\|\nabla \cdot \hat{\mathbf{u}}\| = \|\nabla \cdot (\mathbf{u} - \hat{\mathbf{u}})\| \leq \sqrt{d} \|\nabla(\mathbf{u} - \hat{\mathbf{u}})\| \leq \sqrt{d} \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V.$$

Then we get the left inequality of (3.7). \square

4 System Diagonalization

As stated, the computation of $(\hat{\mathbf{e}}_u, \hat{e}_p)$ requires the formation and solution of a global system, so one might naturally be concerned that this approach is too expensive for practical consideration. Generally speaking, the hierarchical basis for W_{k+d} is typically made up of highly oscillatory functions with compact support, therefore we may approximate the stiffness matrix by a diagonal matrix, which reduces the cost of computation.

4.1 Diagonalization with respect to Velocity

Let $\{\phi_j\}_{j=1}^N$ be the bases for W_{k+d} , i.e.

$$W_{k+d} = \text{span}\{\phi_j\}_{j=1}^N.$$

Let $\{\varphi_j\}_{j=1}^{N_v}$ and $\{\psi_j\}_{j=1}^{N_p}$ be the bases in W_{k+d} for velocity and pressure, respectively. It is clear that $N = N_v + N_p$ and $\{\phi_j\}_{j=1}^N = \{\varphi_j\}_{j=1}^{N_v} \cup \{\psi_j\}_{j=1}^{N_p}$.

Define an $N_v \times N_v$ matrix A by $A_{\ell,j} = a(\varphi_j, \varphi_\ell)$ and an $N_v \times N_p$ matrix B by $B_{\ell,j} = -b(\psi_j, \varphi_\ell)$. Then we can rewrite (2.15) in a matrix form

$$\begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_u \\ \mathbf{x}_p \end{bmatrix} = \begin{bmatrix} F_v \\ F_p \end{bmatrix}, \quad (4.1)$$

where \mathbf{x}_u and \mathbf{x}_p are the coefficients of $\hat{\mathbf{e}}_u$ and \hat{e}_p with respect to the bases, respectively; F_v and F_p are the vectors formed by the right-hand function of (2.15) acting on the bases of velocity and pressure, respectively. For any $(\hat{\mathbf{v}}, \hat{q}) = \sum_{j=1}^N x_j \phi_j$, $(\hat{\mathbf{w}}, \hat{r}) = \sum_{j=1}^N y_j \phi_j \in W_{k+d}$, we have

$$a_1((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) = \mathbf{y}^T M \mathbf{x}, \quad (4.2)$$

where

$$\mathbf{x} = (x_1, \dots, x_N)^T, \quad \mathbf{y} = (y_1, \dots, y_N)^T, \quad \text{and} \quad M = \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix}. \quad (4.3)$$

Let \mathbf{x}_v be a vector composed of elements related to velocity in \mathbf{x} , then it holds

$$\|(\hat{\mathbf{v}}, \hat{q})\|_V^2 = |\hat{\mathbf{v}}|_{1,\Omega}^2 + \|\hat{q}\|^2 = \mathbf{x}_v^T A \mathbf{x}_v + \|\hat{q}\|^2.$$

Let D_v be the diagonal matrix with the same diagonal as A and M_v be

$$M_v = \begin{bmatrix} D_v & B \\ -B^T & 0 \end{bmatrix}. \quad (4.4)$$

Define

$$a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) = \mathbf{y}^T M_v \mathbf{x} \quad (4.5)$$

and norms

$$\|(\hat{\mathbf{v}}, \hat{q})\|_D^2 = \mathbf{x}_v^T D_v \mathbf{x}_v + \|\hat{q}\|^2, \quad (4.6)$$

$$|\hat{\mathbf{v}}|_D^2 = \mathbf{x}_v^T D_v \mathbf{x}_v. \quad (4.7)$$

Now, we are at the stage to present **the second error problem**: Find $(\tilde{\mathbf{e}}_u, \tilde{e}_p) \in W_{k+d}$ such that

$$a_2(\tilde{\mathbf{e}}_u, \tilde{e}_p, (\hat{\mathbf{v}}, \hat{q})) = f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q})), \quad \forall (\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}, \quad (4.8)$$

where $a_2(\cdot, \cdot)$ is specified in (4.5).

For any $T \in \mathcal{T}$ and $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}$, denote by $\{\varphi_{T,j}\}_{j=1}^{N_{v,T}}$ the basis functions of velocity related to T , then $\hat{\mathbf{v}}_T := \hat{\mathbf{v}}|_T = \sum_{j=1}^{N_{v,T}} x_{T,j} \varphi_{T,j}$ with $\{x_{T,j}\}_{j=1}^{N_{v,T}}$ being the coefficients. Let $\hat{\mathbf{v}}_{T,j} := x_{T,j} \varphi_{T,j}$, then $\hat{\mathbf{v}}_T = \sum_{j=1}^{N_{v,T}} \hat{\mathbf{v}}_{T,j}$. We can rewrite $|\hat{\mathbf{v}}|_{1,\Omega}$ and $|\hat{\mathbf{v}}|_D$ as follows:

$$|\hat{\mathbf{v}}|_{1,\Omega}^2 = \sum_{T \in \mathcal{T}} |\hat{\mathbf{v}}_T|_{1,T}^2,$$

$$|\hat{\mathbf{v}}|_D^2 = \sum_{T \in \mathcal{T}} \sum_{j=1}^{N_{v,T}} |\hat{\mathbf{v}}_{T,j}|_{1,T}^2.$$

We define the local norm of $|\cdot|_D$ by

$$|\hat{\mathbf{v}}|_{D,T} = \sqrt{\sum_{j=1}^{N_{v,T}} |\hat{\mathbf{v}}_{T,j}|_{1,T}^2}, \quad (4.9)$$

where $N_{v,T}$ is the number of basis functions of velocity in element T .

Lemma 4.1 *There exist two positive constants β_1 and β_2 independent of h such that*

$$\beta_1 \leq \frac{|\hat{\mathbf{w}}|_{1,T}^2}{|\hat{\mathbf{w}}|_{D,T}^2} \leq \beta_2, \quad \beta_1 \leq \frac{|\hat{\mathbf{w}}|_{1,\Omega}^2}{|\hat{\mathbf{w}}|_D^2} \leq \beta_2, \quad (4.10)$$

for all $T \in \mathcal{T}$ and $\hat{\mathbf{w}} \in WV_{k+d+1}$.

Proof We claim that there exist two positive constants β_{1T} and β_{2T} independent of h such that

$$\beta_{1T} \sum_{j=1}^{N_{v,T}} |\hat{\mathbf{w}}_{T,j}|_{1,T}^2 \leq |\hat{\mathbf{w}}_T|_{1,T}^2 \leq \beta_{2T} \sum_{j=1}^{N_{v,T}} |\hat{\mathbf{w}}_{T,j}|_{1,T}^2, \quad T \in \mathcal{T}_h. \quad (4.11)$$

where $N_{v,T}$ is the number of basis functions of velocity in element T .

For the first inequality in (4.11), divide $\Lambda = \{j \in N^+ | 1 \leq j \leq N_{v,T}\}$ into two subsets $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$. From Theorem 1 in [14], it gets that

$$\left(\sum_{j_1 \in \Lambda_1} \nabla \hat{\mathbf{w}}_{T,j_1}, \sum_{j_2 \in \Lambda_2} \nabla \hat{\mathbf{w}}_{T,j_2} \right) \leq \gamma_{v,T} \left| \sum_{j_1 \in \Lambda_1} \hat{\mathbf{w}}_{T,j_1} \right|_{1,T} \left| \sum_{j_2 \in \Lambda_2} \hat{\mathbf{w}}_{T,j_2} \right|_{1,T}, \quad (4.12)$$

where $0 \leq \gamma_{v,T} < 1$ is independent of h . Using the strengthened Cauchy inequality (4.12) and Cauchy-Schwarz inequality, we deduce

$$\begin{aligned}
|\hat{\mathbf{w}}_T|_{1,T}^2 &= \left| \sum_{j=1}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T}^2 = \left(\sum_{j=1}^{N_{v,T}} \nabla \hat{\mathbf{w}}_{T,j}, \sum_{j=1}^{N_{v,T}} \nabla \hat{\mathbf{w}}_{T,j} \right) \\
&= |\hat{\mathbf{w}}_{T,1}|_{1,T}^2 + \left| \sum_{j=2}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T}^2 + 2(\nabla \hat{\mathbf{w}}_{T,1}, \sum_{j=2}^{N_{v,T}} \nabla \hat{\mathbf{w}}_{T,j}) \\
&\geq |\hat{\mathbf{w}}_{T,1}|_{1,T}^2 + \left| \sum_{j=2}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T}^2 - 2\gamma_{v,T} |\hat{\mathbf{w}}_{T,1}|_{1,T} \left| \sum_{j=2}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T} \\
&\geq (1 - \gamma_{v,T}) |\hat{\mathbf{w}}_{T,1}|_{1,T}^2 + (1 - \gamma_{v,T}) \left| \sum_{j=2}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T}^2.
\end{aligned}$$

By a similar argument, we obtain

$$|\hat{\mathbf{w}}_T|_{1,T}^2 = \left| \sum_{j=1}^{N_{v,T}} \hat{\mathbf{w}}_{T,j} \right|_{1,T}^2 \geq \sum_{j=1}^{N_{v,T}} (1 - \gamma_{v,T})^j |\hat{\mathbf{w}}_{T,j}|_{1,T}^2 \geq (1 - \gamma_{v,T})^{N_{v,T}} \sum_{j=1}^{N_{v,T}} |\hat{\mathbf{w}}_{T,j}|_{1,T}^2,$$

which implies the first inequality in (4.11) with $\beta_{1T} = (1 - \gamma_{v,T})^{N_{v,T}}$.

The second inequality in (4.11) follows from the Cauchy-Schwarz inequality with $\beta_{2T} = N_{v,T}$. Therefore, the claim (4.11) holds. Summing up (4.11) overall $T \in \mathcal{T}$ and noting

$$\frac{|\hat{\mathbf{w}}|_W^2}{|\hat{\mathbf{w}}|_D^2} = \frac{\sum_{T \in \mathcal{T}_h} |\hat{\mathbf{w}}_T|_{1,T}^2}{\sum_{T \in \mathcal{T}} \sum_{j=1}^{N_{v,T}} |\hat{\mathbf{w}}_{T,j}|_{1,T}^2},$$

we arrive at the conclusion (4.10) with $\beta_1 = \min_{T \in \mathcal{T}} (1 - \gamma_{v,T})^{N_{v,T}}$ and $\beta_2 = \max_{T \in \mathcal{T}} N_{v,T}$. \square

Lemma 4.2 For any $(\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}) \in W_{k+d}$, we have

$$a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) \leq \mathfrak{C}_2 \|(\hat{\mathbf{w}}, \hat{r})\|_V \|(\hat{\mathbf{v}}, \hat{q})\|_V, \quad (4.13)$$

where \mathfrak{C}_2 is a positive constant.

Proof For any $(\hat{\mathbf{v}}, \hat{q}) = \sum_{j=1}^N x_j \phi_j$, $(\hat{\mathbf{w}}, \hat{r}) = \sum_{j=1}^N y_j \phi_j \in W_{k+d}$, define $\mathbf{x} = (x_1, \dots, x_N)^T$ and $\mathbf{y} = (y_1, \dots, y_N)^T$. Let \mathbf{x}_v and \mathbf{y}_v be vectors composed of elements related to velocity in \mathbf{x} and \mathbf{y} , respectively. Similarly, let \mathbf{x}_p and \mathbf{y}_p be vectors composed of elements related to pressure in \mathbf{x} and \mathbf{y} , respectively. Then using (4.5)~(4.7), Cauchy-Schwarz inequality, and Lemma 4.1, we have

$$\begin{aligned}
a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) &= \mathbf{y}^T M_v \mathbf{x} = [\mathbf{y}_v^T \ \mathbf{y}_p^T] \begin{bmatrix} D_v & B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_v \\ \mathbf{x}_p \end{bmatrix} \\
&= \mathbf{y}_v^T D_v \mathbf{x}_v - \mathbf{y}_p^T B^T \mathbf{x}_v + \mathbf{y}_v^T B \mathbf{x}_p \\
&\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D - (\nabla \cdot \hat{\mathbf{v}}, \hat{r}) + (\nabla \cdot \hat{\mathbf{w}}, \hat{q}) \\
&\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + \|\nabla \cdot \hat{\mathbf{v}}\| \|\hat{r}\| + \|\nabla \cdot \hat{\mathbf{w}}\| \|\hat{q}\| \\
&\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + \sqrt{d} |\hat{\mathbf{v}}|_{1,\Omega} \|\hat{r}\| + \sqrt{d} |\hat{\mathbf{w}}|_{1,\Omega} \|\hat{q}\| \\
&\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + \sqrt{d} \sqrt{\beta_2} |\hat{\mathbf{v}}|_D \|\hat{r}\| + \sqrt{d} \sqrt{\beta_2} |\hat{\mathbf{w}}|_D \|\hat{q}\| \\
&\leq \sqrt{d |\hat{\mathbf{v}}|_D^2 |\hat{\mathbf{w}}|_D^2 + d^2 \beta_2 (\|\hat{\mathbf{v}}\|_D^2 \|\hat{r}\|^2 + \|\hat{\mathbf{w}}\|_D^2 \|\hat{q}\|^2)} \\
&\leq \mathfrak{C}_2 \|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D,
\end{aligned}$$

where $\mathfrak{C}_2 = \max(\sqrt{d}, d\sqrt{\beta_2})$. \square

Lemma 4.3 The bilinear form $a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))$ satisfies the estimate

$$\inf_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D} \geq \frac{(\mu \beta_1)^2}{(1 + \mu \beta_1)^2},$$

where μ and β_1 are the constants in Lemma 2.3 and Lemma 4.1, respectively.

Proof To prove the inequality, we choose an arbitrary but fixed element $(\hat{\mathbf{v}}, \hat{q}) \in WP_{k+d} \setminus \{0\}$. Due to Lemma 2.3, there is a velocity field $\hat{\mathbf{w}}_{\hat{q}} \in WV_{k+d+1}$ with $|\hat{\mathbf{w}}_{\hat{q}}|_D = 1$ such that

$$\sum_{T \in \mathcal{T}} \int_T \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} dx = \int_{\Omega} \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} dx \geq \mu \|\hat{q}\|.$$

By using Cauchy-Schwartz inequality, Lemma 4.1, Lemma 2.3, and noting $|\hat{\mathbf{w}}_{\hat{q}}|_D = 1$, we therefore obtain for every $\delta > 0$,

$$\begin{aligned} & a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) \\ &= a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}}, \hat{q})) - \delta \|\hat{q}\| a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}_{\hat{q}}, 0)) \\ &= |\hat{\mathbf{v}}|_D^2 - \delta \|\hat{q}\| \mathbf{y}_v^T D_v \mathbf{x}_v + \delta \|\hat{q}\| \sum_{T \in \mathcal{T}} \int_T \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} \\ &\geq |\hat{\mathbf{v}}|_D^2 - \delta |\hat{\mathbf{v}}|_D \|\hat{q}\| + \delta \mu \|\hat{q}\|^2 |\hat{\mathbf{w}}_{\hat{q}}|_{1,\Omega} \\ &\geq |\hat{\mathbf{v}}|_D^2 - \delta |\hat{\mathbf{v}}|_D \|\hat{q}\| + \delta \mu \beta_1 \|\hat{q}\|^2 \\ &\geq (1 - \frac{\delta}{2\mu\beta_1}) |\hat{\mathbf{v}}|_D^2 + \frac{1}{2} \delta \mu \beta_1 \|\hat{q}\|^2, \end{aligned}$$

where $\mathbf{x}_v = (x_1, x_2, \dots, x_{N_v})^T$ and $\mathbf{y}_v = (y_1, y_2, \dots, y_{N_v})^T$ are such that $\hat{\mathbf{v}} = \sum_{j=1}^{N_v} x_j \varphi_j$, $\hat{\mathbf{w}}_{\hat{q}} = \sum_{j=1}^{N_v} y_j \varphi_j \in WV_{k+d+1}$.

Similar to the proof in Lemma 2.4, the choice of $\delta = \frac{2\mu\beta_1}{1+(\mu\beta_1)^2}$ yields

$$a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) \geq \frac{(\mu\beta_1)^2}{1 + (\mu\beta_1)^2} \|(\hat{\mathbf{v}}, \hat{q})\|_D^2,$$

and

$$\|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_D \leq \frac{(1 + \mu\beta_1)^2}{1 + (\mu\beta_1)^2} \|(\hat{\mathbf{v}}, \hat{q})\|_D.$$

Then we arrive at

$$\sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D} \geq \frac{a_2((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_D} \geq \frac{(\mu\beta_1)^2}{(1 + \mu\beta_1)^2}.$$

Since $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}$ is arbitrary, this completes the proof. \square

Using Lemma 4.2, Lemma 4.3, and a proof similar to that of Theorem 2.1, we have the following conclusion.

Theorem 4.1 *The finite element scheme (4.8) has a unique solution.*

Lemma 4.4 *Let $(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_p)$ and $(\tilde{\mathbf{e}}_u, \tilde{\mathbf{e}}_p)$ be the solutions of (2.15) and (4.8), respectively.*

$$\frac{(\mu\beta_1)^2}{\mathfrak{C}_1(1 + \mu\beta_1)^2 \sqrt{\beta_2 + 1}} \|(\tilde{\mathbf{e}}_u, \tilde{\mathbf{e}}_p)\|_D \leq \|(\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_{ph})\|_V \leq \frac{\mathfrak{C}_2 \sqrt{1 + \beta_1} (1 + \mu)^2}{\sqrt{\beta_1} \mu^2} \|(\tilde{\mathbf{e}}_u, \tilde{\mathbf{e}}_p)\|_D, \quad (4.14)$$

where $\|\cdot\|_V$ and $\|\cdot\|_D$ are defined in (2.2) and (4.6), respectively. The constants $\mathfrak{C}_1, \mathfrak{C}_2, \beta_1, \beta_2$, and μ are defined in (2.3), (4.13), (4.11), and (2.16).

Proof It follows from (2.15) and (4.8) that

$$a_2((\tilde{\mathbf{e}}_u, \tilde{\mathbf{e}}_p), (\hat{\mathbf{v}}, \hat{q})) = a_1((\hat{\mathbf{e}}_u, \hat{\mathbf{e}}_{ph}), (\hat{\mathbf{v}}, \hat{q})), \quad \forall (\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}. \quad (4.15)$$

Using (4.15), Lemma 4.1, and Lemma 4.3, we obtain

$$\begin{aligned}
\frac{(\mu\beta_1)^2}{(1+\mu\beta_1)^2} \|(\tilde{e}_u, \tilde{e}_p)\|_D &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\tilde{e}_u, \tilde{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\
&= \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\hat{e}_u, \hat{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_1 \|(\hat{e}_u, \hat{e}_p)\|_V \|(\hat{\mathbf{v}}, \hat{q})\|_V}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_1 \|(\hat{e}_u, \hat{e}_p)\|_V \sqrt{|\hat{\mathbf{v}}|_{1,\Omega}^2 + \|\hat{q}\|^2}}{\sqrt{|\hat{\mathbf{v}}|_D^2 + \|\hat{q}\|^2}} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_1 \|(\hat{e}_u, \hat{e}_p)\|_V \sqrt{\beta_2 |\hat{\mathbf{v}}|_D^2 + \|\hat{q}\|^2}}{\sqrt{|\hat{\mathbf{v}}|_D^2 + \|\hat{q}\|^2}} \\
&\leq \mathfrak{C}_1 \sqrt{\beta_2 + 1} \|(\hat{e}_u, \hat{e}_p)\|_V,
\end{aligned}$$

which implies the first inequality in (4.14).

Similarly, using (4.15) and Lemma 2.3, we have

$$\begin{aligned}
\frac{\mu^2}{(1+\mu)^2} \|(\hat{e}_u, \hat{e}_p)\|_V &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_1((\hat{e}_u, \hat{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_V} \\
&= \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\tilde{e}_u, \tilde{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_V} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_2 \|(\tilde{e}_u, \tilde{e}_p)\|_D \|(\hat{\mathbf{v}}, \hat{q})\|_D}{\|(\hat{\mathbf{v}}, \hat{q})\|_V} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_2 \|(\tilde{e}_u, \tilde{e}_p)\|_D \sqrt{|\hat{\mathbf{v}}|_D^2 + \|\hat{q}\|^2}}{\sqrt{|\hat{\mathbf{v}}|_{1,\Omega}^2 + \|\hat{q}\|^2}} \\
&\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_2 \|(\tilde{e}_u, \tilde{e}_p)\|_D \sqrt{\frac{|\hat{\mathbf{v}}|_{1,\Omega}^2}{\beta_1} + \|\hat{q}\|^2}}{\sqrt{|\hat{\mathbf{v}}|_{1,\Omega}^2 + \|\hat{q}\|^2}} \\
&\leq \frac{\mathfrak{C}_2 \sqrt{1+\beta_1}}{\sqrt{\beta_1}} \|(\tilde{e}_u, \tilde{e}_p)\|_D,
\end{aligned}$$

which implies the second inequality in (4.14). \square

Combining Theorem 3.1 and Lemma 4.4, we obtain the following lower and upper bounds related to $\|(\tilde{e}_u, \tilde{e}_p)\|_D$.

Theorem 4.2 *Let (\mathbf{u}, p) , $(\hat{\mathbf{u}}, \hat{p})$, and $(\tilde{e}_u, \tilde{e}_p)$ be the solutions of (2.1), (2.15), and (4.8), respectively. There are constants $\tilde{\mathfrak{C}}_* = \frac{\mu^4 \beta_1^2}{2\mathfrak{C}_1^2(1+\mu)^2(1+\beta_1\mu)^2\sqrt{1+\beta_2}}$ and $\tilde{\mathfrak{C}}^* = \frac{\mathfrak{C}_1 \mathfrak{C}_2 \sqrt{1+\beta_1}(1+\mu)^2}{\mathfrak{c}_1 \sqrt{\beta_1} \mu^2}$ such that*

$$\tilde{\mathfrak{C}}_* \|(\tilde{e}_u, \tilde{e}_p)\|_D + \frac{1}{2\sqrt{d}} \|\nabla \cdot \hat{\mathbf{u}}\| \leq \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V \leq \tilde{\mathfrak{C}}^* \|(\tilde{e}_u, \tilde{e}_p)\|_D + \frac{1}{\mathfrak{c}_1} \|\nabla \cdot \hat{\mathbf{u}}\| + \frac{C_{\mathcal{T}}}{\mathfrak{c}_1} \text{osc}(\mathbf{f}), \quad (4.16)$$

where $\|\cdot\|_V$ and $\|\cdot\|_D$ are defined in (2.2) and (4.6), respectively. The constants $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{c}_1, \mu, \beta_1, \beta_2$, and $C_{\mathcal{T}}$ are defined in (2.3), (4.13) (2.4), (2.16), (4.11), and Lemma 3.1, respectively.

4.2 Diagonalization with respect to Pressure

Recall that $\{\varphi_j\}_{j=1}^{N_v}$ and $\{\psi_j\}_{j=1}^{N_p}$ are the bases in space W_{k+d} for velocity and pressure, respectively. For $\tilde{e}_u = \sum_{j=1}^{N_v} \tilde{x}_{u,j} \varphi_j$ and $\tilde{e}_p = \sum_{j=1}^{N_p} \tilde{x}_{p,j} \psi_j$, rewrite (4.8) in a matrix form

$$\begin{bmatrix} D_v & B \\ -B^T & 0 \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{x}}_u \\ \tilde{\mathbf{x}}_p \end{bmatrix} = \begin{bmatrix} F_v \\ F_p \end{bmatrix}, \quad (4.17)$$

where

$$\begin{aligned}\tilde{\mathbf{x}}_u &= (\tilde{x}_{u,1}, \tilde{x}_{u,2}, \dots, \tilde{x}_{u,N_v})^T, & F_v &= (F_{v,1}, F_{v,2}, \dots, F_{v,N_v})^T, \\ \tilde{\mathbf{x}}_p &= (\tilde{x}_{p,1}, \tilde{x}_{p,2}, \dots, \tilde{x}_{p,N_p})^T, & F_p &= (F_{p,1}, F_{p,2}, \dots, F_{p,N_p})^T.\end{aligned}$$

Here, $F_{v,j}$ and $F_{p,j}$ are defined by

$$F_{v,j} = f(\varphi_j) - a_1((\hat{\mathbf{u}}, \hat{p}), (\varphi_j, 0)), \quad j = 1, 2, \dots, N_v, \quad (4.18)$$

$$F_{p,j} = -a_1((\hat{\mathbf{u}}, \hat{p}), (0, \psi_j)), \quad j = 1, 2, \dots, N_p. \quad (4.19)$$

After a simple calculation, we have

$$D_v \tilde{\mathbf{x}}_u + B \tilde{\mathbf{x}}_p = F_v, \quad (4.20)$$

$$B^T D_v^{-1} B \tilde{\mathbf{x}}_p = F_p + B^T D_v^{-1} F_v. \quad (4.21)$$

The inverse of the matrix D_v is easy to calculate because it is a diagonal matrix. If we get $\tilde{\mathbf{x}}_p$ by solving (4.21), $\tilde{\mathbf{x}}_u$ is easy to get by (4.20). Let $D_p = \text{diag}(B^T D_v^{-1} B)$, which is the diagonal matrix with the same diagonal as $B^T D_v^{-1} B$. Let $c_s = \max_{T \in \mathcal{T}} N_{p,T}$, which is the maximum number of basis functions of pressure for each element. Then replacing $B^T D_v^{-1} B$ with $c_s D_p$ in (4.21), we get

$$D_v \bar{\mathbf{x}}_u + B \bar{\mathbf{x}}_p = F_v, \quad (4.22)$$

$$c_s D_p \bar{\mathbf{x}}_p = F_p + B^T D_v^{-1} F_v, \quad (4.23)$$

where $\bar{\mathbf{x}}_u = (\bar{x}_{u,1}, \bar{x}_{u,2}, \dots, \bar{x}_{u,N_v})$ and $\bar{\mathbf{x}}_p = (\bar{x}_{p,1}, \bar{x}_{p,2}, \dots, \bar{x}_{p,N_p})$.

Equations (4.22) and (4.23) are equivalent to

$$\begin{aligned}D_v \bar{\mathbf{x}}_u + B \bar{\mathbf{x}}_p &= F_v, \\ -B^T \bar{\mathbf{x}}_u + (c_s D_p - B^T D_v^{-1} B) \bar{\mathbf{x}}_p &= F_p,\end{aligned}$$

whose matrix form is

$$\begin{bmatrix} D_v & B \\ -B^T & c_s D_p - B^T D_v^{-1} B \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_u \\ \bar{\mathbf{x}}_p \end{bmatrix} = \begin{bmatrix} F_v \\ F_p \end{bmatrix}.$$

For any $(\hat{\mathbf{v}}, \hat{q}) = \sum_{j=1}^N x_j \phi_j$ and $(\hat{\mathbf{w}}, \hat{r}) = \sum_{j=1}^N y_j \phi_j \in W_{k+d}$, we define

$$a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) = \mathbf{y}^T M_{vp} \mathbf{x}, \quad (4.24)$$

where

$$\mathbf{y} = (y_1, \dots, y_N)^T, \quad M_{vp} = \begin{bmatrix} D_v & B \\ -B^T & c_s D_p - B^T D_v^{-1} B \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = (x_1, \dots, x_N)^T.$$

It is time to present **the third error problem**: Find $\{\bar{\mathbf{e}}_u, \bar{\mathbf{e}}_p\} \in W_{k+d}$ with $\bar{\mathbf{e}}_u = \sum_{j=1}^{N_v} \bar{x}_{u,j} \varphi_j$ and $\bar{\mathbf{e}}_p = \sum_{j=1}^{N_p} \bar{x}_{p,j} \psi_j$ such that

$$a_3((\bar{\mathbf{e}}_u, \bar{\mathbf{e}}_p), (\hat{\mathbf{v}}, \hat{q})) = f(\hat{\mathbf{v}}) - a_1((\hat{\mathbf{u}}, \hat{p}), (\hat{\mathbf{v}}, \hat{q})), \quad \forall (\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}. \quad (4.25)$$

Remark 4.1 Equations (4.22) and (4.23) are equivalent to (4.25), but they are used in different ways. Obviously, (4.22) and (4.23) are easier to calculate. In section 5, the global and local estimators will be generated from (4.22) and (4.23). However, (4.25) is essential in the proof of equivalence. Therefore, we use (4.22) and (4.23) for the numerical computation and (4.25) for the theoretical analysis.

Because matrix D_v and D_p are diagonal matrices in (4.22) and (4.23), the existence and uniqueness of finite element scheme (4.25) are obvious.

Theorem 4.3 *The finite element scheme (4.25) has a unique solution.*

Next, we turn our attention to the discrete pressure space. Two new norms will be defined. We still use $\{\psi_j\}_{j=1}^{N_p}$ to denote the basis functions of pressure in W_{k+d} . For $\hat{q} = \sum_{j=1}^{N_p} x_p^j \psi_j$ and $\hat{r} = \sum_{j=1}^{N_p} y_p^j \psi_j$, define two bilinear forms

$$E_{31}(\hat{q}, \hat{r}) = \mathbf{y}_p^T B^T D_v^{-1} B \mathbf{x}_p, \quad E_{32}(\hat{q}, \hat{r}) = \mathbf{y}_p^T D_p \mathbf{x}_p,$$

and norms

$$\|\hat{q}\|_B^2 = \sqrt{E_{31}(\hat{q}, \hat{q})}, \quad \|\hat{q}\|_P^2 = \sqrt{E_{32}(\hat{q}, \hat{q})},$$

where $\mathbf{x}_p = (x_{p,1}, \dots, x_{p,N_p})^T$ and $\mathbf{y}_p = (y_{p,1}, \dots, y_{p,N_p})^T$. The next two lemmas will establish some inequalities related to the three pressure norms $\|\cdot\|_B$, $\|\cdot\|_P$, and $\|\cdot\|$.

Lemma 4.5 *There exist two positive constants c_i and c_s independent of h such that*

$$c_i \leq \frac{\|\hat{q}\|_B^2}{\|\hat{q}\|_P^2} \leq c_s, \quad \forall \hat{q} \in WP_{k+d}, \quad (4.26)$$

where c_s is the same as in (4.23).

Proof For any $T \in \mathcal{T}$, denote by $\{\psi_{T,j}\}_{j=1}^{N_{p,T}}$ the basis functions of pressure related to T , then $\hat{q}_T := \hat{q}|_T = \sum_{j=1}^{N_{p,T}} x_{T,j} \psi_{T,j}$ with $\{x_{T,j}\}_{j=1}^{N_{p,T}}$ being the coefficients. Let $\hat{q}_{T,j} := x_{T,j} \psi_{T,j}$, then $\hat{q}_T = \sum_{j=1}^{N_{p,T}} \hat{q}_{T,j}$. We claim that there exist two positive constants c_{iT} and c_{sT} , independent of h , such that

$$c_{iT} \sum_{j=1}^{N_{p,T}} \|\hat{q}_{T,j}\|_B^2 \leq \|\hat{q}_T\|_B^2 \leq c_{sT} \sum_{j=1}^{N_{p,T}} \|\hat{q}_{T,j}\|_B^2, \quad T \in \mathcal{T}. \quad (4.27)$$

For the first inequality in (4.27), divide $\Lambda = \{j \in N^+ \mid 1 \leq j \leq N_{p,T}\}$ into two subsets $\Lambda = \Lambda_1 \cup \Lambda_2$ with $\Lambda_1 \cap \Lambda_2 = \emptyset$. From Theorem 1 in [14], it gets that

$$E_{31}\left(\sum_{j \in \Lambda_1} \hat{q}_{T,j}, \sum_{\ell \in \Lambda_2} \hat{q}_{T,\ell}\right) \leq \gamma_{p,T} \left\| \sum_{j \in \Lambda_1} \hat{q}_{T,j} \right\|_B \left\| \sum_{\ell \in \Lambda_2} \hat{q}_{T,\ell} \right\|_B, \quad (4.28)$$

where $0 \leq \gamma_{p,T} < 1$ is independent of h . Using the strengthened Cauchy inequality (4.28), we deduce

$$\begin{aligned} \left\| \sum_{j=1}^{N_{p,T}} \hat{q}_{T,j} \right\|_B^2 &= \|\hat{q}_T\|_B^2 = E_{31}\left(\sum_{j=1}^{N_{p,T}} \hat{q}_{T,j}, \sum_{j=1}^{N_{p,T}} \hat{q}_{T,j}\right) \\ &= \|\hat{q}_{T,1}\|_B^2 + \left\| \sum_{j=2}^{N_{p,T}} \hat{q}_{T,j} \right\|_B^2 + 2E_{31}\left(\hat{q}_{T,1}, \sum_{j=2}^{N_{p,T}} \hat{q}_{T,j}\right) \\ &\geq \|\hat{q}_{T,1}\|_B^2 + \left\| \sum_{j=2}^{N_{p,T}} \hat{q}_{T,j} \right\|_B^2 - 2\gamma_{p,T} \|\hat{q}_{T,1}\|_B \left\| \sum_{j=2}^{N_{p,T}} \hat{q}_{T,j} \right\|_B \\ &\geq (1 - \gamma_{p,T}) \|\hat{q}_{T,1}\|_B^2 + (1 - \gamma_{p,T}) \left\| \sum_{j=2}^{N_{p,T}} \hat{q}_{T,j} \right\|_B^2. \end{aligned}$$

By a similar argument, we obtain

$$\|\hat{q}_T\|_B^2 = \left\| \sum_{j=1}^{N_{p,T}} \hat{q}_{T,j} \right\|_B^2 \geq \sum_{j=1}^{N_{p,T}} (1 - \gamma_{p,T})^j \|\hat{q}_{T,j}\|_B^2 \geq (1 - \gamma_{p,T})^{N_{p,T}} \sum_{j=1}^{N_{p,T}} \|\hat{q}_{T,j}\|_B^2,$$

which implies the first inequality in (4.27) with $c_{iT} = (1 - \gamma_{p,T})^{N_{p,T}}$.

The second inequality in (4.11) follows from the Cauchy-Schwarz inequality with $c_{sT} = N_{p,T}$. Therefore, the claim (4.11) holds. Summing up (4.11) over all $T \in \mathcal{T}$ and noting

$$\frac{\|\hat{q}\|_B^2}{\|\hat{q}\|_P^2} = \frac{\sum_{T \in \mathcal{T}} \|\hat{q}_T\|_B^2}{\sum_{T \in \mathcal{T}} \sum_{j=1}^{N_{p,T}} \|\hat{q}_{T,j}\|_B^2},$$

we arrive at the conclusion (4.10) with $c_i = \min_{T \in \mathcal{T}} (1 - \gamma_{p,T})^{N_{p,T}}$ and $c_s = \max_{T \in \mathcal{T}} N_{p,T}$. \square

Lemma 4.6 For any $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}$, we have

$$\|\hat{q}\|_B \leq d(d+1)\beta_2\|\hat{q}\|$$

where d is the dimension and β_2 is defined in (4.10).

Proof We continue to use $\{\varphi_j\}_{j=1}^{N_v}$ and $\{\psi_j\}_{j=1}^{N_p}$ as the bases in W_{k+d} for velocity and pressure, respectively. Define a diagonal matrix \tilde{D} whose elements are the square roots of the corresponding elements of D_v^{-1} , and it is clear that $D_v^{-1} = \tilde{D}\tilde{D}$. Let $q = \sum_{j=1}^{N_p} x_j \psi_j$ and $\mathbf{x} = (x_1, x_2, \dots, x_{N_p})^T$, then

$$\|\hat{q}\|_B^2 = \mathbf{x}^T B^T D_v^{-1} B \mathbf{x} = \mathbf{x}^T B^T \tilde{D} \tilde{D} B \mathbf{x}. \quad (4.29)$$

Let $\{d_j\}_{j=1}^{N_v}$ denote the diagonal elements of the matrix \tilde{D} whose dimension is N_v . Let \mathbf{d}_j denote the j -th column of matrix \tilde{D} and $\hat{\mathbf{v}}_j = d_j \varphi_j$ and denoted by T_1, \dots, T_{j_T} the elements related to φ_j respectively. Then

$$\sum_{i=1}^{j_T} |\hat{\mathbf{v}}_j|_{D, T_i}^2 = |\hat{\mathbf{v}}_j|_D^2 = \mathbf{d}_j^T D_v \mathbf{d}_j = 1. \quad (4.30)$$

From (4.29), (4.30), Cauchy-Schwartz inequality, and Lemma 4.1, we have

$$\begin{aligned} \|\hat{q}\|_B^2 &= \sum_{j=1}^{N_v} (\mathbf{x}^T B^T \mathbf{d}_j) (\mathbf{d}_j^T B \mathbf{x}) = \sum_{j=1}^{N_v} (b(\hat{\mathbf{v}}_j, \hat{q}))^2 = \sum_{j=1}^{N_v} \left(\sum_{T \in \mathcal{T}_h} (\nabla \cdot \hat{\mathbf{v}}_j, \hat{q})_T \right)^2 \\ &= \sum_{j=1}^{N_v} \left(\sum_{i=1}^{j_T} (\nabla \cdot \hat{\mathbf{v}}_j, \hat{q})_{T_i} \right)^2 \leq \sum_{j=1}^{N_v} \sum_{i=1}^{j_T} \|\nabla \cdot \hat{\mathbf{v}}_j\|_{0, T_i}^2 \|\hat{q}\|_{0, T_i}^2 \\ &\leq \sum_{j=1}^{N_v} \sum_{i=1}^{j_T} d |\hat{\mathbf{v}}_j|_{1, T_i}^2 \|\hat{q}\|_{0, T_i}^2 \leq \sum_{j=1}^{N_v} \sum_{i=1}^{j_T} d \beta_2 |\hat{\mathbf{v}}_j|_{D, T_i}^2 \|\hat{q}\|_{0, T_i}^2 \\ &\leq \sum_{j=1}^{N_v} \sum_{i=1}^{j_T} d \beta_2 \|\hat{q}\|_{0, T_i}^2 = d(d+1)\beta_2 \sum_{T \in \mathcal{T}} \|\hat{q}\|_{0, T}^2 = d(d+1)\beta_2 \|\hat{q}\| \end{aligned}$$

□

Lemma 4.7 The bi-linear form $a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))$ satisfies the estimates

$$\inf_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D} \geq \frac{(\mu\beta_1)^2}{(1 + \mu\beta_1)^2},$$

where μ and β_1 are the constants in Lemma 2.3 and Lemma 4.1, respectively.

Proof To prove the inequality, we choose an arbitrary but fixed element $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}$. Due to Lemma 2.3, there is a velocity field $\hat{\mathbf{w}}_{\hat{q}} \in WP_{k+d}$ with $|\hat{\mathbf{w}}_{\hat{q}}|_D = 1$ such that

$$\sum_{T \in \mathcal{T}} \int_T \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} = \int_{\Omega} \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} \geq \mu \|\hat{q}\|.$$

By using Cauchy-Schwartz inequality, Lemma 4.1, Lemma 2.3, and Lemma 4.5, we therefore obtain for every $\delta > 0$

$$\begin{aligned} &a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) \\ &= a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}}, \hat{q})) - \delta \|\hat{q}\| a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}_{\hat{q}}, 0)) \\ &= |\hat{\mathbf{v}}|_D^2 + c_s \|\hat{q}\|_p - \|\hat{q}\|_B - \delta \|\hat{q}\| \mathbf{y}_v^T D_v \mathbf{x}_v + \delta \|\hat{q}\| \sum_{T \in \mathcal{T}_h} \int_T \hat{q} \nabla \cdot \hat{\mathbf{w}}_{\hat{q}} \\ &\geq |\hat{\mathbf{v}}|_D^2 - \delta |\hat{\mathbf{v}}|_D \|\hat{q}\| + \delta \mu \|\hat{q}\|^2 |\hat{\mathbf{w}}_{\hat{q}}|_{1, \Omega} \\ &\geq |\hat{\mathbf{v}}|_D^2 - \delta |\hat{\mathbf{v}}|_D \|\hat{q}\| + \delta \mu \beta_1 \|\hat{q}\|^2 \\ &\geq (1 - \frac{\delta}{2\mu\beta_1}) |\hat{\mathbf{v}}|_D^2 + \frac{1}{2} \delta \mu \beta_1 \|\hat{q}\|^2, \end{aligned}$$

where $\mathbf{x}_v = (x_{v,1}, \dots, x_{v,N_v})^T$, $\mathbf{x}_q = (x_{q,1}, \dots, x_{q,N_p})^T$, and $\mathbf{y}_v = (y_{v,1}, \dots, y_{v,N_v})^T$. Let $\hat{\mathbf{v}} = \sum_{j=1}^{N_v} x_{v,j} \varphi_j$, $\hat{q} = \sum_{j=1}^{N_v} x_{q,j} \psi_j$, and $\hat{\mathbf{w}}_{\hat{q}} = \sum_{j=1}^{N_v} y_{v,j} \varphi_j$.

Similar to the proof of Lemma 2.4, the choice of $\delta = \frac{2\mu\beta_1}{1+(\mu\beta_1)^2}$ yields

$$a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})) \geq \frac{(\mu\beta_1)^2}{1 + (\mu\beta_1)^2} \|(\hat{\mathbf{v}}, \hat{q})\|_D^2,$$

and

$$\|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_D \leq \frac{(1 + \mu\beta_1)^2}{1 + (\mu\beta_1)^2} \|(\hat{\mathbf{v}}, \hat{q})\|_D.$$

Then we arrive at

$$\sup_{(\hat{\mathbf{w}}, \hat{r}) \in W_{k+d} \setminus \{0\}} \frac{a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D} \geq \frac{a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{v}} - \delta \|\hat{q}\| \hat{\mathbf{w}}_{\hat{q}}, \hat{q})\|_D} \geq \frac{(\mu\beta_1)^2}{(1 + \mu\beta_1)^2}.$$

Since $(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}$ is arbitrary, this completes the proof. \square

Lemma 4.8 For any $(\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r}) \in W_{k+d}$, we have

$$a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) \leq \mathfrak{C}_3 \|(\hat{\mathbf{w}}, \hat{r})\|_D \|(\hat{\mathbf{v}}, \hat{q})\|_D, \quad (4.31)$$

where \mathfrak{C}_3 is a positive constant independent of h .

Proof We continue to use $\{\varphi_j\}_{j=1}^{N_v}$ and $\{\psi_j\}_{j=1}^{N_p}$ as the bases in space W_{k+d} for velocity and pressure, respectively.

Let

$$\begin{aligned} \hat{\mathbf{v}} &= \sum_{j=1}^{N_v} x_{v,j} \varphi_j, & \mathbf{x}_v &= (x_{v,1}, x_{v,2}, \dots, x_{v,N_v})^T, \\ \hat{q} &= \sum_{j=1}^{N_p} x_{p,j} \psi_j, & \mathbf{x}_p &= (x_{p,1}, x_{p,2}, \dots, x_{p,N_p})^T, \\ \hat{\mathbf{w}} &= \sum_{j=1}^{N_v} y_{v,j} \varphi_j, & \mathbf{y}_v &= (y_{v,1}, y_{v,2}, \dots, y_{v,N_v})^T, \\ \hat{r} &= \sum_{j=1}^{N_p} y_{p,j} \psi_j, & \mathbf{y}_p &= (y_{p,1}, y_{p,2}, \dots, y_{p,N_p})^T. \end{aligned}$$

Then, using (4.24), Cauchy-Schwarz inequality, Lemma 4.5, Lemma 4.1, and Lemma 4.6

$$\begin{aligned} &a_3((\hat{\mathbf{v}}, \hat{q}), (\hat{\mathbf{w}}, \hat{r})) \\ &= \mathbf{y}_v^T D_v \mathbf{x}_v + \mathbf{y}_p^T B \mathbf{x}_p - \mathbf{y}_p^T B^T \mathbf{x}_v + \mathbf{y}_p^T (c_s D_p - B^T D_v^{-1} B) \mathbf{x}_p \\ &\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + \|\hat{q}\| \|\nabla \cdot \hat{\mathbf{w}}\| + \|\hat{r}\| \|\nabla \cdot \hat{\mathbf{v}}\| + \|\hat{q}\|_P \|\hat{r}\|_P + \|\hat{q}\|_B \|\hat{r}\|_B \\ &\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + d \|\hat{q}\| \|\hat{\mathbf{w}}\|_{1,\Omega} + d \|\hat{r}\| \|\hat{\mathbf{v}}\|_{1,\Omega} + (c_i^{-1} + 1) \|\hat{q}\|_B \|\hat{r}\|_B \\ &\leq |\hat{\mathbf{v}}|_D |\hat{\mathbf{w}}|_D + d \beta_2 \|\hat{q}\| \|\hat{\mathbf{w}}\|_D + d \beta_2 \|\hat{r}\| \|\hat{\mathbf{v}}\|_D + (c_i^{-1} + 1) d(d+1) \beta_2 \|\hat{q}\| \|\hat{r}\| \\ &\leq \mathfrak{C}_3 \|(\hat{\mathbf{v}}, \hat{q})\|_D \|(\hat{\mathbf{w}}, \hat{r})\|_D, \end{aligned}$$

with $\mathfrak{C}_3 = \sqrt{2} \max\{1, d\beta_2, (c_i^{-1} + 1)d(d+1)\beta_2\}$. \square

Lemma 4.9 Let $(\tilde{\mathbf{e}}_u, \tilde{e}_p)$ and $(\bar{\mathbf{e}}_u, \bar{e}_p)$ be the solutions of (4.8) and (4.22)-(4.23), respectively. Then,

$$\frac{(\mu\beta_1)^2}{\mathfrak{C}_2(1 + \mu\beta_1)^2} \|(\bar{\mathbf{e}}_u, \bar{e}_p)\|_D \leq \|(\tilde{\mathbf{e}}_u, \tilde{e}_p)\|_D \leq \frac{\mathfrak{C}_3(1 + \mu\beta_1)^2}{(\mu\beta_1)^2} \|(\bar{\mathbf{e}}_u, \bar{e}_p)\|_D, \quad (4.32)$$

where $\|\cdot\|_D$ is defined in (4.6). The constants $\mathfrak{C}_2, \mathfrak{C}_3, \beta_1$, and μ are defined in (4.13), (4.31), (4.11), and (2.16), respectively.

Proof It follows from (4.8) and (4.25) that

$$a_3((\bar{e}_u, \bar{e}_p), (\hat{\mathbf{v}}, \hat{q})) = a_2((\tilde{e}_u, \tilde{e}_p), (\hat{\mathbf{v}}, \hat{q})), \quad \forall (\hat{\mathbf{v}}, \hat{q}) \in W_{k+d}. \quad (4.33)$$

Using (4.33) and Lemma 4.7, we obtain

$$\begin{aligned} \frac{(\mu\beta_1)^2}{(1+\mu\beta_1)^2} \|(\bar{e}_u, \bar{e}_p)\|_D &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_3((\bar{e}_u, \bar{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &= \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\tilde{e}_u, \tilde{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_2 \|(\tilde{e}_u, \tilde{e}_p)\|_D \|(\hat{\mathbf{v}}, \hat{q})\|_D}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &= \mathfrak{C}_2 \|(\tilde{e}_u, \tilde{e}_p)\|_D, \end{aligned}$$

which implies the first inequality in (4.32).

Similarly, using (4.33) and Lemma 4.3, we have

$$\begin{aligned} \frac{(\mu\beta_1)^2}{(1+\mu\beta_1)^2} \|(\tilde{e}_u, \tilde{e}_p)\|_D &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_2((\tilde{e}_u, \tilde{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &= \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{a_3((\bar{e}_u, \bar{e}_p), (\hat{\mathbf{v}}, \hat{q}))}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &\leq \sup_{(\hat{\mathbf{v}}, \hat{q}) \in W_{k+d} \setminus \{0\}} \frac{\mathfrak{C}_3 \|(\bar{e}_u, \bar{e}_p)\|_D \|(\hat{\mathbf{v}}, \hat{q})\|_D}{\|(\hat{\mathbf{v}}, \hat{q})\|_D} \\ &= \mathfrak{C}_3 \|(\bar{e}_u, \bar{e}_p)\|_D, \end{aligned}$$

which implies the second inequality in (4.32). \square

Combining Theorem 4.2 and Lemma 4.9, we obtain the following lower and upper bounds for the estimator $\|(\bar{e}_u, \bar{e}_p)\|_D$.

Theorem 4.4 *Let $(\mathbf{u}, p), (\hat{\mathbf{u}}, \hat{p})$ and (\bar{e}_u, \bar{e}_p) be the solutions of (2.1), (2.15), and (4.25), respectively. There are constants $\bar{\mathfrak{C}}_* = \frac{\mu^6 \beta_1^4}{2\mathfrak{C}_1^2 \mathfrak{C}_2 (1+\mu)^2 (1+\beta_1 \mu)^4 \sqrt{1+\beta_2}}$, and $\bar{\mathfrak{C}}^* = \frac{\mathfrak{C}_1 \mathfrak{C}_2 \mathfrak{C}_3 \sqrt{1+\beta_1} (1+\mu)^2 (1+\mu\beta_1)^2}{\mathfrak{C}_1 \beta_1^4 \sqrt{\beta_1 \mu^4}}$ such that*

$$\bar{\mathfrak{C}}_* \|(\bar{e}_u, \bar{e}_p)\|_D + \frac{1}{2\sqrt{d}} \|\nabla \cdot \hat{\mathbf{u}}\| \leq \|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V \leq \bar{\mathfrak{C}}^* \|(\bar{e}_u, \bar{e}_p)\|_D + \frac{1}{\mathfrak{C}_1} \|\nabla \cdot \hat{\mathbf{u}}\| + \frac{C_{\mathcal{T}}}{\mathfrak{C}_1} \text{osc}(\mathbf{f}), \quad (4.34)$$

where $\|\cdot\|_V$ and $\|\cdot\|_D$ are defined in (2.2) and (4.6), respectively. The constants $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3, \mathfrak{C}_1, \mu, \beta_1, \beta_2$, and $C_{\mathcal{T}}$ are defined in (2.3), (4.13), (4.31), (2.4), (2.16), (4.11), and Lemma 3.1, respectively.

5 Adaptive Algorithm

In this section, we construct an adaptive FEM to solve (1.1)-(1.3) based on the local and global *a posteriori* error estimators, denoted by $\eta_{L,T}$ and $\eta_G(\mathcal{T}_m)$, defined in (5.1) and (5.2), which produce a sequence of discrete solutions $(\hat{\mathbf{u}}_m, \hat{p}_m)$ in nested spaces $V_{k,m}$ over triangulation \mathcal{T}_m . The index m indicates the underlying mesh with size h_m . Assume that an initial mesh \mathcal{T}_0 , a Döfler parameter $\theta \in (0, 1)$, and a targeted tolerance ε are given.

Actually, a common adaptive refinement scheme involves a loop structure of the form:

SOLVE \rightarrow ESTIMATE \rightarrow MARK \rightarrow REFINE

with the initial triangulation \mathcal{T}_0 of Ω (cf. [19, 18]). **SOLVE** refers to solving the FEM scheme (2.8) on a relatively coarse mesh \mathcal{T}_m . **ESTIMATE** relies on an efficient and reliable *a posteriori* error estimate, and the local and global estimators are defined in (5.1) and (5.2). With the help of the error estimators, **MARK** determines the elements to be refined, hence creating a subset \mathcal{S}_m of \mathcal{T}_m for refinement. Finally, **REFINE** generates a finer triangulation \mathcal{T}_{m+1} by dividing those elements in \mathcal{S}_m , and an updated numerical solution will be computed on \mathcal{T}_{m+1} .

For the first and the last step, there have been rapid advances for solving the linear system (2.8) and refinement implementation, respectively in recent years, and we refer to [8, 9, 25] for the details. Here, we focus on the interplay between the error estimator and the marking strategy. The error estimator consists of local and global estimates for a given triangulation. The local estimator provides the information for

the marking strategy to determine the triangles to be refined, while the global error estimator provides the measure for the reliable stop condition of the loops.

Recall that $\{\varphi_j\}_{j=1}^{N_v}$ and $\{\psi_j\}_{j=1}^{N_p}$ are the bases in W_{k+d} for velocity and pressure, respectively. The matrix form of the third error problem is: Find $(\bar{e}_u, \bar{e}_p) \in W_{k+d}$ with $\bar{e}_u = \sum_{j=1}^{N_v} \bar{x}_{u,j} \varphi_j$ and $\bar{e}_p = \sum_{j=1}^{N_p} \bar{x}_{p,j} \psi_j$ satisfying (4.22) and (4.23). The definitions of matrix D_v and B are similar to (4.17). The elements of D_v and B are as follows

$$(D_v)_{j,j} = a(\varphi_j, \varphi_j), \quad j = 1, \dots, N_v, \\ B_{\ell,j} = -b(\psi_j, \varphi_\ell), \quad \ell = 1, \dots, N_v \text{ and } j = 1, \dots, N_p.$$

Let D_p be the diagonal matrix with the same diagonal as $B^T D_v^{-1} B$. Then the elements of D_p are

$$(D_p)_{j,j} = \sum_{\ell=1}^{N_v} \frac{B_{\ell,j}^2}{(D_v)_{\ell,\ell}}, \quad j = 1, \dots, N_p.$$

From (4.23), we can get

$$\bar{x}_{p,j} = \frac{F_{p,j} + \sum_{\ell=1}^{N_v} \frac{B_{\ell,j} F_{v,\ell}}{(D_v)_{\ell,\ell}}}{c_s \sum_{\ell=1}^{N_v} \frac{B_{\ell,j}^2}{(D_v)_{\ell,\ell}}}, \quad j = 1, \dots, N_p,$$

where $F_{v,\ell}$ and $F_{p,j}$ can be find in (4.18) and (4.19). For any $T \in \mathcal{T}_m$, set $\Lambda_T^p = \{j \mid \text{supp}(\psi_j) \cap T \neq \emptyset, j = 1, \dots, N_p\}$, then

$$\bar{e}_{p,T} := (\bar{e}_p)|_T = \sum_{j \in \Lambda_T^p} \bar{x}_{p,j} \psi_j.$$

Following [25], the local error estimator for pressure can be defined as

$$\eta_{L,T}^p = \|\bar{e}_{p,T}\|_{0,T} = \left\| \sum_{j \in \Lambda_T^p} \bar{x}_{p,j} \psi_j \right\|_T, \quad T \in \mathcal{T}_m.$$

From (4.22), we can get

$$\bar{x}_{u,\ell} = \frac{1}{(D_v)_{\ell,\ell}} (F_{v,\ell} - \sum_{j=1}^{N_p} B_{\ell,j} \bar{x}_{p,j}), \quad \ell = 1, \dots, N_v.$$

Set $\Lambda_T^v = \{j \mid \text{supp}(\varphi_j) \cap T \neq \emptyset, j = 1, \dots, N_v\}$, then

$$\bar{e}_{u,T} := (\bar{e}_u)|_T = \sum_{j \in \Lambda_T^v} \bar{x}_{u,j} \varphi_j.$$

From (4.9), the local error estimator for velocity can be defined as

$$\eta_{L,T}^v = |\bar{e}_{u,T}|_{D,T} = \sqrt{\sum_{j \in \Lambda_T^v} |\bar{x}_{u,j} \varphi_j|_{1,T}^2}, \quad T \in \mathcal{T}_m.$$

The local error estimator for divergence term can be defined as

$$\eta_{L,T}^d = \|\nabla \cdot \hat{\mathbf{u}}\|_{0,T}.$$

Then, the local error estimator can be defined as

$$\eta_{L,T} = \sqrt{(\eta_{L,T}^p)^2 + (\eta_{L,T}^v)^2 + (\eta_{L,T}^d)^2}. \quad (5.1)$$

Recall the third error problem (4.25) and the associated norm (4.6), we define the global error estimator as

$$\eta_G(\mathcal{T}_m) = \sqrt{\sum_{k \in \mathcal{T}_m} \eta_{L,T}^2} = \sqrt{\|(\bar{\mathbf{e}}_u, \bar{\mathbf{e}}_p)\|_D^2 + \|\nabla \cdot \hat{\mathbf{u}}\|^2}. \quad (5.2)$$

Based on Theorem 4.4, the global error estimator $\eta_G(\mathcal{T}_m)$ provides an estimate of the discretization error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$, which is frequently used to judge the quality of the underlying discretization. The local error estimator $\eta_{L,T}$ is an estimate of the error on element T . All elements $T \in \mathcal{T}_m$ are marked for refinement, if $\eta_{L,T}$ exceeds the certain tolerance. Denote the set of all marked elements by $\mathcal{S}_m \subset \mathcal{T}_m$. The global error estimator associated with \mathcal{S}_m is denoted by $\eta_G(\mathcal{S}_m)$.

The algorithm of adaptive FEM is listed in Algorithm 5.1.

Algorithm 5.1 Adaptive FEM

Input: Construct an initial mesh \mathcal{T}_0 . Choose a parameter $0 < \theta < 1$ and a tolerance ε .

Output: Final triangulation \mathcal{T}_M and the finite element approximation $(\hat{\mathbf{u}}_M, \hat{p}_M)$ on \mathcal{T}_M .

Set $m = 0$ and $\eta_G(\mathcal{T}_m) = 1$.

While $\eta_G(\mathcal{T}_m) > \varepsilon$

1. (SOLVE) Solve the FEM scheme (2.8) on \mathcal{T}_m .
2. (ESTIMATE) Compute the local error estimator as defined in (5.1) for all elements $T \in \mathcal{T}_m$.
3. (MARK) Construct a subset $\mathcal{S}_m \subset \mathcal{T}_m$ with least number of elements such that

$$\eta_G^2(\mathcal{S}_m) \geq \theta \eta_G^2(\mathcal{T}_m).$$

4. (REFINE) Refine elements in \mathcal{S}_m together with the elements, which must be refined to make \mathcal{T}_{m+1} conforming.
5. Set $m = m + 1$.

End

Set $(\hat{\mathbf{u}}_M, \hat{p}_M) = (\hat{\mathbf{u}}_m, \hat{p}_m)$ and $\mathcal{T}_M = \mathcal{T}_m$.

In the MARK step, we adopt the Dörfler marking strategy which is a mature strategy and is widely used in the adaptive algorithm [12]. Recently, it has been shown that Dörfler marking with minimal cardinality is a linear complexity problem [20]. In this marking strategy the local error estimators $\{\eta_{L,T}\}_{T \in \mathcal{T}_m}$ are sorted in descending order. The sorted local error estimators are denoted by $\{\tilde{\eta}_{L,T}\}_{T \in \mathcal{T}_m}$. Then, the set of elements marked for refinement is given by $\{\tilde{\eta}_{L,T}\}_{T \in \mathcal{S}_m}$, where \mathcal{S}_m contains the least number of elements such that

$$\eta_G^2(\mathcal{S}_m) = \sum_{T \in \mathcal{S}_m} \tilde{\eta}_{L,T}^2 \geq \theta \eta_G^2(\mathcal{T}_m).$$

Generally speaking, a small value of θ leads to a small set \mathcal{S}_m , while a large value of θ leads to a large set \mathcal{S}_m . In [12], θ is suggested to be adopted in $[0.5, 0.8]$. We emphasize that many auxiliary elements are refined to eliminate the hanging nodes, which may have been created in the MARK step. There are many mature toolkits to process the hanging nodes [25].

Finally, to show the effectiveness of the global error estimator defined in (5.2), we introduce the effective index as follows

$$\kappa_{eff} = \frac{\eta_G(\mathcal{T}_m)}{\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V}, \quad (5.3)$$

which is the ratio between the global error estimator and the FEM approximation error. According to Theorem 4.4, the effective index is bounded from both above and below.

6 Numerical experiments

In this section, we present two-dimensional numerical examples to demonstrate the efficiency and reliability of our adaptive FEM. All these simulations have been implemented on a 3.2GHz quad-core processor with 16GB RAM by Matlab.

Example 1. This example is taken from page 113 in [24]. The solution is singular at the origin. Let Ω be the L-shape domain $(-1, 1)^2 \setminus [0, 1) \times (-1, 0]$, and select $\mathbf{f} = 0$. Then, use (r, φ) to denote the polar coordinates. We impose an appropriate inhomogeneous boundary condition for \mathbf{u} so that

$$\begin{aligned}
u_1(r, \varphi) &= r^\lambda((1 + \lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi)), \\
u_2(r, \varphi) &= r^\lambda(\sin(\varphi)\Psi'(\varphi) - (1 + \lambda)\cos(\varphi)\Psi(\varphi)), \\
p(r, \varphi) &= -r^{\lambda-1}[(1 + \lambda)^2\Psi'(\varphi) + \Psi''']/(1 - \lambda),
\end{aligned}$$

where

$$\begin{aligned}
\Psi(\varphi) &= \sin((1 + \lambda)\varphi)\cos(\lambda\omega)/(1 + \lambda) - \cos((1 + \lambda)\varphi) - \sin((1 - \lambda)\varphi)\cos(\lambda\omega)/(1 - \lambda) + \cos((1 - \lambda)\varphi), \\
\omega &= \frac{3\pi}{2}.
\end{aligned}$$

The exponent λ is the smallest positive solution of

$$\sin(\lambda\omega) + \lambda\sin(\omega) = 0,$$

thereby, $\lambda \approx 0.54448373678246$.

We emphasize that (\mathbf{u}, p) is analytic in $\overline{\Omega} \setminus \{0\}$, but both $\nabla \mathbf{u}$ and p are singular at the origin; indeed, here $\mathbf{u} \notin [H^2(\Omega)]^2$ and $p \notin H^1(\Omega)$. This example reflects the typical (singular) behavior that solutions of the two-dimensional Stokes problem exhibit in the vicinity of reentrant corners in the computational domain.

We denote the finite element spaces by V_1 and W_3 in the approximation problem and the error problem, respectively. The finite element space V_1 consists of velocity space and pressure space. The velocity space is the space of continuous piecewise quadratic polynomials and the pressure space is the space of continuous piecewise linear polynomials associated with \mathcal{T} . It is characterized in terms of Lagrange basis. The hierarchical basis of any component with respect to velocity in any element $T \in \mathcal{T}$ will be

$$\lambda_1\lambda_2\lambda_3, \lambda_2^2\lambda_3, \lambda_2\lambda_3^2, \lambda_1^2\lambda_3, \lambda_1\lambda_3^2, \lambda_1^2\lambda_2, \lambda_1\lambda_2^2, \lambda_2^2\lambda_3^2, \lambda_1^2\lambda_3^2, \lambda_1^2\lambda_2^2, \lambda_1^2\lambda_2\lambda_3, \lambda_1\lambda_2^2\lambda_3, \lambda_1\lambda_2\lambda_3^2.$$

The hierarchical basis of pressure in any element $T \in \mathcal{T}$ will be $\lambda_1\lambda_2\lambda_3$, where $\lambda_i (i = 1, 2, 3)$ are Lagrange bases of the three vertices in T , respectively. These bases of W_3 in any element are shown in Figure 6.1.

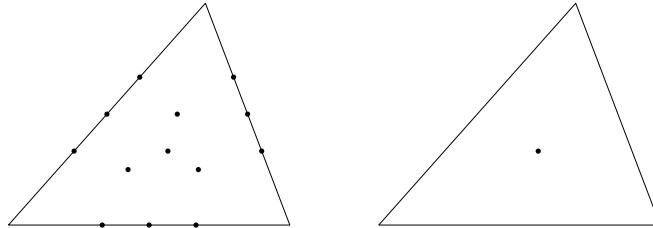


Fig. 6.1 The basis of velocity(left) and pressure(right) in any element for Example 1.

Figure 6.2(a) shows that for Example 1, $\eta_G(\mathcal{T}_m)$ has different convergent rates with respect to the degrees of freedom (*d.o.f*) for different θ . Table 6.1 shows the computation cost for different θ when $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V < 0.25$. From the comparison in Table 6.1, adaptive FEM is much faster than the uniform refinement. In the MARK step, we set the Döfler parameter as $\theta = 0.7$. In the REFINE step, the refinement process is implemented using the MATLAB function REFINEMESH. The key is dividing the marked element into four parts by regular refinement (dividing all edges of the selected triangles in half).

Figure 6.2(b) shows the convergent rates of $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ and $\eta_G(\mathcal{T}_m)$ for Algorithm 5.1. The x -axes denotes the *d.o.f* in log scale, while y -axes denotes the errors in log scale. The squared line denotes the error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ of the uniform refinement. The asterisk line and the circled line denote the error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ and $\eta_G(\mathcal{T}_m)$ of adaptive FEM, respectively. It is obvious that $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$ and $\eta_G(\mathcal{T}_m)$ have the same convergence order and have a higher convergence order than the uniform refinement.

Table 6.2 shows the error $\|(\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\|_V$, the global error estimator $\eta_G(\mathcal{T}_m)$, and the effective index κ_{eff} of the adaptive FEM as the *d.o.f* increases. The results of effective index κ_{eff} defined in (5.3) are shown in the sixth column. The effective index κ_{eff} is between 0.5 and 0.7 with adaptive refinement,

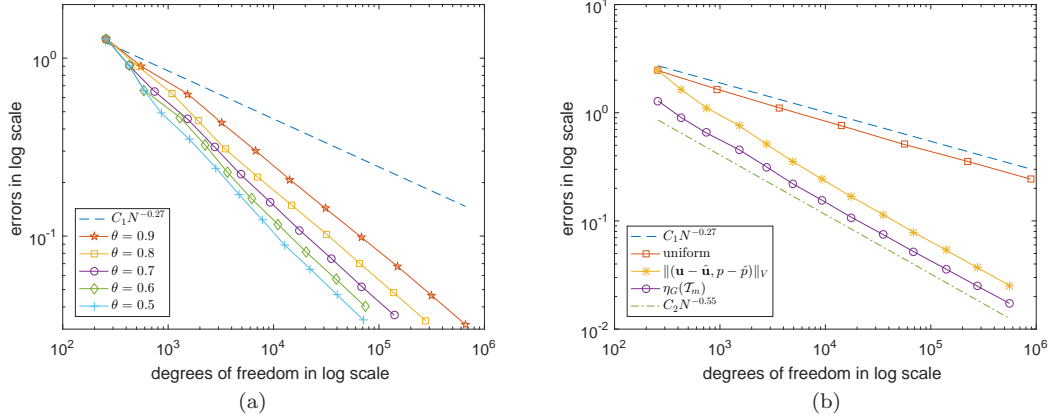


Fig. 6.2 The convergent rates of adaptive FEM for Example 1.

Table 6.1 Computation cost for different θ in Example 1.

θ	refinement steps	$d.o.f$	$\ (\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\ _V$	time(s)
0.9	7	31063	0.243	6.533
0.8	7	14935	0.243	4.339
0.7	7	9250	0.244	3.299
0.6	7	6148	0.244	3.018
0.5	7	4738	0.245	2.308
<i>uniform</i>	7	887299	0.243	130.463

Table 6.2 The errors, the global error estimator, and the effective index of the adaptive FEM in Example 1.

$d.o.f$	$\ (\mathbf{u} - \hat{\mathbf{u}}, p - \hat{p})\ _V$	order	$\eta_G(\mathcal{T}_m)$	order	κ_{eff}
259	2.452E0	—	1.286E0	—	0.524
426	1.644E0	0.803	9.068E-1	0.702	0.551
738	1.116E0	0.705	6.526E-1	0.598	0.584
1523	7.595E-1	0.531	4.550E-1	0.497	0.599
2778	5.101E-1	0.662	3.151E-1	0.611	0.617
4871	3.559E-1	0.640	2.215E-1	0.627	0.622
9250	2.436E-1	0.590	1.557E-1	0.549	0.639
17707	1.670E-1	0.581	1.078E-1	0.565	0.645
35236	1.145E-1	0.547	7.492E-2	0.529	0.653
68949	7.846E-2	0.564	5.194E-2	0.545	0.661
138420	5.381E-2	0.535	3.590E-2	0.524	0.667
277820	3.687E-2	0.548	2.492E-2	0.529	0.675
557663	2.529E-2	0.540	1.728E-2	0.525	0.683

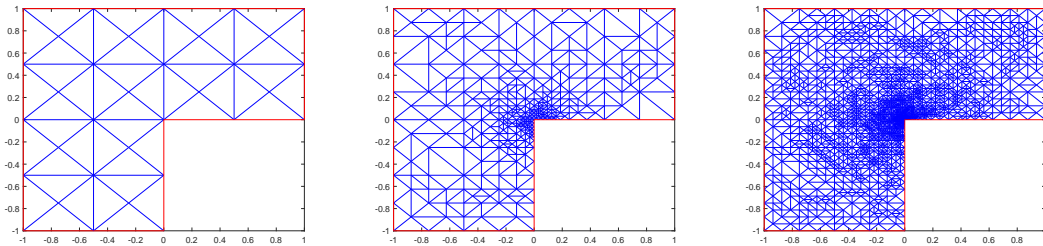


Fig. 6.3 The meshes with $d.o.f = 259$ (left), 2778 (middle), and 17707 (right) for Example 1.

which shows the adaptive FEM is reliable. Figure 6.3 shows the initial mesh with $d.o.f = 259$, fourth refinement mesh with $d.o.f = 2778$, and seventh refinement mesh with $d.o.f = 17707$. It has inserted refinement elements around the singularity at $(x, y) = (0, 0)$ as $d.o.f$ increases to reduce the global error.

Example 2. In this case, we test the lid-driven cavity problem. The domain is taken as the square $\Omega = (0, 1) \times (0, 1)$, we set $\mathbf{f} = \mathbf{0}$, and the boundary conditions $\mathbf{u} = \mathbf{0}$ on $[\{0\} \times (0, 1)] \cup [(0, 1) \times \{0\}] \cup [\{1\} \times (0, 1)]$ and $\mathbf{u} = (1, 0)^T$ on $(0, 1) \times \{1\}$. This problem has a corner singularity. The tangential component of velocity $\mathbf{u} \cdot \boldsymbol{\tau}$ has a discontinuity at the two top corners, where $\boldsymbol{\tau}$ denotes the unit tangential vector on the boundary. We use the proposed adaptive FEM algorithm to solve this problem. The finite element space, Döfler parameter, and refinement criterion are the same as Example 1. Figure 6.4 shows that the

refinement of mesh focuses on the two top corners. In Figure 6.5, we depict the discrete pressure field obtained using the initial and adapted meshes where we note the improvement in the quality of the computed solution since the singular nature of the pressure is better captured in the adapted mesh.

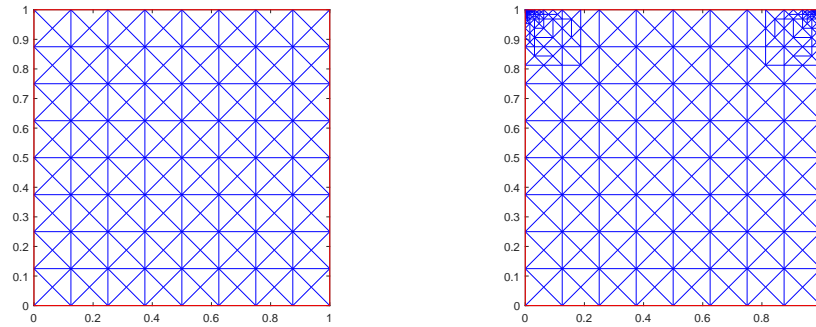


Fig. 6.4 The initial mesh and tenth refinement mesh for Example 2.

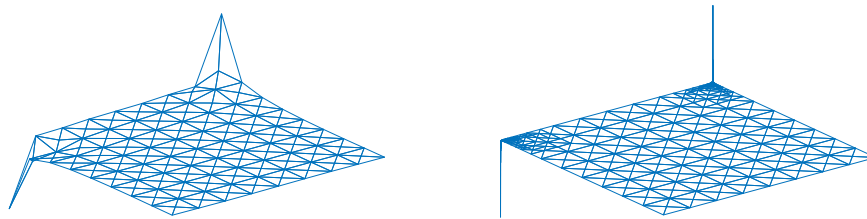


Fig. 6.5 The pressures in initial mesh and tenth refinement mesh for Example 2.

7 Conclusion

In this paper, we present an adaptive FEM for solving the Stokes problem with Dirichlet boundary condition. Based on auxiliary subspace techniques, we proposed a hierarchical basis *a posteriori* error estimator, which is most efficient and robust. We need to solve only two global diagonal linear systems. In theory, The estimator is proved to have global upper and lower bounds without saturation assumption. Numerical experiments are shown to illustrate the efficiency and reliability of our adaptive algorithm.

Acknowledgments

The work of J.C. Zhang was supported by the Natural Science Foundation of Jiangsu Province (grant no.BK20210540) , the Natural Science Foundation of the Jiangsu Higher Education Institutions of China (grant no.21KJB110015). The work of R. Zhang was supported by the National Key Research and Development Program of China (grant no.2020YFA0713601).

Declarations

Conflict of interest The authors have no conflicts of interest to declare.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A.

The proof of Lemma 2.2.

Proof The idea of proof is similar to [8] for $d = 2$ and [9] for $d = 3$. Next, we will give proofs for $d = 2$ and $d = 3$, respectively.

Case $d = 2$: The idea is to consider a macroelement partition of the domain Ω in such a way that each macroelement contains exactly three triangles. By virtue of Remark 3.3 in [8], it suffices to prove the inf-sup condition for only one macroelement. We consider a macroelement $\Omega_i = a \cup b \cup c$ as in Figure A.1

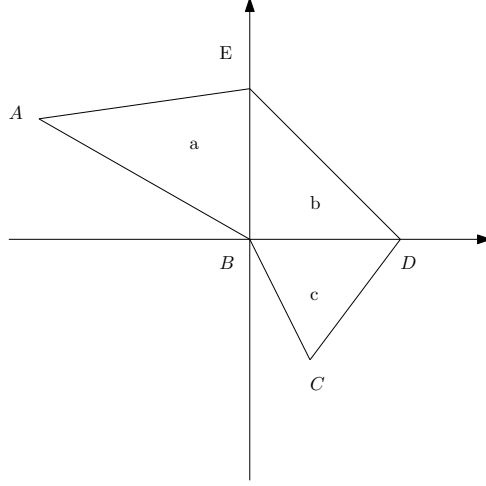


Fig. A.1 The macroelement partition containing three triangles.

Let us introduce some notations. We denote by λ_{AB}^a by the barycentric coordinate related to the triangle a, which vanishes on the edge AB (analogous notations for the other cases). we denote by $L_{i,x}^a$ the i -th Legendre polynomial with respect to the measure $\mu_{a,x}$ defined as

$$\int_{x_A}^0 f(x) d\mu_{a,x} = \int_a \lambda_{AB}^a \lambda_{AE}^a f(x) dx dy \quad \forall f(x) : [x_A, 0] \rightarrow \mathbb{R}, \quad (\text{A.1})$$

where x_A is the x -coordinate of the vertex A . A similar definition will hold for $L_{i,y}^c$ using $\lambda_{BC} \lambda_{CD}$. On the triangle b we shall use both $L_{i,x}^b$ (using $\lambda_{ED} \lambda_{BD}$) and $L_{i,y}^b$ (using $\lambda_{BE} \lambda_{ED}$). These Legendre polynomials are defined up to a constant factor so that we can normalize them by imposing that they assume the same value at the origin. This is possible by virtue of Proposition 2.1 in [8].

Our approach to the stability condition will be related to the modified inf-sup condition that can be written as

$$\sup_{\hat{\mathbf{v}} \in \overline{WV}_{k+j+1}} \frac{b(\hat{\mathbf{v}}, \hat{q})}{\|\hat{\mathbf{v}}\|} \geq \mu \|\nabla \hat{q}\|, \quad \forall \hat{q} \in \overline{P}_{k+j},$$

which implies the standard one [23].

For every fixed $\hat{q} \in \overline{P}_{k+j}$ we want to construct $\hat{\mathbf{v}} \in \overline{WV}_{k+j+1}$ such that

$$-\int_{\Omega} \hat{\mathbf{v}} \cdot \nabla \hat{q} dx \geq c_1 \|\nabla \hat{q}\|^2, \quad (\text{A.2})$$

$$\|\hat{\mathbf{v}}\|_{0,\Omega} \leq c_2 \|\nabla \hat{q}\|. \quad (\text{A.3})$$

Define:

$$\begin{aligned}
\hat{\mathbf{v}}(x, y) &= (\hat{v}_1(x, y), \hat{v}_2(x, y)), \\
\hat{v}_1(x, y)|_a &= -\lambda_{AB}^a \lambda_{AE}^a \|\nabla \hat{q}\| L_{k-1,x}^a \cdot \text{sign}(H_a), \\
\hat{v}_2(x, y)|_a &= -\lambda_{AB}^a \lambda_{AE}^a \frac{\partial \hat{q}}{\partial y}, \\
\hat{v}_1(x, y)|_b &= -\lambda_{ED}^b \lambda_{BD}^b \|\nabla \hat{q}\| L_{k+d-1,x}^b \cdot \text{sign}(H_b) - \lambda_{ED}^b \lambda_{EB}^b \frac{\partial \hat{q}}{\partial x}, \\
\hat{v}_2(x, y)|_b &= -\lambda_{EB}^b \lambda_{BD}^b \frac{\partial \hat{q}}{\partial y} - \lambda_{ED}^b \lambda_{EB}^b \|\nabla \hat{q}\| L_{k+d-1,y}^b \cdot \text{sign}(K_b), \\
\hat{v}_1(x, y)|_c &= -\lambda_{BC}^c \lambda_{CD}^c \frac{\partial \hat{q}}{\partial x}, \\
\hat{v}_2(x, y)|_c &= -\lambda_{BC}^c \lambda_{CD}^c \|\nabla \hat{q}\| L_{k+d-1,y}^c \cdot \text{sign}(K_c),
\end{aligned}$$

where $\text{sign}(x)$ is sign function defined as

$$\text{sign}(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

and

$$\begin{aligned}
H_a &= \int_a \lambda_{AB}^a \lambda_{AE}^a L_{k+d-1,x} \cdot \frac{\partial \hat{q}}{\partial x}, & H_b &= \int_b \lambda_{ED}^b \lambda_{BD}^b L_{k+d-1,x} \cdot \frac{\partial \hat{q}}{\partial x}, \\
K_a &= \int_b \lambda_{EB}^b \lambda_{ED}^b L_{k+d-1,y} \cdot \frac{\partial \hat{q}}{\partial y}, & K_b &= \int_c \lambda_{BC}^c \lambda_{CD}^c L_{k+d-1,y} \cdot \frac{\partial \hat{q}}{\partial y}.
\end{aligned}$$

First of all, we observe that $\hat{\mathbf{v}}$ is an element of \overline{WV}_{k+j+1} , by the virtue of the fact that the tangential components of $\nabla \hat{q}$ along the interface EB and BD are continuous.

It is easy to verify that $\hat{\mathbf{v}}$ satisfies (A.3). In order to check the validity of (A.2), define $\|\nabla \hat{q}\| = -\int_\Omega \hat{\mathbf{v}} \cdot \nabla \hat{q}$. Then

$$\begin{aligned}
\|\nabla \hat{q}\|^2 &= \int_a \lambda_{AB}^a \lambda_{AE}^a \left(\frac{\partial \hat{q}}{\partial y}\right)^2 + \|\nabla \hat{q}\|(|H_a| + |H_b|) \\
&\quad + \int_b (\lambda_{ED}^b \lambda_{EB}^b \left(\frac{\partial \hat{q}}{\partial x}\right)^2 + \lambda_{ED}^b \lambda_{BD}^b \left(\frac{\partial \hat{q}}{\partial y}\right)^2) \\
&\quad + \|\nabla \hat{q}\|(|K_a| + |K_b|) + \int_c \lambda_{BC}^c \lambda_{CD}^c \left(\frac{\partial \hat{q}}{\partial x}\right)^2.
\end{aligned} \tag{A.4}$$

We verify that the expression $\|\frac{\partial \hat{q}}{\partial x}\|_H := |H_a| + |H_b|$ is a norm of $\frac{\partial \hat{q}}{\partial x}$ in $a \cup b$ and $\|\frac{\partial \hat{q}}{\partial y}\|_K := |K_a| + |K_b|$ is a norm of $\frac{\partial \hat{q}}{\partial y}$ in $b \cup c$.

Step 1. We will show $|H_a| + |H_b|$ vanishes only when $\frac{\partial \hat{q}}{\partial x}$ equals zero. From (A.1)

$$0 = \|\frac{\partial \hat{q}}{\partial x}\|_H = \left| \int_{x_A}^0 L_{k+d-1,x} \cdot \frac{\partial \hat{q}}{\partial x} \right| + \left| \int_0^1 L_{k+d-1,x} \cdot \frac{\partial \hat{q}}{\partial x} \right|$$

From the orthogonality of Legendre polynomials $L_{i,x}^a, L_{i,x}^b$ and noting that $\frac{\partial \hat{q}}{\partial x}$ is a homogeneous polynomial of degree $k+d-1$, we have $\frac{\partial \hat{q}}{\partial x} = 0$ in $a \cup b$.

Step 2. We will get $\|k \frac{\partial \hat{q}}{\partial x}\|_H = |k| \|\frac{\partial \hat{q}}{\partial x}\|_H$ from

$$\|k \frac{\partial \hat{q}}{\partial x}\|_H = \left| \int_{x_A}^0 L_{k+d-1,x} \cdot k \frac{\partial \hat{q}}{\partial x} \right| + \left| \int_0^1 L_{k+d-1,x} \cdot k \frac{\partial \hat{q}}{\partial x} \right|.$$

Step 3. We will show that $\|\frac{\partial \hat{q}_1}{\partial x} + \frac{\partial \hat{q}_2}{\partial x}\|_H \leq \|\frac{\partial \hat{q}_1}{\partial x}\|_H + \|\frac{\partial \hat{q}_2}{\partial x}\|_H$.

$$\begin{aligned}
\left\| \frac{\partial \hat{q}_1}{\partial x} + \frac{\partial \hat{q}_2}{\partial x} \right\|_H &= \left| \int_{x_A}^0 L_{k+d-1,x} \cdot \left(\frac{\partial \hat{q}_1}{\partial x} + \frac{\partial \hat{q}_2}{\partial x} \right) \right| + \left| \int_0^1 L_{k+d-1,x} \cdot \left(\frac{\partial \hat{q}_1}{\partial x} + \frac{\partial \hat{q}_2}{\partial x} \right) \right| \\
&\leq \left| \int_{x_A}^0 L_{k+d-1,x} \cdot \frac{\partial \hat{q}_1}{\partial x} \right| + \left| \int_{x_A}^0 L_{k+d-1,x} \cdot \frac{\partial \hat{q}_2}{\partial x} \right| + \left| \int_0^1 L_{k+d-1,x} \cdot \frac{\partial \hat{q}_2}{\partial x} \right| + \left| \int_0^1 L_{k+d-1,x} \cdot \frac{\partial \hat{q}_1}{\partial x} \right| \\
&= \left\| \frac{\partial \hat{q}_1}{\partial x} \right\|_H + \left\| \frac{\partial \hat{q}_2}{\partial x} \right\|_H.
\end{aligned}$$

Similarly, $\left\| \frac{\partial \hat{q}}{\partial y} \right\|_K = |K_b| + |K_c|$ is a norm of $\frac{\partial \hat{q}}{\partial y}$ in $b \cup c$. From the equivalence of norms on a finite dimensional space, there exists a constant $C_a, C_b, C_c, C_H, C_K > 0$ such that

$$\begin{aligned}
\int_a \lambda_{AB}^a \lambda_{AE}^a \left(\frac{\partial \hat{q}}{\partial y} \right)^2 &\geq C_a \int_a \left(\frac{\partial \hat{q}}{\partial y} \right)^2, \quad \left\| \frac{\partial \hat{q}}{\partial x} \right\|_H \geq C_H \sqrt{\int_a \left(\frac{\partial \hat{q}}{\partial x} \right)^2 + \int_b \left(\frac{\partial \hat{q}}{\partial x} \right)^2}, \\
\int_b (\lambda_{ED}^b \lambda_{EB}^b \left(\frac{\partial \hat{q}}{\partial x} \right)^2 + \lambda_{ED}^b \lambda_{BD}^b \left(\frac{\partial \hat{q}}{\partial y} \right)^2) &\geq C_b \int_b \left(\left(\frac{\partial \hat{q}}{\partial x} \right)^2 + \left(\frac{\partial \hat{q}}{\partial y} \right)^2 \right) \\
\int_c \lambda_{BC}^c \lambda_{CD}^c \left(\frac{\partial \hat{q}}{\partial x} \right)^2 &\geq C_c \int_c \left(\frac{\partial \hat{q}}{\partial x} \right)^2, \quad \left\| \frac{\partial \hat{q}}{\partial y} \right\|_H \geq C_K \sqrt{\int_b \left(\frac{\partial \hat{q}}{\partial y} \right)^2 + \int_c \left(\frac{\partial \hat{q}}{\partial y} \right)^2}.
\end{aligned}$$

Set $c_1 = \min\{C_a, C_b, C_c, C_H, C_K\}$ and obtain (A.2).

Case $d = 3$: We use the macroelement described by Stenberg in [22] in order to check the inf-sup condition. Let \mathcal{M} be a macroelement partition of the domain decomposition of \mathcal{T} . For a macroelement $M \in \mathcal{M}$ we introduce the following usual notation:

$$\begin{aligned}
WV_M &= \{\hat{\mathbf{v}}|_M \mid \hat{\mathbf{v}} \in W_{k+d+1}\} \cap [H_0^1(M)]^3, \\
WP_M &= \{\hat{q}|_M \mid \hat{q} \in WP_{k+d}\}.
\end{aligned}$$

Consider a generic macroelement $M \in \mathcal{M}$. Let $T_0 \in \mathcal{T}$ be a tetrahedron of M and denote by x_0 the internal vertex of T_0 which also belongs to the other element of M . There are three edges $e_i, i = 1, \dots, 3$ of T_0 meeting at x_0 . Thanks to the fact that x_0 is internal, none of the edges e_i lie on the boundary $\partial\Omega$.

Let $\hat{q} \in WP_M$ be given and suppose that

$$\int_M \hat{q} \nabla \cdot \hat{\mathbf{v}} = 0, \quad \forall \hat{\mathbf{v}} \in WV_M. \tag{A.5}$$

We shall prove that $\nabla \hat{q}$ vanishes on T_0 , thus obtaining H1 condition described in Theorem 2.1 in [9] by virtue of the fact that T_0 is arbitrary and q is continuous.

First, we concentrate our attention on the edge e_1 and fix an (x, y, z) -coordinate system in such a way that e_1 lies in the direction of the x -axis. We consider the collection $\mathcal{A} = \{T_0, \dots, T_n\}$ of those elements of \mathcal{T} which share the edge e_1 in common with T_0 (including T_0 itself). It is clear that $T_i \in M$ and that exactly two faces of T_i touch other elements of \mathcal{A} of every i .

Define $\hat{\mathbf{v}}$ in the following way:

$$\begin{aligned}
\hat{\mathbf{v}}|_{T_i} &= (\lambda_i \kappa_i \frac{\partial \hat{q}}{\partial x}, 0, 0), \\
\hat{\mathbf{v}}|_T &= (0, 0, 0), \quad \text{if } T \in \mathcal{T}, T \neq T_i, \forall i,
\end{aligned}$$

where λ_i and κ_i are the equations of the two faces of T_i which are not in common with any other element of \mathcal{A} , normalized in order to assume the same value at the opposite vertex. It is worthwhile to observe that these two vertices are x_0 and the other extreme of the edge e_1 .

It is easily verified that $\hat{\mathbf{v}}$ is a polynomial of degree $k+1$ and that it is continuous in M . The continuity of \hat{q} in M ensures that the gradient of \hat{q} is continuous between two elements in all the directions which are contained in the plane of the interface; the x -axis is the direction of e_1 which is the edge of all common faces among the elements of \mathcal{A} . Moreover, $\hat{\mathbf{v}}$ vanishes at the boundary of M ; hence, the following inclusion holds:

$$\hat{\mathbf{v}} \in WV_M.$$

Suppose now that (A.5) hold.

$$0 = \int_M \hat{q} \nabla \hat{\mathbf{v}} = - \int_M \nabla \hat{q} \cdot \hat{\mathbf{v}} = - \sum_{i=1}^n \int_{T_i} \lambda_i \kappa_i \left(\frac{\partial \hat{q}}{\partial x} \right)^2.$$

It follows that the component of $\nabla \hat{q}$ in the direction of the x -axis vanishes in T_i for every i .

The same argument applies to the edge e_2 and e_3 , giving the result that $\nabla \hat{q}$ vanishes on T_0 in the direction of e_i , for $i = 1, \dots, 3$. These three directions being independent, the final result

$$\nabla \hat{q} = (0, 0, 0), \quad \text{in } T_0$$

is obtained and the lemma is proved. Then the H1 condition of Theorem 2.1 in [9] is proved and the H2-H3 conditions follow immediately from the regularity assumption of \mathcal{T} . \square

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