

Inductive k -independent graphs and c -colorable subgraphs in scheduling: A review

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Abstract Inductive k -independent graphs generalize chordal graphs and have recently been advocated in the context of interference-avoiding wireless communication scheduling. The NP-hard problem of finding maximum-weight induced c -colorable subgraphs, which is a generalization of finding maximum independent sets, naturally occurs when selecting c sets of pairwise non-conflicting jobs (modeled as graph vertices). We investigate the parameterized complexity of this problem on inductive k -independent graphs. We show that the MAXIMUM INDEPENDENT SET problem is W[1]-hard even on 2-simplicial 3-minoes—a subclass of inductive 2-independent graphs. In contrast, we prove that the more general MAX-WEIGHT c -COLORABLE SUBGRAPH problem is fixed-parameter tractable on edge-wise unions of cluster and chordal graphs, which are 2-simplicial. In both cases, the parameter is the solution size. Aside from this, we survey other graph classes between inductive 1-independent and inductive 2-independent graphs with applications in scheduling.

Keywords independent set · job interval selection · interval graphs · chordal graphs · inductive k -independent graphs · NP-hard problems · parameterized complexity

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1 Introduction

Finding sets of “independent” (that is, pairwise non-conflicting) jobs is of central importance in many scheduling scenarios. In particular, the NP-hard MAXIMUM INDEPENDENT SET problem and its generalization of finding maximum-weight induced c -colorable subgraphs, both in inductive k -independent graphs, have recently been identified as key tools in the development of polynomial-time approximation algorithms in interference-avoiding wireless communication scheduling (Ásgeirsson et al 2017; Halldórsson 2016; Halldórsson and Tonoyan 2015).

We conduct a deeper study of the computational complexity of these problems on subclasses of inductive k -independent graphs. We also exhibit a rich fine structure of known graph classes that have applications in scheduling and are inductive 2-independent.

A graph is inductive k -independent if its vertices can be ordered from left to right so that the “right-hand” neighborhood of each vertex contains no independent vertex set of size greater than k . The problem we study on these graphs is MAXIMUM (WEIGHT) c -COLORABLE SUBGRAPH¹—find a maximum (weight) vertex subset that induces a subgraph allowing for coloring the vertices in c colors so that no adjacent vertices have the same color. Clearly, for $c = 1$, we obtain the classical MAXIMUM (WEIGHT) INDEPENDENT SET problem.

Halldórsson and Tonoyan (2015) found that, in interference-avoiding wireless communication scheduling, one typically faces these problems in inductive k -independent graphs with $k \leq 12$. Unfortunately, we will see that already unweighted MAXIMUM INDEPENDENT SET is hard for subclasses of inductive 2-independent graphs. Notably, however, clas-

¹ Ásgeirsson et al (2017) are interested in maximum-weight unions of c independent sets. In the graph theory literature the problem is known as MAX-WEIGHT c -COLORABLE SUBGRAPH; we prefer to stick to the established graph theory notion.

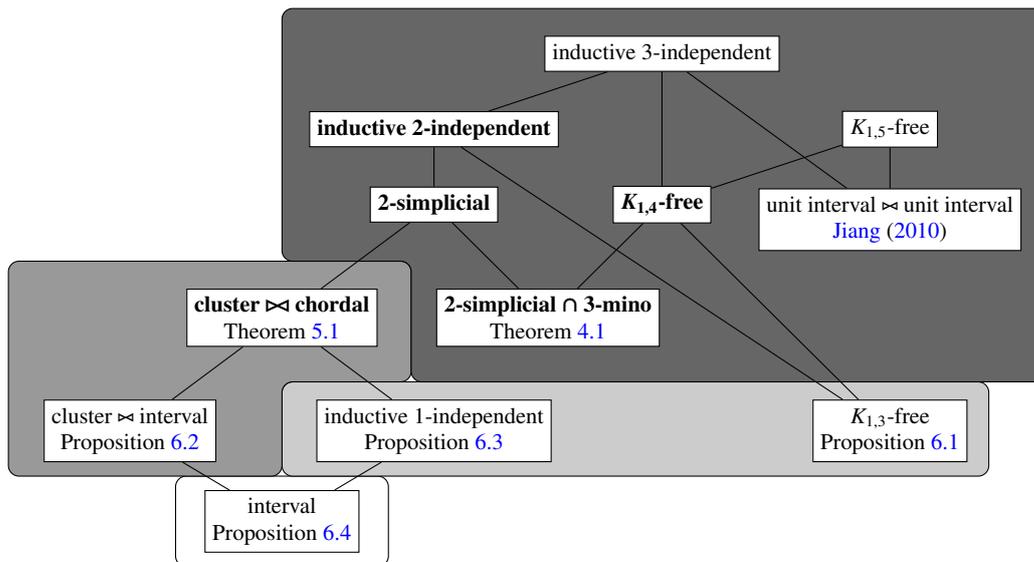


Fig. 1.1: For definitions of graph classes, see Section 2. Our new results are in bold.

White — MAX-WEIGHT c -COLORABLE SUBGRAPH is solvable in polynomial time.

Light Gray — MAX-WEIGHT INDEPENDENT SET is polynomial-time solvable, MAXIMUM c -COLORABLE SUBGRAPH is NP-hard, MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by the solution size is fixed-parameter tractable.

Gray — MAXIMUM INDEPENDENT SET is NP-hard. MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by solution size is fixed-parameter tractable.

Dark Gray — MAXIMUM INDEPENDENT SET is NP-hard and W[1]-hard parameterized by the solution size.

sis scheduling problems have been studied for a number of subclasses of inductive 2-independent graphs, including interval graphs (Kolen et al 2007), strip graphs (Halldórsson and Karlsson 2006), their superclass 2-track interval graphs (Höhn et al 2011; van Bevern et al 2015), and claw-free graphs (Gaur and Krishnamurti 2003; Köse et al 2017; Köse and Médard 2017).

Our contributions and organization of this work. Our main contributions are as follows. We refer to Section 2 for definitions of graph and complexity classes.

In Section 3, we explore the fine structure of graph classes below inductive 3-independent graphs, discuss relations to known graph classes (often already appearing in scheduling applications), and also survey the recognition complexities of the respective graph classes.

In Section 4, we show that already (unweighted) MAXIMUM INDEPENDENT SET parameterized by the solution size (number of vertices) is W[1]-hard on 2-simplicial 3-minoes, a proper subclass of inductive 2-independent graphs. Thus, there is no reason to hope for exact fixed-parameter algorithms to find even small independent sets in these graphs.

In contrast, in Section 5 we show that MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by the solution size is fixed-parameter tractable on a class that lies properly between inductive 1-independent and inductive 2-independent graphs and generalizes the class of strip graphs, which Halldórsson

and Karlsson (2006) used to model the JOB INTERVAL SELECTION problem.

Finally, in Section 6, we briefly survey complexity results on MAX-WEIGHT c -COLORABLE SUBGRAPH in the other graph classes discussed in Section 3.

We refer to Figure 1.1 for an overview of results on the parameterized complexity of MAXIMUM INDEPENDENT SET and MAX-WEIGHT c -COLORABLE SUBGRAPH on graph classes mainly below inductive 3-independent graphs.

Related work. It is well known that, on general graphs, already (unweighted) MAXIMUM INDEPENDENT SET is a notoriously hard problem from the viewpoint of polynomial-time approximability as well as from the viewpoint of parameterized complexity. Hence, it is natural to study MAXIMUM INDEPENDENT SET and related problems on special graph classes.

Our results improve on or complement the following known results on MAXIMUM INDEPENDENT SET and MAX-WEIGHT c -COLORABLE SUBGRAPH on inductive k -independent graph classes.

- MAXIMUM INDEPENDENT SET parameterized by solution size is fixed-parameter tractable on strip graphs (van Bevern et al 2015); strip graphs are equivalent to the class of cluster \bowtie interval graphs (see Figure 1.1), a proper subclass of inductive 2-independent graphs.
- MAX-WEIGHT INDEPENDENT SET parameterized by solution size is W[1]-hard already on unit 2-track interval graphs

(Jiang 2010); unit 2-track interval graphs form a subclass of inductive 3-independent graphs (Ye and Borodin 2012).

- (Unweighted) MAXIMUM c -COLORABLE SUBGRAPH parameterized by solution size is fixed-parameter tractable on inductive 1-independent (that is, chordal) graphs (Misra et al 2013).

2 Preliminaries

This section introduces basic notation and concepts that we will use throughout this work.

Fixed-parameter algorithms. The essential idea behind fixed-parameter algorithms is to accept exponential running times, which are seemingly inevitable in solving NP-hard problems, but to restrict them to one aspect of the problem, the *parameter* (Cygan et al 2015; Downey and Fellows 2013; Flum and Grohe 2006; Niedermeier 2006).

Thus, formally, an instance of a *parameterized problem* $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is a pair (x, k) consisting of the *input* x and the *parameter* k . A parameterized problem Π is *fixed-parameter tractable (FPT)* with respect to a parameter k if there is an algorithm solving any instance of Π with size n in $f(k) \cdot \text{poly}(n)$ time for some computable function f . Such an algorithm is called a *fixed-parameter algorithm*. It is potentially efficient for small values of k , in contrast to an algorithm that is merely running in polynomial time for each fixed k (thus allowing the degree of the polynomial to depend on k). FPT is the complexity class of fixed-parameter tractable parameterized problems.

Parameterized intractability To show that a problem is presumably not fixed-parameter tractable, there is a parameterized analog of NP-hardness theory. The parameterized analog of $P \subseteq NP$ is a hierarchy of complexity classes $\text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \dots$, where all inclusions are assumed to be proper. Analogously to the class coNP , for $t \in \mathbb{N}$, one defines the classes $\text{coW}[t]$ containing all parameterized problems $\Pi \subseteq \Sigma^* \times \mathbb{N}$ for which the complement $(\Sigma^* \times \mathbb{N}) \setminus \Pi$ is in $\text{W}[t]$. A parameterized problem Π with parameter k is *(co)W[t]-hard* for some $t \in \mathbb{N}$ if any problem in $(\text{co})\text{W}[t]$ has a parameterized reduction to Π : a *parameterized reduction* from a parameterized problem Π_1 to a parameterized problem Π_2 is an algorithm mapping an instance (x, k) to an instance (x', k') in $f(k) \cdot \text{poly}(|x|)$ time such that $k' \leq g(k)$ and $(x, k) \in \Pi_1 \iff (x', k') \in \Pi_2$, where f and g are arbitrary computable functions. By definition, no $(\text{co})\text{W}[t]$ -hard problem is fixed-parameter tractable unless $\text{FPT} = \text{W}[t]$.

Graph theory. We consider undirected, finite, simple graphs $G = (V, E)$, where V is the set of *vertices* and $E \subseteq \{\{v, w\} \mid v \neq w \text{ and } v, w \in V\}$ is the set of *edges*.

If the vertices have *weights* $w: V \rightarrow \mathbb{N}$, then we denote the weight of a vertex subset $S \subseteq V$ as $w(S) := \sum_{v \in S} w(v)$.

The (*open*) *neighborhood* $N_G(v) := \{u \in V \mid \{u, v\} \in E\}$ of a vertex $v \in V$ is the set of vertices *adjacent to* v in G . The *closed neighborhood* of v is $N_G[v] := N_G(v) \cup \{v\}$. When there is no ambiguity, we drop the subscript G . An *independent set* is a set of pairwise nonadjacent vertices.

For a vertex subset $S \subseteq V$, the *subgraph* $G[S]$ of G *induced by* S is the graph with vertex set S and edge set $\{\{u, v\} \in E \mid u \in S, v \in S\}$. We say that G is *F-free* for some graph F if G does not contain induced subgraphs isomorphic to F . A $K_{n,m}$ is a complete bipartite graph with n vertices on the one side and m vertices on the other.

A graph is c -colorable if one can assign each vertex one of c colors so that the endpoints of each edge have distinct colors.

We study the parameterized complexity of the following problem.

Problem 2.1 (MAX-WEIGHT c -COLORABLE SUBGRAPH)

Input: A graph $G = (V, E)$ with vertex weights $w: V \rightarrow \mathbb{N}$.
Task: Find a set $S \subseteq V$ of maximum weight $w(S)$ such that $G[S]$ is c -colorable.

For $c = 1$, we obtain the classic MAX-WEIGHT INDEPENDENT SET problem.

Problem 2.2 (MAX-WEIGHT INDEPENDENT SET)

Input: A graph $G = (V, E)$ with vertex weights $w: V \rightarrow \mathbb{N}$.
Task: Find a set $S \subseteq V$ of pairwise nonadjacent vertices and maximum weight $w(S)$.

If all vertices have weight one, then we call the problems MAXIMUM c -COLORABLE SUBGRAPH and MAXIMUM INDEPENDENT SET, respectively.

Graph classes. The motivation, applications, and recognition complexity of the following graph classes are discussed in detail in Section 3. We first list a number of classic graph classes and then introduce the central graph class of this work, inductive k -independent graphs (Definition 2.3).

We use $A \cap B$ to denote the class of graphs that belong to both classes A and B .

A graph is *Hamiltonian* if it contains a cycle using each vertex of the graph exactly once.

A graph is *cubic* if each vertex has exactly three neighbors.

A graph is *chordal* if it does not contain cycles of length at least four as induced subgraphs.

A graph is an *interval graph* if its vertices can be represented as intervals of the real line such that two vertices are adjacent if and only if their corresponding intervals intersect. If the vertices can be represented by intervals of equal length, then the graph is a *unit interval graph*.

Definition 2.3 (inductive k -independent) A graph G is *inductive k -independent* if there is a *k -independence ordering* v_1, v_2, \dots, v_n of its vertices such that all independent sets of $G[N[v_i] \cap \{v_i, v_{i+1}, \dots, v_n\}]$ have size at most k for each $i \in \{1, 2, \dots, n\}$.

Ye and Borodin (2012) survey inductive k -independent graphs.

When dropping the order constraint from the definition of inductive k -independent graphs, that is, if one requires the neighborhood of each vertex to contain independent sets of size at most k , then one obtains the $K_{1,k+1}$ -free graphs—a proper subclass.

Definition 2.4 ($K_{1,k}$ -free graphs) A graph is *$K_{1,k}$ -free* if it does not contain a $K_{1,k}$ (a tree with one internal node and k leaves) as an induced subgraph.

Another proper subclass of inductive k -independent graphs are *k -simplicial graphs* (Ye and Borodin 2012), also studied by e. g. Jamison and Mulder (2000), Kammer et al (2010), and Halldórsson and Tonoyan (2015).

Definition 2.5 (k -simplicial) A graph is *k -simplicial* if there is an ordering of its vertices v_1, v_2, \dots, v_n such that $G[N[v_i] \cap \{v_i, v_{i+1}, \dots, v_n\}]$ can be partitioned into at most k cliques for each $i \in \{1, 2, \dots, n\}$.

When, again, dropping the order constraint, one gets *k -minoes* as introduced by Metelsky and Tyshkevich (2003), which are a proper subclass of $K_{1,k+1}$ -free graphs:

Definition 2.6 (k -mino) A graph is called a *k -mino* if each vertex is contained in at most k maximal cliques.

In several scheduling works (see Section 3), one encounters variants of MAX-WEIGHT INDEPENDENT SET in graphs of the following form:²

Definition 2.7 ($A \bowtie B$) For two graph classes A and B , we denote by $A \bowtie B$ the class of graphs $G = (V, E)$ such that $E = E_1 \cup E_2$ for a graph $G_1 = (V, E_1)$ in class A and a graph $G_2 = (V, E_2)$ in class B .

3 Graph classes related to scheduling

Before delving deeper into problem-specific considerations, in this section we collect a number of mostly simple properties of and relations between graph classes in the range from cluster graphs (a special case of interval graphs) on the “bottom end” and inductive 2-independent graphs on the “top end”. In this way, we lay the foundations for subsequent studies mainly related to independent set problems (with their

² We are not aware that $A \bowtie B$ or an alike notation has been used before. Rather, previous work has been coming up with ad-hoc names for the classes $A \bowtie B$ for various A and B , often referring to one and the same class by several names.

connections to scheduling). More specifically, we exhibit the inclusion relationships between various graph classes, discuss the time complexities of their recognition problems (that is, given a graph, how hard is it to decide whether it belongs to the particular graph class), and indicate whether the graph class has already been investigated in the context of scheduling. This section might also be of independent interest to researchers outside of scheduling contexts.

First, we will discuss some kind of backbone structure of graph classes (Section 3.1) and then we will enrich the picture by further natural graph classes (Section 3.2).

3.1 Backbone graph classes

To start with, consider Figure 2.1a, which depicts the fundamental backbone structure of our graph classes. Basically, it shows the relations between various levels of inductive k -independent and $K_{1,k}$ -free graphs, enriched by interval graphs. Notably, here and also in Figure 2.1b, all shown containments are proper. Next, we individually discuss the most relevant graph classes.

Interval graphs. Interval graphs certainly are among the most fundamental graph classes studied in the context of scheduling. Among other things, they are used to model *interval scheduling* problems, where not only the lengths but also the start times of jobs are given and the task is to minimize the number of required machines or to select most profitable set of jobs (Kolen et al 2007). Other uses include the problem of maximizing the weighted number of *just-in-time jobs* (exactly meeting their deadlines) on parallel identical machines (Sung and Vlach 2005), *mutual exclusion scheduling* (Baker and Coffman 1996; Gardi 2009), or selecting the optimal start times of jobs from finite sets (van Bevern et al 2015). Interval graphs can be recognized in linear time (Booth and Lueker 1976), contain cluster graphs, and are special cases of chordal graphs (Dirac 1961), which are the same as inductive 1-independent graphs.

Cluster graphs. These are disjoint unions of cliques (often occurring in the context of graph-based data clustering (Bansal et al 2004; Shamir et al 2004) and are special cases of unit interval graphs. They are exactly the $K_{1,2}$ -free graphs (where $K_{1,2}$ is a path on three vertices). It is easy to see that they can be recognized in polynomial time. In scheduling, cluster graphs naturally occur as conflict graphs when, for example, modeling several variants of one job as multiple vertices and only one of them is allowed to be in a solution (for example, a job might be available at various starting times or with various processing times, which influences its profit or weight) (Halldórsson and Karlsson 2006; van Bevern et al 2015).

As observed by [Ye and Borodin \(2012\)](#), having the degree in the polynomial of the running time of the recognition algorithm for inductive k -independent graphs depending on k is unavoidable unless $\text{FPT} = \text{W}[1]$. Since [Ye and Borodin \(2012\)](#) omitted the formal proof, for the sake of completeness we provide it here.

Proposition 3.1 *Deciding whether a graph is inductive k -independent is $\text{coW}[1]$ -hard with respect to k .*

Proof We reduce any instance $(G = (V, E), k+1)$ of MAXIMUM INDEPENDENT SET to an instance $G' = (V', E')$ such that G contains an independent set of size $k+1$ if and only if G' is not inductive k -independent. Since MAXIMUM INDEPENDENT SET is $\text{W}[1]$ -hard parameterized by k , this shows that recognizing non-inductive k -independent graphs is $\text{W}[1]$ -hard. Thus, recognizing inductive k -independent graphs is $\text{coW}[1]$ -hard.

The reduction works as follows. Let $V = \{v_1, v_2, \dots, v_n\}$. Then, $V' := V \cup \{u_1, u_2, \dots, u_{k+1}\}$ for $k+1$ new vertices u_1, u_2, \dots, u_{k+1} and $E' := E \cup \{\{u_i, v_j\} \mid 1 \leq i \leq k+1, 1 \leq j \leq n\}$. This completes our reduction.

Now, assume that a maximum independent set in G has size at most k . We show that G' is inductive k -independent. Put each of the newly introduced vertices u_i for $i \in \{1, 2, \dots, k+1\}$ first within the k -independence ordering and then all vertices $v \in V$ in an arbitrary order. Since $N_{G'}(u_i) = V$, the neighborhood of any vertex (ignoring vertices that come before it in the k -independence ordering) induces a subgraph of G . Thus it trivially holds that this neighborhood only contains independent sets of size at most k .

Now, assume that G contains an independent set of size at least $k+1$. Then, G' is not inductive k -independent: no vertex u_i for $i \in \{1, 2, \dots, k+1\}$ can be the first in a k -independence ordering since $N(u_i) = V$. Moreover, no vertex v_j for $j \in \{1, 2, \dots, n\}$ can be the first in a k -independence ordering since $N(v_j) \supseteq \{u_1, u_2, \dots, u_{k+1}\}$, which are pairwise nonadjacent. \square

$K_{1,k}$ -free graphs for $k \geq 4$. These are obvious superclasses of claw-free graphs and, by definition, each $K_{1,k}$ -free graph is inductive $(k-1)$ -independent. Again, there is a straightforward polynomial-time recognition algorithm; however, the degree of the polynomial depends on k and thus the algorithm is impractical already for small k . A folklore reduction shows that this is unavoidable unless $\text{FPT} = \text{W}[1]$.

Proposition 3.2 *Recognizing $K_{1,k}$ -free graphs is $\text{coW}[1]$ -hard with respect to k .*

Proof We will give a parameterized reduction from MAXIMUM INDEPENDENT SET, which is $\text{W}[1]$ -hard parameterized by solution size k . Let $(G = (V, E), k)$ be an instance of MAXIMUM INDEPENDENT SET. We construct a graph G' that is not $K_{1,k}$ -free if and only if G contains an independent set of size k .

Let u be a vertex not in V and $G' = (V \cup \{u\}, E \cup \{\{u, v\} \mid v \in V\})$. If G contains an independent set I of size at least k , then $G'[I \cup \{u\}]$ is a $K_{1,k}$. Thus, G' is not $K_{1,k}$ -free.

Now, assume that G does not contain an independent set of size at least k . Then, G' does not contain such an independent set either since u is adjacent to all other vertices of G' . Thus, no vertex in G' can have k pairwise nonadjacent neighbors in G' and, thus, G' is $K_{1,k}$ -free. \square

3.2 Further graph classes

Now, we present a number of graph classes that enrich our backbone structure; some of the new graph classes have prominently appeared in the context of scheduling. Adding these further graph classes to [Figure 2.1a](#) leads to the “richer” [Figure 2.1b](#). We remark that even though we could not spot literature references for some of the subsequent graph classes in the context of scheduling, we advocate their relevance in the scheduling context because they naturally generalize or specialize established “scheduling graph classes”; thus, considering these classes may lead to strengthenings of some known results.

Strip graphs. These are equivalent to the class of cluster \bowtie interval graphs and thus form an obvious superclass of interval graphs. Their recognition is NP-hard; they find applications in classic scheduling ([Halldórsson and Karlsson 2006](#)), in particular in modeling the JOB INTERVAL SELECTION problem introduced by [Nakajima and Hakimi \(1982\)](#).

Cluster \bowtie chordal graphs. These graphs form an obvious superclass of strip graphs and of chordal graphs. We will see that their recognition is NP-hard as well (under Turing reductions; see [Proposition 3.6](#)). We are not aware of a direct scheduling application.

We will now show that cluster \bowtie chordal graphs form a proper subclass of 2-simplicial graphs and then prove the hardness of recognizing cluster \bowtie chordal graphs.

Proposition 3.3 *Cluster \bowtie chordal graphs are a proper subclass of 2-simplicial graphs.*

Proof Consider a cluster \bowtie chordal graph G . We show that any of its induced subgraphs G' contains a vertex whose neighborhood can be covered by two cliques. Repeatedly deleting such vertices from G yields an ordering as required by [Definition 2.5](#).

Note that G' , like its supergraph G , is also an edge-wise union of a chordal graph G_1 and a cluster graph G_2 on the same vertex set. Since G_1 is chordal, it contains a vertex v whose neighborhood is a clique ([Blair and Peyton 1993](#)). Its neighborhood in G_2 is also a clique. Thus, the neighborhood of v in G' can be covered by two cliques. This concludes

the proof that cluster \bowtie chordal graphs form a subclass of 2-simplicial graphs.

Next we show that this inclusion is strict. To this end, we show that $K_{2,4}$ is 2-simplicial but not a cluster \bowtie chordal graph. A $K_{2,4}$ is 2-simplicial since every induced subgraph contains a vertex whose neighborhood can be covered using two cliques. It is, however, not cluster \bowtie chordal: if $K_{2,4}$ was the edge-wise union of a chordal graph G_1 and a cluster graph G_2 , then G_2 would contain at most two edges of $K_{2,4}$. Independently of the choice of these two edges, the remaining edges contain an induced C_4 and thus cannot be part of G_1 . \square

Next, we show that recognizing cluster \bowtie chordal graphs is NP-hard. We will basically use the same reduction from the NP-hard HAMILTONIAN CYCLE problem on triangle-free cubic graphs that Halldórsson and Karlsson (2006) used to show that recognizing strip graphs is NP-hard. Yet we need to adapt their proof, particularly concerning the Hamiltonicity of triangle-free cubic graphs (Lemma 3.5). For the sake of self-containedness, we provide all details, including also parts that coincide with Halldórsson and Karlsson (2006).

Problem 3.4 (HAMILTONIAN CYCLE)

Input: A graph $G = (V, E)$

Question: Is there a cycle in G that passes through every vertex exactly once?

We start with an auxiliary lemma.

Lemma 3.5 *A triangle-free cubic graph $G = (V, E)$ is Hamiltonian if and only if, for each vertex v , there is an edge e incident to v such that $G' = (V, E \setminus \{e\})$ is a cluster \bowtie chordal graph.*

Proof “ \Rightarrow ” Assume that G contains a Hamiltonian cycle $C = (c_1, c_2, \dots, c_n)$. Then, $\{c_i, c_{i+1}\} \in E$ for $i \in \{1, 2, \dots, n-1\}$ and $\{c_n, c_1\} \in E$. Without loss of generality, let $v = c_n$. We will prove that $E \setminus \{c_n, c_1\}$ can be partitioned into two sets E_1 and E_2 such that (V, E_1) is a cluster graph and (V, E_2) is chordal. First, we define $E_2 = \{\{c_i, c_{i+1}\} \mid 1 \leq i \leq n-1\}$. The graph (V, E_2) is a path and, hence, chordal. Now, observe that each vertex has degree exactly one in G_1 . Thus, G_1 is a perfect matching and, thus, a cluster graph.

“ \Leftarrow ” Assume that, for an arbitrary but fixed vertex v , there is an edge $\{v, w\}$ such that $G' = (V, E \setminus \{\{v, w\}\})$ is a cluster \bowtie chordal graph. We will show that G contains a Hamiltonian cycle. Fix two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E_1 \cup E_2 = E$, where G_1 is a cluster graph and G_2 is a chordal graph. Since G is triangle-free, it holds that G_1 and G_2 are triangle-free, too. Every vertex has degree at most one in G_1 since, otherwise, there would be a connected component of size at least three in G_1 , which is a contradiction to the fact that G_1 is a triangle-free cluster graph. Since G is a cubic graph, it holds that each vertex except for v and w has degree at least two in G_2 . Since G_2 is chordal, it has an

1-independence ordering, that is, for each vertex u , the set of succeeding neighbors of u induce a clique. Thus, each vertex has at most one succeeding neighbor; otherwise, G_2 would contain a triangle. Since there is an ordering of the vertices of G_2 such that each vertex has at most one succeeding neighbor and only v and w can have degree one in G_2 , we get that G_2 is a path through all vertices with v and w as endpoints. Adding $\{v, w\}$ to this path gives a Hamiltonian cycle for G . \square

Using Lemma 3.5, we can now easily show the following.

Proposition 3.6 *Recognizing cluster \bowtie chordal graphs is NP-hard.*

Proof Assume that cluster \bowtie chordal graphs are recognizable in polynomial time. Assuming this, we derive a polynomial-time algorithm for the NP-hard problem of checking whether a triangle-free cubic graph G is Hamiltonian (West and Shmoys 1984), which implies $P = NP$.

Let v be any vertex of G . Since G is cubic, there are three edges incident to v in G . If G is Hamiltonian, then, by Lemma 3.5, deleting one out of the three edges will turn G into a cluster \bowtie chordal graph. Our algorithm simply tries out all three possibilities and, for each of them, checks whether we get a cluster \bowtie chordal graph in polynomial time. \square

2-simplicial graphs. These graphs form a superclass of cluster \bowtie chordal graphs (Proposition 3.3) and, by definition, are inductive 2-independent.

They can be easily recognized in polynomial time by the following algorithm: as long as possible, find and delete a vertex whose neighborhood can be covered by two cliques, which is equivalent to checking whether the complement of the neighborhood is 2-colorable. The input graph is 2-simplicial if and only if this process terminates with the empty graph as result. We are not aware of a direct scheduling application.

3-minoes. In 3-minoes every vertex is contained in at most three maximal cliques. It is straightforward to see that they form a subclass of $K_{1,4}$ -free graphs; they can be recognized in polynomial time (Metelsky and Tyshkevich 2003). We are not aware of a direct scheduling application.

Interval \bowtie interval graphs. This graph class is also known as 2-track interval graphs and as 2-union graphs. They trivially contain interval graphs. The corresponding recognition problem is NP-hard (Gyárfás and West 1995). They have strong scheduling applications in industrial steel manufacturing (Höhn et al 2011).

Unit interval \bowtie unit interval graphs. These are special cases of 2-track interval graphs and are also called 2-track unit interval graphs. They are contained in the class of inductive 3-independent graphs (Ye and Borodin 2012) and in the class of $K_{1,5}$ -free graphs because unit interval graphs are $K_{1,3}$ -free (Wegner 1961). Their recognition problem is NP-hard (Jiang 2013). We are not aware of a direct scheduling application.

4 W[1]-hardness on 2-simplicial 3-minoes

MAXIMUM INDEPENDENT SET parameterized by the solution size ℓ is W[1]-hard on inductive 3-independent graphs; Jiang (2010) showed this on unit interval \bowtie unit interval graphs, which are a subclass of inductive 3-independent graphs (Ye and Borodin 2012).

In contrast, the more general MAXIMUM c -COLORABLE SUBGRAPH problem parameterized by the solution size ℓ is fixed-parameter tractable on inductive 1-independent (that is, chordal) graphs (Misra et al 2013). In Section 5, we will generalize this tractability result to cluster \bowtie chordal graphs, a non-chordal proper subclass of inductive 2-independent graphs (cf. Figure 2.1b).

Strengthening the result by Jiang (2010), we show that MAX-WEIGHT c -COLORABLE SUBGRAPH in inductive 2-independent graphs is W[1]-hard. More precisely, we show that already MAXIMUM INDEPENDENT SET parameterized by the solution size ℓ is W[1]-hard even on 2-simplicial 3-minoes (see Definitions 2.5 and 2.6). Since these are $K_{1,4}$ -free (see Figure 1.1), our result also contrasts the fact that MAX-WEIGHT INDEPENDENT SET is solvable in $O(n^3)$ time on $K_{1,3}$ -free graphs (Faenza et al 2011).

Theorem 4.1 MAXIMUM INDEPENDENT SET parameterized by solution size ℓ is W[1]-hard on 2-simplicial 3-minoes.

It may be tempting to prove Theorem 4.1 using a parameterized reduction from MAXIMUM INDEPENDENT SET on graphs of small degree; however such efforts must be in vein since MAXIMUM INDEPENDENT SET parameterized by solution size is fixed-parameter tractable on graphs even with logarithmic degree via a simple search tree algorithm.

To prove Theorem 4.1, we use a parameterized reduction from the MULTICOLORED CLIQUE problem, which is W[1]-hard with respect to k (Fellows et al 2009).

Problem 4.2 (MULTICOLORED CLIQUE)

Input: A graph G whose vertex set is partitioned into k independent sets $V_1 \uplus V_2 \uplus \dots \uplus V_k$.

Task: Does G contain a complete subgraph of order k ?

We will also refer to the sets V_i for $i \in \{1, 2, \dots, k\}$ as *color classes*. We now describe our parameterized reduction from MULTICOLORED CLIQUE to MAXIMUM INDEPENDENT SET, which is illustrated in Figure 4.1. Subsequently, we prove its correctness and that it creates 2-simplicial 3-minoes.

Construction 4.3 Given a MULTICOLORED CLIQUE instance $G = (V_1 \uplus V_2 \uplus \dots \uplus V_k, E)$, we create an instance G' of MAXIMUM INDEPENDENT SET that has a solution of size $\ell := k^2 + \binom{k}{2}$ if and only if G has a clique of size k .

For each color class $V_i = \{v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(n_i)}\}$ of G , graph G' contains a *vertex selection gadget* U_i , which consists of k cliques $U_{i \rightarrow j}$, where $j \in \{1, 2, \dots, k\}$. For each $j \in \{1, 2, \dots, k\}$, clique $U_{i \rightarrow j}$ will be used to connect gadget U_i to gadget U_j and consists of newly introduced vertices $\{u_{i \rightarrow j}^{(1)}, u_{i \rightarrow j}^{(2)}, \dots, u_{i \rightarrow j}^{(n_i)}\}$, each vertex $u_{i \rightarrow j}^{(p)}$ with $p \in \{2, 3, \dots, n_i\}$ of which is adjacent to each vertex $u_{i \rightarrow l}^{(q)}$ with $q \in \{1, 2, \dots, p-1\}$ and $l = (j \bmod k) + 1$.

Furthermore, for each two color classes $V_i = \{v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(n_i)}\}$ and $V_j = \{v_j^{(1)}, v_j^{(2)}, \dots, v_j^{(n_j)}\}$ of G such that $1 \leq i < j \leq k$, graph G' contains a *verification gadget* $E_{i < j}$, which is a clique on the vertices $e_{i < j}^{p, q}$, newly introduced for each edge $\{v_i^{(p)}, v_j^{(q)}\} \in E$ of G .

We connect the gadgets so that choosing a vertex $e_{i < j}^{p, q}$ of a verification gadget $E_{i < j}$ into an independent set enforces that $v_i^{(p)}$ of V_i and $v_j^{(q)}$ of V_j are also in the independent set, where i is the smaller color index (which is reflected in the naming convention). To this end, for each edge $\{v_i^{(p)}, v_j^{(q)}\} \in E$ with $1 \leq i < j \leq k$ of G , vertex $u_{i \rightarrow j}^{(p)}$ is adjacent to all vertices $e_{i < j}^{p', q'}$ such that $p \neq p'$ and $u_{j \rightarrow i}^{(q)}$ is adjacent to all vertices $e_{i < j}^{p', q'}$ such that $q \neq q'$. \square

We now prove the correctness of Construction 4.3. Thereafter, it remains to show that it creates 2-simplicial 3-minoes.

Lemma 4.4 A MULTICOLORED CLIQUE instance $G = (V_1 \uplus V_2 \uplus \dots \uplus V_k, E)$ has a clique of order k if and only if the graph G' created by Construction 4.3 has an independent set of size $\ell := k^2 + \binom{k}{2}$.

Proof (\Rightarrow) Let S be a clique of order k in G . It contains exactly one vertex of each color class. We describe an independent set S' of size ℓ for G' . To this end, denote each color class V_i for $i \in \{1, 2, \dots, k\}$ of G as $V_i = \{v_i^{(1)}, v_i^{(2)}, \dots, v_i^{(n_i)}\}$.

For each vertex $v_i^{(p)}$ in S , set S' contains the k vertices $u_{i \rightarrow j}^{(p)}$ with $j \in \{1, 2, \dots, k\}$ of the vertex selection gadget U_i . Thus, S' contains exactly one vertex of each clique $U_{i \rightarrow j}$ and these vertices are pairwise nonadjacent: there are neither edges between vertices $u_{i \rightarrow j}^{(p)}$ and $u_{i \rightarrow l}^{(p)}$ for any $j \neq l$ with $\{j, l\} \subseteq \{1, 2, \dots, k\}$ nor are there edges between distinct vertex selection gadgets U_i and U_j in G' .

For each edge $\{v_i^{(p)}, v_j^{(q)}\}$ with $1 \leq i < j \leq k$ of S , set S' contains the vertex $e_{i < j}^{p, q}$ of the verification gadget $E_{i < j}$. Since there are no edges between verification gadgets, these vertices are pairwise nonadjacent. Moreover, note that $e_{i < j}^{p, q}$ has neither edges to $u_{i \rightarrow j}^{(p)}$ nor to $u_{j \rightarrow i}^{(q)}$, which are the only vertices of $U_{i \rightarrow j}$ and $U_{j \rightarrow i}$ in S' , nor has it edges to any vertex in $U_{i' \rightarrow j'}$ for $\{i', j'\} \neq \{i, j\}$. Thus, S' is an independent set.

Finally, S' has size $\ell = k^2 + \binom{k}{2}$: it contains k vertices for each of k color classes of G and one vertex for each edge of the clique S .

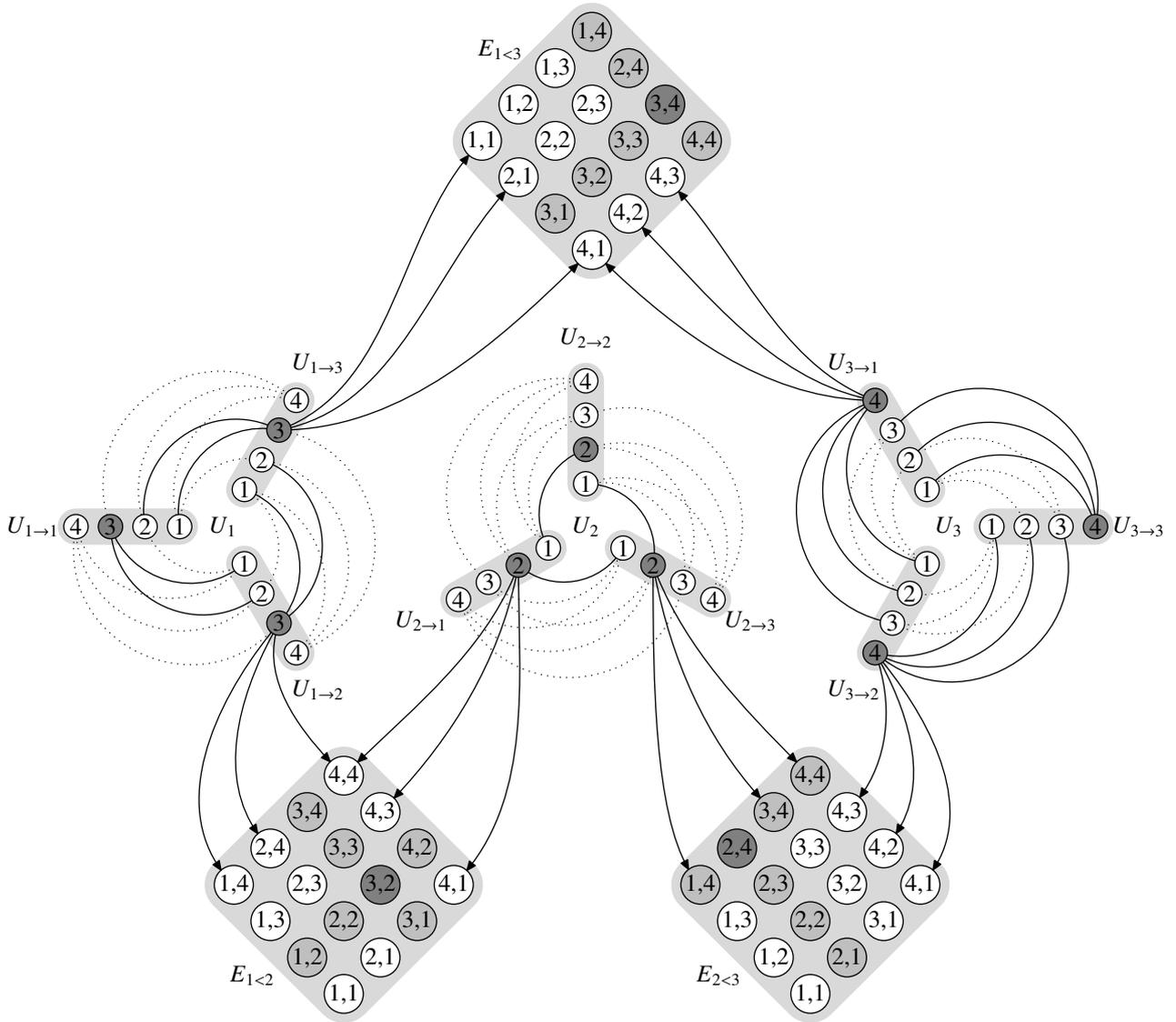


Fig. 4.1: A MAXIMUM INDEPENDENT SET instance created by Construction 4.3 from a MULTICOLORED CLIQUE instance with three color classes $V_1, V_2,$ and V_3 of four vertices each, where all edges between vertices in V_i and V_j for $i \neq j$ exist. Greysthaded areas are cliques. Dotted edges are ordinary edges and are dotted merely to unclutter the figure. The dark grey vertices form an independent set of maximum size (one vertex of each clique). Connections from vertex selection gadgets to verification gadgets $E_{i<j}$ are only shown for the dark grey vertices in $U_{i \rightarrow j}$ and $U_{j \rightarrow i}$: a directed arc means that the tail of the arc has edges to all vertices in the row or column pointed to by the head. Light grey vertices in the verification gadgets $E_{i<j}$ are those blocked by only one arc pointing at $E_{i<j}$. The labels of the vertices indicate their superscript in the corresponding clique used in Construction 4.3.

(\Leftarrow) Let S' be an independent set of size ℓ for G' . We describe a clique of order k in G . Since each vertex of G' is in one of the ℓ pairwise vertex-disjoint cliques $U_{i \rightarrow j}$ with $\{i, j\} \subseteq \{1, 2, \dots, k\}$ and $E_{i<j}$ with $1 \leq i < j \leq k$, S' contains exactly one vertex of each of them. Let S be the set of vertices $v_i^{(p)}$ of G for $i \in \{1, 2, \dots, k\}$ such that $u_{i \rightarrow i}^{(p)}$ of $U_{i \rightarrow i}$ is in S' . We will prove that the vertices in S are pairwise adjacent in G .

To this end, we first prove that $p = q$ for any two vertices $u_{i \rightarrow j}^{(p)}$ and $u_{i \rightarrow j}^{(q)}$ in S' , where $\{i, j, l\} \subseteq \{1, 2, \dots, k\}$. To this

end, note that G' contains edges from vertex $u_{i \rightarrow j}^{(p)}$ to each vertex $u_{i \rightarrow l}^{(q')}$ with $q' \in \{1, 2, \dots, p-1\}$ and $l' = (j \bmod k) + 1$. Thus, for the vertex $u_{i \rightarrow j}^{(p)}$ in S' , we have $q' \geq p$. Iterating the argument, we get $q \geq q' \geq p$, and, iterating further, $p \geq q \geq q' \geq p$, and thus $p = q$.

We now prove that two arbitrary vertices $v_i^{(p)}$ and $v_j^{(q)}$ with $1 \leq i < j \leq k$ in S are adjacent in G . By choice of S , we have that $u_{i \rightarrow i}^{(p)}$ and $u_{j \rightarrow j}^{(q)}$ are in S' . As shown in the previous paragraph, we thus also have that $u_{i \rightarrow j}^{(p)}$ and $u_{j \rightarrow i}^{(q)}$ are in S' . Since

$u_{i \rightarrow j}^{(p)}$ is adjacent to all vertices $e_{i < j}^{p', q'}$ of $E_{i, j}$ such that $p \neq p'$ and $u_{j \rightarrow i}^{(q)}$ is adjacent to all vertices $u_{i < j}^{p', q'}$ such that $q \neq q'$, we get that S' contains vertex $e_{i < j}^{p, q}$ of $E_{i < j}$. The existence of this vertex in $E_{i < j}$ shows that there is an edge $\{v_i^{(p)}, v_j^{(q)}\} \in E$ in G . \square

We will use the following lemma to show that Construction 4.3 generates 2-inductive 3-minoes by showing that, in any induced subgraph of the graphs generated by Construction 4.3, we can find a vertex for which at least one of the three sets in the following lemma are empty or contain only one vertex:

Lemma 4.5 *Let $u_{i \rightarrow j}^{(p)}$ with $1 \leq i \leq j \leq k$ be a vertex in the graph created by Construction 4.3 from an instance $G = (V_1 \uplus V_2 \uplus \dots \uplus V_k, E)$ of MULTICOLORED CLIQUE.*

Then, the neighborhood of $u_{i \rightarrow j}^{(p)}$ can be covered by at most three cliques, consisting of

- i) $\{u_{i \rightarrow j}^{(q)} \mid q \geq p\} \cup \{u_{i \rightarrow l}^{(q)} \mid q < p, l = (j \bmod k) + 1\}$,
- ii) $\{u_{i \rightarrow j}^{(q)} \mid q \leq p\} \cup \{u_{i \rightarrow l}^{(q)} \mid q > p, j = (l \bmod k) + 1\}$, and
- iii) of $u_{i \rightarrow j}^{(p)}$ and its neighbors in either $E_{i < j}$ or $E_{j < i}$, if $i \neq j$.

Proof We first show that each of the sets (i)–(iii) is a clique. This easily follows for (iii) since $E_{i < j}$ for $1 \leq i < j \leq k$ is a clique by Construction 4.3.

We now prove that (i) and (ii) are cliques. By Construction 4.3, each vertex $u_{i \rightarrow j}^{(p)}$ is adjacent to each vertex $u_{i \rightarrow j'}^{(q')}$ with $q' \in \{1, 2, \dots, p' - 1\}$ and $l' = (j' \bmod k) + 1$. Herein, intuitively, we can think of l' as the “successor” to j' in the cycle $(1, 2, \dots, k, 1)$. By setting $p' := p$ and $j' := j$, we immediately get that (i) is a clique. By choosing $p' := p + 1$ and $l' := j$, we get that (ii) is a clique, since in (ii) the left set of the union now plays the role of the “successor” of the right set.

It remains to prove that the sets (i)–(iii) cover the whole neighborhood of $u_{i \rightarrow j}^{(p)}$. Vertex $u_{i \rightarrow j}^{(p)}$ has neighbors in all sets $U_{i \rightarrow l}$ such that $l = j, l = (j \bmod k) + 1$, or $j = (l \bmod k) + 1$. If $i \neq j$, then it might also have neighbors in either $E_{i < j}$ or $E_{j < i}$. Thus, the union of the sets in (i)–(iii) covers all neighbors in $E_{i < j}$, $E_{j < i}$, and $U_{i \rightarrow j}$. Moreover, set (i) covers all neighbors in $U_{i \rightarrow l}$ for $l = (j \bmod k) + 1$ by Construction 4.3. Finally, if set (ii) did not cover all neighbors in $U_{i \rightarrow l}$ with $j = (l \bmod k) + 1$, then this would mean that $u_{i \rightarrow j}^{(p)}$ had a neighbor $u_{i \rightarrow l}^{(q)}$ with $q \leq p$. This is impossible since, by Construction 4.3, $u_{i \rightarrow l}^{(q)}$ is adjacent to vertices $u_{i \rightarrow j}^{(p)}$ only for $p < q$. \square

Using Lemma 4.5, the following lemma is easy to prove and finishes the proof of Theorem 4.1.

Lemma 4.6 *Construction 4.3 creates 2-simplicial 3-minoes.*

Proof We first show that Construction 4.3 creates 3-minoes. By Lemma 4.5, the neighborhood of each vertex in vertex selection gadgets can be covered by three cliques. Also a vertex of a verification gadget $E_{i < j}$ has neighbors only in the three sets $E_{i < j}$, $U_{i \rightarrow j}$, and $U_{j \rightarrow i}$, each of which is a clique.

We now show that the graphs G created by Construction 4.3 are 2-simplicial. To this end, it is enough to show that each induced subgraph G' of G contains a vertex whose neighborhood can be covered by two cliques. Then, subsequently deleting a vertex whose neighborhood can be covered by two cliques gives an ordering as required by Definition 2.5.

If G' contains only vertices of verification gadgets, then the neighborhood of each vertex in G' can be covered using one clique. If G' contains a vertex $u_{i \rightarrow i}^{(p)}$ of G for any $i \in \{1, 2, \dots, k\}$, then, by Lemma 4.5, its neighborhood can be covered by two cliques. Otherwise, there are $j \in \{1, 2, \dots, k\}$ and $l = (j \bmod k) + 1$ such that G' contains vertices of $U_{i \rightarrow j}$ but no vertices of $U_{i \rightarrow l}$. Let $u_{i \rightarrow j}^{(p)}$ be the vertex of $U_{i \rightarrow j}$ with maximum p in G' . For this vertex, the set (i) in Lemma 4.5 only contains $u_{i \rightarrow j}^{(p)}$ and is therefore contained in the other two sets. Thus, its neighborhood can be covered by two cliques—sets (ii) and (iii). \square

Theorem 4.1 now follows from Lemmas 4.4 and 4.6, the fact that Construction 4.3 runs in polynomial time, and that MULTICOLORED CLIQUE is W[1]-hard with respect to k (Fellows et al 2009).

5 Fixed-parameter tractability on cluster \bowtie chordal graphs

In this section, we prove that MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by the solution size ℓ is fixed-parameter tractable on cluster \bowtie chordal graphs. This complements the negative result of Section 4.

Theorem 5.1 *A maximum-weight c -colorable subgraph with at most ℓ vertices of a cluster \bowtie chordal graph can be computed in $2^{\ell+c} \cdot (c + e + 3)^\ell \cdot \ell^{O(\log \ell)} \cdot n^2 \cdot \log^3 n$ time if the decomposition of the input graph into a cluster and a chordal graph is given. Herein, e is Euler’s number.*

Note that $c \leq \ell$ holds in all nontrivial cases. Thus, Theorem 5.1 shows that MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by ℓ is fixed-parameter tractable in cluster \bowtie chordal graphs. Moreover, note that it also shows that MAX-WEIGHT c -COLORABLE SUBGRAPH parameterized by the weight W of the sought solution is fixed-parameter tractable, since if there is an independent set of weight W , then there is also one consisting of at most W vertices.

On the one hand, Theorem 5.1 is a generalization of a result of Misra et al (2013), who showed that MAXIMUM c -COLORABLE SUBGRAPH parameterized by the solution size ℓ is fixed-parameter tractable on chordal graphs. On the other hand, it generalizes a fixed-parameter tractability result on the JOB INTERVAL SELECTION problem of van Bevern et al (2015), who showed that MAXIMUM INDEPENDENT SET parameterized by the solution size ℓ is fixed-parameter tractable on cluster \bowtie interval graphs.

The proof of Theorem 5.1 works in three steps. First, in Section 5.1, we use the color coding technique due to Alon et al (1995) to reduce MAX-WEIGHT c -COLORABLE SUBGRAPH to MAX-WEIGHT INDEPENDENT SET. Then, in Section 5.2, we again use color coding to reduce MAX-WEIGHT INDEPENDENT SET in cluster \bowtie chordal graphs to the problem of finding a maximum-weight independent set of pairwise distinct colors in chordal graphs whose vertices are colored in ℓ colors. Finally, in Section 5.3, we use dynamic programming to find this colorful independent set.

5.1 FROM MAX-WEIGHT c -COLORABLE SUBGRAPH TO MAX-WEIGHT INDEPENDENT SET

In order to prove Theorem 5.1, we show that it is enough to give a fixed-parameter algorithm for MAX-WEIGHT INDEPENDENT SET parameterized by the solution size ℓ .

To this end, we use the color coding technique of Alon et al (1995). This technique has been used by Misra et al (2013) to show that MAXIMUM c -COLORABLE SUBGRAPH parameterized by the solution size ℓ is fixed-parameter tractable in chordal graphs. Their approach is neither limited to the unweighted problem nor to chordal graphs. Note, however, that the approach requires some tuning to work in the weighted setting. We describe it here in the most general form. To this end, we need the following definition.

Definition 5.2 (hereditary graph class) A graph class C is *hereditary* if every induced subgraph of a graph in C also belongs to C .

All graph classes considered in this work are hereditary.

Lemma 5.3 *Let C be a hereditary graph class. Moreover, let $t: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a function, nondecreasing in both arguments, such that a maximum-weight independent set of size at most ℓ in C can be computed in $t(\ell, n)$ time.*

Then, a maximum-weight induced c -colorable subgraph with at most ℓ vertices of a graph in C is computable in $2^{\ell+c} c^\ell \ell^{O(\log \ell)} \log^2 n \cdot t(\ell, n)$ time.

Proof Our algorithm makes use of (n, ℓ, c) -universal sets (Misra et al 2013)—sets \mathcal{F} of functions $f: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, c\}$ such that, for any set $I \subseteq \{1, 2, \dots, n\}$ of size $|I| = \ell$ and any function $g: \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, c\}$, there is a function $f \in \mathcal{F}$ with $f(i) = g(i)$ for each $i \in I$.

Our algorithm computes a maximum-weight c -colorable subgraph $G[S]$ with at most ℓ vertices of a graph $G = (\{v_1, v_2, \dots, v_n\}, E)$ in C as follows. First, in $c^\ell \ell^{O(\log \ell)} n \log^2 n$ time, compute an (n, ℓ, c) -universal set \mathcal{F} of size $|\mathcal{F}| \in c^\ell \ell^{O(\log \ell)} \cdot \log^2 n$ (Misra et al 2013). Then, iterate over each $f \in \mathcal{F}$ and each of the $\binom{\ell+c}{c} \leq 2^{\ell+c}$ possible partitions $\ell_1 + \ell_2 + \dots + \ell_c \leq \ell$ such that $\ell_i \geq 0$ for each $i \in \{1, 2, \dots, c\}$. In each iteration, do the following: Compute a maximum-weight independent

set S_i of size at most ℓ_i in the subgraph G_i induced by the vertices v_k with $f(k) = i$ for each $i \in \{1, 2, \dots, c\}$. Return the maximum-weight set $S' = S_1 \uplus S_2 \uplus \dots \uplus S_c$ found in any iteration. Note that the independent set S_i in each graph G_i for $i \in \{1, 2, \dots, c\}$ can be found in $t(\ell, n)$ time since C is a hereditary graph class and, thus, $G_i \in C$.

It is obvious that $G[S']$ has at most ℓ vertices and is c -colorable since it is the union of c independent sets. Thus, $w(S') \leq w(S)$, where $G[S]$ is a maximum-weight c -colorable subgraph with at most ℓ vertices in G .

It remains to show $w(S') \geq w(S)$. Let $\text{col}: V \rightarrow \mathbb{N}$ be a proper c -coloring of $G[S]$. Since $|S| \leq \ell$, there is a function $f \in \mathcal{F}$ such that $\text{col}(v_k) = f(k)$ for each $v_k \in S$. For each $i \in \{1, 2, \dots, c\}$, let $S_i := \{v_k \in S \mid f(k) = i\}$ and $\ell_i := |S_i|$. Since $G[S]$ is a maximum-weight c -colorable subgraph, for each $i \in \{1, 2, \dots, c\}$, S_i is a maximum-weight independent set with at most ℓ_i vertices in the subgraph G_i induced by the vertices v_k with $f(k) = i$. Since the algorithm iterated over this f and these $\ell_1 + \ell_2 + \dots + \ell_c \leq \ell$, we get that $w(S') \geq w(S)$. \square

We point out that, for an easier and faster implementation of the algorithm in the proof of Lemma 5.3, it makes sense to trade the factor $\ell^{(O \log \ell)} \log^2 n$ in the running time for a small error probability ε by simply choosing $f(k) \in \{1, \dots, c\}$ uniformly at random for each vertex v_k of G instead of using universal sets. With probability at least $c^{-\ell}$, f will be a proper coloring for a maximum-weight solution and the algorithm will thus find it. Repeating the algorithm with $O(c^\ell \ln 1/\varepsilon)$ random assignments f thus allows for finding a maximum-weight solution with error probability at most $\varepsilon > 0$.

5.2 FROM MAX-WEIGHT INDEPENDENT SET TO MAX-WEIGHT COLORFUL INDEPENDENT SET

In order to show Theorem 5.1, by Lemma 5.3, it is enough to show that MAX-WEIGHT INDEPENDENT SET parameterized by the solution size ℓ is fixed-parameter tractable in cluster \bowtie chordal graphs. We do this using an algorithm for the following auxiliary problem.

Problem 5.4 (MAX-WEIGHT COLORFUL INDEPENDENT SET)

Input: A graph $G = (V, E)$ with vertex weights $w: V \rightarrow \mathbb{N}$ and a coloring $\text{col}: V \rightarrow \{1, 2, \dots, c\}$ of its vertices.

Task: Find a maximum-weight independent set whose vertices have pairwise distinct colors.

Colorful independent sets of interval graphs have been studied by van Bevern et al (2015) in order to show that MAXIMUM INDEPENDENT SET parameterized by the solution size ℓ is fixed-parameter tractable on cluster \bowtie interval graphs, thus strengthening a result on the JOB INTERVAL SELECTION problem of Halldórsson and Karlsson (2006).

In Section 5.3, we will study MAX-WEIGHT COLORFUL INDEPENDENT SET on chordal graphs. Together with the following lemma, this will give a fixed-parameter algorithm for MAX-WEIGHT INDEPENDENT SET on cluster \bowtie chordal graphs.

Lemma 5.5 *Let C be a graph class and $t: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be a function such that a maximum-weight colorful independent set in graphs in C is computable in $t(c, n)$ time.*

Then, a maximum-weight independent set of size at most ℓ in cluster $\bowtie C$ graphs is computable in $e^\ell \ell^{O(\log \ell)} \log n \cdot t(\ell, n)$ time, assuming that the decomposition of the input graph into a cluster graph and a graph of C is given.

Proof Let $G = (V, E_1 \cup E_2)$ such that $G_1 = (V, E_1)$ is a cluster graph and $G_2 = (V, E_2) \in C$. Let d be the number of clusters in G_1 . Our algorithm uses a (d, ℓ) -perfect hash family \mathcal{F} —a family of functions $f: \{1, 2, \dots, d\} \rightarrow \{1, 2, \dots, \ell\}$ such that, for each subset $C \subseteq \{1, 2, \dots, d\}$ of size ℓ , at least one of the functions in \mathcal{F} is a bijection between C and $\{1, 2, \dots, \ell\}$.

We find a maximum-weight independent set with at most ℓ vertices in G as follows. First, in $e^\ell \ell^{O(\log \ell)} d \log d$ time, construct a (d, ℓ) -perfect hash family \mathcal{F} with $|\mathcal{F}| \in e^\ell \ell^{O(\log \ell)} \log d$ (Cygan et al 2015, Definition 5.15–Theorem 5.18). Then, iterate over all functions $f \in \mathcal{F}$. In each iteration, consider the coloring col_f that colors the vertices in each cluster $i \in \{1, 2, \dots, d\}$ of G_1 using color $f(i) \in \{1, 2, \dots, \ell\}$ and compute a maximum-weight colorful independent set in G_2 with respect to col_f in $t(\ell, n)$ time. Return the colorful independent set S' of maximum weight found in any iteration.

We show that S' is an independent set for G with $w(S') = w(S)$, where S is a maximum-weight independent set of size at most ℓ in G .

“ \leq ”: Observe that S' , being an independent set for G_2 , contains no edges of G_2 . Moreover, being colorful for some coloring col_f , S' neither contains edges of G_1 : the endpoints of such an edge belong to the same cluster i and thus get the same color $\text{col}_f(i)$. Thus, S' is an independent set of size at most ℓ for G and, therefore, $w(S') \leq w(S)$.

“ \geq ”: Since S uses at most one vertex of each cluster of G_1 , the set C_S of clusters that contain vertices of S has size at most ℓ . Thus, \mathcal{F} contains a bijection $f: C_S \rightarrow \{1, 2, \dots, |C_S|\}$. Note that S is a colorful independent set for G_2 with respect to col_f . Thus, $w(S') \geq w(S)$. \square

As in the case with Lemma 5.3, for an easier and faster implementation of the algorithm in the proof of Lemma 5.5, it makes sense to trade the factor $\ell^{O(\log \ell)} \log d$ in the running time for a small error probability by randomly coloring the clusters instead of using perfect hash families: the vertices of maximum-weight solution will have pairwise distinct colors with probability at least $\ell!/\ell^\ell \in \Theta(e^{-\ell})$. Repeating the algorithm with $O(e^\ell \ln 1/\varepsilon)$ random colorings will thus find a maximum-weight solution with error probability at most $\varepsilon > 0$.

5.3 MAX-WEIGHT

COLORFUL INDEPENDENT SET in chordal graphs

To complete our proof of Theorem 5.1, we finally show the following proposition, which, together with Lemmas 5.3 and 5.5, yields Theorem 5.1.

Proposition 5.6 *MAX-WEIGHT COLORFUL INDEPENDENT SET in chordal graphs is solvable in $O(3^c n^2)$ time.*

Proposition 5.6 generalizes a result of Halldórsson and Karlsson (2006), who showed that MAX-WEIGHT COLORFUL INDEPENDENT SET parameterized by the number c of colors is fixed-parameter tractable in interval graphs. Whereas the algorithm of Halldórsson and Karlsson (2006) is based on dynamic programming over the interval representation of interval graphs, we will exploit dynamic programming over *tree decompositions* of chordal graphs:

Definition 5.7 (tree decomposition) *A tree decomposition of a graph $G = (V, E)$ is a pair (\mathcal{X}, T) , where $\mathcal{X} \subseteq 2^V$ and T is a tree on \mathcal{X} , such that*

- (i) $\bigcup_{X \in \mathcal{X}} X = V$, the sets $X \in \mathcal{X}$ are called *bags*,
- (ii) for each edge $e \in E$, there is a bag $X \in \mathcal{X}$ with $e \subseteq X$,
- (iii) for any three bags $X, Y, Z \in \mathcal{X}$ such that Y is on the path between X and Z in T , one has $X \cap Z \subseteq Y$.

Chordal graphs have tree decompositions in which each bag induces a clique and such tree decomposition can be computed in linear time (Blair and Peyton 1993).

We will additionally assume that the tree decomposition is rooted at an arbitrary bag and that each bag X with more than one child in T has exactly two children Y and Z in T such that $X = Y \cup Z$. This property can be established step-by-step as follows: if some bag X has more than two children or its children are not all equal to X , then we can attach two copies X_1 and X_2 of X to X as children in T , attach one original child of X to X_1 and the other children of X to X_2 . This procedure, in the worst case, triples the number of bags in the tree decomposition and works in linear time.

Using this linear-time computable tree decomposition and the following lemma with $\alpha = 1$ yields Proposition 5.6, which, together with Lemmas 5.3 and 5.5, concludes the proof of Theorem 5.1.

Lemma 5.8 *Given a tree decomposition, each bag of which has at most α pairwise nonadjacent vertices, MAX-WEIGHT COLORFUL INDEPENDENT SET can be solved in $O(3^c \cdot n^{\alpha+1})$ time.*

Proof Let $G = (V, E)$ be a graph and let (\mathcal{X}, T) be a tree decomposition such that each bag of \mathcal{X} contains at most α pairwise nonadjacent vertices. Root \mathcal{T} at an arbitrary bag $R \in \mathcal{X}$. For each bag $X \in \mathcal{X}$, let $V_X \subseteq V$ denote the set of vertices in X and its descendant bags in T . Moreover, let $G_X := G[V_X]$ and $\text{col}(S) := \{\text{col}(v) \mid v \in S\}$.

For each bag $X \in \mathcal{X}$, each $S \subseteq X$, each subset $C \subseteq \{1, 2, \dots, c\}$ of colors, let $T[X, C, S]$ denote the maximum weight of an independent set I of G_X with $I \cap X = S$ whose vertices have pairwise distinct colors from C . Then, by Definition 5.7(i), the solution to MAX-WEIGHT COLORFUL INDEPENDENT SET is $\max_{S \subseteq R} T[R, \{1, 2, \dots, c\}, S]$. We now show how to compute it in $O(3^c n^{\alpha+1})$ time, considering each bag type separately.

Infeasible solutions. Since no bag has an independent set with more than α vertices, for each $X \in \mathcal{X}$, each $S \subseteq X$, and each $C \subseteq \{1, 2, \dots, c\}$, we have $T[X, C, S] = -\infty$ if $|S| > \alpha$, or if S is not independent, or if the vertices of S do not have pairwise distinct colors from C .

In the following, we will thus assume that $S \subseteq X$ is always an independent set and that its vertices have pairwise distinct colors from $C \subseteq \{1, 2, \dots, c\}$.

Bags without children. For each bag $X \in \mathcal{X}$ without children,

$$T[X, C, S] = w(S). \quad (5.1)$$

Bags with a single child. Consider a bag $X \in \mathcal{X}$ that has one child Y . We show that

$$T[X, C, S] = \max \left\{ \begin{array}{l} T[Y, C \setminus \text{col}(S \setminus S'), S'] + w(S \setminus S') \\ \text{for each independent set } S' \subseteq Y \\ \text{such that } S \cap X \cap Y = S' \cap X \cap Y \end{array} \right\}. \quad (5.2)$$

Herein, the condition $S \cap X \cap Y = S' \cap X \cap Y$ ensure that independent sets for G_X and G_Y agree on the vertices that are both in the bag X and its child Y . Next, we show validity of (5.2).

“ \leq ”: Consider a maximum-weight independent set I for G_X with $I \cap X = S$ whose vertices have pairwise distinct colors from C . Then $I' := I \cap V_Y$ is an independent set for G_Y . For $S' := I' \cap Y$, we have that $S' \cap X \cap Y = I' \cap X \cap Y = I \cap X \cap Y = S \cap X \cap Y$. Since $I \setminus I' \subseteq X$ and, by Definition 5.7(iii), $I' \cap X \subseteq Y$, one has $I \setminus I' = (I \cap X) \setminus (I' \cap X) = (I \cap X) \setminus (I' \cap X \cap Y) = (I \cap X) \setminus (I' \cap Y) = S \setminus S'$. Thus, the vertices of I have pairwise distinct colors from $C' := C \setminus \text{col}(S \setminus S')$ and

$$\begin{aligned} T[X, C, S] &= w(I) = w(I') + w(S \setminus S') \\ &\leq T[Y, C \setminus \text{col}(S \setminus S'), S'] + w(S \setminus S'). \end{aligned}$$

“ \geq ”: Consider any independent set $S' \subseteq Y$ with $S \cap X \cap Y = S' \cap X \cap Y$. Let I' be a maximum-weight independent set for G_Y with $I' \cap Y = S'$ whose vertices use pairwise distinct colors from $C' := C \setminus \text{col}(S \setminus S')$.

Then, $I := I' \cup S = I' \cup (S \setminus S')$ uses pairwise distinct colors in C . Moreover, by Definition 5.7(iii), $I' \cap X \subseteq V_Y \cap X \subseteq Y$ and, therefore, $I \cap X = (I' \cup S) \cap X = (I' \cap X) \cup (S \cap X) = (I' \cap X \cap Y) \cup S = (S' \cap X \cap Y) \cup S = (S \cap X \cap Y) \cup S = S$.

We show that I is independent. Towards a contradiction, assume that there is an edge $e = \{x, y\} \subseteq I = I' \cup S$. Since

$e \not\subseteq I'$ and $e \not\subseteq S$, we have $x \in S \setminus I' \subseteq X$ and $y \in I' \setminus S \subseteq V_Y$. Since $x \notin I' \supseteq S' \cap X \cap Y = S \cap X \cap Y$ but $x \in S = I \cap X$, we get $x \notin Y$. Consequently, $x \notin V_Y$ by Definition 5.7(iii) and $\{x, y\} \subseteq X$ by Definition 5.7(ii). Thus, we showed that $\{x, y\} \subseteq X \cap I = S$, a contradiction.

Finally since $I' \cap X \subseteq I' \cap Y = S'$, sets I' and $S \setminus S'$ are disjoint and we get

$$T[X, C, S] \geq w(I) = w(I') + w(S \setminus S') = T[Y, C', S'] + w(S \setminus S').$$

Bags with two children. For each bag $X \in \mathcal{X}$ with two children $Y = Z = X$, we show that

$$T[X, C, S] = \max \left\{ \begin{array}{l} T[Y, C_Y, S] + T[Z, C_Z, S] - w(S) \\ \text{such that } C_Y \cup C_Z = C \\ \text{and } C_Y \cap C_Z = \text{col}(S) \end{array} \right\}. \quad (5.3)$$

“ \leq ”: Let I be a maximum-weight independent set for G_X with $X \cap I = S$ whose vertices use pairwise distinct colors from C . Then, $I_Y := I \cap V_Y$ and $I_Z := I \cap V_Z$ are independent sets for G_Y and G_Z , respectively. One has $Y \cap I_Y = Y \cap I \cap V_Y = Y \cap I = X \cap I = S$ and, likewise, $Z \cap I_Z = S$.

By Definition 5.7(iii), one has $I_Y \cap I_Z \subseteq X$ and thus $I_Y \cap I_Z = I_Y \cap I_Z \cap X = S$. Since I contains only one vertex of each color from C , we also get $\text{col}(I_Y) \cap \text{col}(I_Z) = \text{col}(S)$. Thus, the vertices of I_Y have pairwise distinct colors in $C_Y := \text{col}(I_Y)$ and the vertices of I_Z have pairwise distinct colors in $C_Z := (C \setminus C_Y) \cup \text{col}(S)$. One has $C_Y \cup C_Z = C$ and $C_Y \cap C_Z = \text{col}(S)$. Thus,

$$\begin{aligned} T[X, C, S] &= w(I) = w(I_Y) + w(I_Z) - w(I_Y \cap I_Z) \\ &= w(I_Y) + w(I_Z) - w(S) \\ &\leq T[Y, C_Y, S] + T[Z, C_Z, S] - w(S). \end{aligned}$$

“ \geq ”: Let C_Y and C_Z be color sets such that $C_Y \cap C_Z = \text{col}(S)$ and $C_Y \cup C_Z = C$. Let I_Y and I_Z be maximum-weight independent sets for G_Y and G_Z with $I_Y \cap Y = I_Z \cap Z = S$ and whose vertices use pairwise distinct colors from C_Y and C_Z , respectively. Then, for $I := I_Y \cup I_Z$, one has $I \cap X = (I_Y \cup I_Z) \cap X = (I_Y \cap X) \cup (I_Z \cap X) = S$.

We show that I is an independent set. Assume towards a contradiction that there is an edge $e = \{y, z\} \subseteq I = I_Y \cup I_Z$. Since $e \not\subseteq I_Y$ and $e \not\subseteq I_Z$, we have $y \in I_Y$ and $z \in I_Z$. Definition 5.7(iii) gives $V_Y \cap V_Z \subseteq X$. Thus, by Definition 5.7(ii), $e \subseteq V_Z$ or $e \subseteq V_Y$. By symmetry, assume that $e \in V_Z$. Then, $y \in V_Y \cap V_Z \subseteq X$. Since $y \in X \cap I_Y = X \cap I_Z$, it follows that $\{y, z\} \subseteq I_Z$, a contradiction to I_Z being an independent set.

We show that the colors of the vertices in I are pairwise distinct. Let there be two vertices $\{u, v\} \subseteq I$ with $\text{col}(u) = \text{col}(v) = c^*$. Then $c^* \in \text{col}(S)$ since $C_Y \cap C_Z = \text{col}(S)$. Thus, there is a vertex $w \in S \subseteq I_Y \cap I_Z$ with $\text{col}(w) = c^*$. Since I_Y and I_Z contain only one vertex of each color, it holds

that $w = u = v$. Hence I only contains one vertex with color c^* . Finally, Definition 5.7(iii) yields $I_Y \cap I_Z \subseteq X$. Thus,

$$\begin{aligned} T[X, C, S] &\geq w(I) = w(I_Y) + w(I_Z) - w(I_Y \cap I_Z) \\ &= w(I_Y) + w(I_Z) - w(I_Y \cap I_Z \cap X) \\ &= w(I_Y) + w(I_Z) - w(S) \\ &= T[Y, C_Y, S] + T[Z, C_Z, S] - w(S). \end{aligned}$$

The algorithm and its running time. There are $O(n)$ bags. The algorithm first computes (5.1) for all leaf bags. Then, for each bag whose children have been processed, it computes (5.2) and (5.3).

For each leaf bag X , we compute (5.1) as follows. First, for each of the at most n^α subsets $S \subseteq X$ with $|S| \leq \alpha$, we once compute $w(S)$ and check the independence of S in $O(\alpha^2)$ time. We then use this information in the computation for all subsets $C \subseteq \{1, 2, \dots, c\}$. Thus, all leaf nodes are processed in $O(n \cdot (2^c + \alpha^2 \cdot n^\alpha))$ time.

For each bag $X \in \mathcal{X}$ with two children, we compute (5.3) as follows. First, for each of the at most n^α subsets $S \subseteq X$ with $|S| \leq \alpha$, we once compute $w(S)$ and check the independence of S in $O(\alpha^2)$ time. We then use this information in the computation for all subsets $C \subseteq \{1, 2, \dots, c\}$. To this end, iterate over all subsets $C_Y \subseteq C$, whereby C_Z can be computed from C_Y in $O(c)$ time. Note that, in total, one iterates over at most 3^c subsets $C_Y \subseteq C \subseteq \{1, 2, \dots, c\}$: each color is either not in C , or in C but not in C_Y , or in C_Y . Thus, bags with two children can be processed in total time $O(n \cdot (3^c c + \alpha^2 \cdot n^\alpha))$.

For each bag $X \in \mathcal{X}$ with a single child Y , we compute (5.2) as follows. First, for each of the at most n^α subsets $S' \subseteq Y$ with $|S'| \leq \alpha$, we once compute the intersection $S' \cap X$ and $w(S' \cap X)$ in $O(\alpha^2)$ time. During this computation, for each encountered intersection S^* and color set $C \subseteq \{1, 2, \dots, c\}$, remember the maximum value $T_{S^*}[Y, C]$ taken by $T[Y, C, S']$ for any $S' \subseteq Y$ with $S^* = S' \cap X$. This works in $O(2^c \cdot \alpha)$ time. We then iterate over all subsets $C \subseteq \{1, 2, \dots, c\}$, each of the encountered intersections $S^* \subseteq X \cap Y$, and each independent set S with $S^* \subseteq S \subseteq X$ and compute $T[Y, C \setminus \text{col}(S \setminus S^*), S']$ as the maximum value $T_{S^*}[C \setminus \text{col}(S \setminus S^*)] - w(S \setminus S^*)$ encountered for any intersection S^* that yielded S . For each $\alpha' \leq \alpha$ and each of $n^{\alpha'}$ intersections S^* of size exactly α' , we enumerate at most $n^{\alpha - \alpha'}$ independent sets S such that $S^* \subseteq S \subseteq X$. Thus, bags with a single child can be processed in $O(n \cdot (\alpha^2 + 2^c \alpha + 2^c n^\alpha))$ total time.

Altogether, the overall algorithm runs in $O(3^c n^{\alpha+1})$ time. \square

6 Parameterized complexity on other graph classes

In Section 4, we have shown that MAXIMUM INDEPENDENT SET is W[1]-hard parameterized by the solution size even on 2-simplicial 3-minoes. In contrast, in Section 5, we have

shown that MAX-WEIGHT c -COLORABLE SUBGRAPH is fixed-parameter tractable parameterized by the solution size on cluster \bowtie chordal graphs.

In this section, we survey the parameterized complexity of MAX-WEIGHT c -COLORABLE SUBGRAPH on the neighboring graph classes discussed in Section 3, thus completing the computational complexity picture given in Figure 1.1. Most results in this section are known or easy to obtain. Yet since they are scattered throughout the literature using terminology from quite diverse subject fields, we summarize them here.

$K_{1,3}$ -free graphs. Section 4 has shown that MAXIMUM INDEPENDENT SET is W[1]-hard parameterized by the solution size on 3-minoes, which are a proper subclass of $K_{1,4}$ -free graphs. This is complemented as follows.

Proposition 6.1 *On $K_{1,3}$ -free graphs,*

- i) MAXIMUM c -COLORABLE SUBGRAPH is NP-hard for each fixed $c \geq 3$ and
- ii) a maximum-weight c -colorable subgraph with at most ℓ vertices is computable in $2^{\ell+c} \cdot c^\ell \cdot \ell^{O(\log \ell)} \cdot n^3 \log^2 n$ time.

Proof Proposition 6.1(i) follows from the fact that checking 3-colorability of line graphs is NP-complete, where line graphs are $K_{1,3}$ -free.

Proposition 6.1(ii) follows directly from Lemma 5.3 using the fact that $K_{1,3}$ -free graphs are a hereditary graph class and that MAX-WEIGHT INDEPENDENT SET is solvable in $O(n^3)$ time on $K_{1,3}$ -free graphs (Faenza et al 2011). \square

Note that $c \leq \ell$ holds in all nontrivial cases. Thus, Proposition 6.1 shows that MAX-WEIGHT c -COLORABLE SUBGRAPH is fixed-parameter tractable parameterized by ℓ on $K_{1,3}$ -free graphs. The complexity for the case $c = 2$ seems to be open.

Cluster \bowtie interval graphs. These graphs form a subclass of cluster \bowtie chordal graphs and therefore Theorem 5.1 applies to them. Yet for cluster \bowtie interval graphs, one can give a slightly better running time.

Proposition 6.2 *On cluster \bowtie interval graphs,*

- i) MAX-WEIGHT INDEPENDENT SET is NP-hard and
- ii) a maximum-weight c -colorable subgraph with at most ℓ vertices is computable in $2^{\ell+c} \cdot (c+e+2)^\ell \cdot \ell^{O(\log \ell)} \cdot (n+m) \cdot \log^3 n$ time if the decomposition of the input graph into a cluster graph and an interval graph is given.

Herein, e is Euler's number.

Proposition 6.2(i) is due to Halldórsson and Karlsson (2006).

Towards proving Proposition 6.2(ii), van Bevern et al (2015) noted that Halldórsson and Karlsson (2006) showed that a maximum-weight independent set of vertices of pairwise distinct colors in an interval graph whose vertices are colored in c colors is computable in $O(2^c \cdot n)$ time if an interval representation with sorted intervals is given, which

can be computed in $O(n + m)$ time. Since cluster \bowtie interval graphs are a hereditary graph class, Proposition 6.2(ii) follows from this result of Halldórsson and Karlsson (2006) and Lemmas 5.3 and 5.5.

Chordal graphs. Theorem 4.1 shows that MAXIMUM INDEPENDENT SET is W[1]-hard parameterized by the solution size on inductive 2-independent graphs. Chordal graphs are exactly the inductive 1-independent graphs (Blair and Peyton 1993; Ye and Borodin 2012).

Proposition 6.3 *In chordal graphs, MAX-WEIGHT c -COLORABLE SUBGRAPH is*

- i) NP-hard,
- ii) polynomial-time solvable for each fixed c yet W[2]-hard parameterized by c , and
- iii) a maximum-weight c -colorable subgraph with at most ℓ vertices is computable in $2^{\ell+c} \cdot c^\ell \cdot \ell^{O(\log \ell)} \cdot (n + m) \cdot \log^3 n$ time.

Proposition 6.3(i) and (ii) were shown by Yannakakis and Gavril (1987). Proposition 6.3(iii) follows from the fact that MAX-WEIGHT INDEPENDENT SET is linear-time solvable in chordal graphs (Frank 1975) and using Lemma 5.3. This was shown for the unweighted variant by Misra et al (2013) and our Lemma 5.3 is, essentially, just a slightly adapted weighted variant of this result.

Interval graphs. Yannakakis and Gavril (1987) noted that MAX-WEIGHT c -COLORABLE SUBGRAPH in interval graphs is solvable in polynomial time by modelling it as a totally unimodular linear program.

Using a flow formulation, Arkin and Silverberg (1987) proved that the equivalent problem of scheduling a maximum-weight subset of intervals on c parallel identical machines such that the intervals on each machine are pairwise nonintersecting is solvable in $O(n^2 \log n)$ time. Since the interval representation of an interval graph is computable in the same time (Corneil et al 2009), one gets the following result.

Proposition 6.4 *MAX-WEIGHT c -COLORABLE SUBGRAPH is solvable in $O(n^2 \log n)$ time on interval graphs.*

Inductive k -independent graphs. Of course, our hardness result of Theorem 4.1 also holds for MAXIMUM INDEPENDENT SET in inductive k -independent graphs.

On the positive side, Ye and Borodin (2012) showed that MAX-WEIGHT INDEPENDENT SET is polynomial-time k -approximable in inductive k -independent graphs. The algorithm requires the k -independence ordering of Definition 2.3, which cannot be computed efficiently (cf. Proposition 3.1), yet sometimes is given by the application data (Ásgeirsson et al 2017; Halldórsson and Tonoyan 2015).

7 Conclusion

Motivated by recent work on wireless scheduling (Ásgeirsson et al 2017; Halldórsson and Tonoyan 2015), we performed an extensive analysis of problems related to independent sets on graphs that lie between interval graphs on the “bottom end” and inductive 2-independent graphs on the “top end”. Some of our (computational hardness) results might be discouraging given that, in real-world scheduling applications, one often has to expect inductive k -independent graphs with constant $k > 2$. This negative impression is alleviated by the following observations.

First, there are several tractability results for classic scheduling problems that can be modeled as independent set problems at this fairly low level in the graph classes hierarchy.

Second, our parameterized complexity studies are basically focused on the parameter solution size. In the spirit of multivariate complexity analysis (Fellows et al 2013; Komusiewicz and Niedermeier 2012; Niedermeier 2010) this encourages the studies of further and also “combined” parameterizations. Indeed, with real-world data at hand, one might first measure numerous parameter values in the instance (say characteristics such as maximum degree, treewidth, degree distribution, feedback edge and vertex numbers) that, if small, altogether might be exploited to get strong fixed-parameter tractability results.

Third, we point out that combining approximation algorithms (there are several results in the literature including Ásgeirsson et al (2017); Halldórsson and Tonoyan (2015); and Ye and Borodin (2012)) with problem-specific parameterizations might lead to parameterized approximation algorithms with improved approximation factors and/or running times in relevant application scenarios, see van Bevern et al (2017) for a concrete example.

We remark that our main positive result (fixed-parameter tractability of finding maximum-weight c -colorable subgraphs on cluster \bowtie chordal graphs, Theorem 5.1) indeed also holds for the so far seemingly neglected class of graphs with a given tree decomposition where the independent sets in each bag have size at most α : it can be solved in $O(3^c \cdot n^{\alpha+1})$ time (Lemma 5.8). We feel that this class of graphs might be of independent interest.

Finally, Section 3 with its classification of graph classes between interval graphs and inductive 3-independent graphs might prove useful even for (graph-algorithmic) studies not touching scheduling and independent set problems. On a different note, we advocate that parameterized complexity analysis may prove useful for further studies on hard scheduling problems—so far the number of studies in this direction is fairly small and leaves numerous challenges for future work (Mnich and van Bevern 2018).

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