# Forward-partial inverse-forward splitting for solving monotone inclusions * 

Luis M. Briceño-Arias ${ }^{1}$<br>${ }^{1}$ Universidad Técnica Federico Santa María<br>Departamento de Matemática<br>Santiago, Chile


#### Abstract

In this paper we provide a splitting method for finding a zero of the sum of a maximally monotone operator, a lipschitzian monotone operator, and a normal cone to a closed vectorial subspace of a real Hilbert space. The problem is characterized by a simpler monotone inclusion involving only two operators: the partial inverse of the maximally monotone operator with respect to the vectorial subspace and a suitable lipschitzian monotone operator. By applying the Tseng's method in this context we obtain a splitting algorithm that exploits the whole structure of the original problem and generalizes partial inverse and Tseng's methods. Connections with other methods available in the literature and applications to inclusions involving $m$ maximally monotone operators, to primal-dual composite monotone inclusions, and to zero-sum games are provided.


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## 1 Introduction

This paper is concerned with the numerical resolution of the problem of finding a zero of the sum of a set-valued maximally monotone operator $A$, a lipschitzian monotone operator $B$, and a normal cone $N_{V}$, where $V$ is a closed vectorial subspace of a real Hilbert space $\mathcal{H}$. This problem arises in a wide range of areas such as optimization [17, 36], variational inequalities [26, 38, 39], monotone operator theory [20, 25, 32, 35], partial differential equations [26, 27, 43], economics [24, 29], signal and image processing [5, 15, 19], evolution inclusions [4, 23, 34], traffic theory [8, 21, 33], and game theory [22, 44], among others.

In the particular case when the operator $B$ is zero, the problem is studied in [35] and it is solved via the method of partial inverses. On the other hand, when $V=\mathcal{H}$, the normal cone is zero and our problem reduces to find a zero of $A+B$. In this case, the problem is studied in [40], where the forward-backward-forward splitting or Tseng's method is proposed for solving this problem (see also [13] and the references therein). In addition, two methods are proposed in [12] for finding a zero of $A+B+N_{V}$ in the particular case when $B$ is cocoercive.

In the general case, several algorithms are available in the literature for finding a zero of $A+B+$ $N_{V}$, but any of them exploits the intrinsic structure of the problem. The forward-backward-forward splitting introduced in [40] can be applied to the general case, but it needs to compute the resolvent of $A+N_{V}$, which is not always easy to compute. It is preferable to activate $A$ and $N_{V}$ separately. Other ergodic approaches for solving the problem can be found in [9, 28]. A disadvantage of these methods is the presence of vanishing parameters, which usually lead to numerical instabilities. The algorithms proposed in $[13,16,35]$ permit to find a zero of the sum of finitely many maximally monotone operators by activating them independently and without considering vanishing parameters. However, these methods involve implicit steps on $B$ by using its resolvent, which is not easy to compute in general. An algorithm proposed in [18] overcome this difficulty by activating explicitly the operator $B$. However, this method does not take advantage of the vector subspace involved and, as a consequence, it needs to store additional auxiliary variables at each iteration, which can be difficult for high dimensional problems.

In this paper we propose a fully split method for finding a zero of $A+B+N_{V}$ by exploiting each of its intrinsic properties. The proposed algorithm computes, at each iteration, explicit steps on $B$ and the resolvent of the partial inverse of $A$ with respect to $V$ [35], which can be explicitly found in several cases. In a particular instance, this resolvent becomes a Douglas-Rachford step [25, 37], which activates separately $A$ and $N_{V}$. Hence, in this case our method can be perceived as a forward-Douglas-Rachford-forward splitting. The proposed algorithm generalizes partial inverse and Tseng's methods in the particular instances when $B=0$ and $V=\mathcal{H}$, respectively. We also provide connections with other methods in the literature and we illustrate the flexibility of this framework via some applications to inclusions involving $m$ maximally monotone operators, to primal-dual composite monotone inclusions, and to zero-sum games. In the application to primal-dual inclusions we introduce a new operation between set-valued operators, called partial sum with respect to a closed vectorial subspace, which preserves monotonicity and takes a central role in the problem and algorithm. On the other hand, in continuous zero-sum games, we provide an interesting splitting algorithm for calculating a Nash equilibrium that avoids the computation of the projection onto mixed strategy spaces in infinite dimensions by performing simpler projections alternately.

The paper is organized as follows. In Section 2 we provide the notation and some preliminaries. We
also obtain a relaxed version of Tseng's method [40], which is interesting in its own right. In Section 3 a characterization of Problem 3.1 in terms of two appropriate monotone operators is given and a method for solving this problem is derived from the relaxed version of Tseng's algorithm. Moreover, we provide connections with other methods in the literature. Finally, in Section 4 we apply our method to the problem of finding a zero of a sum of $m$ maximally monotone operators and a lipschitzian monotone operator, to a primal-dual composite monotone inclusion, and to continuous zero-sum games. The methods derived in each instance generalize and improve available algorithms in the literature in each context.

## 2 Notation and Preliminaries

Throughout this paper, $\mathcal{H}$ is a real Hilbert space with scalar product denoted by $\langle\cdot \mid \cdot\rangle$ and associated norm $\|\cdot\|$. The symbols $\rightharpoonup$ and $\rightarrow$ denote, respectively, weak and strong convergence and Id denotes the identity operator. The indicator function of a subset $C$ of $\mathcal{H}$ is $\iota_{C}$, which takes the value 0 in $C$ and $+\infty$ in $\mathcal{H} \backslash C$. If $C$ is non-empty, closed, and convex, the projection of $x$ onto $C$, denoted by $P_{C} x$, is the unique point in $\operatorname{Argmin}_{y \in C}\|x-y\|$, and the normal cone to $C$ is the maximally monotone operator

$$
N_{C}: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \begin{cases}\{u \in \mathcal{H} \mid(\forall y \in C)\langle y-x \mid u\rangle \leq 0\}, & \text { if } x \in C  \tag{2.1}\\ \varnothing, & \text { otherwise }\end{cases}
$$

An operator $T: \mathcal{H} \rightarrow \mathcal{H}$ is $\beta$-cocoercive for some $\beta \in] 0,+\infty[$ if, for every $x \in \mathcal{H}$ and $y \in \mathcal{H}$, $\langle x-y \mid T x-T y\rangle \geq \beta\|T x-T y\|^{2}$, it is $\chi$-lipschitzian if, for every $x \in \mathcal{H}$ and $y \in \mathcal{H},\|T x-T y\| \leq$ $\chi\|x-y\|$, it is non expansive if it is 1 -lipschitzian, and the set of fixed points of $T$ is given by Fix $T=\{x \in \mathcal{H} \mid T x=x\}$.

We denote by gra $A=\{(x, u) \in \mathcal{H} \times \mathcal{H} \mid u \in A x\}$ the graph of a set-valued operator $A$ : $\mathcal{H} \rightarrow 2^{\mathcal{H}}$, by $\operatorname{dom} A=\{x \in \mathcal{H} \mid A x \neq \varnothing\}$ its domain, by zer $A=\{x \in \mathcal{H} \mid 0 \in A x\}$ its set of zeros, by ran $A=$ $\{u \in \mathcal{H} \mid(\exists x \in \mathcal{H}) u \in A x\}$ its range, and by $J_{A}=(\operatorname{Id}+A)^{-1}$ its resolvent. If $A$ is monotone, i.e., for every $(x, u)$ and $(y, v)$ in gra $A,\langle x-y \mid u-v\rangle \geq 0$, then $J_{A}$ is single-valued and non expansive. In addition, if $\operatorname{ran}(\operatorname{Id}+A)=\mathcal{H}, A$ is maximally monotone and $\operatorname{dom} J_{A}=\mathcal{H}$. Let $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be maximally monotone. The reflection operator of $A$ is $R_{A}=2 J_{A}-\mathrm{Id}$, which is non expansive. The partial inverse of $A$ with respect to a vector subspace $V$ of $\mathcal{H}$, denoted by $A_{V}$, is defined by

$$
\begin{equation*}
\left(\forall(x, y) \in \mathcal{H}^{2}\right) \quad y \in A_{V} x \quad \Leftrightarrow \quad\left(P_{V} y+P_{V^{\perp}} x\right) \in A\left(P_{V} x+P_{V^{\perp} y}\right) . \tag{2.2}
\end{equation*}
$$

Note that $A_{\mathcal{H}}=A$ and $A_{\{0\}}=A^{-1}$. The following properties of the partial inverse will be useful throughout this paper.

Proposition 2.1 Let $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be a set-valued operator and let $V$ be a vector subspace of $\mathcal{H}$. Then the following hold.
(i) $\left(A_{V}\right)^{-1}=\left(A^{-1}\right)_{V}=A_{V^{\perp}}$.
(ii) $P_{V}\left(A+N_{V}\right)^{-1} P_{V}=P_{V}\left(A_{V^{\perp}}+N_{V}\right) P_{V}$.

Proof. (i): Let $(x, u) \in \mathcal{H}^{2}$. We have from (2.2) that

$$
\begin{align*}
u \in\left(A_{V}\right)^{-1} x & \Leftrightarrow x \in A_{V} u \\
& \Leftrightarrow P_{V} x+P_{V^{\perp}} u \in A\left(P_{V} u+P_{V^{\perp}} x\right)  \tag{2.3}\\
& \Leftrightarrow P_{V} u+P_{V^{\perp}} x \in A^{-1}\left(P_{V} x+P_{V^{\perp}} u\right) \\
& \Leftrightarrow x \in\left(A^{-1}\right)_{V} u . \tag{2.4}
\end{align*}
$$

On the other hand, it follows from (2.3) and (2.2) that $u \in\left(A_{V}\right)^{-1} x$ is equivalent to $u \in A_{V^{\perp} x}$. (ii): Let $(x, u) \in \mathcal{H}^{2}$. We deduce from (i) and (2.2) that

$$
\begin{align*}
u \in P_{V}\left(A+N_{V}\right)^{-1}\left(P_{V} x\right) & \Leftrightarrow(u \in V) \quad P_{V} x \in A u+N_{V} u \\
& \Leftrightarrow(u \in V)\left(\exists y \in V^{\perp}\right) \quad P_{V} x-y \in A u \\
& \Leftrightarrow(u \in V)\left(\exists y \in V^{\perp}\right) \quad u-y \in A_{V^{\perp}}\left(P_{V} x\right) \\
& \Leftrightarrow u \in P_{V}\left(A_{V^{\perp}}+N_{V}\right)\left(P_{V} x\right), \tag{2.5}
\end{align*}
$$

which yields the result.
The following result is a relaxed version of the method originally proposed in [40] over some modifications developed in $[11,13]$.

Proposition 2.2 Let $\mathcal{A}: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be maximally monotone and let $\mathcal{B}: \mathcal{H} \rightarrow \mathcal{H}$ be monotone and $\eta-$ lipschitzian such that $\operatorname{zer}(\mathcal{A}+\mathcal{B}) \neq \varnothing$. Moreover, let $z_{0} \in \mathcal{H}$, let $\left.\varepsilon \in\right] 0, \max \{1,1 / 2 \eta\}\left[\right.$, let $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon,(1 / \eta)-\varepsilon]$, let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$, and iterate, for every $n \in \mathbb{N}$,

$$
\left\lvert\, \begin{align*}
& r_{n}=z_{n}-\delta_{n} \mathcal{B} z_{n}  \tag{2.6}\\
& s_{n}=J_{\delta_{n} \mathcal{A}} r_{n} \\
& t_{n}=s_{n}-\delta_{n} \mathcal{B} s_{n} \\
& z_{n+1}=z_{n}+\lambda_{n}\left(t_{n}-r_{n}\right) .
\end{align*}\right.
$$

Then, $z_{n} \rightharpoonup \bar{z}$ for some $\bar{z} \in \operatorname{zer}(\mathcal{A}+\mathcal{B})$ and $z_{n+1}-z_{n} \rightarrow 0$.
Proof. First note that (2.6) yields

$$
\begin{equation*}
(\forall n \in \mathbb{N}) \quad \delta_{n}^{-1}\left(r_{n}-s_{n}\right) \in \mathcal{A} s_{n} \tag{2.7}
\end{equation*}
$$

Let $z \in \operatorname{zer}(\mathcal{A}+\mathcal{B})$ and fix $n \in \mathbb{N}$. It follows from [40, Lemma 3.1] and (2.6) that

$$
\begin{align*}
&\left\|z_{n+1}-z\right\|^{2}=\left\|\left(1-\lambda_{n}\right)\left(z_{n}-z\right)+\lambda_{n}\left(s_{n}-\delta_{n}\left(\mathcal{B} s_{n}-\mathcal{B} z_{n}\right)-z\right)\right\|^{2} \\
&=\left(1-\lambda_{n}\right)\left\|z_{n}-z\right\|^{2}+\lambda_{n}\left\|s_{n}-\delta_{n}\left(\mathcal{B} s_{n}-\mathcal{B} z_{n}\right)-z\right\|^{2}-\lambda_{n}\left(1-\lambda_{n}\right)\left\|t_{n}-r_{n}\right\|^{2} \\
& \leq\left(1-\lambda_{n}\right)\left\|z_{n}-z\right\|^{2}+\lambda_{n}\left(\left\|z_{n}-z\right\|^{2}+\delta_{n}^{2}\left\|\mathcal{B} s_{n}-\mathcal{B} z_{n}\right\|^{2}-\left\|s_{n}-z_{n}\right\|^{2}\right) \\
& \quad-\lambda_{n}\left(1-\lambda_{n}\right)\left\|t_{n}-r_{n}\right\|^{2} \\
& \leq\left\|z_{n}-z\right\|^{2}-\left(1-\left(\delta_{n} \eta\right)^{2}\right)\left\|s_{n}-z_{n}\right\|^{2}-\lambda_{n}\left(1-\lambda_{n}\right)\left\|t_{n}-r_{n}\right\|^{2} . \tag{2.8}
\end{align*}
$$

Hence, since $\delta_{n}<1 / \eta$ and $0<\lambda_{n} \leq 1$, we obtain $\left\|z_{n+1}-z\right\|^{2} \leq\left\|z_{n}-z\right\|^{2}$, which yields the boundedness of the sequence $\left(z_{k}\right)_{k \in \mathbb{N}}$. Moreover, we deduce from (2.8) and [13, Lemma 2.1] that $\left(\left\|s_{k}-z_{k}\right\|^{2}\right)_{k \in \mathbb{N}}$ and $\left(\left\|t_{k}-r_{k}\right\|^{2}\right)_{k \in \mathbb{N}}$ are summable and, in particular,

$$
\begin{equation*}
s_{k}-z_{k} \rightarrow 0 \quad \text { and } \quad t_{k}-r_{k} \rightarrow 0 \tag{2.9}
\end{equation*}
$$

which yields $z_{k+1}-z_{k}=\lambda_{k}\left(t_{k}-r_{k}\right) \rightarrow 0$. By setting, for every $k \in \mathbb{N}, u_{k}=\delta_{k}^{-1}\left(r_{k}-t_{k}\right)$, it follows from (2.6), (2.7), and (2.9) that

$$
\begin{equation*}
(\forall k \in \mathbb{N}) \quad 0 \leftarrow u_{k}=\delta_{k}^{-1}\left(r_{k}-s_{k}\right)+\mathcal{B} s_{k} \in(\mathcal{A}+\mathcal{B}) s_{k} \tag{2.10}
\end{equation*}
$$

Now let us take $w \in \mathcal{H}$ be any sequential weak cluster point of $\left(z_{k}\right)_{k \in \mathbb{N}}$, say $z_{k_{\ell}} \rightharpoonup w$. Then, it follows from (2.9) and (2.10) that

$$
\begin{equation*}
s_{k_{\ell}} \rightharpoonup w, u_{k_{\ell}} \rightarrow 0, \quad \text { and } \quad\left(s_{k_{\ell}}, u_{k_{\ell}}\right) \in \operatorname{gra}(\mathcal{A}+\mathcal{B}) \tag{2.11}
\end{equation*}
$$

Since $\mathcal{B}$ is monotone and continuous, it is maximally monotone [7, Corollary 20.25]. Moreover, since $\operatorname{dom} \mathcal{B}=\mathcal{H}$, we deduce from [7, Corollary $24.4(\mathrm{i})]$ that $\mathcal{A}+\mathcal{B}$ is maximally monotone and, hence, its graph is sequentially closed in $\mathcal{H}^{\text {weak }} \times \mathcal{H}^{\text {strong }}$ [7, Proposition 20.33(ii)]. Therefore, we conclude from (2.10) that $w \in \operatorname{zer}(\mathcal{A}+\mathcal{B})$ and from [13, Lemma 2.2] we deduce that there exists $\bar{z} \in \operatorname{zer}(\mathcal{A}+\mathcal{B})$ such that $z_{n} \rightharpoonup \bar{z}$ which yields the result. $[$

Remark 2.3 As in [13, Theorem 2.5], absolutely summable errors can be incorporated in each step of the algorithm in (2.6). However, for ease of presentation throughout the document, we only provide the error free version.

For complements and further background on monotone operator theory and algorithms, the reader is referred to $[4,7,35,43]$.

## 3 Forward-Partial Inverse-Forward Splitting

We aim at solving the following problem.
Problem 3.1 Let $\mathcal{H}$ be a real Hilbert space and let $V$ be a closed vector subspace of $\mathcal{H}$. Let $A: \mathcal{H} \rightarrow$ $2^{\mathcal{H}}$ be a maximally monotone operator and let $B: \mathcal{H} \rightarrow \mathcal{H}$ be a monotone and $\chi$-lipschitzian operator. The problem is to

$$
\begin{equation*}
\text { find } x \in \mathcal{H} \quad \text { such that } \quad 0 \in A x+B x+N_{V} x \tag{3.1}
\end{equation*}
$$

under the assumption $\operatorname{zer}\left(A+B+N_{V}\right) \neq \varnothing$.
In this section we provide our method for solving Problem 3.1. We first provide a characterization of the solutions to Problem 3.1, which motivates our algorithm. Its convergence to a solution to Problem 3.1 is then proved.

### 3.1 Characterization

The following result provides a characterization of the solutions to Problem 3.1 in terms of two suitable monotone operators.

Proposition 3.2 Let $\gamma \in] 0,+\infty[$ and $\mathcal{H}, A, B$, and $V$ be as in Problem 3.1. Define

$$
\left\{\begin{array}{l}
\mathcal{A}_{\gamma}=(\gamma A)_{V}: \mathcal{H} \rightarrow 2^{\mathcal{H}}  \tag{3.2}\\
\mathcal{B}_{\gamma}=\gamma P_{V} \circ B \circ P_{V}: \mathcal{H} \rightarrow V .
\end{array}\right.
$$

Then the following hold.
(i) $\mathcal{A}_{\gamma}$ is maximally monotone and, for every $\left.\delta \in\right] 0,+\infty\left[\right.$ and $x \in \mathcal{H}, J_{\delta \mathcal{A}_{\gamma}} x=P_{V} p+\gamma P_{V} \perp q$, where $p$ and $q$ in $\mathcal{H}$ are such that $x=p+\gamma q$ and

$$
\begin{equation*}
\frac{P_{V} q}{\delta}+P_{V \perp} q \in A\left(P_{V} p+\frac{P_{V \perp} p}{\delta}\right) \tag{3.3}
\end{equation*}
$$

In particular, for every $x \in \mathcal{H}, J_{\mathcal{A}_{\gamma}} x=2 P_{V} J_{\gamma A}-J_{\gamma A}+\mathrm{Id}-P_{V}=\left(\operatorname{Id}+R_{N_{V}} R_{\gamma A}\right) / 2$.
(ii) $\mathcal{B}_{\gamma}$ is $\gamma \chi$-lipschitzian and monotone.
(iii) Let $x \in \mathcal{H}$. Then $x$ is a solution to Problem 3.1 if and only if $x \in V$ and

$$
\begin{equation*}
\left(\exists y \in V^{\perp} \cap(A x+B x)\right) \text { such that } x+\gamma\left(y-P_{V^{\perp}} B x\right) \in \operatorname{zer}\left(\mathcal{A}_{\gamma}+\mathcal{B}_{\gamma}\right) \tag{3.4}
\end{equation*}
$$

Proof. (i): Since $\gamma A$ is maximally monotone, $\mathcal{A}_{\gamma}$ inherits this property [35, Proposition 2.1]. In addition, it follows from (2.2) that, for every $(p, q, x) \in \mathcal{H}^{3}$ such that $p+\gamma q=x$ and every $\left.\delta \in\right] 0,+\infty[$,

$$
\begin{align*}
\frac{P_{V} q}{\delta}+P_{V \perp} q \in A\left(P_{V} p+\frac{P_{V} \perp}{\delta}\right) & \Leftrightarrow \frac{\gamma P_{V} q}{\delta}+\gamma P_{V \perp} q \in \gamma A\left(P_{V} p+\frac{P_{V} \perp p}{\delta}\right) \\
& \Leftrightarrow \frac{\gamma P_{V} q}{\delta}+\frac{P_{V^{\perp}} p}{\delta} \in \mathcal{A}_{\gamma}\left(P_{V} p+\gamma P_{V^{\perp}} q\right) \\
& \Leftrightarrow \gamma P_{V} q+P_{V^{\perp}} p \in \delta \mathcal{A}_{\gamma}\left(P_{V} p+\gamma P_{V^{\perp}} q\right) \\
& \Leftrightarrow P_{V} p+\gamma P_{V \perp} q=J_{\delta \mathcal{A}_{\gamma}}(p+\gamma q) \\
& \Leftrightarrow P_{V} p+\gamma P_{V^{\perp}} q=J_{\delta \mathcal{A}_{\gamma}} x . \tag{3.5}
\end{align*}
$$

In particular, if $\delta=1,(3.3)$ reduces to $p=J_{\gamma A}(p+\gamma q)=J_{\gamma A} x$ and, hence,

$$
\begin{align*}
J_{\mathcal{A}_{\gamma}} x & =P_{V}\left(J_{\gamma A} x\right)+P_{V}\left(x-J_{\gamma A} x\right) \\
& =2 P_{V} J_{\gamma A} x-J_{\gamma A} x+x-P_{V} x \\
& =\frac{1}{2}\left(x+2 P_{V}\left(2 J_{\gamma A} x-x\right)-2 J_{\gamma A} x+x\right) \\
& =\frac{1}{2}\left(x+R_{N_{V}} R_{\gamma A} x\right) \tag{3.6}
\end{align*}
$$

(ii): Let $(x, y) \in \mathcal{H}^{2}$. We have from (3.2), the monotonicity of $B$, the fact that $P_{V}$ is linear, and $P_{V}^{*}=$ $P_{V}$ that $\left\langle x-y \mid \mathcal{B}_{\gamma} x-\mathcal{B}_{\gamma} y\right\rangle=\gamma\left\langle P_{V} x-P_{V} y \mid B\left(P_{V} x\right)-B\left(P_{V} y\right)\right\rangle \geq 0$, and, from the lipschitzian property on $B$ and (3.2) we obtain $\left\|\mathcal{B}_{\gamma} x-\mathcal{B}_{\gamma} y\right\| \leq \gamma\left\|B\left(P_{V} x\right)-B\left(P_{V} y\right)\right\| \leq \gamma \chi\left\|P_{V} x-P_{V} y\right\| \leq \gamma \chi \| x-$ $y \|$. (iii): Let $x \in \mathcal{H}$ be a solution to Problem 3.1. We have $x \in V$ and there exists $y \in V^{\perp}=N_{V} x$ such that $y \in A x+B x$. Since $B$ is single valued and $P_{V}$ is linear, it follows from (2.2) that

$$
\begin{align*}
y \in A x+B x & \Leftrightarrow \gamma y-\gamma B x \in \gamma A x \\
& \Leftrightarrow-\gamma P_{V}(B x) \in(\gamma A)_{V}\left(x+\gamma\left(y-P_{V^{\perp}} B x\right)\right) \\
& \Leftrightarrow 0 \in(\gamma A)_{V}\left(x+\gamma\left(y-P_{V^{\perp}} B x\right)\right)+\gamma P_{V}\left(B\left(P_{V}\left(x+\gamma\left(y-P_{V^{\perp}} B x\right)\right)\right)\right) \\
& \Leftrightarrow x+\gamma\left(y-P_{V^{\perp}} B x\right) \in \operatorname{zer}\left(\mathcal{A}_{\gamma}+\mathcal{B}_{\gamma}\right) \tag{3.7}
\end{align*}
$$

which yields the result.
Remark 3.3 Note that the characterization in Proposition 3.2 (iii) yields $Z=P_{V}\left(\operatorname{zer}\left(\mathcal{A}_{\gamma}+\mathcal{B}_{\gamma}\right)\right)$.

### 3.2 Algorithm and convergence

In the following result we propose our algorithm and we prove its convergence to a solution to Problem 3.1. Since Proposition 3.2 asserts that Problem 3.1 can be solved via a monotone inclusion involving a maximally monotone operator and a single-valued lipschitzian monotone operator, our method is obtained as a consequence of Proposition 2.2, which is inspired from [13, 40].

Theorem 3.4 Let $\mathcal{H}, V, A$, and $B$, be as in Problem 3.1, let $\gamma \in] 0,+\infty[$, let $\varepsilon \in] 0, \max \left\{1, \frac{1}{2 \gamma \chi}\right\}[$, let $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $\left[\varepsilon, \frac{1}{\gamma \chi}-\varepsilon\right]$, and let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Moreover, let $x_{0} \in V$, let $y_{0} \in V^{\perp}$, and iterate, for every $n \in \mathbb{N}$,

1. find $\left(p_{n}, q_{n}\right) \in \mathcal{H}^{2}$ such that $x_{n}-\delta_{n} \gamma P_{V} B x_{n}+\gamma y_{n}=p_{n}+\gamma q_{n}$

$$
\begin{equation*}
\text { and } \frac{P_{V} q_{n}}{\delta_{n}}+P_{V} q_{n} \in A\left(P_{V} p_{n}+\frac{P_{V^{\perp}} p_{n}}{\delta_{n}}\right) . \tag{3.8}
\end{equation*}
$$

2. set $x_{n+1}=x_{n}+\lambda_{n}\left(P_{V} p_{n}+\delta_{n} \gamma P_{V}\left(B x_{n}-B P_{V} p_{n}\right)-x_{n}\right)$ and $y_{n+1}=y_{n}+\lambda_{n}\left(P_{V^{\perp}} q_{n}-y_{n}\right)$. Go to 1 .

Then, the sequences $\left(x_{n}\right)_{n \in \mathbb{N}}$ and $\left(y_{n}\right)_{n \in \mathbb{N}}$ are in $V$ and $V^{\perp}$, respectively, $x_{n} \rightharpoonup \bar{x}$ and $y_{n} \rightharpoonup \bar{y}$ for some solution $\bar{x} \in \operatorname{zer}\left(A+B+N_{V}\right)$ and $\bar{y} \in V^{\perp} \cap\left(A \bar{x}+P_{V} B \bar{x}\right), x_{n+1}-x_{n} \rightarrow 0$, and $y_{n+1}-y_{n} \rightarrow 0$.

Proof. Since $x_{0} \in V$ and $y_{0} \in V^{\perp}$, (3.8) yields $\left(x_{n}\right)_{n \in \mathbb{N}} \subset V$ and $\left(y_{n}\right)_{n \in \mathbb{N}} \subset V^{\perp}$. Thus, for every $n \in \mathbb{N}$, it follows from (3.8) and Proposition 3.2(i) that

$$
\begin{equation*}
P_{V} p_{n}+\gamma P_{V^{\perp}} q_{n}=J_{\delta_{n}(\gamma A)_{V}}\left(x_{n}+\gamma y_{n}-\delta_{n} \gamma P_{V} B x_{n}\right) . \tag{3.9}
\end{equation*}
$$

For every $n \in \mathbb{N}$, denote by $z_{n}=x_{n}+\gamma y_{n}$ and by

$$
\begin{equation*}
s_{n}=J_{\delta_{n}(\gamma A)_{V}}\left(x_{n}+\gamma y_{n}-\delta_{n} \gamma P_{V} B x_{n}\right)=J_{\delta_{n}(\gamma A)_{V}}\left(z_{n}-\delta_{n} \gamma P_{V} B P_{V} z_{n}\right)=J_{\delta_{n} \mathcal{A}_{\gamma}}\left(z_{n}-\delta_{n} \mathcal{B}_{\gamma} z_{n}\right) . \tag{3.10}
\end{equation*}
$$

Hence, it follows from (3.9) that $P_{V} p_{n}=P_{V} s_{n}, \gamma P_{V^{\perp}} q_{n}=P_{V^{\perp}} s_{n}$, and, from (3.8), we obtain

$$
\left\{\begin{array}{l}
x_{n+1}=x_{n}+\lambda_{n}\left(P_{V} s_{n}+\delta_{n} \gamma P_{V}\left(B x_{n}-B P_{V} s_{n}\right)-x_{n}\right)  \tag{3.11}\\
\gamma y_{n+1}=\gamma y_{n}+\lambda_{n}\left(P_{V^{\perp}} s_{n}-\gamma y_{n}\right) .
\end{array}\right.
$$

By adding the latter equations we deduce that the algorithm described in (3.8) can be written equivalently as

$$
(\forall n \in \mathbb{N}) \quad\left[\begin{array}{l}
r_{n}=z_{n}-\delta_{n} \mathcal{B}_{\gamma} z_{n}  \tag{3.12}\\
s_{n}=J_{\delta_{n} \mathcal{A}_{\gamma}} r_{n} \\
t_{n}=s_{n}-\delta_{n} \mathcal{B}_{\gamma} s_{n} \\
z_{n+1}=z_{n}+\lambda_{n}\left(t_{n}-r_{n}\right)
\end{array}\right.
$$

which is a particular instance of (2.6) when $\mathcal{B}=\mathcal{B}_{\gamma}$ and $\mathcal{A}=\mathcal{A}_{\gamma}$. Therefore, it follows from Proposition 3.2(i)\&(ii) and Proposition 2.2 that $z_{n} \rightharpoonup \bar{z} \in \operatorname{zer}\left(\mathcal{A}_{\gamma}+\mathcal{B}_{\gamma}\right)$ and $z_{n+1}-z_{n} \rightarrow 0$. By defining $\bar{x}:=P_{V} \bar{z} \in Z$ and $\bar{y}:=P_{V^{\perp}} \bar{z} / \gamma \in(A \bar{x}+B \bar{x})-P_{V^{\perp}} B \bar{x}=A \bar{x}+P_{V} B \bar{x}$, the results follow from Proposition 3.2(iii) and Proposition 2.2.

## Remark 3.5

(i) It is known that the forward-backward-forward splitting admits errors in the computations of the operators involved [11, 13]. In our algorithm these inexactitudes have not been considered for simplicity.
(ii) In the particular case when $\lambda_{n} \equiv 1$ and $B \equiv 0(\chi=0)$, (3.8) reduces to the classical partial inverse method proposed in [35] for finding $x \in V$ such that there exists $y \in V^{\perp}$ satisfying $y \in A x$.
(iii) As in [13], under further assumptions on the operators $\mathcal{A}_{\gamma}$ and/or $\mathcal{B}_{\gamma}$, e.g., as demiregularity (see [2, Definition 2.3\&Proposition 2.4]), strong convergence can be achieved.

The sequence $\left(\delta_{n}\right)_{n \in \mathbb{N}}$ in Theorem 3.4 can be manipulated in order to accelerate the algorithm. However, as in [35], Step 1 in Theorem 3.4 is not always easy to compute. The following result show us a particular case of our method in which Step 1 can be obtained explicitly when the resolvent of $A$ is computable. The method can be seen as a forward-Douglas-Rachford-forward splitting for solving Problem 3.2.

Corollary 3.6 Let $\mathcal{H}, V, A$, and $B$, be as in Problem 3.1, let $\gamma \in] 0,1 / \chi[$, let $\varepsilon \in] 0,1[$, and let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Moreover, let $z_{0} \in \mathcal{H}$, and iterate, for every $n \in \mathbb{N}$,

$$
\left\lvert\, \begin{align*}
& r_{n}=z_{n}-\gamma P_{V} B P_{V} z_{n}  \tag{3.13}\\
& p_{n}=J_{\gamma A} r_{n} \\
& s_{n}=2 P_{V} p_{n}-p_{n}+r_{n}-P_{V} r_{n} \\
& t_{n}=s_{n}-\gamma P_{V} B P_{V} s_{n} \\
& z_{n+1}=z_{n}+\lambda_{n}\left(t_{n}-r_{n}\right) .
\end{align*}\right.
$$

Then, by setting, for every $n \in \mathbb{N}$, $x_{n}=P_{V} z_{n}$ and $y_{n}=P_{V^{\perp}} z_{n} / \gamma$, we have $x_{n} \rightharpoonup \bar{x}$ and $y_{n} \rightharpoonup \bar{y}$ for some $\bar{x} \in \operatorname{zer}\left(A+B+N_{V}\right)$ and $\bar{y} \in V^{\perp} \cap\left(A \bar{x}+P_{V} B \bar{x}\right), x_{n+1}-x_{n} \rightarrow 0$, and $y_{n+1}-y_{n} \rightarrow 0$.

Proof. Indeed, it follows from the proof of Theorem 3.4 that (3.8) is equivalent to (3.12), where, for every $n \in \mathbb{N}, z_{n}=x_{n}+\gamma y_{n}$. In the particular case when $\left.\delta_{n} \equiv 1 \in\right] 0,1 /(\gamma \chi)[$, it follows from Proposition 3.2(i) that (3.12) reduces to (3.13). Hence, the results follow from Theorem 3.4.

## Remark 3.7

(i) Note that, when $V=\mathcal{H}$ and $\lambda_{n} \equiv 1$, we have $V^{\perp}=\{0\}, P_{V}=\mathrm{Id}$, $\left(\operatorname{Id}+R_{N_{V}} R_{\gamma A}\right) / 2=J_{\gamma A}$, and, therefore, (3.13) reduces to

$$
(\forall n \in \mathbb{N}) \quad\left[\begin{array}{l}
r_{n}=x_{n}-\gamma B x_{n}  \tag{3.14}\\
s_{n}=J_{\gamma A} r_{n} \\
t_{n}=s_{n}-\gamma B s_{n} \\
x_{n+1}=x_{n}+t_{n}-r_{n},
\end{array}\right.
$$

which is a version with constant step size of the modified forward-backward splitting [40] for finding a zero of $A+B$.
(ii) On the other hand, when $B \equiv 0$, (3.13) reduces to

$$
(\forall n \in \mathbb{N}) \quad\left[\begin{array}{l}
s_{n}=\left(z_{n}+R_{N_{V}} R_{\gamma A} z_{n}\right) / 2  \tag{3.15}\\
z_{n+1}=z_{n}+\lambda_{n}\left(s_{n}-z_{n}\right),
\end{array}\right.
$$

which is the Douglas-Rachford splitting method $[25,37]$ for finding $x \in \mathcal{H}$ such that $x \in N_{V} x+$ $A x$. It coincides with Spingarn's partial inverse method with constant step size.

## 4 Applications

In this section we study three applications of our algorithm. We first apply Theorem 3.4 to the problem of finding a zero of the sum of $m$ maximally monotone operators and a monotone lipschitzian operator. Secondly, we study a primal-dual composite monotone inclusion involving normal cones and we obtain from Theorem 3.4 a primal-dual method for solving this problem. Finally, we study the application of our method in the framework of continuous zero-sum games. Connections with other methods in each framework are also provided.

### 4.1 Inclusion Involving the Sum of $m$ Monotone Operators

Let us consider the following problem.
Problem 4.1 Let $(\mathrm{H},|\cdot|)$ be a real Hilbert space, for every $i \in\{1, \ldots, m\}$, let $\mathrm{A}_{i}: \mathrm{H} \rightarrow 2^{\mathrm{H}}$ be a maximally monotone operator, and let $\mathrm{B}: \mathrm{H} \rightarrow \mathrm{H}$ be a monotone and $\chi$-lipschitzian operator. The problem is to

$$
\begin{equation*}
\text { find } \quad \mathrm{x} \in \mathrm{H} \text { such that } 0 \in \sum_{i=1}^{m} \mathrm{~A}_{i} \mathrm{x}+\mathrm{Bx}, \tag{4.1}
\end{equation*}
$$

under the assumption that solutions exist.
Problem 4.1 has several applications in image processing, principally in the variational setting (see, e.g., $[17,30]$ and the references therein), variational inequalities [38, 39], partial differential equations [27, 43], and economics [24, 29], among others. In [30, 41], Problem 4.1 is solved by a fully split algorithm in the particular case when B is cocoercive. Nevertheless, this approach does not seem to work in the general case. In [18] a method for solving a more general problem than Problem 4.1 is proposed. However, this approach stores and updates at each iteration $m$ dual variables in order to solve (4.1) and its dual simultaneously. This generality does not allow to exploit the intrinsic properties of Problem 4.1, which may be unfavourable in large scale systems. Our method is obtained as a consequence of Theorem 3.4 for a suitable closed vectorial subspace and exploits the whole structure of the problem.

Let us first provide a connection between Problem 4.1 and Problem 3.1 via product space techniques. Let $\left(\omega_{i}\right)_{1 \leq i \leq m}$ be real numbers in $] 0,1\left[\right.$ such that $\sum_{i=1}^{m} \omega_{i}=1$, let $\mathcal{H}$ be the real Hilbert space obtained by endowing the Cartesian product $\mathrm{H}^{m}$ with the scalar product and associated norm respectively defined by

$$
\begin{equation*}
\langle\cdot \mid \cdot\rangle:(x, y) \mapsto \sum_{i=1}^{m} \omega_{i}\left\langle\mathrm{x}_{i} \mid \mathrm{y}_{i}\right\rangle \quad \text { and } \quad\|\cdot\|: x \mapsto \sqrt{\sum_{i=1}^{m} \omega_{i}\left|\mathrm{x}_{i}\right|^{2}} \tag{4.2}
\end{equation*}
$$

where $x=\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m}$ is a generic element of $\mathcal{H}$.
Proposition 4.2 Let $\mathrm{H},\left(\mathrm{A}_{i}\right)_{1 \leq i \leq m}$, and B be as in Problem 4.1, and define

$$
\left\{\begin{array}{l}
V=\left\{x=\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m} \in \mathcal{H} \mid \mathrm{x}_{1}=\cdots=\mathrm{x}_{m}\right\}  \tag{4.3}\\
j: \mathrm{H} \rightarrow V \subset \mathcal{H}: \mathrm{x} \mapsto(\mathrm{x}, \ldots, \mathrm{x}) \\
A: \mathcal{H} \rightarrow 2^{\mathcal{H}}: x \mapsto \frac{1}{\omega_{1}} \mathrm{~A}_{1} \mathrm{x}_{1} \times \cdots \times \frac{1}{\omega_{m}} \mathrm{~A}_{m} \mathrm{x}_{m} \\
B: \mathcal{H} \rightarrow \mathcal{H}: x \mapsto\left(\mathrm{Bx}_{1}, \ldots, \mathrm{~B} \mathrm{x}_{m}\right) .
\end{array}\right.
$$

Then the following hold.
(i) $V$ is a closed vector subspace of $\mathcal{H}, P_{V}:\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m} \mapsto j\left(\sum_{i=1}^{m} \omega_{i} \mathrm{x}_{i}\right)$, and

$$
N_{V}: x \mapsto \begin{cases}V^{\perp}=\left\{x=\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m} \in \mathcal{H} \mid \sum_{i=1}^{m} \omega_{i} \mathrm{x}_{i}=0\right\}, & \text { if } x \in V  \tag{4.4}\\ \varnothing, & \text { otherwise } .\end{cases}
$$

(ii) $j: \mathrm{H} \rightarrow V$ is a bijective isometry and $j^{-1}:(\mathrm{x}, \ldots, \mathrm{x}) \mapsto \mathrm{x}$.
(iii) $A$ is a maximally monotone operator and, for every $\gamma \in] 0,+\infty\left[, J_{\gamma A}:\left(\mathrm{x}_{i}\right)_{1 \leq i \leq m} \mapsto\left(J_{\gamma \mathrm{A}_{i} / \omega_{i}} \mathrm{x}_{i}\right)\right.$.
(iv) $B$ is monotone and $\chi$-lipschitzian, $B(j(\mathrm{x}))=j(\mathrm{Bx})$, and $B(V) \subset V$.
(v) For every $\mathrm{x} \in \mathrm{H}, \mathrm{x}$ is a solution to Problem 4.1 if and only if $j(\mathrm{x}) \in \operatorname{zer}\left(A+B+N_{V}\right)$.

Proof. (i)\&(ii): They follow from (2.1) and easy computations. (iii): See [7, Proposition 23.16]. (iv): They follow from straightforward computations by using (4.3), (4.2), and the properties on B. (v): Let $x \in H$. We have

$$
\begin{align*}
0 \in \sum_{i=1}^{m} \mathrm{~A}_{i} \mathrm{x}+\mathrm{Bx} & \Leftrightarrow\left(\exists\left(\mathrm{y}_{i}\right)_{1 \leq i \leq m} \in{\left.\underset{i=1}{\times} \mathrm{A}_{i} \mathrm{x}\right) \quad 0=\sum_{i=1}^{m} \mathrm{y}_{i}+\mathrm{Bx}}\right. \\
& \Leftrightarrow\left(\exists\left(\mathrm{y}_{i}\right)_{1 \leq i \leq m} \in{\underset{i=1}{X}}^{m} \mathrm{~A}_{i} \mathrm{x}\right) \quad 0=\sum_{i=1}^{m} \omega_{i}\left(-\mathrm{y}_{i} / \omega_{i}-\mathrm{Bx}\right) \\
& \Leftrightarrow\left(\exists\left(\mathrm{y}_{i}\right)_{1 \leq i \leq m} \in{\left.\underset{i=1}{\times} \mathrm{A}_{i} \mathrm{x}\right)-\left(\mathrm{y}_{1} / \omega_{1}, \ldots, \mathrm{y}_{m} / \omega_{m}\right)-j(\mathrm{Bx}) \in V^{\perp}}\right. \\
& \Leftrightarrow 0 \in A(j(\mathrm{x}))+B(j(\mathrm{x}))+N_{V}(j(\mathrm{x})) \\
& \Leftrightarrow j(\mathrm{x}) \in \operatorname{zer}\left(A+B+N_{V}\right), \tag{4.5}
\end{align*}
$$

which yields the result.
The following result provides a method for solving Problem 4.1. It is a direct consequence of Corollary 3.6 applied to the equivalent monotone inclusion in Proposition 4.2(v).

Theorem 4.3 Let $\mathrm{H},\left(\mathrm{A}_{i}\right)_{1 \leq i \leq m}$, and B be as in Problem 4.1, let $\left.\gamma \in\right] 0,1 / \chi[$, let $\varepsilon \in] 0,1[$, and let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Moreover, let $\left(\mathrm{z}_{i, 0}\right)_{1 \leq i \leq m} \in \mathrm{H}^{m}$ and iterate, for every $n \in \mathbb{N}$,

$$
\begin{align*}
& \mathrm{x}_{n}=\sum_{j=1}^{m} \omega_{j} \mathbf{z}_{j, n} \\
& \text { For } i=1, \ldots, m \\
& \left\lvert\, \begin{array}{l}
\mathbf{r}_{i, n}=\mathbf{z}_{i, n}-\gamma \mathbf{B x}_{n} \\
\mathbf{p}_{i, n}=J_{\gamma \mathbf{A}_{i} / \omega_{i}} \mathbf{r}_{i, n}
\end{array}\right. \\
& \mathbf{q}_{n}=\sum_{j=1}^{m} \omega_{j} \mathbf{p}_{j, n}  \tag{4.6}\\
& \text { For } i=1, \ldots, m \\
& \left\lvert\, \begin{array}{l}
\mathbf{s}_{i, n}=2 \mathbf{q}_{n}-\mathbf{p}_{i, n}+\mathbf{z}_{i, n}-\mathbf{x}_{n} \\
\mathbf{t}_{i, n}=\mathbf{s}_{i, n}-\gamma \mathbf{B q}_{n} \\
\mathbf{z}_{i, n+1}=\mathbf{z}_{i, n}+\lambda_{n}\left(\mathbf{t}_{i, n}-\mathbf{r}_{i, n}\right) .
\end{array}\right.
\end{align*}
$$

Then, $\mathrm{x}_{n} \rightharpoonup \overline{\mathrm{x}}$ for some solution $\overline{\mathrm{x}}$ to Problem 4.1 and $\mathrm{x}_{n+1}-\mathrm{x}_{n} \rightarrow 0$.
Proof. Set, for every $n \in \mathbb{N}, x_{n}=j\left(\mathrm{x}_{n}\right), q_{n}=j\left(\mathrm{q}_{n}\right), s_{n}=\left(\mathrm{s}_{i, n}\right)_{1 \leq i \leq m}, z_{n}=\left(\mathrm{z}_{i, n}\right)_{1 \leq i \leq m}$, and $p_{n}=\left(\mathrm{p}_{i, n}\right)_{1 \leq i \leq m}$. It follows from Proposition 4.2(i) and (4.6) that, for every $n \in \mathbb{N}, x_{n}=P_{V} z_{n}$ and $q_{n}=P_{V} p_{n}=P_{V} s_{n}$. Hence, it follows from (4.3) and Proposition 4.2 that (4.6) can be written equivalently as (3.13). Altogether, Corollary 3.6 and Proposition $4.2(\mathrm{v})$ yield the results.

Remark 4.4 In the particular case when $m=2, B=0$, and $\omega_{1}=\omega_{2}=1 / 2$, the method proposed in Theorem 4.3 reduces to

$$
(\forall n \in \mathbb{N}) \quad \left\lvert\, \begin{align*}
& \mathrm{x}_{n}=\left(\mathrm{z}_{1, n}+\mathrm{z}_{2, n}\right) / 2  \tag{4.7}\\
& \mathrm{p}_{1, n}=J_{2 \gamma \mathrm{~A}_{1}}\left(\mathrm{z}_{1, n}\right) \\
& \mathrm{p}_{2, n}=J_{2 \gamma \mathrm{~A}_{2}}\left(\mathrm{z}_{2, n}\right) \\
& \mathrm{z}_{1, n+1}=\mathbf{z}_{1, n}+\lambda_{n}\left(\mathrm{p}_{2, n}-\mathrm{x}_{n}\right) \\
& \mathrm{z}_{2, n+1}=\mathbf{z}_{2, n}+\lambda_{n}\left(\mathrm{p}_{1, n}-\mathrm{x}_{n}\right),
\end{align*}\right.
$$

which is exactly the method proposed in [12, Remark 6.2(ii)] for finding a zero of the sum of two maximally monotone operators $A_{1}$ and $A_{2}$. In the case when these resolvents are hard to calculate, (4.7) provides an alternative method which computes them in parallel.

### 4.2 Primal-Dual Monotone Inclusions

This section is devoted to the numerical resolution of a very general composite primal-dual monotone inclusion involving vectorial subspaces. A difference of the method in Section 4.1, the algorithm proposed in this section deals with monotone operators composed with linear transformations and solves simultaneously primal and dual inclusions.

Let us introduce a partial sum of two set-valued operators with respect a closed vectorial subspace. This notion is a generalization of the parallel sum (see, e.g., [10] and the references therein).

Definition 4.5 Let $\mathcal{H}$ be a real Hilbert space, let $U \subset \mathcal{H}$ be a closed vectorial subspace, and let $A: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ and $B: \mathcal{H} \rightarrow 2^{\mathcal{H}}$ be two non linear operators. The partial sum of $A$ and $B$ with respect to $U$ is defined by

$$
\begin{equation*}
A \square_{U} B=\left(A_{U}+B_{U}\right)_{U} . \tag{4.8}
\end{equation*}
$$

In particular, we have $A \square_{\mathcal{H}} B=A+B$ and $A \square_{\{0\}} B=A \square B=\left(A^{-1}+B^{-1}\right)^{-1}$.

Note that, since the operation $A \mapsto A_{U}$ preserves monotonicity [35], if $A$ and $B$ are monotone then $A \square_{U} B$ is monotone as well. In this section we are interested in the following problem.

Problem 4.6 Let $\mathrm{H},\left(\mathrm{G}_{i}\right)_{1 \leq i \leq m}$ be real Hilbert spaces, for every $i \in\{1, \ldots, m\}$, let $\mathrm{U} \subset \mathrm{H}$ and $\mathrm{V}_{i} \subset \mathrm{G}_{i}$ be closed vectorial spaces, let $\mathrm{A}: \mathrm{H} \rightarrow 2^{\mathrm{H}}$ and $\mathrm{B}_{i}: \mathrm{G}_{i} \rightarrow 2^{\mathrm{G}_{i}}$ be maximally monotone, let $\mathrm{L}_{i}: \mathrm{H} \rightarrow \mathrm{G}_{i}$ be linear and bounded, let $\mathrm{D}_{i}: \mathrm{G}_{i} \rightarrow 2^{\mathrm{G}_{i}}$ be monotone such that $\left(\mathrm{D}_{i}\right)_{\mathrm{V}_{i}}$ is $\nu_{i}$-lipschitzian for some $\left.\nu_{i} \in\right] 0,+\infty[$, let $\mathrm{C}: \mathrm{H} \rightarrow \mathrm{H}$ be monotone and $\mu$-lipschitzian for some $\mu \in] 0,+\infty[$, let $\mathbf{z} \in \mathrm{H}$, and let $\mathrm{b}_{i} \in \mathrm{G}_{i}$. The problem is to solve the primal inclusion

$$
\begin{equation*}
\text { find } \mathrm{x} \in \mathrm{H} \quad \text { such that } \quad \mathrm{z} \in \mathrm{Ax}+N_{\mathrm{U}} \mathrm{x}+\sum_{i=1}^{m}\left(\mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}}\left(\mathrm{~B}_{i} \square_{\mathrm{V}_{i}^{\perp}} \mathrm{D}_{i}+N_{\mathrm{V}_{i}}\right) P_{\mathrm{V}_{i}}\left(\mathrm{~L}_{i} \mathrm{x}-\mathrm{b}_{i}\right)\right)+\mathrm{C} \mathrm{x} \tag{4.9}
\end{equation*}
$$

together with the dual inclusion
find $\mathrm{u}_{1} \in \mathrm{G}_{1}, \ldots, \mathrm{u}_{m} \in \mathrm{G}_{m}$ such that

$$
(\exists \mathrm{x} \in \mathrm{H})\left\{\begin{array}{l}
\mathrm{z}-\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i} \in \mathrm{Ax}+\mathrm{Cx}+N_{\mathrm{Ux}}  \tag{4.10}\\
(\forall i \in\{1, \ldots, m\}) \mathrm{u}_{i} \in P_{\mathrm{V}_{i}}\left(\mathrm{~B}_{i} \square_{\mathrm{V}_{i}} \mathrm{D}_{i}+N_{\mathrm{V}_{i}}\right) P_{\mathrm{V}_{i}}\left(\mathrm{~L}_{i} \mathrm{x}-\mathrm{b}_{i}\right) .
\end{array}\right.
$$

The set of solutions to (4.9) is denoted by $\mathcal{P}$ and the set of solutions to (4.10) by $\mathcal{D}$, which are assumed to be nonempty.

In the particular case when $\mathrm{U}=\mathrm{H}$ and, for every $i \in\{1, \ldots, m\}, \mathrm{V}_{i}=\mathrm{G}_{i}$, Problem 4.6 reduces to the problem solved in [18], where a convergent primal-dual algorithm activating separately each involved operator is proposed. In the case when, for every $i \in\{1, \ldots, m\}, \mathrm{V}_{i}=\mathrm{G}_{i}$, Problem 4.6 reduces to the problem addressed in [9], where a splitting method with ergodic convergence is provided. A disadvantage of this algorithm is the presence of vanishing parameters which may lead to numerical instabilities together with additionally conditions difficult to be verified in general. At the best of our knowledge, the general case has not been tackled in the literature via splitting methods.

Problem 4.6 requires a lipschitzian condition on $\left(\mathrm{D}_{i \mathrm{~V}_{i}^{\perp}}\right)_{1 \leq i \leq m}$. In the simplest case when, for every $i \in\{1, \ldots, m\}, \mathrm{V}_{i}=\mathrm{G}_{i}$, this condition reduces to the lipschitzian property on $\mathrm{D}_{i}{ }^{-1}$, which is trivially satisfied, e.g., when $\mathrm{D}_{i} 0=\mathrm{G}_{i}$ and, for every $y \neq 0, \mathrm{D}_{i} y=\varnothing$. The following proposition furnishes other non-trivial instances in which the partial inverse of a monotone operator with respect to a closed vectorial subspace is lipschitzian.

Proposition 4.7 Let V be a closed vectorial subspace of a real Hilbert space H and suppose that one of the following holds.
(i) $\mathrm{D}: \mathrm{H} \rightarrow \mathrm{H}$ is $\beta$-strongly monotone and $\nu$-cocoercive.
(ii) $\mathrm{D}=\nabla \mathrm{f}$, where $\mathrm{f}: \mathrm{H} \rightarrow]-\infty,+\infty]$ is differentiable, $\beta$-strongly convex, and $\nabla \mathrm{f}$ is $\nu^{-1}$-lipschitzian.
(iii) D is linear bounded operator satisfying, for every $\mathrm{x} \in \mathrm{H},\langle\mathrm{x} \mid \mathrm{Dx}\rangle \geq \beta\|\mathrm{x}\|^{2}$, and $\nu=\beta /\|\mathrm{D}\|^{2}$.

Then $\mathrm{D}_{\mathrm{V}}$ is $\alpha$-cocoercive and $\alpha$-strongly monotone with $\alpha=\min \{\beta, \nu\} / 2$. In particular, $\mathrm{D}_{\mathrm{V}}$ is $\alpha^{-1}$ lipschitzian.

Proof. (i): Let ( $\mathrm{x}, \mathrm{u}$ ) and ( $\mathrm{y}, \mathrm{v}$ ) in gra $(\mathrm{D} \mathrm{V})$. Then it follows from (2.2) that $\left(P_{\mathrm{V}}+P_{\mathrm{V} \perp \mathrm{u}}, P_{\mathrm{V}} \mathrm{u}+P_{\mathrm{V} \perp \mathrm{x}}\right)$ and $\left(P_{\vee} \mathrm{y}+P_{\mathrm{V}} \perp \mathrm{V}, P_{\mathrm{V}}+P_{\mathrm{V}} \perp \mathrm{y}\right)$ are in $\operatorname{gra}(\mathrm{D})$, and, from the strong monotonicity assumption on D , we have

$$
\begin{align*}
\langle\mathrm{x}-\mathrm{y} \mid \mathrm{u}-\mathrm{v}\rangle & =\left\langle P_{\mathrm{V}}(\mathrm{x}-\mathrm{y}) \mid P_{\mathrm{V}}(\mathrm{u}-\mathrm{v})\right\rangle+\left\langle P_{\mathrm{V}^{\perp}}(\mathrm{u}-\mathrm{v}) \mid P_{\mathrm{V}^{\perp}}(\mathrm{x}-\mathrm{y})\right\rangle \\
& =\left\langle P_{\mathrm{V}}+P_{\mathrm{V}^{\perp}} \mathrm{u}-\left(P_{\mathrm{V}} \mathrm{y}+P_{\mathrm{V} \perp \mathrm{v}}\right) \mid P_{\mathrm{V}} \mathrm{u}+P_{\mathrm{V}^{\perp}} \mathrm{x}-\left(P_{\mathrm{V} \mathrm{v}}+P_{\mathrm{V} \perp \mathrm{y}}\right)\right\rangle \\
& \geq \beta\left\|P_{\mathrm{V}}+P_{\mathrm{V}^{\perp}} \mathrm{u}-\left(P_{\mathrm{V} \mathrm{y}}+P_{\mathrm{V} \perp \mathrm{v}}\right)\right\|^{2} \\
& =\beta\left(\left\|P_{\mathrm{V}}(\mathrm{x}-\mathrm{y})\right\|^{2}+\left\|P_{\mathrm{V}^{\perp}}(\mathrm{u}-\mathrm{v})\right\|^{2}\right) . \tag{4.11}
\end{align*}
$$

Analogously, the cocoercivity assumption on D yields $\langle\mathrm{x}-\mathrm{y} \mid \mathrm{u}-\mathrm{v}\rangle \geq \nu\left(\left\|P_{\mathrm{V}}(\mathrm{u}-\mathrm{v})\right\|^{2}+\left\|P_{\mathrm{V} \perp}(\mathrm{x}-\mathrm{y})\right\|^{2}\right)$. Hence, it follows from (4.11) that

$$
\begin{equation*}
\langle\mathrm{x}-\mathrm{y} \mid \mathrm{u}-\mathrm{v}\rangle \geq \frac{\beta}{2}\left(\left\|P_{\mathrm{V}}(\mathrm{x}-\mathrm{y})\right\|^{2}+\left\|P_{\mathrm{V}^{\perp}}(\mathrm{u}-\mathrm{v})\right\|^{2}\right)+\frac{\nu}{2}\left(\left\|P_{\mathrm{V}}(\mathrm{u}-\mathrm{v})\right\|^{2}+\left\|P_{\mathrm{V} \perp}(\mathrm{x}-\mathrm{y})\right\|^{2}\right) \tag{4.12}
\end{equation*}
$$

which yields $\langle x-y \mid u-v\rangle \geq \alpha\left(\|x-y\|^{2}+\|u-v\|^{2}\right)$ and the result follows. (ii): From the strong convexity of f we have that $\mathrm{D}=\nabla \mathrm{f}$ is $\beta$-strongly monotone and it follows from $[6]$ that D is $\nu$-cocoercive. Hence, the result follows from (i). (iii): Since $D$ is linear and bounded we have $\|x\|^{2} \geq\|D x\|^{2} /\|D\|^{2}$. Then D is $\beta$-strongly monotone and $\nu$-cocoercive and the result follows from (i).

The following proposition gives a connection between Problem 4.6 and Problem 3.1.
Proposition 4.8 In the real Hilbert space $\mathcal{H}=\mathrm{H} \oplus \mathrm{G}_{1} \oplus \cdots \oplus \mathrm{G}_{m}$ set

$$
\left\{\begin{array}{l}
A: \mathcal{H} \rightarrow 2^{\mathcal{H}}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto(-\mathrm{z}+\mathrm{Ax}) \times\left(P_{\mathrm{V}_{1}} \mathrm{~b}_{1}+\left(\mathrm{B}_{1}\right)_{\mathrm{V}_{1}^{\perp}} \mathrm{u}_{1}\right) \times \cdots \times\left(P_{\mathrm{V}_{m}} \mathrm{~b}_{m}+\left(\mathrm{B}_{m}\right)_{\mathrm{V}_{m}^{\perp}} \mathrm{u}_{m}\right) \\
L: \mathcal{H} \rightarrow \mathcal{H}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto\left(\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i},-P_{\mathrm{V}_{1}} \mathrm{~L}_{1} \mathrm{x}, \ldots,-P_{\mathrm{V}_{m}} \mathrm{~L}_{m} \mathrm{x}\right) \\
C: \mathcal{H} \rightarrow \mathcal{H}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto\left(\mathrm{Cx},\left(\mathrm{D}_{1}\right)_{\mathrm{V}_{1}^{\perp}} \mathrm{u}_{1}, \ldots,\left(\mathrm{D}_{m}\right)_{\mathrm{V}_{m}^{\perp}} \mathrm{u}_{m}\right) \\
B: \mathcal{H} \rightarrow \mathcal{H}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto(C+L)\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \\
W=\mathrm{U} \times \mathrm{V}_{1} \times \cdots \times \mathrm{V}_{m} \\
\chi=\max \left\{\mu, \nu_{1}, \ldots, \nu_{m}\right\}+\sqrt{\sum_{i=1}^{m}\left\|\mathrm{~L}_{i}\right\|^{2}} \tag{4.13}
\end{array}\right.
$$

Then the following hold.
(i) $A$ is maximally monotone and, for every $\gamma \in] 0,+\infty[$,

$$
\begin{equation*}
J_{\gamma A}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto\left(J_{\gamma \mathrm{A}}(\mathrm{x}+\mathrm{z}), J_{\gamma\left(\mathrm{B}_{1}\right)_{\mathrm{v}_{1}^{1}}}\left(\mathrm{u}_{1}-P_{\mathrm{V}_{1}} \mathrm{~b}_{1}\right), \ldots, J_{\gamma\left(\mathrm{B}_{m}\right)_{\mathrm{v}_{m}^{\prime}}}\left(\mathrm{u}_{m}-P_{\mathrm{V}_{m}} \mathrm{~b}_{m}\right)\right) \tag{4.14}
\end{equation*}
$$

(ii) $L$ is a linear bounded operator, $L^{*}=-L$, and $\|L\| \leq \sqrt{\sum_{i=1}^{m}\left\|\mathrm{~L}_{i}\right\|^{2}}$.
(iii) $B$ is monotone and $\chi$-lipschitzian.
(iv) $W$ is a closed vectorial subspace of $\mathcal{H}, N_{W}:\left(\times, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto N_{\mathrm{U}} \times \times N_{\mathrm{V}_{1}} \mathrm{u}_{1} \times \cdots \times N_{\mathrm{V}_{m}} \mathbf{u}_{m}$, and $P_{W}:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto\left(P_{\mathrm{U}}, P_{\mathrm{V}_{1}} \mathrm{u}_{1}, \ldots, P_{\mathrm{V}_{m}} \mathrm{u}_{m}\right)$.
(v) $\operatorname{zer}\left(A+B+N_{W}\right) \subset \mathcal{P} \times \mathcal{D}$.
(vi) $\mathcal{P} \neq \varnothing \Leftrightarrow \operatorname{zer}\left(A+B+N_{W}\right) \neq \varnothing \Leftrightarrow \mathcal{D} \neq \varnothing$.

Proof. (i): Since, for every $i \in\{1, \ldots, m\},\left(\mathrm{B}_{i}\right)_{\mathrm{V}_{i}^{\perp}}$ is maximally monotone, the result follows from [7, Proposition 23.15 and Proposition 23.16]. (ii): Let us define $M: \mathrm{G}_{1} \oplus \cdots \oplus \mathrm{G}_{m} \rightarrow \mathrm{H}$ by $\mathrm{M}:\left(\mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto \sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i}$. Since $\left(\mathrm{L}_{i}\right)_{1 \leq i \leq m}$ and $\left(P_{\mathrm{V}_{i}}\right)_{1 \leq i \leq m}$ are linear bounded operators, it is easy to check that M is linear and bounded, $\mathrm{M}^{*}: \mathrm{x} \mapsto\left(P_{\mathrm{V}_{1}} \mathrm{~L}_{1} \mathrm{x}, \ldots, P_{\mathrm{V}_{m}} \mathrm{~L}_{m} \mathrm{x}\right)$, and that we can rewrite $L$ as $L:\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \mapsto\left(\mathrm{M}\left(\mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right),-\mathrm{M}^{*} \mathrm{x}\right)$. Hence, we deduce from [13, Proposition 2.7 (ii)] that $L$ is linear and bounded, that $L^{*}=-L$, and that $\|L\|=\|M\|$. Now, for every $\left(\mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \in \mathrm{G}_{1} \oplus \cdots \oplus \mathrm{G}_{m}$, we have from triangle and Hölder inequalities $\left\|M\left(\mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right)\right\| \leq$ $\sum_{i=1}^{m}\left\|\mathrm{~L}_{i}\right\|\left\|P_{\mathrm{V}_{i}}\right\|\left\|\mathrm{u}_{i}\right\| \leq \sum_{i=1}^{m}\left\|\mathrm{~L}_{i}\right\|\left\|\mathrm{u}_{i}\right\| \leq \sqrt{\sum_{i=1}^{m}\left\|\mathrm{~L}_{i}\right\|^{2}} \sqrt{\sum_{i=1}^{m}\left\|\mathrm{u}_{i}\right\|^{2}}$, which yields the last assertion.
(iii): It follows from (ii) that $L$ is linear, bounded, and skew. Therefore, it is monotone and $\|L\|-$ lipschitzian. On the other hand, since C and $\left(\mathrm{D}_{i}\right)_{\mathrm{V}_{\dot{i}}}$ are monotone and lipschitzian, $C$ is monotone and $\max \left\{\mu, \nu_{1}, \ldots, \nu_{m}\right\}$-lipschitzian. Altogether, it follows from (ii) that $B=C+L$ is monotone and $\chi$-lipschitzian. (iv): Clear. (v): Let $\left(x, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \in \mathrm{H} \times \mathrm{G}_{1} \times \cdots \mathrm{G}_{m}$. We have from (4.13) and Proposition 2.1(ii) that

$$
\begin{align*}
& \left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \in \operatorname{zer}\left(A+B+N_{W}\right) \\
& \Leftrightarrow\left\{\begin{array}{l}
0 \in-\mathrm{z}+\mathrm{Ax}+\mathrm{Cx}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i}+N_{\mathrm{UX}} \\
0 \in P_{\mathrm{V}_{1}} \mathrm{~b}_{1}+\left(\mathrm{B}_{1}\right)_{\mathrm{V}_{1}} \mathrm{u}_{1}+\left(\mathrm{D}_{1}\right)_{\mathrm{V}_{\frac{1}{1}}} \mathrm{u}_{1}-P_{\mathrm{V}_{1}} \mathrm{~L}_{1} \mathrm{x}+N_{\mathrm{V}_{1}} \mathrm{u}_{1} \\
\vdots \\
0 \in P_{\mathrm{V}_{m}} \mathrm{~b}_{m}+\left(\mathrm{B}_{m}\right)_{\mathrm{V}_{\frac{1}{m}} \mathrm{u}_{m}+\left(\mathrm{D}_{m}\right)_{\mathrm{V}_{\frac{1}{m}}^{\prime}} \mathrm{u}_{m}-P_{\mathrm{V}_{m}} \mathrm{~L}_{m} \mathrm{x}+N_{\mathrm{V}_{m}} \mathrm{u}_{m}}
\end{array}\right. \\
& \Leftrightarrow\left\{\begin{array}{l}
0 \in-\mathrm{z}+\mathrm{Ax}+\mathrm{Cx}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i}+N_{\mathrm{U}} \mathrm{x} \\
P_{\mathrm{V}_{1}}\left(\mathrm{~L}_{1} \mathrm{x}-\mathrm{b}_{1}\right) \in\left(\left(\mathrm{B}_{1}\right)_{\mathrm{V}_{1}^{\perp}}+\left(\mathrm{D}_{1}\right)_{\mathrm{V}_{1}^{\perp}}+N_{\mathrm{V}_{1}}\right) \mathrm{u}_{1}, \mathrm{u}_{1} \in \mathrm{~V}_{1} \\
\vdots \\
P_{\mathrm{V}_{m}}\left(\mathrm{~L}_{m} \mathrm{x}-\mathrm{b}_{m}\right) \in\left(\left(\mathrm{B}_{m}\right)_{\mathrm{V}_{ \pm}}+\left(\mathrm{D}_{m}\right)_{\mathrm{V}_{ \pm}^{\prime}}+N_{\mathrm{V}_{m}}\right) \mathrm{u}_{m}, \mathrm{u}_{m} \in \mathrm{~V}_{m}
\end{array}\right. \\
& \Leftrightarrow\left\{\begin{array}{l}
0 \in-\mathrm{z}+\mathrm{Ax}+\mathrm{Cx}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i}+N_{\mathrm{U}} \\
\mathrm{u}_{1} \in P_{\mathrm{V}_{1}}\left(\left(\mathrm{~B}_{1}\right)_{\mathrm{V}_{1}^{\perp}}+\left(\mathrm{D}_{1}\right)_{\mathrm{V}_{\frac{1}{1}}^{\perp}}+N_{\mathrm{V}_{1}}\right)^{-1} P_{\mathrm{V}_{1}}\left(\mathrm{~L}_{1} \mathrm{x}-\mathrm{b}_{1}\right) \\
\quad \vdots \\
\mathrm{u}_{m} \in P_{\mathrm{V}_{m}}\left(\left(\mathrm{~B}_{m}\right)_{\mathrm{V}_{\frac{1}{m}}}+\left(\mathrm{D}_{m}\right)_{\mathrm{V}_{\frac{1}{m}}}+N_{\mathrm{V}_{m}}\right)^{-1} P_{\mathrm{V}_{m}}\left(\mathrm{~L}_{m} \mathrm{x}-\mathrm{b}_{m}\right)
\end{array}\right. \\
& \Leftrightarrow\left\{\begin{array}{l}
\mathrm{z}-\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{u}_{i} \in \mathrm{Ax}+\mathrm{Cx}+N_{\mathrm{Ux}} \\
\mathrm{u}_{1} \in P_{\mathrm{V}_{1}}\left(\mathrm{~B}_{1} \square_{\mathrm{V}_{1}^{\prime}} \mathrm{D}_{1}+N_{\mathrm{V}_{1}}\right) P_{\mathrm{V}_{1}}\left(\mathrm{~L}_{1} \mathrm{x}-\mathrm{b}_{1}\right) \\
\quad \vdots \\
\mathrm{u}_{m} \in P_{\mathrm{V}_{m}}\left(\mathrm{~B}_{m} \square_{\mathrm{V}_{m}^{\prime}} \mathrm{D}_{m}+N_{\mathrm{V}_{m}}\right) P_{\mathrm{V}_{m}}\left(\mathrm{~L}_{m} \mathrm{x}-\mathrm{b}_{m}\right)
\end{array}\right.  \tag{4.15}\\
& \Rightarrow \mathrm{z} \in \mathrm{Ax}+N_{\mathrm{U}}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}}\left(\mathrm{~B}_{i} \square_{\mathrm{V}_{i}^{\perp}} \mathrm{D}_{i}+N_{\mathrm{v}_{i}}\right) P_{\mathrm{V}_{i}}\left(\mathrm{~L}_{i} \mathrm{x}-\mathrm{b}_{i}\right)+\mathrm{Cx}, \tag{4.16}
\end{align*}
$$

which yields $\mathrm{x} \in \mathcal{P}$. Moreover, (4.15) yields $\left(\mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \in \mathcal{D}$.
(vi): We will prove $\mathcal{P} \neq \varnothing \Rightarrow \mathcal{D} \neq \varnothing \Rightarrow \operatorname{zer}\left(A+B+N_{W}\right) \neq \varnothing \Rightarrow \mathcal{P} \neq \varnothing$. If $\mathrm{x} \in \mathcal{P}$, there exist $\left(u_{1}, \ldots, u_{m}\right)$ such that (4.15) holds and, hence, $\left(u_{1}, \ldots, u_{m}\right) \in \mathcal{D}$. Now, if $\left(u_{1}, \ldots, u_{m}\right) \in \mathcal{D}$, there exists $\mathrm{x} \in \mathrm{H}$ such that (4.15) holds and we deduce from the equivalences in (4.15) that $\left(\mathrm{x}, \mathrm{u}_{1}, \ldots, \mathrm{u}_{m}\right) \in$ $\operatorname{zer}\left(A+B+N_{W}\right)$. The last implication follows from (v).

Theorem 4.9 In the setting of Problem 4.6, let $\gamma \in] 0,1 / \chi[$ where $\chi$ is defined in (4.13), and let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Moreover, let $\mathrm{x}_{0} \in \mathrm{H}$, let $\left(\mathrm{u}_{i, 0}\right)_{1 \leq i \leq m} \in \mathrm{G}_{1} \times \cdots \times \mathrm{G}_{m}$, and iterate, for every $n \in \mathbb{N}$,

$$
\begin{align*}
& \mathrm{r}_{1, n}=\mathrm{x}_{n}-\gamma P_{\mathrm{U}}\left(C P_{\mathrm{U}} \mathrm{x}_{n}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathbf{u}_{i, n}\right) \\
& \mathrm{p}_{1, n}=J_{\gamma \mathrm{A}}\left(\mathrm{r}_{1, n}+\gamma \mathrm{z}\right) \\
& \mathrm{s}_{1, n}=2 P_{\mathrm{UP}}^{1, n} 1-\mathrm{p}_{1, n}+\mathrm{r}_{1, n}-P_{\mathrm{U}} \mathrm{r}_{1, n} \\
& \text { For } i=1, \ldots, m \\
& \mathrm{r}_{2, i, n}=\mathrm{u}_{i, n}-\gamma P_{\mathrm{V}_{i}}\left(\mathrm{D}_{i \mathrm{~V}_{i}^{\perp}} P_{\mathrm{V}_{i}} \mathrm{u}_{i, n}-\mathrm{L}_{i} P_{\mathrm{U}} \mathrm{x}_{n}\right) \\
& \mathrm{p}_{2, i, n}=J_{\gamma \mathrm{B}_{\mathrm{V}_{\mathrm{V}_{i}^{\prime}}}}\left(\mathrm{r}_{2, i, n}-\gamma P_{\mathrm{V}_{i}} \mathrm{~b}_{i}\right)  \tag{4.17}\\
& \mathrm{s}_{2, i, n}=2 P_{\mathrm{V}_{i}} \mathrm{p}_{2, i, n}-\mathrm{p}_{2, i, n}+\mathrm{r}_{2, i, n}-P_{\mathrm{V}_{i}} \mathrm{r}_{2, i, n} \\
& \mathrm{t}_{2, i, n}=\mathrm{s}_{2, i, n}-\gamma P_{\mathrm{V}_{i}}\left(\mathrm{D}_{i \mathrm{~V}_{i}^{\perp}} P_{\mathrm{V}_{i}} \mathrm{~s}_{2, i, n}-\mathrm{L}_{i} P_{\mathrm{U}} \mathrm{~s}_{1, n}\right) \\
& \mathrm{u}_{i, n+1}=\mathrm{u}_{i, n}+\lambda_{n}\left(\mathrm{t}_{2, i, n}-\mathrm{r}_{2, i, n}\right) \\
& \begin{array}{l}
\mathrm{t}_{1, n}=\mathrm{s}_{1, n}-\gamma P_{\mathrm{U}}\left(C P_{\mathbf{U}} \mathrm{s}_{1, n}+\sum_{i=1}^{m} \mathrm{~L}_{i}^{*} P_{\mathrm{V}_{i}} \mathrm{~s}_{2, i, n}\right) \\
\mathrm{x}_{n+1}=\mathrm{x}_{n}+\lambda_{n}\left(\mathrm{t}_{1, n}-\mathrm{r}_{1, n}\right) .
\end{array}
\end{align*}
$$

Then, $\mathrm{x}_{n} \rightharpoonup \overline{\mathrm{x}} \in \mathrm{H}$ and, for every $i \in\{1, \ldots, m\}, \mathrm{u}_{i, n} \rightharpoonup \overline{\mathrm{u}}_{i} \in \mathrm{G}_{i}$, and $\left(P_{\mathrm{U}} \overline{\mathrm{x}}, P_{\mathrm{V}_{1}} \overline{\mathrm{u}}_{1}, \ldots, P_{\mathrm{V}_{m}} \overline{\mathrm{u}}_{m}\right)$ is a solution to Problem 4.6. Moreover, $\mathrm{x}_{n+1}-\mathrm{x}_{n} \rightarrow 0$ and, for every $i \in\{1, \ldots, m\}, \mathrm{u}_{i, n+1}-\mathrm{u}_{i, n} \rightarrow 0$.

Proof. For every $n \in \mathbb{N}$, denote by $z_{n}=\left(x_{n}, \mathbf{u}_{1, n}, \ldots, \mathbf{u}_{m, n}\right), r_{n}=\left(\mathrm{r}_{1, n}, \mathrm{r}_{2,1, n}, \ldots, \mathrm{r}_{2, m, n}\right)$, $p_{n}=$ $\left(\mathrm{p}_{1, n}, \mathrm{p}_{2,1, n}, \ldots, \mathrm{p}_{2, m, n}\right), s_{n}=\left(\mathrm{s}_{1, n}, \mathrm{~s}_{2,1, n}, \ldots, \mathrm{~s}_{2, m, n}\right)$, and $t_{n}=\left(\mathrm{t}_{1, n}, \mathrm{t}_{2,1, n}, \ldots, \mathrm{t}_{2, m, n}\right)$. Then, it follows from Proposition 4.8 that (4.17) is a particular instance of (3.13). Hence, the results follow from Corollary 3.6 and Proposition 4.8(v).

## Remark 4.10

(i) Even if Problem 4.1 can be seen as a particular case of Problem 4.6, the methods in (4.17) and (4.7) have different structures. Indeed, in (4.17) dual variables are updated at each iteration, which may be numerically costly in large scale problems, while only primal variables are updated in Theorem 4.3.
(ii) Algorithm (4.17) activates independently each operator involved in Problem 4.6. The algorithm is explicit in each step if the resolvents of A and $\left(\mathrm{B}_{i \mathrm{~V}_{\perp}}\right)_{1 \leq i \leq m}$ can be computed explicitly. Observe that the resolvent of the partial inverse of a maximally monotone operator can be explicitly found via Proposition 3.2(i).
(iii) Note that, when $\lambda_{n} \equiv 1, \mathrm{U}=\mathrm{H}$, and, for every $i \in\{1, \ldots, m\}, \mathrm{V}_{i}=\mathrm{G}_{i}$, the method in Theorem 4.9 reduces to the algorithm proposed in [18, Theorem 3.1] with constant step-size
(iv) In the simplest case when $m=2, \mathrm{z}=\mathrm{A}=\mathrm{C}=\mathrm{b}_{1}=\mathrm{b}_{2}=0, \mathrm{~L}_{1}=\mathrm{L}_{2}=\mathrm{Id}, \mathrm{U}=\mathrm{H}, \mathrm{V}_{1} \equiv \mathrm{G}_{1}$, $\mathrm{V}_{2} \equiv \mathrm{G}_{2}, \mathrm{D}_{1} 0=\mathrm{G}_{1}, \mathrm{D}_{2} 0=\mathrm{G}_{2}$, and for every $y \neq 0, \mathrm{D}_{1} y=\mathrm{D}_{2} y=\varnothing$, we have, for every $i \in\{1,2\}, \mathrm{D}_{i V_{i}^{\perp}}=\mathrm{D}_{i\{0\}}=\mathrm{D}_{i}^{-1}: y \mapsto 0$ and Problem 4.6 reduces to find a zero of $\mathrm{B}_{1}+\mathrm{B}_{2}$. In this case (4.17) becomes

$$
(\forall n \in \mathbb{N}) \quad \left\lvert\, \begin{align*}
& \mathrm{p}_{1, n}=J_{\gamma \mathrm{B}_{1}^{-1}}\left(\mathrm{u}_{1, n}+\gamma \mathrm{x}_{n}\right)  \tag{4.18}\\
& \mathrm{p}_{2, n}=J_{\gamma \mathrm{B}_{2}^{-1}}\left(\mathrm{u}_{2, n}+\gamma \mathrm{x}_{n}\right) \\
& \mathrm{x}_{n+1}=\mathrm{x}_{n}-\gamma \lambda_{n}\left(\mathrm{p}_{1, n}+\mathrm{p}_{2, n}\right) \\
& \mathrm{u}_{1, n+1}=\left(1-\lambda_{n}\right) \mathrm{u}_{1, n}+\lambda_{n}\left(\mathrm{p}_{1, n}-\gamma^{2}\left(\mathrm{u}_{1, n}+\mathrm{u}_{2, n}\right)\right) \\
& \mathrm{u}_{2, n+1}=\left(1-\lambda_{n}\right) \mathrm{u}_{2, n}+\lambda_{n}\left(\mathrm{p}_{2, n}-\gamma^{2}\left(\mathrm{u}_{1, n}+\mathrm{u}_{2, n}\right)\right) .
\end{align*}\right.
$$

A difference of the method derived in Remark 4.4 for solving this problem, (4.18) update primal and dual variables and solve the primal and dual inclusion, simultaneously.

### 4.3 Zero-Sum Games

Our last application focus in the problem of finding a Nash equilibrium in continuous zero sum games. Some comments on finite zero-sum games are also provided. This problem can be formulated in the form of Problem 3.1 and solved via an algorithm derived from Theorem 3.4.

Problem 4.11 For every $i \in\{1,2\}$, let $\mathrm{H}_{i}$ and $\mathrm{G}_{i}$ be real Hilbert spaces, let $\mathrm{C}_{i}$ be a closed convex subset of $\mathrm{H}_{i}$, let $\mathrm{L}_{i}: \mathrm{H}_{i} \rightarrow \mathrm{G}_{i}$ be a linear bounded operator with closed range, let $\mathrm{S}_{i}=\left\{\mathrm{x} \in \mathrm{C}_{i} \mid\right.$ $\left.\mathrm{L}_{i} \mathrm{x}=\mathrm{b}_{i}\right\}$, where $\mathrm{b}_{i}=\mathrm{L}_{i} \mathrm{e}_{i}$ for some $\mathrm{e}_{i} \in \mathrm{H}_{i}$, let $\left.\chi \in\right] 0,+\infty$, and let $\mathrm{f}: \mathrm{H}_{1} \times \mathrm{H}_{2} \rightarrow \mathbb{R}$ be a differentiable function with a $\chi$-lipschitzian gradient such that, for every $z_{1} \in H_{1}, f\left(z_{1}, \cdot\right)$ is concave and, for every $z_{2} \in H_{2}, f\left(\cdot, z_{2}\right)$ is convex. Moreover suppose that $\operatorname{int}\left(C_{1}-e_{1}\right) \cap \operatorname{ker} L_{1} \neq \varnothing$ and $\operatorname{int}\left(C_{2}-e_{2}\right) \cap \operatorname{ker} L_{2} \neq \varnothing$. The problem is to

$$
\text { find } x_{1} \in S_{1} \quad \text { and } x_{2} \in S_{2} \quad \text { such that }\left\{\begin{array}{l}
x_{1} \in \underset{z_{1} \in S_{1}}{\operatorname{Argmin}} f\left(z_{1}, x_{2}\right)  \tag{4.19}\\
x_{2} \in \underset{z_{2} \in S_{2}}{\operatorname{Argmax}} f\left(x_{1}, z_{2}\right),
\end{array}\right.
$$

under the assumption that solutions exist.
Problem 4.11 is a generic zero-sum game in which the sets $S_{1}$ and $S_{2}$ are usually convex bounded sets representing mixed strategy spaces. For example, if, for every $i \in\{1,2\}, H_{i}=\mathbb{R}^{N_{i}}, \mathrm{C}_{i}$ is the positive orthant, $\mathrm{G}_{i} \equiv \mathbb{R}, \mathrm{~b}_{i} \equiv 1$, and $\mathrm{L}_{i}$ is the sum of the components in the space $\mathbb{R}^{N_{i}}, \mathrm{~S}_{i}$ is the simplex in $\mathbb{R}^{N_{i}}$. In that case, for a bilinear function $f$, Problem 4.11 reduces to a finite zero-sum game. Beyond this particular case, Problem 4.11 covers continuous zero-sum games in which mixed strategies are distributions and $L_{1}$ and $L_{2}$ are integral operators.

As far as we know, some attempts for solving (4.19) are proposed in [1,3] in particular cases when the function $f$ has a special separable structure with specific coupling schemes. In this particular context they propose alternating methods for finding a Nash equilibrium. On the other hand, a method proposed in [14] can solve (4.19) when the projections onto $S_{1}$ and $S_{2}$ are computable. However, in infinite dimension this projections are not always easy to compute, as we will discuss in Example 4.14 below. The following result provides an algorithm for solving Problem 4.11 in the general case, which is obtained as a consequence of Corollary 3.6. The method avoids the projections onto $\mathrm{S}_{1}$ and $\mathrm{S}_{2}$ by alternating simpler projections onto $\mathrm{C}_{1}, \mathrm{C}_{2}, \operatorname{ker}\left(\mathrm{~L}_{1}\right)$, and $\operatorname{ker}\left(\mathrm{L}_{2}\right)$. Let us first introduce the generalized Moore-Penrose inverse of a bounded linear operator $\mathrm{L}: \mathrm{H} \rightarrow \mathrm{G}$ with closed range, defined by $\mathrm{L}^{\dagger}: \mathrm{G} \rightarrow \mathrm{H}: \mathrm{y} \mapsto P_{C_{y}} 0$, where, for every $\mathrm{y} \in \mathrm{G}, C_{\mathrm{y}}=\left\{\mathrm{x} \in \mathrm{H} \mid \mathrm{L}^{*} \mathrm{Lx}=\mathrm{L}^{*} \mathrm{y}\right\}$. The operator $\mathrm{L}^{\dagger}$ is also linear and bounded and, in the particular case when $L^{*} L$ is invertible, $L^{\dagger}=\left(L^{*} L\right)^{-1} L^{*}$. For further details and properties the reader is referred to [7, Section 3].

Theorem 4.12 Under the notation and assumptions of Problem 4.11, let $\varepsilon \in] 0,1[$, let $\gamma \in] 0,1 / \chi[$, and let $\left(\lambda_{n}\right)_{n \in \mathbb{N}}$ be a sequence in $[\varepsilon, 1]$. Moreover, let $\left(\mathbf{z}_{1,0}, \mathbf{z}_{2,0}\right) \in \mathrm{H}_{1} \oplus \mathrm{H}_{2}$, and iterate, for every $n \in \mathbb{N}$,

$$
\left\lvert\, \begin{align*}
& \mathrm{u}_{1, n}=\mathrm{z}_{1, n}-\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \mathbf{z}_{1, n} \\
& \mathrm{u}_{2, n}=\mathrm{z}_{2, n}-\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{* \dagger} \mathrm{z}_{2, n} \\
& \mathrm{~g}_{1, n}=\nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{u}_{2, n}\right)\right)\left(\mathrm{e}_{1}+\mathrm{u}_{1, n}\right)-\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{u}_{2, n}\right)\right)\left(\mathrm{e}_{1}+\mathrm{u}_{1, n}\right) \\
& \mathrm{g}_{2, n}=-\nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{u}_{1, n}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{u}_{2, n}\right)+\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{* \dagger} \nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{u}_{1, n}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{u}_{2, n}\right) \\
& \mathrm{r}_{1, n}=\mathrm{z}_{1, n}-\gamma \mathrm{g}_{1, n} \\
& \mathrm{r}_{2, n}=\mathrm{z}_{2, n}-\gamma \mathrm{g}_{2, n} \\
& \mathrm{p}_{1, n}=P_{\mathrm{C}_{1}}\left(\mathrm{r}_{1, n}+\mathrm{e}_{1}\right)-\mathrm{e}_{1} \\
& \mathrm{p}_{2, n}=P_{\mathrm{C}_{2}}\left(\mathrm{r}_{2, n}+\mathrm{e}_{2}\right)-\mathrm{e}_{2}  \tag{4.20}\\
& \mathrm{v}_{1, n}=\mathrm{p}_{1, n}-\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \mathrm{p}_{1, n} \\
& \mathrm{v}_{2, n}=\mathrm{p}_{2, n}-\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{+\dagger} \mathrm{p}_{2, n} \\
& \mathrm{~s}_{1, n}=2 \mathbf{v}_{1, n}-\mathrm{p}_{1, n}+\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \mathrm{r}_{1, n} \\
& \mathrm{~s}_{2, n}=2 \mathbf{v}_{2, n}-\mathrm{p}_{2, n}+\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{* \dagger} \mathrm{r}_{2, n} \\
& \mathrm{~h}_{1, n}=\nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{v}_{2, n}\right)\right)\left(\mathrm{e}_{1}+\mathrm{v}_{1, n}\right)-\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{v}_{2, n}\right)\right)\left(\mathrm{e}_{1}+\mathrm{v}_{1, n}\right) \\
& \mathrm{h}_{2, n}=-\nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{v}_{1, n}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{v}_{2, n}\right)+\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{* \dagger} \nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{v}_{1, n}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{v}_{2, n}\right) \\
& \mathrm{t}_{1, n}=\mathrm{s}_{1, n}-\gamma \mathrm{h}_{1, n} \\
& \mathrm{t}_{2, n}=\mathrm{s}_{2, n}-\gamma \mathrm{h}_{2, n} \\
& \mathrm{z}_{1, n+1}=\mathrm{z}_{1, n}+\lambda_{n}\left(\mathrm{t}_{1, n}-\mathrm{r}_{1, n}\right) \\
& \mathrm{z}_{2, n+1}=\mathrm{z}_{2, n}+\lambda_{n}\left(\mathrm{t}_{2, n}-\mathrm{r}_{2, n}\right) .
\end{align*}\right.
$$

Then there exists a solution $\left(\bar{x}_{1}, \bar{x}_{2}\right)$ to Problem 4.11 such that $\mathbf{z}_{1, n}+\mathrm{e}_{1} \rightharpoonup \bar{x}_{1}$ and $\mathbf{z}_{2, n}+\mathrm{e}_{2} \rightharpoonup \bar{x}_{2}$.
Proof. It follows from [7, Theorem 16.2] that Problem 4.11 can be written equivalently as the problem of finding $\mathrm{x}_{1}$ and $\mathrm{x}_{2}$ such that $0 \in \partial\left(\iota \mathrm{~s}_{1}+\mathrm{f}\left(\cdot, \mathrm{x}_{2}\right)\right)\left(\mathrm{x}_{1}\right)$ and $0 \in \partial\left(\iota \mathrm{~s}_{2}-\mathrm{f}\left(\mathrm{x}_{1}, \cdot\right)\right)\left(\mathrm{x}_{2}\right)$, which, because of [7, Corollary 16.38], is equivalent to

$$
\left\{\begin{array}{l}
0 \in N_{\mathrm{S}_{1}}\left(\mathrm{x}_{1}\right)+\nabla\left(\mathrm{f}\left(\cdot, \mathrm{x}_{2}\right)\right)\left(\mathrm{x}_{1}\right)  \tag{4.21}\\
0 \in N_{\mathrm{S}_{2}}\left(\mathrm{x}_{2}\right)-\nabla\left(\mathrm{f}\left(\mathrm{x}_{1}, \cdot\right)\right)\left(\mathrm{x}_{2}\right) .
\end{array}\right.
$$

Now since, for every $i \in\{1,2\}, \mathrm{S}_{i}=\mathrm{C}_{i} \cap \mathrm{~L}_{i}^{-1}\left(\mathrm{~b}_{i}\right)=\mathrm{C}_{i} \cap\left(\mathrm{e}_{i}+\operatorname{ker}^{\mathrm{L}} \mathrm{L}_{i}\right)$, it follows from qualification conditions assumed in Problem 4.11 that (4.21) is equivalent to

$$
\left\{\begin{array}{l}
0 \in N_{\mathrm{C}_{1}}\left(\mathrm{e}_{1}+\mathrm{z}_{1}\right)+N_{\text {ker } \mathrm{L}_{1}}\left(\mathrm{z}_{1}\right)+\nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{z}_{2}\right)\right)\left(\mathrm{e}_{1}+\mathrm{z}_{1}\right)  \tag{4.22}\\
0 \in N_{\mathrm{C}_{2}}\left(\mathrm{e}_{2}+\mathrm{z}_{2}\right)+N_{\text {ker } \mathrm{L}_{2}}\left(\mathrm{z}_{2}\right)-\nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{z}_{1}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{z}_{2}\right),
\end{array}\right.
$$

where $z_{1}=x_{1}-e_{1}$ and $z_{2}=x_{2}-e_{2}$. Hence, by defining

$$
\left\{\begin{array}{l}
V=\operatorname{ker}\left(\mathrm{L}_{1}\right) \times \operatorname{ker}\left(\mathrm{L}_{2}\right)  \tag{4.23}\\
A: \mathrm{H}_{1} \times \mathrm{H}_{2} \rightarrow 2^{\mathrm{H}_{1} \times \mathrm{H}_{2}}:\left(\mathrm{z}_{1}, \mathrm{z}_{2}\right) \mapsto N_{\mathrm{C}_{1} \times \mathrm{C}_{2}}\left(\mathrm{e}_{1}+\mathrm{z}_{1}, \mathrm{e}_{2}+\mathrm{z}_{2}\right) \\
B: \mathrm{H}_{1} \times \mathrm{H}_{2} \rightarrow \mathrm{H}_{1} \times \mathrm{H}_{2}:\left(\mathrm{z}_{1}, \mathrm{z}_{2}\right) \mapsto\binom{\nabla\left(\mathrm{f}\left(\cdot, \mathrm{e}_{2}+\mathrm{z}_{2}\right)\right)\left(\mathrm{e}_{1}+\mathrm{z}_{1}\right)}{-\nabla\left(\mathrm{f}\left(\mathrm{e}_{1}+\mathrm{z}_{1}, \cdot\right)\right)\left(\mathrm{e}_{2}+\mathrm{z}_{2}\right)}
\end{array}\right.
$$

Problem 4.11 is equivalent to find $\mathrm{z}_{1} \in \mathrm{H}_{1}$ and $\mathrm{z}_{2} \in \mathrm{H}_{2}$ such that $0 \in A\left(\mathrm{z}_{1}, \mathrm{z}_{2}\right)+B\left(\mathrm{z}_{1}, \mathrm{z}_{2}\right)+N_{V}\left(\mathrm{z}_{1}, \mathrm{z}_{2}\right)$, where $V$ is clearly a closed vectorial subspace of $\mathrm{H}_{1} \times \mathrm{H}_{2}, A$ is maximally monotone [7, Proposition 20.22], and $B$ is monotone ([7, Proposition 20.22] and [31]). Moreover, since $\nabla \mathrm{f}$ is $\chi$ lipschitzian, $B$ is also $\chi$-lipschitzian. On the other hand, it follows from [7, Proposition 3.28(iii)]
and [7, Proposition 23.15 (iii)] that $P_{V}:\left(\mathbf{z}_{1}, \mathbf{z}_{2}\right) \mapsto\left(\mathbf{z}_{1}-\mathrm{L}_{1}^{*} \mathrm{~L}_{1}^{* \dagger} \mathrm{z}_{1}, \mathrm{z}_{2}-\mathrm{L}_{2}^{*} \mathrm{~L}_{2}^{* \dagger} \mathrm{z}_{2}\right), J_{\gamma A}:\left(\mathbf{z}_{1}, \mathrm{z}_{2}\right) \mapsto$ $\left(P_{\mathrm{C}_{1}}\left(\mathrm{z}_{1}+\mathrm{e}_{1}\right)-\mathrm{e}_{1}, P_{\mathrm{C}_{2}}\left(\mathrm{z}_{2}+\mathrm{e}_{2}\right)-\mathrm{e}_{2}\right)$ and we deduce that (4.20) is a particular case of (3.13) when $A$, $B$, and $V$ are defined by (4.23). Altogether, the result follows from Corollary 3.6.

Remark 4.13 Note that the proposed method does not need the projection onto $S_{1}$ and $S_{2}$ at each iteration, but it converges to solution strategies belonging to these sets. This new feature is very useful in cases in which the projection onto $S_{1}$ and $S_{2}$ are not available or are not easy to compute as the following example illustrates.

Example 4.14 We consider a 2-player zero-sum game in which $X_{1} \subset \mathbb{R}^{N_{1}}$ is bounded and represents the set of pure strategies of player 1 , and $S_{1}=\left\{f \in L^{2}\left(X_{1}\right) \mid f \geq 0\right.$ a.e., $\left.\int_{X_{1}} f(x) d x=1\right\}$ is her set of mixed strategies, which are distributions of probability in $L^{2}\left(X_{1}\right)\left(X_{2}, N_{2}\right.$, and $S_{2}$ are defined likewise). We recall that $L^{2}(X)$ stands for the set of square-integrable functions $\left.\left.f: X \subset \mathbb{R}^{n} \rightarrow\right]-\infty,+\infty\right]$. Moreover, let $F \in L^{2}\left(X_{1} \times X_{2}\right)$ be a function representing the payoff for player 1 and let $-F$ be the payoff of player 2 . The problem is to
find $f_{1} \in S_{1}$ and $f_{2} \in S_{2}$ such that $\left\{\begin{array}{l}f_{1} \in \underset{g_{1} \in S_{1}}{\operatorname{Argmin}} \int_{X_{1}} \int_{X_{2}} F\left(x_{1}, x_{2}\right) g_{1}\left(x_{1}\right) f_{2}\left(x_{2}\right) d x_{2} d x_{1} \\ f_{2} \in \underset{g_{2} \in S_{2}}{\operatorname{Argmax}} \int_{X_{1}} \int_{X_{2}} F\left(x_{1}, x_{2}\right) f_{1}\left(x_{1}\right) g_{2}\left(x_{2}\right) d x_{2} d x_{1} .\end{array}\right.$
Note that $S_{1}$ and $S_{2}$ are closed convex sets in $L^{2}\left(X_{1}\right)$ and $L^{2}\left(X_{2}\right)$, respectively. Hence, the projection of any square-integrable function onto $S_{1}$ or $S_{2}$ is well defined. However, these projections are not easy to compute. A possible way to avoid the explicit computation of these projections is to split $S_{1}$ and $S_{2}$ in $S_{1}=\mathrm{C}_{1} \cap\left(\mathrm{e}_{1}+\operatorname{ker} \mathrm{L}_{1}\right)$ and $S_{2}=\mathrm{C}_{2} \cap\left(\mathrm{e}_{2}+\operatorname{ker} \mathrm{L}_{2}\right)$ as in the proof of Theorem 4.12 , where, for every $i \in\{1,2\}, \mathrm{C}_{i}=\left\{f \in L^{2}\left(X_{i}\right) \mid f \geq 0 \quad\right.$ a.e. $\}, \mathrm{e}_{i} \equiv\left(m_{i}\left(X_{i}\right)\right)^{-1}, \mathrm{~L}_{i}: f \mapsto \int_{X_{i}} f(x) d x$, and $m_{i}\left(X_{i}\right)$ stands for the Lebesgue measure of the set $X_{i}$. Note that $\mathrm{e}_{1} \in \operatorname{int} \mathrm{C}_{1} \cap\left(\mathrm{e}_{1}+\operatorname{ker} \mathrm{L}_{1}\right)$ and $\mathrm{e}_{2} \in \operatorname{int} \mathrm{C}_{2} \cap\left(\mathrm{e}_{2}+\operatorname{ker} \mathrm{L}_{2}\right)$, which yield the qualification condition in Problem 4.11. For every $i \in\{1,2\}$, let $\mathrm{H}_{i}=L^{2}\left(X_{i}\right)$ and define the function $\mathrm{f}: \mathrm{H}_{1} \times \mathrm{H}_{2} \rightarrow \mathbb{R}:\left(f_{1}, f_{2}\right) \mapsto \int_{X_{1}} \int_{X_{2}} F\left(x_{1}, x_{2}\right) f_{1}\left(x_{1}\right) f_{2}\left(x_{2}\right) d x_{2} d x_{1}$, which is bilinear, differentiable, and it follows from $F \in L^{2}\left(X_{1} \times X_{2}\right)$ that

$$
\begin{equation*}
\nabla \mathrm{f}:\left(f_{1}, f_{2}\right) \mapsto\left(\int_{X_{2}} F\left(\cdot, x_{2}\right) f_{2}\left(x_{2}\right) d x_{2}, \int_{X_{1}} F\left(x_{1}, \cdot\right) f_{1}\left(x_{1}\right) d x_{1}\right) \in \mathrm{H}_{1} \times \mathrm{H}_{2} \tag{4.25}
\end{equation*}
$$

and that $\nabla \mathrm{f}$ is $\chi$-lipschitzian with $\chi=\|F\|_{L^{2}\left(X_{1} \times X_{2}\right)}$. Thus, by defining $\mathrm{G}_{i}=\mathbb{R}$, (4.24) is a particular instance of Problem 4.11. Note that, for every $i \in\{1,2\}, \mathrm{L}_{i}^{*}: \mathbb{R} \rightarrow L^{2}\left(X_{i}\right): \xi \mapsto \delta_{\xi}$, where, for every $\xi \in \mathbb{R}, \delta_{\xi}: x \mapsto \xi$ is the constant function. Moreover, the operator $\mathrm{L}_{i} \circ \mathrm{~L}_{i}^{*}: \xi \rightarrow m_{i}\left(X_{i}\right) \xi$ is invertible with $\left(\mathrm{L}_{i} \circ \mathrm{~L}_{i}^{*}\right)^{-1}: \xi \mapsto \xi / m_{i}\left(X_{i}\right)$, which yields $\mathrm{L}_{i}^{* \dagger}=\left(\mathrm{L}_{i} \circ \mathrm{~L}_{i}^{*}\right)^{-1} \mathrm{~L}_{i}=\mathcal{M}_{i}$, where $\mathcal{M}_{i}: f \mapsto \delta_{\bar{f}}$ and $\bar{f}=\int_{X_{i}} f(x) d x / m_{i}\left(X_{i}\right)$ is the mean value of $f$. In addition, for every $i \in\{1,2\}, P_{C_{i}}: f \mapsto f+: \mapsto$
$\max \{0, f(t)\}$. Altogether, (4.20) reduces to

$$
\left\lvert\, \begin{align*}
& u_{1, n}=z_{1, n}-\mathcal{M}_{1}\left(z_{1, n}\right) \\
& u_{2, n}=z_{2, n}-\mathcal{M}_{2}\left(z_{2, n}\right) \\
& g_{1, n}=G_{1}+\left(\int_{X_{2}} F\left(\cdot, x_{2}\right) u_{2, n}\left(x_{2}\right) d x_{2}-\mathcal{M}_{1}\left(\int_{X_{2}} F\left(\cdot, x_{2}\right) u_{2, n}\left(x_{2}\right) d x_{2}\right)\right) \\
& g_{2, n}=-G_{2}-\left(\int_{X_{1}} F\left(x_{1}, \cdot\right) u_{1, n}\left(x_{1}\right) d x_{1}-\mathcal{M}_{2}\left(\int_{X_{1}} F\left(x_{1}, \cdot\right) u_{1, n}\left(x_{1}\right) d x_{1}\right)\right) \\
& r_{1, n}=z_{1, n}-\gamma g_{1, n} \\
& r_{2, n}=z_{2, n}-\gamma g_{2, n} \\
& p_{1, n}=\left[r_{1, n}+m_{1}\left(X_{1}\right)^{-1}\right]_{1}-m_{1}\left(X_{1}\right)^{-1} \\
& p_{2, n}=\left[r_{2, n}+m_{2}\left(X_{2}\right)^{-1}\right]_{+}-m_{2}\left(X_{2}\right)^{-1} \\
& v_{1, n}=p_{1, n}-\mathcal{M}_{1}\left(p_{1, n}\right)  \tag{4.26}\\
& v_{2, n}=p_{2, n}-\mathcal{M}_{2}\left(p_{2, n}\right) \\
& s_{1, n}=2 v_{1, n}-p_{1, n}+\mathcal{M}_{1}\left(r_{1, n}\right) \\
& s_{2, n}=2 v_{2, n}-p_{2, n}+\mathcal{M}_{2}\left(r_{2, n}\right) \\
& h_{1, n}=G_{1}+\left(\int_{X_{2}} F\left(\cdot, x_{2}\right) v_{2, n}\left(x_{2}\right) d x_{2}-\mathcal{M}_{1}\left(\int_{X_{2}} F\left(\cdot, x_{2}\right) v_{2, n}\left(x_{2}\right) d x_{2}\right)\right) \\
& h_{2, n}=-G_{2}-\left(\int_{X_{1}} F\left(x_{1}, \cdot\right) v_{1, n}\left(x_{1}\right) d x_{1}-\mathcal{M}_{2}\left(\int_{X_{1}} F\left(x_{1}, \cdot\right) v_{1, n}\left(x_{1}\right) d x_{1}\right)\right) \\
& t_{1, n}=s_{1, n}-\gamma h_{1, n} \\
& t_{2, n}=s_{2, n}-\gamma h_{2, n} \\
& z_{1, n+1}=z_{1, n}+\lambda_{n}\left(t_{1, n}-r_{1, n}\right) \\
& z_{2, n+1}=z_{2, n}+\lambda_{n}\left(t_{2, n}-r_{2, n}\right),
\end{align*}\right.
$$

where

$$
\left\{\begin{array}{l}
G_{1}: z_{1} \mapsto \mathcal{M}_{2}\left(F\left(z_{1}, \cdot\right)\right)-\frac{1}{m_{1}\left(X_{1}\right) m_{2}\left(X_{2}\right)} \int_{X_{1}} \int_{X_{2}} F\left(x_{1}, x_{2}\right) d x_{2} d x_{1}  \tag{4.27}\\
G_{2}: z_{2} \mapsto \mathcal{M}_{1}\left(F\left(\cdot, z_{2}\right)\right)-\frac{1}{m_{1}\left(X_{1}\right) m_{2}\left(X_{2}\right)} \int_{X_{1}} \int_{X_{2}} F\left(x_{1}, x_{2}\right) d x_{2} d x_{1}
\end{array}\right.
$$

and $\gamma \in] 0,1 / \chi\left[\right.$. Altogether, Theorem 4.12 asserts that the sequences $\left(z_{1, n}+m_{1}\left(X_{1}\right)^{-1}\right)_{n \in \mathbb{N}}$ and $\left(z_{2, n}+m_{2}\left(X_{2}\right)^{-1}\right)_{n \in \mathbb{N}}$ converge to $\bar{f}_{1} \in \mathrm{H}_{1}$ and $\bar{f}_{2} \in \mathrm{H}_{2}$, respectively, where $\left(\bar{f}_{1}, \bar{f}_{2}\right)$ is a solution to (4.24).

Note that, in the particular case when $X_{1}$ and $X_{2}$ are finite sets of actions (or pure strategies), $S_{1}$ and $S_{2}$ are finite dimensional simplexes, and $F:\left(x_{1}, x_{2}\right) \mapsto x_{1}^{\top} \mathrm{F} x_{2}$ is a payoff matrix. In this case (4.26) provides a first order method for finding Nash equilibria in the finite zero-sum game (for complements and background on finite games, see [42])

$$
\text { find } x_{1} \in S_{1} \text { and } x_{2} \in S_{2} \quad \text { such that }\left\{\begin{array}{l}
x_{1} \in \underset{y_{1} \in S_{1}}{\operatorname{Argmin}} x_{1}^{\top} \mathrm{F} x_{2}  \tag{4.28}\\
x_{2} \in \underset{y_{2} \in S_{2}}{\operatorname{Argmax}} x_{1}^{\top} \mathrm{F} x_{2} .
\end{array}\right.
$$

When a large number of pure actions are involved (e.g., Texas Hold'em poker) classical linear programming methods for solving (4.24) are enormous and unsolvable via standard algorithms as simplex. Other attempts using acceleration schemes for obtaining good convergence rates are provided in [22]. However, the proposed method does not guarantee the convergence of the iterates. Other methods need to compute a Nash equilibrium at each iteration, which is costly numerically [44]. The method obtained from (4.26) is an explicit convergent method that solves (4.28) overcoming previous difficulties. Numerical simulations and comparisons with other methods in the literature are part of further research.

## 5 Conclusions

We provide a fully split algorithm for finding a zero of $A+B+N_{V}$. The proposed method exploits the intrinsic properties of each of the operators involved by activating explicitly the single-valued operator $B$ and by computing the resolvent of $A$ and projections onto $V$. Weak convergence to a zero of $A+B+N_{V}$ is guaranteed and applications to monotone inclusions involving $m$ maximally monotone operators, to primal-dual composite inclusions involving partial sums of monotone operators, and continuous zero-sum games are provided. In addition, the partial sum of two set-valued operators with respect to a closed vectorial subspace is introduced. This operation preserves monotonicity and a further study will be done in a future work. Furthermore, in the zero-sum games context, a splitting method is provided for computing Nash equilibria. The algorithm replaces the projections onto mixed strategy spaces (infinite dimensional simplexes) by alternate simpler projections.

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[^0]:    ${ }^{*}$ Contact author: L. M. Briceño-Arias, luis.briceno@usm.cl, phone: +56 24326662 . This work was supported by CONICYT under grants FONDECYT 3120054, ECOS-CONICYT C13E03, Anillo ACT 1106, Math-Amsud N 13MATH01, and by "Programa de financiamiento basal" from the Center for Mathematical Modeling, Universidad de Chile.

