# Global Uniqueness and Solvability for Tensor Complementarity Problems * 

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#### Abstract

Recently, the tensor complementarity problem (TCP for short) has been investigated in the literature. An important question involving the property of global uniqueness and solvability (GUS-property) for a class of TCPs was proposed by Song and Qi in their paper "Properties of Some Classes of Structured Tensors". In the present paper, we give an answer to this question by constructing two counterexamples. We also show that the solution set of this class of TCPs is nonempty and compact. In particular, we introduce a class of related structured tensors, and show that the corresponding TCP has the GUS-property.


Key words: Tensor complementarity problem; Nonlinear complementarity problem; Global uniqueness and solvability; $P$ tensor Strong $P$ tensor

Mathematics Subject Classifications(2000): 90C33; 65K10; 15A18; 15A69; 65F15; 65F10

[^0]
## 1 Introduction

As a generalization of the linear complementarity problem [1], the tensor complementarity problem has been introduced and investigated in [2, 3, 4, [5, 6], which is a specific class of nonlinear complementarity problems [7, 8, , 9]. By using properties of structured tensors, many good results for the tensor complementarity problem have been obtained in the literature.

It is well-known that a linear complementarity problem has the property of global uniqueness and solvability (GUS-property) if and only if the matrix involved in the concerned problem is a $P$-matrix [10]. It is also well-known that such a result can not be generalized to the nonlinear complementarity problem [8, 9]. A natural question is whether such a result can be generalized to the tensor complementarity problem or not? i.e., whether the result that a tensor complementarity problem has the GUS-property if and only if the tensor involved in the problem is a $P$ tensor holds or not? Such a question was proposed by Song and Qi (see, Question 6.3 in [2]). In this paper, we show that the answer to this question is negative by constructing two counter-examples.

It has been shown that the tensor complementarity problem with a $P$ tensor has a solution; while our counter-example demonstrates that it is possible such a problem has more than one solution. Thus, a natural question is what more we can say about the solution set of such a complementarity problem; and another natural question is that for which kind of tensor, the corresponding tensor complementarity problem has the GUS-property. For the first question, we will show that the solution set of the tensor complementarity problem is nonempty and compact when the involved tensor is a $P$ tensor; and for the second question, we will introduce a new class of tensors, called the strong $P$ tensor, and show that the corresponding tensor complementarity problem has the GUS-property. We also show that the set consists of all strong $P$ tensors is a proper subset of the set consists of all $P$ tensors; and hence, many results obtained for the case of the $P$ tensor are still satisfied for the case of the strong $P$ tensor.

The rest of this paper is organized as follows. In the next section, we first briefly review some basic concepts and results which are useful in the subsequent analysis. In Sect. 3, we give a negative answer to the result that a tensor complementarity problem has the GUS-property if and only if the tensor involved in the problem is a $P$ tensor, and show that the solution set of the tensor complementarity problem with a $P$ tensor is compact. In Sect. 4, we introduce the concept of the strong $P$ tensor and discuss its related properties. Conclusions are given in Sect. 5.

## 2 Preliminaries

The complementarity problem, denoted by $\mathrm{CP}(F)$, is to find a point $x \in \mathbb{R}^{n}$ such that

$$
x \geq 0, \quad F(x) \geq 0, \quad x^{T} F(x)=0 .
$$

When $F(x)=A x+q$ with given $A \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^{n}, \mathrm{CP}(F)$ reduces to the linear complementarity problem (denoted by $\operatorname{LCP}(q, A)$ ), which is to find a point $x \in \mathbb{R}^{n}$ such that

$$
x \geq 0, \quad A x+q \geq 0, \quad x^{T}(A x+q)=0
$$

and when $F(x)=\mathcal{A} x^{m-1}+q$ with given $\mathcal{A}=\left(a_{i_{1} i_{2} \cdots i_{m}}\right) \in T_{m, n}$ (the set of all real $m$ th order $n$-dimensional tensors) and $q \in \mathbb{R}^{n}, \mathrm{CP}(F)$ reduces to the tensor complementarity problem (denoted by $\operatorname{TCP}(q, \mathcal{A})$ ) [2, 3, 4, 5, 6], which is to find a point $x \in \mathbb{R}^{n}$ such that

$$
x \geq 0, \quad \mathcal{A} x^{m-1}+q \geq 0, \quad x^{T}\left(\mathcal{A} x^{m-1}+q\right)=0
$$

where $\mathcal{A} x^{m-1} \in \mathbb{R}^{n}$ defined by

$$
\left(\mathcal{A} x^{m-1}\right)_{i}:=\sum_{i_{2}, \cdots, i_{m}=1}^{n} a_{i i_{2} \cdots i_{m}} x_{i_{2}} \cdots x_{i_{m}}, \quad \forall i \in\{1,2, \ldots, n\} .
$$

It is easy to see that

$$
\mathcal{A} x^{m}=x^{T}\left(\mathcal{A} x^{m-1}\right)=\sum_{i_{1}, i_{2}, \cdots, i_{m}=1}^{n} a_{i_{1} i_{2} \cdots i_{m}} x_{i_{1}} x_{i_{2}} \cdots x_{i_{m}} .
$$

Throughout this paper, for any positive integer $n$, we denote $[n]:=\{1,2, \ldots, n\}$ and $\mathbb{R}_{+}^{n}:=\left\{x \in \mathbb{R}^{n}: x \geq 0\right\}$. For any $x \in \mathbb{R}^{n}$, we denote

$$
[x]_{+}:=\left(\max \left\{x_{1}, 0\right\}, \ldots, \max \left\{x_{n}, 0\right\}\right)^{T} .
$$

The eigenvalue of tensor is initially studied by Qi [11] and Lim [12]. If there is a nonzero vector $x \in \mathbb{R}^{n}$ and a scalar $\lambda \in \mathbb{R}$ such that

$$
\left(\mathcal{A} x^{m-1}\right)_{i}=\lambda x_{i}^{m-1}, \quad \forall i \in[n]
$$

then $\lambda$ is called an $H$-eigenvalue of $\mathcal{A}$ and $x$ is called an $H$-eigenvector of $\mathcal{A}$ associated with $\lambda$; and if there is a nonzero vector $x \in \mathbb{R}^{n}$ and a scalar $\lambda \in \mathbb{R}$ such that

$$
\mathcal{A} x^{m-1}=\lambda x, \forall i \in\{1,2, \ldots, n\} \quad \text { and } \quad x^{T} x=1
$$

then $\lambda$ is called a $Z$-eigenvalue of $\mathcal{A}$ and $x$ is called a $Z$-eigenvector of $\mathcal{A}$ associated with $\lambda$.

Recently, many classes of structured tensors are introduced and the related properties are studied [2, 3, 4, 5, 6, 13, 14, 15, 16, 17, 18, 19]. In this paper, we need the following concepts of several structured tensors.

Definition 2.1 Let $\mathcal{A}=\left(a_{i_{1} \cdots i_{m}}\right) \in T_{m, n}$. We say that $\mathcal{A}$ is
(i) a strictly semi-positive tensor iff for each $x \in \mathbb{R}_{+}^{n} \backslash\{0\}$, there exists an index $i \in[n]$ such that $x_{i}>0$ and $\left(\mathcal{A} x^{m-1}\right)_{i}>0$;
(ii) a $P$ tensor iff for each $x \in \mathbb{R}^{n} \backslash\{0\}$, there exists an index $i \in[n]$ such that $x_{i}\left(\mathcal{A} x^{m-1}\right)_{i}>0 ;$
(iii) an $R$-tensor iff there is no $(x, t) \in\left(\mathbb{R}_{+}^{n} \backslash\{0\}\right) \times \mathbb{R}_{+}$such that for any $i \in[n]$,

$$
\left\{\begin{array}{l}
\left(\mathcal{A} x^{m-1}\right)_{i}+t=0 \text { if } x_{i}>0 \\
\left(\mathcal{A} x^{m-1}\right)_{i}+t \geq 0 \text { if } x_{i}=0
\end{array}\right.
$$

Obviously, every $P$ tensor is a strictly semi-positive tensor and an $R$-tensor.
In this paper, we also need the following concepts of functions.
Definition 2.2 ([3]) Let mapping $F: K \subseteq \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$. We say that $F$ is
(i) a P-function if for all pairs of distinct vectors $x$ and $y$ in $K$,

$$
\max _{i \in[n]}\left(x_{i}-y_{i}\right)\left(F_{i}(x)-F_{i}(y)\right)>0
$$

(ii) a uniform $P$-function if there exists a constant $\mu>0$ such that for all pairs of vectors $x$ and $y$ in $K$,

$$
\max _{i \in[n]}\left(x_{i}-y_{i}\right)\left(F_{i}(x)-F_{i}(y)\right) \geq \mu\|x-y\|^{2}
$$

Obviously, every uniform $P$-function is a $P$-function. In addition, it is easy to see from Definition 2.1 and Definition 2.2 that if the mapping $\mathcal{A} x^{m-1}+q$ with any given $q \in \mathbb{R}^{n}$ is a $P$-function, then $\mathcal{A}$ is a $P$ tensor.

Recall that $\operatorname{LCP}(q, A)$ is said to have the GUS-property if $\operatorname{LCP}(q, A)$ has a unique solution for every $q \in \mathbb{R}^{n}$. Similarly, we say that $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property if $T C P(q, \mathcal{A})$ has a unique solution for every $q \in \mathbb{R}^{n}$. For the solvability of $\operatorname{LCP}(q, A)$, an important result is that $\operatorname{LCP}(q, A)$ has the GUS-property iff the matrix $A$ is a $P$-matrix. In fact, the GUS-property has been extensively discussed for various complementarity problems, including nonlinear complementarity problems [20], linear complementarity problems over symmetric cones [21] and Lorentz cone linear complementarity problems on Hilbert spaces [22]. A natural question is given by

Q1 Whether or not $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property iff the tensor $\mathcal{A}$ is a $P$ tensor?
Such a question was proposed by Song and Qi (see, Question 6.3 in [2]). In the next section, we will answer this question.

## 3 Answer to Q1

First, we construct a $\operatorname{TCP}(q, \mathcal{A})$ which has a unique solution for every $q \in \mathbb{R}^{2}$.
Example 3.1 Let $\mathcal{A}=\left(a_{i_{1} i_{2} i_{3}}\right) \in T_{3,2}$, where $a_{111}=1, a_{222}=1$ and all other $a_{i_{1} i_{2} i_{3}}=0$. Then,

$$
\mathcal{A} x^{2}=\binom{x_{1}^{2}}{x_{2}^{2}}
$$

In this case, $\operatorname{TCP}(q, \mathcal{A})$ is to find $x \in \mathbb{R}^{2}$ such that

$$
\left\{\begin{array} { l } 
{ x _ { 1 } \geq 0 , }  \tag{3.1}\\
{ x _ { 2 } \geq 0 , }
\end{array} \quad \left\{\begin{array} { l } 
{ x _ { 1 } ^ { 2 } + q _ { 1 } \geq 0 , } \\
{ x _ { 2 } ^ { 2 } + q _ { 2 } \geq 0 , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
x_{1}\left(x_{1}^{2}+q_{1}\right)=0 \\
x_{2}\left(x_{2}^{2}+q_{2}\right)=0
\end{array}\right.\right.\right.
$$

For any $q \in \mathbb{R}^{2}$, let $x^{q}:=\left(x_{1}^{q}, x_{2}^{q}\right) \in \mathbb{R}^{2}$ be given by

$$
x_{i}^{q}:= \begin{cases}0, & \text { if } q_{i} \geq 0 \\ \sqrt{-q_{i}}, & \text { otherwise }\end{cases}
$$

It is easy to see that for every $q \in \mathbb{R}^{2}, \operatorname{TCP}(q, \mathcal{A})$ given by (3.1) has a unique solution $x^{q}$. Thus, we obtain that $\operatorname{TCP}(q, \mathcal{A})$ given in this example has the GUS-property.

It is proved in [23, Proposition 2.1] that there dose not exist an odd order $P$ tensor; and hence, the tensor given in Example 3.1 is not a $P$ tensor. This, together with Example 3.1, implies that one does not obtain that tensor $\mathcal{A}$ is a $P$ tensor under the assumption that $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property.

Second, we construct the following $\operatorname{TCP}(q, \mathcal{A})$ where $\mathcal{A} \in T_{4,2}$ is a $P$ tensor, but it has two distinct solutions for some $q \in \mathbb{R}^{2}$.

Example 3.2 Let $\mathcal{A}=\left(a_{i_{1} i_{2} i_{3} i_{4}}\right) \in T_{4,2}$, where $a_{1111}=1, a_{1112}=-2, a_{1122}=1, a_{2222}=1$ and all other $a_{i_{1} i_{2} i_{3} i_{4}}=0$. Then,

$$
\mathcal{A} x^{3}=\binom{x_{1}^{3}-2 x_{1}^{2} x_{2}+x_{1} x_{2}^{2}}{x_{2}^{3}}
$$

and

$$
x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}^{4}-2 x_{1}^{3} x_{2}+x_{1}^{2} x_{2}^{2}, \quad x_{2}\left(\mathcal{A} x^{3}\right)_{2}=x_{2}^{4}
$$

For any $x \in \mathbb{R}^{2} \backslash\{0\}$, it is easy see that

- when $x_{2} \neq 0$, it follows that $x_{2}\left(\mathcal{A} x^{3}\right)_{2}>0$; and
- when $x_{2}=0$, it follows that $x_{1} \neq 0$ since $x \neq 0$, and in this case, we have $x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}^{4}>0$.

Thus, for any $x \in \mathbb{R}^{2} \backslash\{0\}$, there is at least one index $i \in\{1,2\}$ such that $x_{i}\left(\mathcal{A} x^{3}\right)_{i}>0$. So, we obtain that tensor $\mathcal{A}$ given in this example is a $P$ tensor by Definition 2.1(ii).

Taking $q=(0,-1)^{T}$, we consider $\operatorname{TCP}(q, \mathcal{A})$ of finding $x \in \mathbb{R}^{2}$ such that

$$
\left\{\begin{array} { l } 
{ x _ { 1 } \geq 0 , }  \tag{3.2}\\
{ x _ { 2 } \geq 0 , }
\end{array} \quad \left\{\begin{array} { l } 
{ x _ { 1 } ^ { 3 } - 2 x _ { 1 } ^ { 2 } x _ { 2 } + x _ { 1 } x _ { 2 } ^ { 2 } \geq 0 , } \\
{ x _ { 2 } ^ { 3 } - 1 \geq 0 , }
\end{array} \quad \text { and } \quad \left\{\begin{array}{l}
x_{1}\left(x_{1}^{3}-2 x_{1}^{2} x_{2}+x_{1} x_{2}^{2}\right)=0, \\
x_{2}\left(x_{2}^{3}-1\right)=0 .
\end{array}\right.\right.\right.
$$

It is easy to see that both $x=(0,1)^{T}$ and $x=(1,1)^{T}$ are the solutions to $\operatorname{TCP}(q, \mathcal{A})$ given by (3.2).

From Example 3.2, we obtain that $\operatorname{TCP}(q, \mathcal{A})$ with $\mathcal{A}$ being a $P$ tensor does not possess the GUS-property.

Note that $\operatorname{TCP}(q, \mathcal{A})$ with $\mathcal{A}$ being a $P$ tensor has a solution for every $q \in \mathbb{R}^{n}$ by [3, Corollary 3.3], and that it is possible that this class of complementarity problems has more than one solution from Example 3.2. What more can we say about the solution set of this class of complementarity problems? In the following, we show that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ with $\mathcal{A}$ being a $P$ tensor is compact.

Theorem 3.1 For any $q \in \mathbb{R}^{n}$ and a $P$ tensor $\mathcal{A} \in T_{m, n}$, the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is nonempty and compact.

Proof. Since $\mathcal{A}$ is a $P$ tensor, it follows from [3, Corollary 3.3] that $\operatorname{TCP}(q, \mathcal{A})$ has a solution for every $q \in \mathbb{R}^{n}$. So we only need to show that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is compact. We divide the proof into two parts.

Part 1. We show the boundedness of the solution set. To this end, we first show the following result:
$\mathbf{R 1} 1$ If there is a sequence $\left\{x^{k}\right\} \subset \mathbb{R}_{+}^{n}$ satisfying

$$
\begin{equation*}
\left\|x^{k}\right\| \rightarrow \infty \quad \text { and } \quad \frac{\left[-\mathcal{A}\left(x^{k}\right)^{m-1}-q\right]_{+}}{\left\|x^{k}\right\|} \rightarrow 0 \quad \text { as } k \rightarrow \infty \tag{3.3}
\end{equation*}
$$

then there exists an $i \in[n]$ such that $x_{i}^{k}\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i}>0$ holds for some $k \geq 0$.

In the following, we assume that the result $\mathbf{R 1}$ does not hold and derive a contradiction. Given an arbitrary sequence $\left\{x^{k}\right\} \subset \mathbb{R}_{+}^{n}$ satisfying (3.3), then since the result $\mathbf{R 1}$ does not hold, we have that

$$
\begin{equation*}
x_{i}^{k}\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i} \leq 0 \quad \forall i \in[n], \forall k \geq 0 \tag{3.4}
\end{equation*}
$$

Since the sequence $\left\{\frac{x^{k}}{\left\|x^{k}\right\|}\right\}$ is bounded, without loss of generality, we can assume $\lim _{k \rightarrow \infty} \frac{x^{k}}{\left\|x^{k}\right\|}=$ $\bar{x} \in \mathbb{R}^{n}$. From $\left\{x^{k}\right\} \subset \mathbb{R}_{+}^{n}$ and $\left\|x^{k}\right\| \rightarrow \infty$ as $k \rightarrow \infty$, we obtain that

$$
\begin{equation*}
\bar{x} \geq 0, \quad \bar{x} \neq 0 \tag{3.5}
\end{equation*}
$$

If $i \in\left\{i \in[n]:\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i} \leq 0\right\}$, then

$$
\left[-\left(\mathcal{A}\left(x^{k}\right)^{m-1}+q\right)_{i}\right]_{+}=-\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i} .
$$

Since $\lim _{k \rightarrow \infty} \frac{q_{i}}{\left\|x^{k}\right\|}=0$ for all $i \in[n]$, we have

$$
\begin{aligned}
0 & =\lim _{k \rightarrow \infty} \frac{\left[-\left(\mathcal{A}\left(x^{k}\right)^{m-1}+q\right)_{i}\right]_{+}}{\left\|x^{k}\right\|}=\lim _{k \rightarrow \infty} \frac{-\left[\mathcal{A}\left(x^{k}\right)^{m-1}\right]_{i}-q_{i}}{\left\|x^{k}\right\|^{m-1}} \\
& =\lim _{k \rightarrow \infty} \frac{-\left[\mathcal{A}\left(x^{k}\right)^{m-1}\right]_{i}}{\left\|x^{k}\right\|^{m-1}}=-\left[\mathcal{A} \bar{x}^{m-1}\right]_{i} ;
\end{aligned}
$$

and if $i \in\left\{i \in[n]:\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i} \geq 0\right\}$, then

$$
0 \leq \lim _{k \rightarrow \infty} \frac{\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i}}{\left\|x^{k}\right\|^{m-1}}=\lim _{k \rightarrow \infty} \frac{\left[\mathcal{A}\left(x^{k}\right)^{m-1}\right]_{i}}{\left\|x^{k}\right\|^{m-1}}=\left[\mathcal{A} \bar{x}^{m-1}\right]_{i} .
$$

Combining these two situations together, we have

$$
\begin{equation*}
\left[\mathcal{A} \bar{x}^{m-1}\right]_{i} \geq 0, \quad \forall i \in[n] . \tag{3.6}
\end{equation*}
$$

In addition, by using (3.4), we have

$$
\bar{x}_{i}\left[\mathcal{A} \bar{x}^{m-1}\right]_{i}=\lim _{k \rightarrow \infty} \frac{x_{i}^{k}}{\left\|x^{k}\right\|} \frac{\left[\mathcal{A}\left(x^{k}\right)^{m-1}\right]_{i}}{\left\|x^{k}\right\|^{m-1}}=\lim _{k \rightarrow \infty} \frac{x_{i}^{k}}{\left\|x^{k}\right\|} \frac{\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i}}{\left\|x^{k}\right\|^{m-1}} \leq 0 .
$$

By using (3.5) and (3.6), we have $\bar{x}\left[\mathcal{A} \bar{x}^{m-1}\right]_{i} \geq 0$. This, together with the above inequality, implies

$$
\begin{equation*}
\bar{x}_{i}\left[\mathcal{A} \bar{x}^{m-1}\right]_{i}=0, \quad \forall i \in[n] . \tag{3.7}
\end{equation*}
$$

Furthermore, by combining (3.5) with (3.6) and (3.7), we obtain that $\bar{x}$ is a nonzero solution of $\operatorname{TCP}(0, \mathcal{A})$. However, since every $P$ tensor is a strictly semi-positive tensor; while if $\mathcal{A}$ is strictly semi-positive, then $\operatorname{TCP}(0, \mathcal{A})$ has a unique solution 0 (see [4, Theorem 3.2]). This derives a contradiction. So the result R1 holds.

Now, suppose that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is unbounded. Then there exists an unbounded solution sequence $\left\{x^{k}\right\}$ of $\operatorname{TCP}(q, \mathcal{A})$ such that $\left\|x^{k}\right\| \rightarrow \infty$ as $k \rightarrow \infty$, and for all $i \in[n], k \geq 0$, it follows that

$$
\begin{equation*}
x^{k} \geq 0, \quad \mathcal{A}\left(x^{k}\right)^{m-1}+q \geq 0, \quad\left(x^{k}\right)^{T}\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]=0 . \tag{3.8}
\end{equation*}
$$

Obviously, $\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i} \geq 0$ implies that

$$
\frac{\left[-\left(\mathcal{A}\left(x^{k}\right)^{m-1}+q\right)_{i}\right]_{+}}{\left\|x^{k}\right\|} \rightarrow 0 \quad \text { as } k \rightarrow \infty
$$

Thus, the solution sequence $\left\{x^{k}\right\}$ satisfies (3.3); and furthermore, by using the result R1, there exist an index $i_{0}$ and a positive integer $k^{*}>0$ such that $x_{i_{0}}^{k^{*}}\left[\mathcal{A}\left(x^{k^{*}}\right)^{m-1}+q\right]_{i_{0}}>0$, which is contrary to that $x_{i}^{k}\left[\mathcal{A}\left(x^{k}\right)^{m-1}+q\right]_{i}=0$ for all $i \in[n]$ and $k \geq 0$. So the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is bounded.

Part 2. We now show that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is closed. Suppose that $\left\{x^{k}\right\}$ is a solution sequence of $\operatorname{TCP}(q, \mathcal{A})$ and

$$
\begin{equation*}
\lim _{k \rightarrow \infty} x_{k}=\bar{x} \tag{3.9}
\end{equation*}
$$

we need to show that $\bar{x}$ solves $\operatorname{TCP}(q, \mathcal{A})$.
Since $\mathcal{A} x^{m-1}+q$ is continuous, by (3.9) we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left[\left(\mathcal{A}\left(x_{k}\right)^{m-1}+q\right]=\mathcal{A}\left(\lim _{k \rightarrow \infty} x_{k}\right)^{m-1}+q=\mathcal{A} \bar{x}^{m-1}+q\right. \tag{3.10}
\end{equation*}
$$

Since $\left\{x^{k}\right\}$ is a solution sequence of $\operatorname{TCP}(q, \mathcal{A})$, we have that (3.8) holds. Furthermore, by using (3.8), (3.9) and (3.10), we can obtain that

$$
\bar{x} \geq 0, \quad \mathcal{A} \bar{x}^{m-1}+q \geq 0, \quad \bar{x}^{T}\left(\mathcal{A} \bar{x}^{m-1}+q\right)=0
$$

So $\bar{x}$ is a solution of $\operatorname{TCP}(q, \mathcal{A})$. Therefore, we obtain that the solution set is closed.
Combining Part 1 with Part 2, we obtain that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is compact. This completes the proof.

## 4 Strong $P$ Tensor and Related Properties

In this section, we consider the question: for which kind of tensor, $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property. For this purpose, we introduce a new class of tensors, called the strong $P$ tensor, which is defined as follows.

Definition 4.3 Let $\mathcal{A}=\left(a_{i_{1} \cdots i_{m}}\right) \in T_{m, n}$. We say that $\mathcal{A}$ is a strong $P$ tensor iff $F(x)=\mathcal{A} x^{m-1}+q$ is a $P$-function.

It is well-known that a matrix $A$ is a $P$-matrix iff $F(x)=A x+q$ is a $P$-function. Thus, the strong $P$ tensor is a generalization of the $P$-matrix from matrix to tensor.

From the definitions of the $P$ tensor and the strong $P$ tensor, it is easy to see that every strong $P$ tensor must be a $P$ tensor. The strong $P$ tensor is defined with the help of the $P$-function, so an advantage of this way is that the related results and methods associated with the $P$-function can be applied to study this class of tensors.

The following result comes from [24, Theorem 2.3].
Lemma 4.1 Let $F: R_{+}^{n} \rightarrow R^{n}$ be a P-function, then the corresponding $C P(F)$ has no more than one solution.

With the help of Lemma 4.1, we show the following result.
Theorem 4.2 Suppose that $\mathcal{A} \in T_{m, n}$ is a strong $P$ tensor, then $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property.

Proof. Since $\mathcal{A}$ is a strong $P$ tensor, it follows that $\mathcal{A}$ is a $P$ tensor. Furthermore, it follows from [3, Corollary 3.3] that $\operatorname{TCP}(q, \mathcal{A})$ has a solution for every $q \in \mathbb{R}^{n}$. Also since $\mathcal{A}$ is a strong $P$ tensor, it follows that $\mathcal{A} x^{m-1}+q$ is a $P$-function; and hence, from Lemma 4.1 it follows that $\operatorname{TCP}(q, \mathcal{A})$ has no more than one solution. Therefore, $\operatorname{TCP}(q, \mathcal{A})$ has a unique solution for every $q \in \mathbb{R}^{n}$, i.e., $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property.

Corollary 4.1 Given $\mathcal{A} \in T_{m, n}$ and $q \in \mathbb{R}^{n}$. Suppose that $F(x)=\mathcal{A} x^{m-1}+q$ is a $P$ function, then the corresponding $C P(F)$ has the GUS-property; and m must be an even number.

In the theory of nonlinear complementarity problems, when the involved function $F$ is a $P$-function, one can only obtain that the corresponding $\mathrm{CP}(F)$ has no more than one solution (see Lemma 4.1); when the involved function $F$ is a uniform $P$-function, one can obtain that the corresponding $\mathrm{CP}(F)$ has the GUS-property (see [25, Corollary 3.2]). From the first result of Corollary 4.1, we see that $\mathrm{CP}(F)$ has the GUS-property when $F(x)=\mathcal{A} x^{m-1}+q$ is a $P$-function. In addition, from the second result of Corollary 4.1, we obtain that a class of functions (i.e., $F(x)=\mathcal{A} x^{m-1}+q$ with $m$ being odd) can not be in the class of the $P$-functions.

In the following, we investigate the relationship between the $P$ tensor and the strong $P$ tensor. Recall that every strong $P$ tensor is a $P$ tensor. The following example demonstrates that the inverse does not hold.

Example 4.3 Let $\mathcal{A}=\left(a_{i_{1} i_{2} i_{3} i_{4}}\right) \in T_{4,2}$, where $a_{1111}=1, a_{1222}=-1, a_{1122}=1, a_{2222}=$ $1, a_{2111}=-1, a_{2211}=1$ and all other $a_{i_{1} i_{2} i_{3} i_{4}}=0$. Obviously,

$$
\mathcal{A} x^{3}=\binom{x_{1}^{3}-x_{2}^{3}+x_{1} x_{2}^{2}}{x_{2}^{3}-x_{1}^{3}+x_{2} x_{1}^{2}}
$$

then

$$
x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}^{4}-x_{1} x_{2}^{3}+x_{1}^{2} x_{2}^{2} \quad \text { and } \quad x_{2}\left(\mathcal{A} x^{3}\right)_{2}=x_{2}^{4}-x_{2} x_{1}^{3}+x_{1}^{2} x_{2}^{2}
$$

We consider the following several cases:

- when $x_{1}=x_{2} \neq 0$, we have $x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}^{4}-x_{1}^{4}+x_{1}^{4}=x_{1}^{4}>0$;
- when only one $x_{i}=0$ for any $i \in\{1,2\}$, then $x_{j}\left(\mathcal{A} x^{3}\right)_{j}=x_{j}^{4}>0$ for $j \neq i$;
- when $x_{1}>x_{2}>0$, we have $x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}\left(x_{1}^{3}-x_{2}^{3}\right)+x_{1}^{2} x_{2}^{2}>0$;
- when $x_{2}>x_{1}>0$, we have $x_{2}\left(\mathcal{A} x^{3}\right)_{2}=x_{2}\left(x_{2}^{3}-x_{1}^{3}\right)+x_{1}^{2} x_{2}^{2}>0$;
- when $0>x_{1}>x_{2}$, we have $x_{2}\left(\mathcal{A} x^{3}\right)_{2}=x_{2}\left(x_{2}^{3}-x_{1}^{3}\right)+x_{1}^{2} x_{2}^{2}>0$;
- when $0>x_{2}>x_{1}$, we have $x_{1}\left(\mathcal{A} x^{3}\right)_{1}=x_{1}\left(x_{1}^{3}-x_{2}^{3}\right)+x_{1}^{2} x_{2}^{2}>0$;
- when $x_{1}>0>x_{2}$, we have $-x_{2} x_{1}^{3}>0$, and so $x_{2}\left(\mathcal{A} x^{3}\right)_{2}>0$;
- when $x_{2}>0>x_{1}$, we have $-x_{1} x_{2}^{3}>0$, and so $x_{1}\left(\mathcal{A} x^{3}\right)_{1}>0$.

Thus, for any $x \in \mathbb{R}^{2} \backslash\{0\}$, there exists an index $i \in\{1,2\}$ such that $x_{i}\left(\mathcal{A} x^{3}\right)_{i}>0$. So $\mathcal{A}$ is a $P$ tensor. However, $\mathcal{A}$ is not a strong $P$ tensor. In fact, if we take $x=(2.1,-1.9)^{T}$ and $y=(2,-2)^{T}$, then we have

$$
\left(x_{1}-y_{1}\right)\left(\left(\mathcal{A} x^{3}\right)_{1}-\left(\mathcal{A} y^{3}\right)_{1}\right)=-0.0299<0
$$

and

$$
\left(x_{2}-y_{2}\right)\left(\left(\mathcal{A} x^{3}\right)_{2}-\left(\mathcal{A} y^{3}\right)_{2}\right)=-0.0499<0
$$

Thus, by Definition 4.3 we obtain that $\mathcal{A}$ is not a strong $P$ tensor.

Example 4.3 demonstrates that the set consists of all strong $P$ tensors is a proper subset of the set consists of all $P$ tensors.

Many properties of the $P$ tensor have been obtained in the literature. Since every strong $P$ tensor is a $P$ tensor, we may easily obtain the following properties of strong $P$ tensor:

Proposition 4.1 If $\mathcal{A} \in T_{m, n}$ is a strong $P$ tensor, then
(i) $\mathcal{A}$ must be strictly semi-positive;
(ii) $\mathcal{A}$ must be an $R$ tensor;
(iii) all of its $H$-eigenvalues and Z-eigenvalues are positive;
(iv) all the diagonal entries of $\mathcal{A}$ are positive;
(v) every principal sub-tensor of $\mathcal{A}$ is still a strong $P$ tensor.

Proof. Since a strong $P$ tensor is a $P$ tensor, the first four results can be easily obtained from [2, 3]. Now we prove the result (v). Let an arbitrary principal subtensor $\mathcal{A}_{r}^{J} \in T_{m, r}$ of the strong $P$ tensor $\mathcal{A} \in T_{m, n}$ be given. We choose any $x=$ $\left(x_{j_{1}}, x_{j_{2}}, \cdots, x_{j_{r}}\right) \in \mathbb{R}^{r} \backslash\{0\}$ and $y=\left(y_{j_{1}}, y_{j_{2}}, \cdots, y_{j_{r}}\right) \in \mathbb{R}^{r} \backslash\{0\}$ with $x \neq y$. Then let $\bar{x}=\left(\bar{x}_{1}, \bar{x}_{2}, \cdots, \bar{x}_{n}\right) \in \mathbb{R}^{n}$ where $\bar{x}_{i}=x_{j_{i}}$ for $i \in J$ and $\bar{x}_{i}=0$ for $i \notin J$. In a similar way, let $\bar{y}=\left(\bar{y}_{1}, \bar{y}_{2}, \cdots, \bar{y}_{n}\right) \in \mathbb{R}^{n}$ where $\bar{y}_{i}=y_{j_{i}}$ for $i \in J$ and $\bar{y}_{i}=0$ for $i \notin J$. Since $\mathcal{A}$ is a strong $P$ tensor, then there exists an index $k \in[n]$ such that

$$
\begin{aligned}
0 & <\max _{k \in[n]}\left(\bar{x}_{k}-\bar{y}_{k}\right)\left(\left(\mathcal{A} \bar{x}^{m-1}\right)_{k}-\left(\mathcal{A} \bar{y}^{m-1}\right)_{k}\right) \\
& =\max _{k \in J}\left(x_{k}-y_{k}\right)\left(\left(\mathcal{A}_{r}^{J} x^{m-1}\right)_{k}-\left(\mathcal{A}_{r}^{J} y^{m-1}\right)_{k}\right) .
\end{aligned}
$$

Thus, $\mathcal{A}_{r}^{J}$ is a strong $P$ tensor.

## 5 Conclusions

By constructing two counter-examples, we proved that $\mathcal{A}$ is a $P$ tensor does not imply that $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property; and that $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property does not imply that $\mathcal{A}$ is a $P$ tensor. These gave a negative answer to Question 6.3 proposed in [2]. We also showed that the solution set of $\operatorname{TCP}(q, \mathcal{A})$ is nonempty and compact when $\mathcal{A}$ is a $P$ tensor.

In order to investigate that for which kind of tensor, the tensor complementarity problem has the GUS-property, we introduced the concept of the strong $P$ tensor, and showed that $\operatorname{TCP}(q, \mathcal{A})$ has the GUS-property when $\mathcal{A}$ is a strong $P$ tensor. We also proved that every strong $P$ tensor is a $P$ tensor; and hence, many known results associated with the $P$ tensor were generalized to the case of the strong $P$ tensor.

Note that the strong $P$ tensor is defined by using the $P$-function, we believe that more properties related to the strong $P$ tensor can be further studied with the help of known methods and results for the $P$-function.

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