# CONVERGENCE RATE FOR A GAUSS COLLOCATION METHOD APPLIED TO UNCONSTRAINED OPTIMAL CONTROL * 

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#### Abstract

A local convergence rate is established for an orthogonal collocation method based on Gauss quadrature applied to an unconstrained optimal control problem. If the continuous problem has a sufficiently smooth solution and the Hamiltonian satisfies a strong convexity condition, then the discrete problem possesses a local minimizer in a neighborhood of the continuous solution, and as the number of collocation points increases, the discrete solution convergences exponentially fast in the sup-norm to the continuous solution. This is the first convergence rate result for an orthogonal collocation method based on global polynomials applied to an optimal control problem.


Key words. Gauss collocation method, convergence rate, optimal control, orthogonal collocation

1. Introduction. A convergence rate is established for an orthogonal collocation method applied to an unconstrained control problem of the form

$$
\begin{array}{cl}
\operatorname{minimize} & C(\mathbf{x}(1)) \\
\text { subject to } & \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad t \in[-1,1]  \tag{1.1}\\
& \mathbf{x}(-1)=\mathbf{x}_{0}
\end{array}
$$

where the state $\mathbf{x}(t) \in \mathbb{R}^{n}, \dot{\mathbf{x}} \equiv \frac{d}{d t} \mathbf{x}$, the control $\mathbf{u}(t) \in \mathbb{R}^{m}, \mathbf{f}: \mathbb{R}^{n} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{n}$, $C: \mathbb{R}^{n} \rightarrow \mathbb{R}$, and $\mathbf{x}_{0}$ is the initial condition, which we assume is given. Assuming the dynamics $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ is nice enough, we can solve for the state $\mathbf{x}$ as a function of the control $\mathbf{u}$, and the control problem reduces to an unconstrained minimization over u.

Let $\mathcal{P}_{N}$ denote the space of polynomials of degree at most $N$ defined on the interval $[-1,+1]$, and let $\mathcal{P}_{N}^{n}$ denote the $n$-fold Cartesian product $\mathcal{P}_{N} \times \ldots \times \mathcal{P}_{N}$. We analyze a discrete approximation to (1.1) of the form

$$
\begin{array}{cl}
\operatorname{minimize} & C(\mathbf{x}(1)) \\
\text { subject to } & \dot{\mathbf{x}}\left(\tau_{i}\right)=\mathbf{f}\left(\mathbf{x}\left(\tau_{i}\right), \mathbf{u}_{i}\right), \quad 1 \leq i \leq N,  \tag{1.2}\\
& \mathbf{x}(-1)=\mathbf{x}_{0}, \quad \mathbf{x} \in \mathcal{P}_{N}^{n}
\end{array}
$$

The collocation points $\tau_{i}, 1 \leq i \leq N$, are where the equation should be satisfied, and $\mathbf{u}_{i}$ is the control approximation at time $\tau_{i}$. The dimension of $\mathcal{P}_{N}$ is $N+1$, while there are $N+1$ equations in (1.2) corresponding to the collocated dynamics at $N$ points and the initial condition. When the discrete dynamics is nice enough, we can solve for the discrete state $\mathbf{x} \in \mathcal{P}_{N}^{n}$ as a function of the discrete controls $\mathbf{u}_{i}, 1 \leq i \leq N$, and the discrete approximation reduces to an unconstrained minimization over the discrete controls.

[^0]We analyze the method developed in [1, 12] where the collocation points are the Gauss quadrature abscissas, or equivalently, the roots of a Legendre polynomial. Other sets of collocation points that have been studied include the Lobatto quadrature points [6, 8, the Chebyshev quadrature points (7, 9, the Radau quadrature points [10, 11, 19, 21, and extrema of Jacobi polynomials [24. The Gauss quadrature points that we analyze are symmetric about $t=0$ and satisfy

$$
-1<\tau_{1}<\tau_{2}<\ldots<\tau_{N}<+1
$$

In addition, we employ two noncollocated points $\tau_{0}=-1$ and $\tau_{N+1}=+1$.
Our goal is to show that if $\left(\mathbf{x}^{*}, \mathbf{u}^{*}\right)$ is a local minimizer for (1.1), then the discrete problem (1.2) has a local minimizer $\left(\mathbf{x}^{N}, \mathbf{u}^{N}\right)$ that converges exponentially fast in $N$ to $\left(\mathbf{x}^{*}, \mathbf{u}^{*}\right)$ at the collocation points. This is the first convergence rate result for an orthogonal collocation method based on global polynomials applied to an optimal control problem. A consistency result for a scheme based on global polynomials and Lobatto collocation is given in 13. Convergence rates have been obtained previously when the approximating space consists of piecewise polynomials as in [2, 3, 5, 4, 14, 18, 22. In these results, convergence is achieved by letting the mesh spacing tend to zero. In our results, on the other hand, convergence is achieved by letting $N$, the degree of the approximating polynomials, tend to infinity.

To state our convergence results in a precise way, we need to introduce a function space setting. Let $\mathcal{C}^{k}\left(\mathbb{R}^{n}\right)$ denote the space of $k$ times continuously differentiable functions $\mathbf{x}:[-1,+1] \rightarrow \mathbb{R}^{n}$ with the sup-norm $\|\cdot\|_{\infty}$ given by

$$
\begin{equation*}
\|\mathbf{x}\|_{\infty}=\sup \{|\mathbf{x}(t)|: t \in[-1,+1]\} \tag{1.3}
\end{equation*}
$$

where $|\cdot|$ is the Euclidean norm. It is assumed that (1.1) has a local minimizer $\left(\mathbf{x}^{*}, \mathbf{u}^{*}\right)$ in $\mathcal{C}^{1}\left(\mathbb{R}^{n}\right) \times \mathcal{C}^{0}\left(\mathbb{R}^{m}\right)$. Given $\mathbf{y} \in \mathbb{R}^{n}$, the ball with center $\mathbf{y}$ and radius $\rho$ is denoted

$$
\mathcal{B}_{\rho}(\mathbf{y})=\left\{\mathbf{x} \in \mathbb{R}^{n}:|\mathbf{x}-\mathbf{y}| \leq \rho\right\}
$$

It is assumed that there exists an open set $\Omega \subset \mathbb{R}^{m+n}$ and $\rho>0$ such that

$$
\mathcal{B}_{\rho}\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t)\right) \subset \Omega \text { for all } t \in[-1,+1]
$$

Moreover, the first two derivative of $f$ and $C$ are continuous on the closure of $\Omega$ and on $\mathcal{B}_{\rho}\left(\mathbf{x}^{*}(1)\right)$ respectively.

Let $\boldsymbol{\lambda}^{*}$ denote the solution of the linear costate equation

$$
\begin{equation*}
\dot{\boldsymbol{\lambda}}^{*}(t)=-\nabla_{x} H\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t), \boldsymbol{\lambda}^{*}(t)\right), \quad \boldsymbol{\lambda}^{*}(1)=\nabla C\left(\mathbf{x}^{*}(1)\right) \tag{1.4}
\end{equation*}
$$

where $H$ is the Hamiltonian defined by $H(\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda})=\boldsymbol{\lambda}^{\top} \mathbf{f}(\mathbf{x}, \mathbf{u})$. Here $\nabla C$ denotes the gradient of $C$. By the first-order optimality conditions (Pontryagin's minimum principle), we have

$$
\begin{equation*}
\nabla_{u} H\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t), \boldsymbol{\lambda}^{*}(t)\right)=\mathbf{0} \tag{1.5}
\end{equation*}
$$

for all $t \in[-1,+1]$.
Since the discrete collocation problem (1.2) is finite dimensional, the first-order optimality conditions (Karush-Kuhn-Tucker conditions) imply that when a constraint qualification holds [20], the gradient of the Lagrangian vanishes. By the analysis in
[12], the gradient of the Lagrangian vanishes if and only if there exists $\boldsymbol{\lambda} \in \mathcal{P}_{N}^{n}$ such that

$$
\begin{align*}
\dot{\boldsymbol{\lambda}}\left(\tau_{i}\right) & =-\nabla_{x} H\left(\mathbf{x}\left(\tau_{i}\right), \mathbf{u}_{i}, \boldsymbol{\lambda}\left(\tau_{i}\right)\right), \quad 1 \leq i \leq N  \tag{1.6}\\
\boldsymbol{\lambda}(+1) & =\nabla C(\mathbf{x}(+1))  \tag{1.7}\\
\mathbf{0} & =\nabla_{u} H\left(\mathbf{x}\left(\tau_{i}\right), \mathbf{u}_{i}, \boldsymbol{\lambda}\left(\tau_{i}\right)\right), \quad 1 \leq i \leq N \tag{1.8}
\end{align*}
$$

The assumptions that play a key role in the convergence analysis are the following: (A1) $\mathbf{x}^{*}$ and $\boldsymbol{\lambda}^{*} \in \mathcal{C}^{\eta+1}$ for some $\eta \geq 3$.
(A2) For some $\alpha>0$, the smallest eigenvalue of the Hessian matrices

$$
\nabla^{2} C\left(\mathbf{x}^{*}(1)\right) \quad \text { and } \quad \nabla_{(x, u)}^{2} H\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t), \boldsymbol{\lambda}^{*}(t)\right)
$$

is greater than $\alpha$, uniformly for $t \in[-1,+1]$.
(A3) The Jacobian of the dynamics satisfies

$$
\left\|\nabla_{x} \mathbf{f}\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t)\right)\right\|_{\infty} \leq 1 / 4 \quad \text { and } \quad\left\|\nabla_{x} \mathbf{f}\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t)\right)^{\boldsymbol{\top}}\right\|_{\infty} \leq 1 / 4
$$

for all $t \in[-1,+1]$ where $\|\cdot\|_{\infty}$ is the matrix sup-norm (largest absolute row sum), and the Jacobian $\nabla_{x} \mathbf{f}$ is an $n$ by $n$ matrix whose $i$-th row is $\left(\nabla_{x} f_{i}\right)^{\top}$.
The smoothness assumption (A1) is used to obtain a bound for the accuracy with which the interpolant of the continuous state $\mathbf{x}^{*}$ satisfies the discrete dynamics. The coercivity assumption (A2) ensures that the solution of the discrete problem is a local minimizer. The condition (A3) does not appear in convergence analysis for (local) piecewise polynomial techniques [2, 3, 5, 4, 14, 18, 22, It arises when we approximate a solution by polynomials defined on the entire interval $[-1,+1]$. More precisely, in the analysis, the dynamics is linearized around ( $\mathbf{x}^{*}, \mathbf{u}^{*}$ ), and (A3) implies that when we perturb the linearized dynamics, the state perturbation is bounded uniformly in $N$ with respect to the perturbation in the dynamics. If the domain $[-1,+1]$ is partitioned into uniform subdomains of width $h$ and a different polynomial is used on each subdomain, then (A3) is replaced by

$$
\left\|\nabla_{x} \mathbf{f}\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t)\right)\right\|_{\infty} \leq 1 /(2 h) \quad \text { and } \quad\left\|\nabla_{x} \mathbf{f}\left(\mathbf{x}^{*}(t), \mathbf{u}^{*}(t)\right)^{\top}\right\|_{\infty} \leq 1 /(2 h)
$$

which is satisfied when $h$ is sufficiently small. In general, (A3) could be replaced by any condition that ensures stability of the linearized dynamics.

In addition to the 3 assumptions, the analysis employs 2 properties of the Gauss collocation scheme. Let $\omega_{j}, 1 \leq j \leq N$, denote the Gauss quadrature weights, and for $1 \leq i \leq N$ and $0 \leq j \leq N$, define

$$
\begin{equation*}
D_{i j}=\dot{L}_{j}\left(\tau_{i}\right), \quad \text { where } L_{j}(\tau):=\prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{\tau-\tau_{i}}{\tau_{j}-\tau_{i}} \tag{1.9}
\end{equation*}
$$

$\mathbf{D}$ is a differentiation matrix in the sense that $(\mathbf{D} \mathbf{p})_{i}=\dot{p}\left(\tau_{i}\right), 1 \leq i \leq N$, where $p \in \mathcal{P}_{N}$ is the polynomial that satisfies $p\left(\tau_{j}\right)=p_{j}$ for $0 \leq j \leq N$. The submatrix $\mathbf{D}_{1: N}$ consisting of the tailing $N$ columns of $\mathbf{D}$, has the following properties:
(P1) $\mathbf{D}_{1: N}$ is invertible and $\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty} \leq 2$.
(P2) If $\mathbf{W}$ is the diagonal matrix containing the quadrature weights $\boldsymbol{\omega}$ on the diagonal, then the rows of the matrix $\left[\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right]^{-1}$ have Euclidean norm bounded by $\sqrt{2}$.

The fact that $\mathbf{D}_{1: N}$ is invertible is established in [12, Prop. 1]. The bounds on the norms in (P1) and (P2), however, are more subtle. We refer to (P1) and (P2) as properties rather than assumptions since the matrices are readily evaluated, and we can check numerically that (P1) and (P2) are always satisfied. In fact, numerically we find that $\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty}=1+\tau_{N}$ where the last Gauss quadrature abscissa $\tau_{N}$ approaches +1 as $N$ tends to $\infty$. On the other hand, we do not yet have a general proof for the properties (P1) and (P2). In contrast, conditions (A1)-(A3) are assumptions that are only satisfied by certain control problems.

If $\mathbf{x}^{N} \in \mathcal{P}_{N}^{n}$ is a solution of (1.2) associated with the discrete controls $\mathbf{u}_{i}, 1 \leq i \leq$ $N$, and if $\boldsymbol{\lambda}^{N} \in \mathcal{P}_{N}^{n}$ satisfies (1.6)-(1.8), then we define

$$
\begin{aligned}
& \mathbf{X}^{N}=\left[\begin{array}{llll}
\mathbf{x}^{N}(-1), & \mathbf{x}^{N}\left(\tau_{1}\right), & \ldots, & \mathbf{x}^{N}\left(\tau_{N}\right), \\
\mathbf{x}^{N}(+1)
\end{array}\right], \\
& \mathbf{X}^{*}=\left[\begin{array}{lllll}
\mathbf{x}^{*}(-1), & \mathbf{x}^{*}\left(\tau_{1}\right), & \ldots, & \mathbf{x}^{*}\left(\tau_{N}\right), & \mathbf{x}^{*}(+1)
\end{array}\right], \\
& \mathbf{U}^{N}=\left[\quad \mathbf{u}_{1}, \quad \ldots, \mathbf{u}_{N}\right], \\
& \mathbf{U}^{*}=\left[\begin{array}{lll}
\mathbf{u}^{*}\left(\tau_{1}\right), & \ldots, & \mathbf{u}^{*}\left(\tau_{N}\right)
\end{array}\right] \text {, } \\
& \boldsymbol{\Lambda}^{N}=\left[\begin{array}{llll}
\boldsymbol{\lambda}^{N}(-1), & \boldsymbol{\lambda}^{N}\left(\tau_{1}\right), & \ldots, & \boldsymbol{\lambda}^{N}\left(\tau_{N}\right),
\end{array} \boldsymbol{\lambda}^{N}(+1)\right], \\
& \boldsymbol{\Lambda}^{*}=\left[\begin{array}{llll}
\boldsymbol{\lambda}^{*}(-1), & \boldsymbol{\lambda}^{*}\left(\tau_{1}\right), & \ldots, & \boldsymbol{\lambda}^{*}\left(\tau_{N}\right),
\end{array} \quad \boldsymbol{\lambda}^{*}(+1) \quad\right] .
\end{aligned}
$$

For any of the discrete variables, we define a discrete sup-norm analogous to the continuous sup-norm in (1.3). For example, if $\mathbf{U}^{N} \in \mathbb{R}^{m N}$ with $\mathbf{U}_{i} \in \mathbb{R}^{m}$, then

$$
\left\|\mathbf{U}^{N}\right\|_{\infty}=\sup \left\{\left|\mathbf{U}_{i}\right|: 1 \leq i \leq N\right\}
$$

The following convergence result is established:
Theorem 1.1. If $\left(\mathbf{x}^{*}, \mathbf{u}^{*}\right)$ is a local minimizer for the continuous problem (1.1) and both (A1)-(A3) and (P1)-(P2) hold, then for $N$ sufficiently large with $N>\eta+1$, the discrete problem (1.2) has a local minimizer $\left(\mathbf{X}^{N}, \mathbf{U}^{N}\right)$ and an associated discrete costate $\boldsymbol{\Lambda}^{N}$ for which

$$
\begin{equation*}
\max \left\{\left\|\mathbf{X}^{N}-\mathbf{X}^{*}\right\|_{\infty},\left\|\mathbf{U}^{N}-\mathbf{U}^{*}\right\|_{\infty},\left\|\boldsymbol{\Lambda}^{N}-\mathbf{\Lambda}^{*}\right\|_{\infty}\right\} \leq c N^{2-\eta} \tag{1.10}
\end{equation*}
$$

where $c$ is independent of $N$.
Although the discrete problem only possesses discrete controls at the collocation points $-1<\tau_{i}<+1,1 \leq i \leq N$, an estimate for the discrete control at $t=$ -1 and $t=+1$ is usually obtained from the minimum principle (1.5) since we do have estimates for the discrete state and costate at the end points. Alternatively, polynomial interpolation could be used to obtain estimates for the control at the end points of the interval.

The paper is organized as follows. In Section 2 the discrete optimization problem (1.2) is reformulated as a nonlinear system of equations obtained from the first-order optimality conditions, and a general approach to convergence analysis is presented. Section 3 obtains an estimate for how closely the solution to the continuous problem satisfies the first-order optimality conditions for the discrete problem. Section 4 proves that the linearization of the discrete control problem around a solution of the continuous problem is invertible. Section 5 establishes an $L^{2}$ stability property for the linearization, while Section 6 strengthens the norm to $L^{\infty}$. This stability property is the basis for the proof of Theorem 1.1 A numerical example illustrating the exponential convergence result is given in Section 7 .

Notation. The meaning of the norm $\|\cdot\|_{\infty}$ is based on context. If $\mathbf{x} \in \mathcal{C}^{0}\left(\mathbb{R}^{n}\right)$, then $\|\mathbf{x}\|_{\infty}$ denotes the maximum of $|\mathbf{x}(t)|$ over $t \in[-1,+1]$, where $|\cdot|$ is the Euclidean norm. If $\mathbf{A} \in \mathbb{R}^{m \times n}$, then $\|\mathbf{A}\|_{\infty}$ is the largest absolute row sum (the matrix norm
induces by the $\ell_{\infty}$ vector norm). If $\mathbf{U} \in \mathbb{R}^{m N}$ is the discrete control with $\mathbf{U}_{i} \in \mathbb{R}^{m}$, then $\|\mathbf{U}\|_{\infty}$ is the maximum of $\left|\mathbf{U}_{i}\right|, 1 \leq i \leq N$. The dimension of the identity matrix $\mathbf{I}$ is often clear from context; when necessary, the dimension of $\mathbf{I}$ is specified by a subscript. For example, $\mathbf{I}_{n}$ is the $n$ by $n$ identity matrix. $\nabla C$ denotes the gradient, a column vector, while $\nabla^{2} C$ denotes the Hessian matrix. Throughout the paper, $c$ denotes a generic constant which has different values in different equations. The value of this constant is always independent of $N . \mathbf{1}$ denotes a vector whose entries are all equal to one, while $\mathbf{0}$ is a vector whose entries are all equal to zero, their dimension should be clear from context.
2. Abstract setting. As shown in [12], the discrete problem (1.2) can be reformulated as the nonlinear programming problem

$$
\begin{array}{ll}
\operatorname{minimize} & C\left(\mathbf{X}_{N+1}\right) \\
\text { subject to } & \sum_{j=0}^{N} D_{i j} \mathbf{X}_{j}=\mathbf{f}\left(\mathbf{X}_{i}, \mathbf{U}_{i}\right), \quad 1 \leq i \leq N, \quad \mathbf{X}_{0}=\mathbf{x}_{0}  \tag{2.1}\\
& \mathbf{X}_{N+1}=\mathbf{X}_{0}+\sum_{j=1}^{N} \omega_{j} \mathbf{f}\left(\mathbf{X}_{j}, \mathbf{U}_{j}\right)
\end{array}
$$

As indicated before Theorem 1.1, $\mathbf{X}_{i}$ corresponds to $\mathbf{x}^{N}\left(\tau_{i}\right)$. Also, [12] shows that the equations obtained by setting the gradient of the Lagrangian to zero are equivalent to the system of equations

$$
\begin{align*}
\sum_{j=1}^{N+1} D_{i j}^{\dagger} \boldsymbol{\Lambda}_{j} & =-\nabla_{x} H\left(\mathbf{X}_{i}, \mathbf{U}_{i}, \boldsymbol{\Lambda}_{i}\right), \quad 1 \leq i \leq N, \quad \boldsymbol{\Lambda}_{N+1}=\nabla C\left(\mathbf{X}_{N+1}\right)  \tag{2.2}\\
\mathbf{0} & =\nabla_{u} H\left(\mathbf{X}_{i}, \mathbf{U}_{i}, \boldsymbol{\Lambda}_{i}\right), \quad 1 \leq i \leq N \tag{2.3}
\end{align*}
$$

where

$$
\begin{align*}
D_{i j}^{\dagger} & =-\left(\frac{\omega_{j}}{\omega_{i}}\right) D_{j i}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq N  \tag{2.4}\\
D_{i, N+1}^{\dagger} & =-\sum_{j=1}^{N} D_{i j}^{\dagger}, \quad 1 \leq i \leq N \tag{2.5}
\end{align*}
$$

Here $\boldsymbol{\Lambda}_{i}$ corresponds to $\boldsymbol{\lambda}^{N}\left(\tau_{i}\right)$. The relationship between the discrete costate $\boldsymbol{\Lambda}_{i}$, the KKT multipliers $\boldsymbol{\lambda}_{i}$ associated with the discrete dynamics, and the multiplier $\boldsymbol{\lambda}_{N+1}$ associated with the equation for $\mathbf{X}_{N+1}$ is

$$
\begin{equation*}
\omega_{i} \boldsymbol{\Lambda}_{i}=\boldsymbol{\lambda}_{i}+\omega_{i} \boldsymbol{\lambda}_{N+1} \quad \text { when } \quad 1 \leq i \leq N, \quad \text { and } \quad \boldsymbol{\Lambda}_{N+1}=\boldsymbol{\lambda}_{N+1} \tag{2.6}
\end{equation*}
$$

The first-order optimality conditions for the nonlinear program (2.1) consist of the equations (2.2) and (2.3), and the constraints in (2.1). This system can be written as $\mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\mathbf{0}$ where

$$
\left(\mathcal{T}_{1}, \mathcal{T}_{2}, \ldots, \mathcal{T}_{5}\right)(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) \in \mathbb{R}^{n N} \times \mathbb{R}^{n} \times \mathbb{R}^{n N} \times \mathbb{R}^{n} \times \mathbb{R}^{m N}
$$

The 5 components of $\mathcal{T}$ are defined as follows:

$$
\begin{aligned}
& \mathcal{T}_{1 i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\left(\sum_{j=0}^{N} D_{i j} \mathbf{X}_{j}\right)-\mathbf{f}\left(\mathbf{X}_{i}, \mathbf{U}_{i}\right), \quad 1 \leq i \leq N \\
& \mathcal{T}_{2}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\mathbf{X}_{N+1}-\mathbf{X}_{0}-\sum_{j=1}^{N} \omega_{j} \mathbf{f}\left(\mathbf{X}_{j}, \mathbf{U}_{j}\right) \\
& \mathcal{T}_{3 i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\left(\sum_{j=1}^{N+1} D_{i j}^{\dagger} \boldsymbol{\Lambda}_{j}\right)+\nabla_{x} H\left(\mathbf{X}_{i}, \mathbf{U}_{i}, \mathbf{\Lambda}_{i}\right), \quad 1 \leq i \leq N, \\
& \mathcal{T}_{4}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\boldsymbol{\Lambda}_{N+1}-\nabla_{x} C\left(\mathbf{X}_{N+1}\right), \\
& \mathcal{T}_{5 i}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\nabla_{u} H\left(\mathbf{X}_{i}, \mathbf{U}_{i}, \mathbf{\Lambda}_{i}\right), \quad 1 \leq i \leq N
\end{aligned}
$$

Note that in formulating $\mathcal{T}$, we treat $\mathbf{X}_{0}$ as a constant whose value is the given starting condition $\mathbf{x}_{0}$. Alternatively, we could treat $\mathbf{X}_{0}$ as an unknown and then expand $\mathcal{T}$ to have a 6 -th component $\mathbf{X}_{0}-\mathbf{x}_{0}$. With this expansion of $\mathcal{T}$, we need to introduce an additional multiplier $\boldsymbol{\Lambda}_{0}$ for the constraint $\mathbf{X}_{0}-\mathbf{x}_{0}$. To achieve a slight simplification in the analysis, we employ a 5 -component $\mathcal{T}$ and treat $\mathbf{X}_{0}$ as a constant, not an unknown.

The proof of Theorem 1.1 reduces to a study of solutions to $\mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\mathbf{0}$ in a neighborhood of $\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \mathbf{\Lambda}^{*}\right)$. Our analysis is based on [5, Proposition 3.1], which we simplify below to take into account the structure of our $\mathcal{T}$. Other results like this are contained in Theorem 3.1 of [3], in Proposition 5.1 of [14], and in Theorem 2.1 of [15.

Proposition 2.1. Let $\mathcal{X}$ be a Banach space and $\mathcal{Y}$ be a linear normed space with the norms in both spaces denoted $\|\cdot\|$. Let $\mathcal{T}: \mathcal{X} \longmapsto \mathcal{Y}$ with $\mathcal{T}$ continuously Fréchet differentiable in $\mathcal{B}_{r}\left(\boldsymbol{\theta}^{*}\right)$ for some $\boldsymbol{\theta}^{*} \in \mathcal{X}$ and $r>0$. Suppose that

$$
\left\|\nabla \mathcal{T}(\boldsymbol{\theta})-\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)\right\| \leq \varepsilon \text { for all } \boldsymbol{\theta} \in \mathcal{B}_{r}\left(\boldsymbol{\theta}^{*}\right)
$$

where $\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)$ is invertible, and define $\mu:=\left\|\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)^{-1}\right\|$. If $\varepsilon \mu<1$ and $\left\|\mathcal{T}\left(\boldsymbol{\theta}^{*}\right)\right\| \leq$ $(1-\mu \varepsilon) r / \mu$, then there exists a unique $\boldsymbol{\theta} \in \mathcal{B}_{r}\left(\boldsymbol{\theta}^{*}\right)$ such that $\mathcal{T}(\boldsymbol{\theta})=\mathbf{0}$. Moreover, we have the estimate

$$
\begin{equation*}
\left\|\boldsymbol{\theta}-\boldsymbol{\theta}^{*}\right\| \leq \frac{\mu}{1-\mu \varepsilon}\left\|\mathcal{T}\left(\boldsymbol{\theta}^{*}\right)\right\| \leq r \tag{2.7}
\end{equation*}
$$

We apply Proposition 2.1 with $\boldsymbol{\theta}^{*}=\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ and $\boldsymbol{\theta}=\left(\mathbf{X}^{N}, \mathbf{U}^{N}, \boldsymbol{\Lambda}^{N}\right)$. The key steps in the analysis are the estimation of the residual $\left\|\mathcal{T}\left(\boldsymbol{\theta}^{*}\right)\right\|$, the proof that $\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)$ is invertible, and the derivation of a bound for $\left\|\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)^{-1}\right\|$ that is independent of $N$. In our context, for the norm in $\mathcal{X}$, we take

$$
\begin{equation*}
\|\boldsymbol{\theta}\|=\|(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})\|_{\infty}=\max \left\{\|\mathbf{X}\|_{\infty},\|\mathbf{U}\|_{\infty},\|\boldsymbol{\Lambda}\|_{\infty}\right\} \tag{2.8}
\end{equation*}
$$

For this norm, the left side of (1.10) and the left side of (2.7) are the same. The norm on $\mathcal{Y}$ enters into the estimation of both the residual $\left\|\mathcal{T}\left(\boldsymbol{\theta}^{*}\right)\right\|$ in (2.7) and the parameter $\mu:=\left\|\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)^{-1}\right\|$. In our context, we think of an element of $\mathcal{Y}$ as a vector with components $\mathbf{y}_{i}, 1 \leq i \leq 3 N+2$, where $\mathbf{y}_{i} \in \mathbb{R}^{n}$ for $1 \leq i \leq 2 N+2$ and $\mathbf{y}_{i} \in \mathbb{R}^{m}$ for $i>2 N+2$. For example, $\mathcal{T}_{1}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}) \in \mathbb{R}^{n N}$ corresponds to the components $\mathbf{y}_{i} \in \mathbb{R}^{n}, 1 \leq i \leq N$. For the norm in $\mathcal{Y}$, we take

$$
\begin{equation*}
\|\mathbf{y}\|_{\infty}=\sup \left\{\left|\mathbf{y}_{i}\right|: 1 \leq i \leq 3 N+2\right\} \tag{2.9}
\end{equation*}
$$

3. Analysis of the residual. We now establish a bound for the residual.

Lemma 3.1. If (A1) holds, then there exits a constant $c$ independent of $N$ such that

$$
\begin{equation*}
\left\|\mathcal{T}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \mathbf{\Lambda}^{*}\right)\right\|_{\infty} \leq c N^{2-\eta} \tag{3.1}
\end{equation*}
$$

for all $N>\eta+1$.
Proof. By the definition of $\mathcal{T}, \mathcal{T}_{4}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)=\mathbf{0}$ since $\mathbf{x}^{*}$ and $\boldsymbol{\lambda}^{*}$ satisfy the boundary condition in (1.4). Likewise, $\mathcal{T}_{5}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)=\mathbf{0}$ since (1.5) holds for all $t \in[-1,+1]$; in particular, (1.5) holds at the collocation points.

Now let us consider $\mathcal{T}_{1}$. By [12, Eq. (7)],

$$
\sum_{j=0}^{N} D_{i j} \mathbf{X}_{j}^{*}=\dot{\mathbf{x}}^{I}\left(\tau_{i}\right), \quad 1 \leq i \leq N
$$

where $\mathbf{x}^{I} \in \mathcal{P}_{N}^{n}$ is the (interpolating) polynomial that passes through $\mathbf{x}^{*}\left(\tau_{i}\right)$ for $0 \leq$ $i \leq N$. Since $\mathbf{x}^{*}$ satisfies the dynamics of (1.1), it follows that $\mathbf{f}\left(\mathbf{X}_{i}^{*}, \mathbf{U}_{i}^{*}\right)=\dot{\mathbf{x}}^{*}\left(\tau_{i}\right)$. Hence, we have

$$
\begin{equation*}
\mathcal{T}_{1 i}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \mathbf{\Lambda}^{*}\right)=\dot{\mathbf{x}}^{I}\left(\tau_{i}\right)-\dot{\mathbf{x}}^{*}\left(\tau_{i}\right) \tag{3.2}
\end{equation*}
$$

We combine Proposition 2.1 and Lemma 2.2 in [16] to obtain for $N>\eta+1$,

$$
\left\|\dot{\mathbf{x}}^{I}-\dot{\mathbf{x}}^{*}\right\|_{\infty} \leq\left(\frac{6 e}{N-1}\right)^{\eta}\left[\left(1+2 N^{2}\right)+6 e N\left(1+c_{1} \sqrt{N}\right)\right]\left(\frac{12\left\|x^{(\eta+1)}\right\|}{\eta+1}\right)
$$

where $\mathbf{x}^{(\eta+1)}$ is the $(\eta+1)$-st derivative of $\mathbf{x}$ and $c_{1} \sqrt{N}$ is a bound for the Lebesgue constant of the point set $\tau_{i}, 0 \leq i \leq N$, given in Theorem 4.1 of [16]. Hence, there exists a constant $c_{2}$, independent of $N$ but dependent on $\eta$, such that

$$
\begin{equation*}
\left\|\dot{\mathbf{x}}^{I}-\dot{\mathbf{x}}^{*}\right\|_{\infty} \leq c_{2} N^{2-\eta} \tag{3.3}
\end{equation*}
$$

Consequently, $\mathcal{T}_{1}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \mathbf{\Lambda}^{*}\right)$ complies with the bound (3.1).
Next, let us consider

$$
\begin{equation*}
\mathcal{T}_{2}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)=\mathbf{x}^{*}(1)-\mathbf{x}^{*}(-1)-\sum_{j=1}^{N} \omega_{j} \mathbf{f}\left(\mathbf{x}^{*}\left(\tau_{j}\right), \mathbf{u}^{*}\left(\tau_{j}\right)\right) \tag{3.4}
\end{equation*}
$$

By the fundamental theorem of calculus and the fact that $N$-point Gauss quadrature is exact for polynomials of degree up to $2 N-1$, we have

$$
\begin{equation*}
\mathbf{0}=\mathbf{x}^{I}(1)-\mathbf{x}^{I}(-1)-\int_{-1}^{1} \dot{\mathbf{x}}^{I}(t) d t=\mathbf{x}^{I}(1)-\mathbf{x}^{I}(-1)-\sum_{j=1}^{N} \omega_{j} \dot{\mathbf{x}}^{I}\left(\tau_{j}\right) \tag{3.5}
\end{equation*}
$$

Subtract (3.5) from (3.4) to obtain

$$
\begin{equation*}
\mathcal{T}_{2}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)=\mathbf{x}^{*}(1)-\mathbf{x}^{I}(1)+\sum_{j=1}^{N} \omega_{j}\left(\dot{\mathbf{x}}^{I}\left(\tau_{j}\right)-\dot{\mathbf{x}}^{*}\left(\tau_{j}\right)\right) \tag{3.6}
\end{equation*}
$$

Since $\omega_{i}>0$ and their sum is 2 , it follows (3.3) that

$$
\begin{equation*}
\left|\sum_{j=1}^{N} \omega_{j}\left(\dot{\mathbf{x}}^{N}\left(\tau_{j}\right)-\dot{\mathbf{x}}^{*}\left(\tau_{j}\right)\right)\right| \leq 2 c_{2} N^{2-\eta} \tag{3.7}
\end{equation*}
$$

By Theorem 15.1 in [23] and Lemma 2.2 and Theorem 4.1 in [16], we have

$$
\begin{align*}
\left|\mathbf{x}^{*}(1)-\mathbf{x}^{I}(1)\right| & \leq\left\|\mathbf{x}^{*}-\mathbf{x}^{I}\right\|_{\infty} \\
& \leq\left(1+c_{1} \sqrt{N}\right)\left(\frac{12}{\eta+2}\right)\left(\frac{6 e}{N}\right)^{\eta+1}\left\|\mathbf{x}^{(\eta+1)}\right\|_{\infty} \tag{3.8}
\end{align*}
$$

We combine (3.6) (3.8) to see that $\mathcal{T}_{2}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ complies with the bound (3.1).
Finally, let us consider $\mathcal{T}_{3}$. By [12, Thm. 1],

$$
\sum_{j=1}^{N+1} D_{i j}^{\dagger} \boldsymbol{\Lambda}_{j}^{*}=\dot{\lambda}^{I}\left(\tau_{i}\right), \quad 1 \leq i \leq N
$$

where $\boldsymbol{\lambda}^{I} \in \mathcal{P}_{N}^{n}$ is the (interpolating) polynomial that passes through $\boldsymbol{\Lambda}_{j}^{*}=\boldsymbol{\lambda}\left(\tau_{j}\right)$ for $1 \leq j \leq N+1$. Since $\boldsymbol{\lambda}^{*}$ satisfies (1.4), it follows that $\dot{\boldsymbol{\lambda}}^{*}\left(\tau_{i}\right)=-\nabla_{x} H\left(\mathbf{X}_{i}^{*}, \mathbf{U}_{i}^{*}, \boldsymbol{\Lambda}_{i}^{*}\right)$. Hence, we have

$$
\mathcal{T}_{3 i}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)=\dot{\boldsymbol{\lambda}}^{I}\left(\tau_{i}\right)-\dot{\boldsymbol{\lambda}}^{*}\left(\tau_{i}\right)
$$

Exactly as we handled $\mathcal{T}_{1}$ in (3.2), we conclude that $\mathcal{T}_{3}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ complies with the bound (3.1). This completes the proof. ㅁ
4. Invertibility. In this section, we show that the derivative $\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)$ is invertible. This is equivalent to showing that for each $\mathbf{y} \in \mathcal{Y}$, there is a unique $\boldsymbol{\theta} \in \mathcal{X}$ such that $\nabla \mathcal{T}\left(\boldsymbol{\theta}^{*}\right)[\boldsymbol{\theta}]=\mathbf{y}$. In our application, $\boldsymbol{\theta}^{*}=\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ and $\boldsymbol{\theta}=(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})$. To simplify the notation, we let $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]$ denote the derivative of $\mathcal{T}$ evaluated at $\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ operating on $(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})$. This derivative involves the following 6 matrices:

$$
\begin{array}{ll}
\mathbf{A}_{i}=\nabla_{x} \mathbf{f}\left(\mathbf{x}^{*}\left(\tau_{i}\right), \mathbf{u}^{*}\left(\tau_{i}\right)\right), & \mathbf{B}_{i}=\nabla_{u} \mathbf{f}\left(\mathbf{x}^{*}\left(\tau_{i}\right), \mathbf{u}^{*}\left(\tau_{i}\right)\right), \\
\mathbf{Q}_{i}=\nabla_{x x} H\left(\mathbf{x}^{*}\left(\tau_{i}\right), \mathbf{u}^{*}\left(\tau_{i}\right), \boldsymbol{\lambda}^{*}\left(\tau_{i}\right)\right), & \mathbf{S}_{i}=\nabla_{u x} H\left(\mathbf{x}^{*}\left(\tau_{i}\right), \mathbf{u}^{*}\left(\tau_{i}\right), \boldsymbol{\lambda}^{*}\left(\tau_{i}\right)\right), \\
\mathbf{R}_{i}=\nabla_{u u} H\left(\mathbf{x}^{*}\left(\tau_{i}\right), \mathbf{u}^{*}\left(\tau_{i}\right), \boldsymbol{\lambda}^{*}\left(\tau_{i}\right)\right), & \mathbf{T}=\nabla^{2} C\left(\mathbf{x}^{*}(1)\right)
\end{array}
$$

With this notation, the 5 components of $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]$ are as follows:

$$
\begin{aligned}
& \nabla \mathcal{T}_{1 i}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=\left(\sum_{j=1}^{N} D_{i j} \mathbf{X}_{j}\right)-\mathbf{A}_{i} \mathbf{X}_{i}-\mathbf{B}_{i} \mathbf{U}_{i}, \quad 1 \leq i \leq N \\
& \nabla \mathcal{T}_{2}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=\mathbf{X}_{N+1}-\sum_{j=1}^{N} \omega_{j}\left(\mathbf{A}_{j} \mathbf{X}_{j}+\mathbf{B}_{j} \mathbf{U}_{j}\right) \\
& \nabla \mathcal{T}_{3 i}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=\left(\sum_{j=1}^{N+1} D_{i j}^{\dagger} \boldsymbol{\Lambda}_{j}\right)+\mathbf{A}_{i}^{\top} \mathbf{\Lambda}_{i}+\mathbf{Q}_{i} \mathbf{X}_{i}+\mathbf{S}_{i} \mathbf{U}_{i}, \quad 1 \leq i \leq N \\
& \nabla \mathcal{T}_{4}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{\Lambda}_{N+1}-\mathbf{T} \mathbf{X}_{N+1} \\
& \nabla \mathcal{T}_{5 i}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=\mathbf{S}_{i}^{\top} \mathbf{X}_{i}+\mathbf{R}_{i} \mathbf{U}_{i}+\mathbf{B}_{i}^{\top} \mathbf{\Lambda}_{i}, \quad 1 \leq i \leq N
\end{aligned}
$$

Notice that $\mathbf{X}_{0}$ does not appear in $\nabla \mathcal{T}^{*}$ since $\mathbf{X}_{0}$ is treated as a constant whose gradient vanishes.

The analysis of invertibility starts with the first component of $\nabla \mathcal{T}^{*}$.
Lemma 4.1. If (P1) and (A3) hold, then for each $\mathbf{q} \in \mathbb{R}^{n}$ and $\mathbf{p} \in \mathbb{R}^{n N}$ with $\mathbf{p}_{i} \in \mathbb{R}^{n}$, the linear system

$$
\begin{align*}
\left(\sum_{j=1}^{N} D_{i j} \mathbf{X}_{j}\right)-\mathbf{A}_{i} \mathbf{X}_{i}=\mathbf{p}_{i} \quad 1 \leq i \leq N  \tag{4.1}\\
\mathbf{X}_{N+1}-\sum_{j=1}^{N} \omega_{j}\left(\mathbf{A}_{j} \mathbf{X}_{j}+\mathbf{B}_{j} \mathbf{U}_{j}\right)=\mathbf{q} \tag{4.2}
\end{align*}
$$

has a unique solution $\mathbf{X}_{j} \in \mathbb{R}^{n}, 1 \leq j \leq N+1$. This solution has the bound

$$
\begin{equation*}
\left\|\mathbf{X}_{j}\right\|_{\infty} \leq 4\|\mathbf{p}\|_{\infty}+\|\mathbf{q}\|_{\infty}, \quad 1 \leq j \leq N+1 \tag{4.3}
\end{equation*}
$$

Proof. Let $\overline{\mathbf{X}}$ be the vector obtained by vertically stacking $\mathbf{X}_{1}$ through $\mathbf{X}_{N}$, let A be the block diagonal matrix with $i$-th diagonal block $\mathbf{A}_{i}, 1 \leq i \leq N$, and define $\overline{\mathbf{D}}=\mathbf{D}_{1: N} \otimes \mathbf{I}_{n}$ where $\otimes$ is the Kronecker product. With this notation, the linear system (4.1) can be expressed $(\overline{\mathbf{D}}-\mathbf{A}) \overline{\mathbf{X}}=\mathbf{p}$. By $(\mathrm{P} 1) \mathbf{D}_{1: N}$ is invertible which implies that $\overline{\mathbf{D}}$ is invertible with $\overline{\mathbf{D}}^{-1}=\mathbf{D}_{1: N}^{-1} \otimes \mathbf{I}_{n}$. Moreover, $\left\|\overline{\mathbf{D}}^{-1}\right\|_{\infty}=\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty} \leq 2$ by (P1). By (A3) $\|\mathbf{A}\|_{\infty} \leq 1 / 4$ and $\left\|\overline{\mathbf{D}}^{-1} \mathbf{A}\right\|_{\infty} \leq\left\|\overline{\mathbf{D}}^{-1}\right\|_{\infty}\|\mathbf{A}\|_{\infty} \leq 1 / 2$. By [17, p. $351], \mathbf{I}-\overline{\mathbf{D}}^{-1} \mathbf{A}$ is invertible and $\left\|\left(\mathbf{I}-\mathbf{D}^{-1} \mathbf{A}\right)^{-1}\right\|_{\infty} \leq 2$. Consequently, $\overline{\mathbf{D}}-\mathbf{A}=$ $\overline{\mathbf{D}}\left(\mathbf{I}-\overline{\mathbf{D}}^{-1} \mathbf{A}\right)$ is invertible, and

$$
\left\|(\overline{\mathbf{D}}-\mathbf{A})^{-1}\right\|_{\infty} \leq\left\|\left(\mathbf{I}-\overline{\mathbf{D}}^{-1} \mathbf{A}\right)^{-1}\right\|_{\infty}\left\|\overline{\mathbf{D}}^{-1}\right\|_{\infty} \leq 4
$$

Thus there exists a unique $\overline{\mathbf{X}}$ such that $(\overline{\mathbf{D}}-\mathbf{A}) \overline{\mathbf{X}}=\mathbf{p}$, and

$$
\begin{equation*}
\left\|\mathbf{X}_{j}\right\|_{\infty} \leq 4\|\mathbf{p}\|_{\infty}, \quad 1 \leq j \leq N \tag{4.4}
\end{equation*}
$$

By (4.2), we have

$$
\begin{equation*}
\left\|\mathbf{X}_{N+1}\right\|_{\infty} \leq\|\mathbf{q}\|_{\infty}+\sum_{j=1}^{N} \omega_{j}\left\|\mathbf{A}_{j}\right\|_{\infty}\left\|\mathbf{X}_{j}\right\|_{\infty} \tag{4.5}
\end{equation*}
$$

Since $\left\|\mathbf{A}_{j}\right\|_{\infty} \leq 1 / 4$ by (A3) and the $\omega_{j}$ are positive and sum to 2 , (4.4) and (4.5) complete the proof of (4.2).

Next, we establish the invertibility of $\nabla \mathcal{T}^{*}$.
Proposition 4.2. If (P1), (A2) and (A3) hold, then $\nabla \mathcal{T}^{*}$ is invertible.
Proof. We formulate a strongly convex quadratic programming problem whose first-order optimality conditions reduce to $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$. Due to the strong convexity of the objective function, the quadratic programming has a solution and there exists $\boldsymbol{\Lambda}$ such that $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$. Since $\mathcal{T}^{*}$ is square and $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$ has a solution for each choice of $\mathbf{y}$, it follows that $\nabla \mathcal{T}^{*}$ is invertible.

The quadratic program is

$$
\left.\begin{array}{ll}
\operatorname{minimize} & \frac{1}{2} \mathcal{Q}(\mathbf{X}, \mathbf{U})+\mathcal{L}(\mathbf{X}, \mathbf{U})  \tag{4.6}\\
\text { subject to } & \sum_{j=1}^{N} D_{i j} \mathbf{X}_{j}=\mathbf{A}_{i} \mathbf{X}_{i}+\mathbf{B}_{i} \mathbf{U}_{i}+\mathbf{y}_{1 i}, \quad 1 \leq i \leq N \\
& \mathbf{X}_{N+1}=\sum_{j=1}^{N} \omega_{j}\left(\mathbf{A}_{j} \mathbf{X}_{j}+\mathbf{B}_{j} \mathbf{U}_{j}\right)+\mathbf{y}_{2}
\end{array}\right\}
$$

where the quadratic and linear terms in the objective are

$$
\begin{align*}
& \mathcal{Q}(\mathbf{X}, \mathbf{U})=\mathbf{X}_{N+1}^{\top} \mathbf{T} \mathbf{X}_{N+1}+\sum_{i=1}^{N} \omega_{i}\left(\mathbf{X}_{i}^{\top} \mathbf{Q}_{i} \mathbf{X}_{i}+2 \mathbf{X}_{i}^{\top} \mathbf{S}_{i} \mathbf{U}_{i}+\mathbf{U}_{i}^{\top} \mathbf{R}_{i} \mathbf{U}_{i}\right)  \tag{4.7}\\
& \mathcal{L}(\mathbf{X}, \mathbf{U})=\mathbf{y}_{4}^{\top} \mathbf{X}_{N+1}-\sum_{i=1}^{N} \omega_{i}\left(\mathbf{y}_{3 i}^{\top} \mathbf{X}_{i}+\mathbf{y}_{5 i}^{\top} \mathbf{U}_{i}\right) \tag{4.8}
\end{align*}
$$

The linear term was chosen so that the first-order optimality conditions for 4.6) reduce to $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$. See [12] for the manipulations needed to obtain the first-order optimality conditions in this form. By (A2), we have

$$
\begin{equation*}
\mathcal{Q}(\mathbf{X}, \mathbf{U}) \geq \alpha\left(\left|\mathbf{X}_{N+1}\right|^{2}+\sum_{i=1}^{N} \omega_{i}\left(\left|\mathbf{X}_{i}\right|^{2}+\left|\mathbf{U}_{i}\right|^{2}\right)\right) \tag{4.9}
\end{equation*}
$$

Since $\alpha$ and $\boldsymbol{\omega}$ are strictly positive, the objective of (4.6) is strongly convex, and by Lemma 4.1, the quadratic programming problem is feasible. Hence, there exists a unique solution to (4.6) for any choice of $\mathbf{y}$, and since the constraints are linear, the first-order conditions hold. Consequently, $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$ has a solution for any choice of $\mathbf{y}$ and the proof is complete.
5. $\omega$-norm bounds. In this section we obtain a bound for the $(\mathbf{X}, \mathbf{U})$ component of the solution to $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$ is terms of $\mathbf{y}$. The bound we derive in this section is in terms of the $\omega$-norms defined by

$$
\begin{equation*}
\|\mathbf{X}\|_{\omega}^{2}=\left|\mathbf{X}_{N+1}\right|^{2}+\sum_{i=1}^{N} \omega_{i}\left|\mathbf{X}_{i}\right|^{2} \quad \text { and } \quad\|\mathbf{U}\|_{\omega}^{2}=\sum_{i=1}^{N} \omega_{i}\left|\mathbf{U}_{i}\right|^{2} \tag{5.1}
\end{equation*}
$$

This defines a norm since the Gauss quadrature weight $\omega_{i}>0$ for each $i$. Since the $(\mathbf{X}, \mathbf{U})$ component of the solution to $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda}]=\mathbf{y}$ is a solution of the quadratic program (4.6), we will bound the solution to the quadratic program.

First, let us think more abstractly. Let $\pi$ be a symmetric, continuous bilinear functional defined on a Hilbert space $\mathcal{H}$, let $\ell$ be a continuous linear functional, let $\phi \in \mathcal{H}$, and consider the quadratic program

$$
\min \left\{\frac{1}{2} \pi(v+\phi, v+\phi)+\ell(v+\phi): v \in \mathcal{V}\right\}
$$

where $\mathcal{V}$ is a subspace of $\mathcal{H}$. If $w$ is a minimizer, then by the first-order optimality conditions, we have

$$
\pi(w, v)+\pi(\phi, v)+\ell(v)=0 \quad \text { for all } v \in \mathcal{V}
$$

Inserting $v=w$ yields

$$
\begin{equation*}
\pi(w, w)=-(\pi(w, \phi)+\ell(w)) \tag{5.2}
\end{equation*}
$$

We apply this observation to the quadratic program (4.6). We identify $\ell$ with the linear functional $\mathcal{L}$ in (4.8), and $\pi$ with the bilinear form associated with the quadratic term (4.7). The subspace $\mathcal{V}$ is the null space of the linear operator in (4.6) and $\phi$ is a particular solution of the linear system. The complete solution of (4.6) is the particular solution plus the minimizer over the null space.

In more detail, let $\chi$ denote the solution to (4.1)-(4.2) given by Lemma 4.1 for $\mathbf{p}=\mathbf{y}_{1}$ and $\mathbf{q}=\mathbf{y}_{2}$. We consider the particular solution $(\mathbf{X}, \mathbf{U})$ of the linear system in (4.6) given by $(\boldsymbol{\chi}, \mathbf{0})$. The relation (5.2) describing the null space component ( $\mathbf{X}, \mathbf{U}$ ) of the solution is

$$
\begin{equation*}
\mathcal{Q}(\mathbf{X}, \mathbf{U})=-\left(\left(\boldsymbol{\chi}_{N}+\mathbf{y}_{4}\right)^{\top} \mathbf{T} \mathbf{X}_{N}+\sum_{i=1}^{N} \omega_{i}\left[\left(\mathbf{Q}_{i} \boldsymbol{\chi}_{i}-\mathbf{y}_{3 i}\right)^{\top} \mathbf{X}_{i}-\mathbf{y}_{5 i}^{\top} \mathbf{U}_{i}\right]\right) \tag{5.3}
\end{equation*}
$$

Here the terms containing $\chi$ are associated with $\pi(w, \phi)$, while the remaining terms are associated with $\ell$ or equivalently, with $\mathcal{L}$. By (A2) we have the lower bound

$$
\begin{equation*}
\mathcal{Q}(\mathbf{X}, \mathbf{U}) \geq \alpha\left(\|\mathbf{X}\|_{\omega}^{2}+\|\mathbf{U}\|_{\omega}^{2}\right) \tag{5.4}
\end{equation*}
$$

All the terms on the right side of (5.3) can be bounded with the Schwarz inequality; for example,

$$
\begin{align*}
\sum_{i=1}^{N} \omega_{i} \mathbf{y}_{3 i}^{\top} \mathbf{X}_{i} & \leq\left(\sum_{i=1}^{N} \omega_{i}\left|\mathbf{y}_{3 i}\right|^{2}\right)^{1 / 2}\left(\sum_{i=1}^{N} \omega_{i}\left|\mathbf{X}_{i}\right|^{2}\right)^{1 / 2} \\
& \leq \sqrt{2}\left\|\mathbf{y}_{3}\right\|_{\infty}\left(\|\mathbf{X}\|_{\omega}^{2}+\|\mathbf{U}\|_{\omega}^{2}\right)^{1 / 2} \tag{5.5}
\end{align*}
$$

The last inequality exploits the fact that the $\omega_{i}$ sum to 2 and $\left|\mathbf{y}_{3 i}\right| \leq\left\|\mathbf{y}_{3}\right\|_{\infty}$. To handle the terms involving $\boldsymbol{\chi}$ in (5.3), we utilize the upper bound $\left\|\chi_{j}\right\|_{\infty} \leq 5\|\mathbf{y}\|_{\infty}$ based on Lemma 4.1 with $\mathbf{p}=\mathbf{y}_{1}$ and $\mathbf{q}=\mathbf{y}_{2}$. Combining upper bounds of the form (5.5) with the lower bound (5.4), we conclude from (5.3) that both $\|\mathbf{X}\|_{\omega}$ and $\|\mathbf{U}\|_{\omega}$ are bounded by a constant times $\|\mathbf{y}\|_{\infty}$. The complete solution of (4.6) is the null space component that we just bounded plus the particular solution $(\boldsymbol{\chi}, \mathbf{0})$. Again, since $\left\|\chi_{j}\right\|_{\infty} \leq 5\|\mathbf{y}\|_{\infty}$, we obtain the following result.

Lemma 5.1. If (A2)-(A3) and ( P 1 ) hold, then there exists a constant $c$, independent of $N$, such that the solution $(\mathbf{X}, \mathbf{U})$ of (4.6) satisfies $\|\mathbf{X}\|_{\omega} \leq c\|\mathbf{y}\|_{\infty}$ and $\|\mathbf{U}\|_{\omega} \leq c\|\mathbf{y}\|_{\infty}$.
6. $\infty$-norm bounds. We now need to convert these $\omega$-norm bounds for $\mathbf{X}$ and $\mathbf{U}$ into $\infty$-norm bounds and at the same time, obtain an $\infty$-norm estimate for $\boldsymbol{\Lambda}$. By Lemma 4.1 the solution to the dynamics in (4.6) can be expressed

$$
\begin{equation*}
\overline{\mathbf{X}}=\left(\mathbf{I}-\overline{\mathbf{D}}^{-1} \mathbf{A}\right)^{-1} \overline{\mathbf{D}}^{-1} \mathbf{B U}+\mathbf{p} \tag{6.1}
\end{equation*}
$$

where $\mathbf{B}$ is the block diagonal matrix with $i$-th diagonal block $\mathbf{B}_{i}$. Taking norms and utilizing the bounds $\|\mathbf{p}\|_{\infty} \leq 4\left\|\mathbf{y}_{1}\right\|_{\infty}$ and $\left\|\left(\mathbf{I}-\overline{\mathbf{D}}^{-1} \mathbf{A}\right)^{-1}\right\|_{\infty} \leq 2$ from Lemma 4.1, we obtain

$$
\begin{equation*}
\|\overline{\mathbf{X}}\|_{\infty} \leq 2\left\|\overline{\mathbf{D}}^{-1} \mathbf{B} \mathbf{U}\right\|_{\infty}+4\left\|\mathbf{y}_{1}\right\|_{\infty} \tag{6.2}
\end{equation*}
$$

We now write

$$
\begin{equation*}
\overline{\mathbf{D}}^{-1} \mathbf{B U}=\left[\mathbf{D}_{1: N}^{-1} \otimes \mathbf{I}_{n}\right] \mathbf{B U}=\left[\left(\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right)^{-1} \otimes \mathbf{I}_{n}\right] \mathbf{B} \mathbf{U}_{\omega} \tag{6.3}
\end{equation*}
$$

where $\mathbf{W}$ is the diagonal matrix with the quadrature weights on the diagonal and $\mathbf{U}_{\omega}$ is the vector whose $i$-th element is $\sqrt{\omega_{i}} \mathbf{U}_{i}$. Note that the $\sqrt{\omega_{i}}$ factors in (6.3) cancel each other. An element of the vector $\overline{\mathbf{D}}^{-1} \mathbf{B U}$ is the dot product between a row of $\left(\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right)^{-1} \otimes \mathbf{I}_{n}$ and the column vector $\mathbf{B} \mathbf{U}_{\omega}$. By (P2) the rows of
$\left(\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right)^{-1} \otimes \mathbf{I}_{n}$ have Euclidean length bounded by $\sqrt{2}$. By the properties of matrix norms induced by vector norms, we have

$$
\left\|\mathbf{B} \mathbf{U}_{\omega}\right\|_{2} \leq\|\mathbf{B}\|_{2}\left\|\mathbf{U}_{\omega}\right\|_{2}=\|\mathbf{B}\|_{2}\|\mathbf{U}\|_{\omega}
$$

It follows that

$$
\begin{equation*}
\left\|\overline{\mathbf{D}}^{-1} \mathbf{B U}\right\|_{\infty} \leq \sqrt{2}\|\mathbf{B}\|_{2}\|\mathbf{U}\|_{\omega} \tag{6.4}
\end{equation*}
$$

Combine Lemma (5.1) with (6.2) and (6.4) to deduce that $\|\overline{\mathbf{X}}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$, where $c$ is independent of $N$. Since $\left|X_{N}\right| \leq c\|\mathbf{y}\|_{\infty}$ by Lemma 5.1 it follows that $\|\mathbf{X}\|_{\infty} \leq$ $c\|\mathbf{y}\|_{\infty}$,

Next, we use the third and fourth components of the linear system $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=$ $\mathbf{y}$ to obtain bounds for $\boldsymbol{\Lambda}$. These equations can be written

$$
\begin{equation*}
\overline{\mathbf{D}}^{\dagger} \overline{\boldsymbol{\Lambda}}+\overline{\mathbf{D}}_{N+1}^{\dagger} \boldsymbol{\Lambda}_{N+1}+\mathbf{A}^{\top} \overline{\boldsymbol{\Lambda}}+\mathbf{Q} \overline{\mathbf{X}}+\mathbf{S U}=\mathbf{y}_{3} \tag{6.5}
\end{equation*}
$$

and

$$
\begin{equation*}
\boldsymbol{\Lambda}_{N+1}-\mathbf{T} \mathbf{X}_{N+1}=\mathbf{y}_{4} \tag{6.6}
\end{equation*}
$$

where $\overline{\boldsymbol{\Lambda}}$ is obtained by vertically stacking $\boldsymbol{\Lambda}_{1}$ through $\boldsymbol{\Lambda}_{N}, \mathbf{Q}$ and $\mathbf{S}$ are block diagonal matrices with $i$-th diagonal blocks $\mathbf{Q}_{i}$ and $\mathbf{S}_{i}$ respectively, $\overline{\mathbf{D}}^{\dagger}=\mathbf{D}_{1: N}^{\dagger} \otimes \mathbf{I}_{n}$, and $\overline{\mathbf{D}}_{N+1}^{\dagger}=\mathbf{D}_{N+1}^{\dagger} \otimes \mathbf{I}_{n}$, where $\mathbf{D}_{N+1}^{\dagger}$ is the $(N+1)$-st column of $\mathbf{D}^{\dagger}$.

We show in Proposition 9.1 of the Appendix that $\mathbf{D}_{1: N}=-\mathbf{J} \mathbf{D}_{1: N}^{\dagger} \mathbf{J}$, where $\mathbf{J}$ is the exchange matrix with ones on its counterdiagonal and zeros elsewhere. It follows that $\mathbf{D}_{1: N}^{-1}=-\mathbf{J}\left(\mathbf{D}_{1: N}^{\dagger}\right)^{-1} \mathbf{J}$. Consequently, the elements in $\mathbf{D}_{1: N}^{-1}$ are the negative of the elements in $\left(\mathbf{D}_{1: N}^{\dagger}\right)^{-1}$, but rearranged. As a result, $\left(\mathbf{D}_{1: N}^{\dagger}\right)^{-1}$ also possesses properties (P1) and (P2), and the analysis of the discrete costate closely parallels the analysis of the state. The main difference is that the costate equation contains the additional $\boldsymbol{\Lambda}_{N+1}$ term along with the additional equation (6.6). By (6.6) and the previously established bound $\|\mathbf{X}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$, it follows that

$$
\begin{equation*}
\left\|\boldsymbol{\Lambda}_{N+1}\right\|_{\infty} \leq c\|\mathbf{y}\|_{\infty} \tag{6.7}
\end{equation*}
$$

where $c$ is independent of $N$. Since $\mathbf{D}^{\dagger} \mathbf{1}=\mathbf{0}$, we deduce that $\left(\mathbf{D}_{1: N}^{\dagger}\right)^{-1} \mathbf{D}_{N+1}=-\mathbf{1}$. It follows that

$$
\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1} \overline{\mathbf{D}}_{N+1}^{\dagger}=\left[\left(\mathbf{D}_{1: N}^{\dagger}\right)^{-1} \otimes \mathbf{I}_{n}\right]\left[\mathbf{D}_{N+1}^{\dagger} \otimes \mathbf{I}_{n}\right]=-\mathbf{1} \otimes \mathbf{I}_{n}
$$

Exploiting this identity, the analogue of (6.1) is

$$
\overline{\boldsymbol{\Lambda}}=\left(\mathbf{I}+\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1} \mathbf{A}^{\top}\right)^{-1}\left[\left(\mathbf{1} \otimes \mathbf{I}_{n}\right) \boldsymbol{\Lambda}_{N+1}+\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1}\left(\mathbf{y}_{3}-\mathbf{Q} \overline{\mathbf{X}}-\mathbf{S U}\right)\right]
$$

Hence, we have

$$
\|\overline{\boldsymbol{\Lambda}}\|_{\infty} \leq 2\left\|\left(\mathbf{1} \otimes \mathbf{I}_{n}\right) \boldsymbol{\Lambda}_{N+1}+\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1}\left(\mathbf{y}_{3}-\mathbf{Q} \overline{\mathbf{X}}-\mathbf{S U}\right)\right\|_{\infty}
$$

Moreover, $\left\|\left(\mathbf{1} \otimes \mathbf{I}_{n}\right) \boldsymbol{\Lambda}_{N+1}\right\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$ by (6.7) and $\left\|\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1} \mathbf{y}_{3}\right\|_{\infty} \leq 2\left\|\mathbf{y}_{3}\right\|_{\infty}$. The terms $\left\|\left(\overline{\mathbf{D}}^{\dagger}\right)^{-1} \mathbf{Q} \overline{\mathbf{X}}\right\|_{\infty}$ and $\left.\|\left(\overline{\mathbf{D}}^{\bar{\dagger}}\right)^{-1} \mathbf{S U}\right) \|_{\infty}$ are handled exactly as the term $\left\|\overline{\mathbf{D}}^{-1} \mathbf{B U}\right\|_{\infty}$ was handled in the state equation (6.1). We again conclude that $\|\boldsymbol{\Lambda}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$ where $c$ is independent of $N$.

Finally, let us examine the fifth component of the linear system $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=$ $y$. These equations can be written

$$
\mathbf{S}_{i}^{\top} \mathbf{X}_{i}+\mathbf{R}_{i} \mathbf{U}_{i}+\mathbf{B}_{i}^{\top} \boldsymbol{\Lambda}_{i}=\mathbf{y}_{5 i}, \quad 1 \leq i \leq N .
$$

By (A2) the smallest eigenvalue of $\mathbf{R}_{i}$ is greater than $\alpha>0$. Consequently, the bounds $\|\mathbf{X}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$ and $\|\boldsymbol{\Lambda}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$ imply the existence of a constant $c$, independent of $N$, such that $\|\mathbf{U}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty}$. In summary, we have the following result:

Lemma 6.1. If (A2)-(A3) and (P1)-(P2) hold, then there exists a constant $c$, independent of $N$, such that the solution of $\nabla \mathcal{T}^{*}[\mathbf{X}, \mathbf{U}, \mathbf{\Lambda}]=\mathbf{y}$ satisfies

$$
\|\mathbf{X}\|_{\infty}+\|\mathbf{U}\|_{\infty}+\|\boldsymbol{\Lambda}\|_{\infty} \leq c\|\mathbf{y}\|_{\infty} .
$$

Let us now prove Theorem 1.1 using Proposition [2.1 By Lemma 6.1 . $\mu=$ $\left\|\nabla \mathcal{T}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)^{-1}\right\|_{\infty}$ is bounded uniformly in $N$. Choose $\varepsilon$ small enough that $\varepsilon \mu<$ 1. When we compute the difference $\nabla \mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})-\nabla \mathcal{T}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ for $(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})$ near $\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$ in the $\infty$-norm, the $\mathbf{D}$ and $\mathbf{D}^{\dagger}$ constant terms cancel, and we are left with terms involving the difference of derivatives of $\mathbf{f}$ or $C$ up to second order at nearby points. By assumption, these second derivative are uniformly continuous on the closure of $\Omega$ and on a ball around $\mathbf{x}^{*}(1)$. Hence, for $r$ sufficiently small, we have

$$
\left\|\nabla \mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})-\nabla \mathcal{T}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)\right\|_{\infty} \leq \varepsilon
$$

whenever

$$
\begin{equation*}
\max \left\{\left\|\mathbf{X}-\mathbf{X}^{*}\right\|_{\infty},\left\|\mathbf{U}-\mathbf{U}^{*}\right\|_{\infty},\left\|\boldsymbol{\Lambda}-\boldsymbol{\Lambda}^{*}\right\|_{\infty}\right\} \leq r . \tag{6.8}
\end{equation*}
$$

By Lemma 3.1 it follows that $\left\|\mathcal{T}\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)\right\| \leq(1-\mu \varepsilon) r / \mu$ for all $N$ sufficiently large. Hence, by Proposition [2.1, there exists a solution to $\mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\mathbf{0}$ satisfying (6.8). Moreover, by (2.7) and (3.1), the estimate (1.10) holds. To complete the proof, we need to show that $(\mathbf{X}, \mathbf{U})$ is a local minimizer for (2.1). After replacing the KKT multipliers by the transformed quantities given by (2.6), the Hessian of the Lagrangian is the following block diagonal matrix:

$$
\operatorname{diag}\left\{\omega_{1} \nabla_{(x, u)}^{2} H\left(\mathbf{X}_{1}, \mathbf{U}_{1}, \boldsymbol{\Lambda}_{1}\right), \ldots, \omega_{N} \nabla_{(x, u)}^{2} H\left(\mathbf{X}_{N}, \mathbf{U}_{N}, \boldsymbol{\Lambda}_{N}\right), \nabla^{2} C\left(\mathbf{X}_{N+1}\right)\right\}
$$

where $H$ is the Hamiltonian. In computing the Hessian, we assume that the $\mathbf{X}$ and $\mathbf{U}$ variables are arranged in the following order: $\mathbf{X}_{1}, \mathbf{U}_{1}, \mathbf{X}_{2}, \mathbf{U}_{2}, \ldots, \mathbf{X}_{N}, \mathbf{U}_{N}$, $\mathbf{X}_{N+1}$. By (A2) the Hessian is positive definite when evaluated at $\left(\mathbf{X}^{*}, \mathbf{U}^{*}, \boldsymbol{\Lambda}^{*}\right)$. By continuity of the second derivative of $C$ and $\mathbf{f}$ and by the convergence result (1.10), we conclude that the Hessian of the Lagrangian, evaluated at the solution of $\mathcal{T}(\mathbf{X}, \mathbf{U}, \boldsymbol{\Lambda})=\mathbf{0}$ satisfying (6.8), is positive definite for $N$ sufficiently large. Hence, by the second-order sufficient optimality condition [20, Thm. 12.6], ( $\mathbf{X}, \mathbf{U}$ ) is a strict local minimizer of (2.1). This completes the proof of Theorem [1.1.
7. Numerical illustration. Although the assumptions (A1)-(A3) are sufficient for exponential convergence, the following example indicates that these assumptions are conservative. Let us consider the unconstrained control problem

$$
\begin{equation*}
\min \left\{-x(2): \dot{x}(t)=\frac{5}{2}\left(-x(t)+x(t) u(t)-u(t)^{2}\right), x(0)=1\right\} . \tag{7.1}
\end{equation*}
$$



FIG. 7.1. The base 10 logarithm of the error in the sup-norm as a function of the number of collocation points.

The optimal solution and associated costate are

$$
\begin{aligned}
& x^{*}(t)=4 / a(t), \quad a(t)=1+3 \exp (2.5 t) \\
& u^{*}(t)=x^{*}(t) / 2 \\
& \lambda^{*}(t)=-a^{2}(t) \exp (-2.5 t) /[\exp (-5)+9 \exp (5)+6]
\end{aligned}
$$

Figure 7.1 plots the logarithm of the sup-norm error in the state, control, and costate as a function of the number of collocation points. Since these plots are nearly linear, the error behaves like $c 10^{-\alpha N}$ where $\alpha \approx 0.6$ for either the state or the control and $\alpha \approx 0.8$ for the costate. In Theorem 1.1 the dependence of the error on $N$ is somewhat complex due to the connection between $m$ and $N$. As we increase $N$, we can also increase $m$ when the solution is infinitely differentiable, however, the norm of the derivatives also enters into the error bound as in (3.3). Nonetheless, in cases where the solution derivatives can be bounded by $c^{m}$ for some constant $c$, it is possible to deduce an exponential decay rate for the error as observed in [12, Sect. 2]. Note that the example problem (7.1) does not satisfy (A2) since $\nabla^{2} C=\mathbf{0}$, which is not positive definite. Nonetheless, the pointwise error decays exponentially fast.
8. Conclusions. A Gauss collocation scheme is analyzed for an unconstrained control problem. For a smooth solution whose Hamiltonian satisfies a strong convexity assumption, we show that the discrete problem has a local minimizer in a neighborhood of the continuous solution, and as the number of collocation points increases, the distance in the sup-norm between the discrete solution and the continuous solution is $O\left(N^{2-\eta}\right)$ when the continuous solution has $\eta+1$ continuous derivatives, $\eta \geq 3$, and the number of collocation points $N$ is sufficiently large. A numerical example is given which exhibits an exponential convergence rate.
9. Appendix. In (2.4) we define a new matrix $\mathbf{D}^{\dagger}$ in terms of the differentiation matrix $\mathbf{D}$. The following proposition shows that the elements of $\mathbf{D}^{\dagger}$ are the negative and a rearrangement of the elements of $\mathbf{D}$.

Proposition 9.1. The entries of the matrices $\mathbf{D}$ and $\mathbf{D}^{\dagger}$ satisfy

$$
D_{i j}=-D_{N+1-i, N+1-j}^{\dagger}, \quad 1 \leq i \leq N, \quad 1 \leq j \leq N
$$

In other words, $\mathbf{D}_{1: N}=-\mathbf{J D}_{1: N}^{\dagger} \mathbf{J}$ where $\mathbf{J}$ is the exchange matrix with ones on its counterdiagonal and zeros elsewhere.

Proof. By (1.9) the elements of $\mathbf{D}$ can be expressed in terms of the derivatives of a set of Lagrange basis functions evaluated at the collocation points:

$$
D_{i j}=\dot{L}_{j}\left(\tau_{i}\right) \quad \text { where } L_{j} \in \mathcal{P}_{N}, \quad L_{j}\left(\tau_{k}\right)= \begin{cases}1 & \text { if } k=j \\ 0 & \text { if } 0 \leq k \leq N, k \neq j\end{cases}
$$

In (1.9) we give an explicit formula for the Lagrange basis functions, while here we express the basis function in terms of polynomials $L_{j}$ that equal one at $\tau_{j}$ and vanish at $\tau_{k}$ where $0 \leq k \leq N, k \neq j$. These $N+1$ conditions uniquely define $L_{j} \in \mathcal{P}_{N}$. Similarly, by [12, Thm. 1], the entries of $\mathbf{D}_{1: N}^{\dagger}$ are given by

$$
D_{i j}^{\dagger}=\dot{M}_{j}\left(\tau_{i}\right) \quad \text { where } M_{j} \in \mathcal{P}_{N}, \quad M_{j}\left(\tau_{k}\right)= \begin{cases}1 & \text { if } k=j \\ 0 & \text { if } 1 \leq k \leq N+1, k \neq j\end{cases}
$$

Observe that $M_{N+1-j}(t)=L_{j}(-t)$ due the symmetry of the quadrature points around $t=0$ :
(a) Since $-\tau_{N+1-j}=\tau_{j}$, we have $L_{j}\left(-\tau_{N+1-j}\right)=L_{j}\left(\tau_{j}\right)=1$.
(b) Since $\tau_{N+1}=1$ and $\tau_{0}=-1$, we have $L_{j}\left(-\tau_{N+1}\right)=L_{j}\left(\tau_{0}\right)=0$.
(c) Since $-\tau_{i}=\tau_{N+1-i}$, we have $L_{j}\left(-\tau_{i}\right)=L_{j}\left(\tau_{N+1-i}\right)=0$ if $i \neq N+1-j$. Since $M_{N+1-j}(t)$ is equal to $L_{j}(-t)$ at $N+1$ distinct points, the two polynomials are equal everywhere. Replacing $M_{N+1-j}(t)$ by $L_{j}(-t)$, we have

$$
D_{N+1-i, N+1-j}^{\dagger}=-\dot{L}_{j}\left(-\tau_{N+1-i}\right)=-\dot{L}_{j}\left(\tau_{i}\right)=-D_{i j}
$$

Tables 9.1 and 9.2 illustrate properties (P1) and (P2) for the differentiation matrix D. In Table 9.1 we observe that $\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty}$ monotonically approaches the upper limit 2. More precisely, it is found that $\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty}=1+\tau_{N}$, where the final collocation point $\tau_{N}$ approaches one as $N$ tends to infinity. In Table 9.2 we show the maximum 2-norm of the rows of $\left[\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right]^{-1}$. It is found that the maximum is attained by the last row of $\left[\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right]^{-1}$, and the maximum monotonically approaches $\sqrt{2}$.

| $N$ | 25 | 50 | 75 | 100 | 125 | 150 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| norm | 1.995557 | 1.998866 | 1.999494 | 1.999714 | 1.999816 | 1.999872 |
| $N$ | 175 | 200 | 225 | 250 | 275 | 300 |
| norm | 1.999906 | 1.999928 | 1.999943 | 1.999954 | 1.999962 | 1.999968 |

Table 9.1
$\left\|\mathbf{D}_{1: N}^{-1}\right\|_{\infty}$

| $N$ | 25 | 50 | 75 | 100 | 125 | 150 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| norm | 1.412201 | 1.413703 | 1.413985 | 1.414085 | 1.414131 | 1.414156 |
| $N$ | 175 | 200 | 225 | 250 | 275 | 300 |
| norm | 1.414171 | 1.414181 | 1.414188 | 1.414193 | 1.414196 | 1.414199 |

TABLE 9.2
Maximum Euclidean norm for the rows of $\left[\mathbf{W}^{1 / 2} \mathbf{D}_{1: N}\right]^{-1}$

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[^0]:    * June 1, 2015, revised August 14, 2015 The authors gratefully acknowledge support by the Office of Naval Research under grants N00014-11-1-0068 and N00014-15-1-2048, and by the National Science Foundation under grants DMS-1522629 and CBET-1404767.
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