# The forward-backward algorithm and the normal problem

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#### Abstract

The forward-backward splitting technique is a popular method for solving monotone inclusions that has applications in optimization. In this paper we explore the behaviour of the algorithm when the inclusion problem has no solution. We present a new formula to define the normal solutions using the forward-backward operator. We also provide a formula for the range of the displacement map of the forward-backward operator. Several examples illustrate our theory.

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#### 1 Introduction

Throughout this paper we work under the assumption that

X is a real Hilbert space,

with inner product  $\langle \cdot, \cdot \rangle$  and induced norm  $\|\cdot\|$ . A (possibly) set-valued operator  $A : X \Rightarrow X$  is *monotone* if any two pairs (x, u) and (y, v) in the graph of A satisfy  $\langle x - y, u - v \rangle \ge 0$ , and is *maximally monotone* if it is monotone and any proper enlargement of the graph of A (in terms of set inclusion) will no longer preserve the monotonicity of A. In the following we assume that

 $A: X \rightrightarrows X$  and  $B: X \rightrightarrows X$  are maximally monotone operators. (1)

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Thanks to the fact that the *subdifferential* operator associated with a convex lower semicontinuous proper function is a maximally monotone operator (see Fact 3.6 below), the notion of monotone operators becomes of significant importance in optimization and nonlinear analysis. For further discussion on monotone operator theory and its connection to optimization see, e.g., the books [8], [17], [19], [21], [44], [45], [49], [50], and [51].

The problem of finding a zero of the sum of two maximally monotone operators *A* and *B* is to find  $x \in X$  such that  $x \in (A + B)^{-1}0$ . When specializing *A* and *B* to subdifferential operators of convex lower semicontinuous proper functions, the problem is equivalent to finding a minimizer of the sum of the two functions, which is a classical optimization problem.

Suppose that *A* is *firmly nonexpansive*<sup>1</sup>(see Section 2). Let  $x_0 \in X$  and let  $T_{FB}$  be the forwardbackward operator associated with the pair (A, B) (see Section 3). When  $(A + B)^{-1}0 \neq \emptyset$  the sequence  $(T_{FB}^n x_0)_{n \in \mathbb{N}}$  produced by iterating the forward-backward operator converges weakly<sup>2</sup> to a point in  $(A + B)^{-1}0 = \text{Fix } T_{FB} = \{x \in X \mid x = T_{FB}x\}$  (see, e.g., [47], [33] or [23]). Applications of this setting appear in convex optimization (see, e.g., [8, Section 27.3]), evolution inclusions (see, e.g., [2]) and inverse problems (see, e.g., [24] and [25]).

The goal of this work is to examine the forward-backward operator in the inconsistent case, i.e., when  $(A + B)^{-1}0 = \emptyset$ , using the framework of the normal problem introduced in [12]. In this case Fix  $T_{FB} = \emptyset$ , and the classical analysis, which uses the advantage of iterating an averaged operator (see Section 2 below) that has a fixed point, is no longer applicable.

Let us summarize the main contributions of the paper:

- **R1** We provide a systematic study of the forward-backward operator when the sum problem is possibly inconsistent. This is mainly illustrated in Proposition 4.1 where we establish the connection between the perturbed problem introduced in [12] and the forward-backward operator.
- **R2** We prove that the range of the displacement operator associated with the forward-backward operator  $T_{\text{FB}}$  coincides with that of the Douglas-Rachford operator  $T_{\text{DR}}$ . Consequently, the minimal displacement vectors associated with  $T_{\text{FB}}$  and  $T_{\text{DR}}$  coincide (see Theorem 4.2). This gives an alternative approach to define the normal problem introduced in [12].
- **R3** A significant consequence of **R2** is that it allows to use the advantage of the self-duality of  $T_{\text{DR}}$  (which does not hold for  $T_{\text{FB}}$  as we illustrate in Example 4.11) to draw more conclusions about  $T_{\text{FB}}$ . In particular, in Theorem 5.3 we provide a formula for the range of the displacement operator in terms of the ranges of the underlying operators using the notion of *near equality*. The result simplifies to more elegant formulae when specializing the operators to subdifferential operators as illustrated in Proposition 5.7. Our results are sharp in the sense that near equality cannot be replaced by equality which we illustrate in Example 5.4.

<sup>&</sup>lt;sup>1</sup>We point out that the assumption of that *A* is firmly nonexpansive can be relaxed to *A* is cocoercive (see Remark 3.1). <sup>2</sup>For general conditions on *strong* convergence of the forward-backward algorithm we refer the reader to [2].

**R4** In the case when *A* and *B* are affine, we prove that, in the consistent case, the sequence produced by iterating  $T_{\text{FB}}$  converges *strongly* to the *nearest* point in the set of zeros of the sum. If *X* is finite-dimensional, we also get *linear* rate of convergence (see Theorem 6.6).

The remainder of this paper is organized as follows: Section 2 provides facts and auxiliary results concerning averaged and (firmly) nonexpansive operators. In Section 3, we provide an overview of the Attouch-Théra duality and formulate the primal and dual solutions using the forward-backward operator. Our main results start in Section 4, which deals with the normal problem and the connection to the forward-backward operator. In Section 5, we explore the range of the displacement operator associated with the forward-backward operator. In Section 6, we study the asymptotic behaviour of *asymptotically regular* affine nonexpansive operators in the possibly fixed point free setting. An application to the forward-backward algorithm is provided as well. Finally in Section 7 we provide some algorithmic consequences.

#### Notation

Let *C* be a nonempty closed convex subset of *X*. We use  $\iota_C$ ,  $N_C$  and  $P_C$  to denote the *indicator function*, the *normal cone* operator and the *projector* (*this is also known as nearest point mapping*) associated with *C*, respectively. Let  $f: X \to ]-\infty, +\infty]$  be convex, lower semicontinuous, and proper. The *subdifferential* of *f* is the (possibly) set-valued operator  $\partial f: X \rightrightarrows X : x \to \{u \in X \mid (\forall y \in X) f(y) \ge f(x) + \langle u, y - x \rangle\}$ . Let  $\mathrm{Id}: X \to X$  be the identity operator. The *resolvent* of *A* is  $J_A := (\mathrm{Id} + A)^{-1}$  and the *reflected resolvent* is  $R_A := 2J_A - \mathrm{Id}$ . Otherwise, the notation we adopt is standard and follows, e.g., [8] and [40].

#### 2 Averaged and (firmly) nonexpansive operators

Let  $T : X \to X$ . Then *T* is *nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad ||Tx - Ty|| \le ||x - y||;$$
(2)

*T* is *firmly nonexpansive* if

$$(\forall x \in X)(\forall y \in X) \quad ||Tx - Ty||^2 + ||(\mathrm{Id} - T)x - (\mathrm{Id} - T)y||^2 \le ||x - y||^2;$$
 (3)

and *T* is *averaged* if there exists  $\alpha \in [0, 1[$  and a nonexpansive operator  $N : X \to X$  such that

$$T = (1 - \alpha) \operatorname{Id} + \alpha N. \tag{4}$$

Fact 2.1. The following hold:

- (i) *J<sub>A</sub>* is single-valued, maximally monotone and firmly nonexpansive.
- (ii) (The inverse resolvent identity)  $J_{A^{-1}} = \text{Id} J_A$ .

Proof. (i): See [35, Corollary on page 344] and [41, Proposition 1(c)]. (ii): See, e.g., [40, Lemma 12.14].

In the sequel we make use of the useful characterization (see, e.g., [31, Equation 11.1 on page 42]):

*T* is firmly nonexpansive  $\Leftrightarrow (\forall x \in X)(\forall y \in X) \quad ||Tx - Ty||^2 \le \langle x - y, Tx - Ty \rangle.$ (5)

**Definition 2.2 (asymptotic regularity of operators vs. sequences).** Let  $T : X \to X$  and let  $(x_n)_{n \in \mathbb{N}}$ be a sequence in X. Then T is asymptotically regular if  $(\forall x \in X) T^n x - T^{n+1} x \to 0$  and  $(x_n)_{n \in \mathbb{N}}$  is asymptotically regular if  $x_n - x_{n+1} \rightarrow 0$ .

**Fact 2.3.** Suppose that  $T: X \to X$  is averaged; in particular, firmly nonexpansive. Then T is asymptotically regular.

Proof. See [20, Corollary 1.1 & Proposition 2.1] or [8, Proposition 5.15(ii) & Corollary 5.16(ii)]. 

**Fact 2.4.** Suppose that  $T: X \to X$  is nonexpansive. Then  $\overline{ran}(Id - T)$  is nonempty closed and convex. Consequently the minimal displacement vector associated with T is the unique well-defined vector

$$v_T := P_{\overline{\operatorname{ran}}(\operatorname{Id} - T)} 0. \tag{6}$$

*Proof.* See [3], [20] or [38].

Unless otherwise stated, throughout this paper we assume that

 $T: X \to X$  is nonexpansive.

The following result is well-known when T is firmly nonexpansive. We include a simple proof, when T is averaged, for the sake of completeness (see also [10, Lemma 3.9]).

**Proposition 2.5.** Suppose that T is averaged and that  $v_T := P_{\overline{ran}(\operatorname{Id} - T)} 0 \in \operatorname{ran}(\operatorname{Id} - T)$ . Let  $x \in X$ . Then the following hold:

- (i)  $\sum_{n=0}^{\infty} ||T^n x T^{n+1} x v_T||^2 < +\infty.$ (ii)  $T^n x T^{n+1} x \to v_T$ , equivalently; the sequence  $(T^n x + nv_T)_{n \in \mathbb{N}}$  is asymptotically regular.

*Proof.* It follows from [23, Lemma 2.1] that  $(\exists \alpha \in [0,1[) \text{ such that } (\forall x \in X) (\forall y \in X))$ 

$$\|(\mathrm{Id} - T)x - (\mathrm{Id} - T)y\|^2 \le \frac{\alpha}{1 - \alpha} \left( \|x - y\|^2 - \|Tx - Ty\|^2 \right).$$
(7)

Moreover [6, Proposition 2.5(vi)] implies that  $(T^n x + nv_T)_{n \in \mathbb{N}}$  is Fejér monotone with respect to Fix( $v_T + T$ ). Now let  $n \in \mathbb{N}$  and let  $y_0 \in Fix(v_T + T)$ . Using [6, Proposition 2.5(iv)] we learn that  $T^n y_0 = y_0 - nv_T$ . It follows from (7) applied with (x, y) replaced by  $(T^n x, T^n y_0)$  that

$$||T^{n}x - T^{n+1}x - v_{T}||^{2} = ||(\mathrm{Id} - T)T^{n}x - (\mathrm{Id} - T)T^{n}y_{0}||^{2}$$
(8a)

$$\leq \frac{\alpha}{1-\alpha} \left( \|T^n x - T^n y_0\|^2 - \|T^{n+1} x - T^{n+1} y_0\|^2 \right).$$
 (8b)

(i): This follows from (8) by telescoping. (ii): This is a direct consequence of (i).

**Proposition 2.6.** Suppose that  $v_T := P_{\overline{ran}(\operatorname{Id} - T)} 0 \in \operatorname{ran}(\operatorname{Id} - T)$  and that  $\operatorname{int} \operatorname{Fix}(v_T + T) \neq \emptyset$ . Then the following hold:

- (i) ∑<sub>n=0</sub><sup>∞</sup> ||T<sup>n</sup>x T<sup>n+1</sup>x v<sub>T</sub>|| < +∞.</li>
   (ii) (T<sup>n</sup>x + nv<sub>T</sub>)<sub>n∈ℕ</sub> converges strongly.

*Proof.* The proof follows along the lines of [8, Proposition 5.10]. (i): Let  $x \in Fix(v_T + T)$  and let r > 0 such that ball $(x; r) \subseteq Fix(v_T + T)$ . Obtain a sequence  $(y_n)_{n \in \mathbb{N}}$  defined as:

$$(\forall n \in \mathbb{N}) \quad y_n = \begin{cases} x, & \text{if } x_{n+1} = x_n; \\ x - r \frac{x_{n+1} - x_n}{\|x_{n+1} - x_n\|}, & \text{otherwise.} \end{cases}$$
(9)

Then  $(y_n)_{n \in \mathbb{N}} \subseteq \text{ball}(x; r)$ . Set  $(\forall n \in \mathbb{N}) x_n := T^n x + nv_T$ . It follows from [6, Proposition 2.5(vi)] that the sequence  $(x_n)_{n \in \mathbb{N}}$  is Fejér monotone with respect to Fix(v + T), therefore  $(\forall n \in \mathbb{N})$  $||x_{n+1} - y_n||^2 \le ||x_n - y_n||^2$ ; equivalently  $(\forall n \in \mathbb{N}) ||x_{n+1} - x + (x - y_n)||^2 \le ||x_n - x + (x - y_n)||^2$ . Expanding and simplifying in view of (9) yield  $(\forall n \in \mathbb{N}) ||x_{n+1} - x||^2 \le ||x_n - x||^2 - 2\langle x_n - x - y_n \rangle$  $x_{n+1}, x - y_n \rangle = ||x_n - x||^2 - 2r||x_n - x_{n+1}||$ . Telescoping yields

$$\sum_{n=0}^{\infty} \|x_n - x_{n+1}\| \le \frac{1}{2r} \|x_0 - x\|^2.$$
(10)

(ii): It follows from (10) that  $(x_n)_{n \in \mathbb{N}} = (T^n x + nv)_{n \in \mathbb{N}}$  is a Cauchy sequence and therefore it converges.

Let S be nonempty subset of X and let  $a \in X$ . Before we proceed further we need the following useful translation formula (see, e.g., [8, Proposition 3.17]).

$$(\forall x \in X) \quad P_{a+S}x = a + P_S(x-a). \tag{11}$$

**Example 2.7.** Let  $n \ge 1$ . Suppose<sup>3</sup> that  $X = \mathbb{R}^n$ , that  $p \in \mathbb{R}^n_{++}$  and that  $T = p + P_{\mathbb{R}^n_+}$ . Then T is (firmly) *nonexpansive*, Fix  $T = \emptyset$ , ran $(Id - T) = -p + \mathbb{R}^n_-$ ,  $v_T = -p \in ran(Id - T)$  and int Fix $(v_T + T) = -p \in ran(Id - T)$  $\mathbb{R}^{n}_{--} \neq \emptyset$ . Consequently  $\sum_{n=0}^{\infty} \left| T^{n}x - T^{n+1}x - v_{T} \right| < +\infty$  and  $(T^{n}x + nv_{T})_{n \in \mathbb{N}}$  converges.

*Proof.* The claim that T is firmly nonexpansive (hence nonexpansive) follows from e.g., [31, Section 3]. Now Id  $-T = Id - p - P_{\mathbb{R}^n} = -p + P_{\mathbb{R}^n}$ , hence  $ran(Id - T) = -p + \mathbb{R}^n_-$  and Fix  $T = \emptyset \Leftrightarrow$  $0 \notin \operatorname{ran}(\operatorname{Id} - T) = -p + \mathbb{R}^n_- \Leftrightarrow p \notin \mathbb{R}^n_-$ , which is true. Using (11) with (a, S) replaced by  $(-p, \mathbb{R}^n_-)$ we have  $v_T = P_{-p+\mathbb{R}^n_-} 0 = -p + P_{\mathbb{R}^n_-} p = -p$ . Consequently  $v_T + T = -p + p + P_{\mathbb{R}^n_+} = P_{\mathbb{R}^n_+}$  and therefore  $\operatorname{Fix}(v_T + T) = \mathbb{R}^n_+$  which implies that  $\operatorname{int} \operatorname{Fix}(v_T + T) = \mathbb{R}^n_{++}$ . Now apply Proposition 2.6.

**Corollary 2.8.** Suppose that  $X = \mathbb{R}$ , that Fix  $T = \emptyset$  and that  $v_T := P_{\overline{ran}(\operatorname{Id} - T)}0 \in \operatorname{ran}(\operatorname{Id} - T)$ . Then int Fix $(v_T + T) \neq \emptyset$  and  $\sum_{n=0}^{\infty} |T^n x - T^{n+1} x - v_T| < +\infty$ . Consequently  $(T^n x + nv_T)_{n \in \mathbb{N}}$  converges.

<sup>&</sup>lt;sup>3</sup>Let  $n \in \mathbb{N}$ . The positive orthant in  $\mathbb{R}^n$  is  $\mathbb{R}^n_+ = [0, +\infty[^n \text{ and the strictly positive orthant in } \mathbb{R}^n \text{ is } \mathbb{R}^n_{++} = ]0, +\infty[^n.$ Likewise we define the *negative orthant* and the *strictly negative orthant*  $\mathbb{R}^n_-$  and  $\mathbb{R}^n_-$ , respectively.

*Proof.* It follows from [6, Proposition 2.5(i)] that Fix(v + T) contains an unbounded interval, and therefore, since  $X = \mathbb{R}$ , we conclude that  $int Fix(v + T) \neq \emptyset$ . Now apply Proposition 2.6. (See also [10, Theorem 3.6]).

### 3 The forward-backward operator and duality

The *primal* problem for the ordered pair (A, B) is

(P) find 
$$x \in X$$
 such that  $0 \in Ax + Bx$ . (12)

The Attouch-Théra *dual* pair [1] for the ordered pair (A, B) is the pair<sup>4</sup>  $(A^{-1}, B^{-0})$  and the corresponding *dual problem* is

(D) find 
$$x \in X$$
 such that  $0 \in A^{-1}x + B^{-\emptyset}x$ . (13)

The sets of primal and dual solutions for the ordered pair (A, B), denoted respectively by *Z* and *K* are

$$Z := (A+B)^{-1}(0) \quad \text{and} \quad K := (A^{-1}+B^{-\emptyset})(0).$$
(14)

From now on we assume that

$$A: X \to X$$
 is firmly nonexpansive. (15)

The forward-backward algorithm to solve (12) iterates the operator

$$T_{\rm FB} := T_{{\rm FB}(A,B)} := J_B({\rm Id} - A).$$
 (16)

On the other hand the Douglas-Rachford algorithm to solve (12) iterates the operator

$$T_{\rm DR} := T_{{\rm DR}(A,B)} := {\rm Id} - J_A + J_B R_A.$$
 (17)

Let  $x \in X$ . If  $Z \neq \emptyset$  then each of the sequences  $(T_{FB}^n x)_{n \in \mathbb{N}}$  (see, e.g., [23, Corollary 6.5] or [8, Section 25.3]) and  $(J_A T_{DR}^n x)_{n \in \mathbb{N}}$  (see, e.g., [46] or [34]) converges weakly to a (possibly different) solution of (12).

**Remark 3.1.** Let  $\alpha > 0$ . Since  $\operatorname{zer}(A + B) = \operatorname{zer}(\alpha A + \alpha B)$ , the assumption that A is firmly nonexpansive could be replaced by A is  $\alpha$ -cocoercive<sup>5</sup>. In this case (16) and (17) can be applied with the ordered pair (A, B) is replaced by  $(\alpha A, \alpha B)$ .

**Definition 3.2** (paramonotone and  $3^*$  monotone operators). Let  $C: X \rightrightarrows X$  be monotone. Then

<sup>&</sup>lt;sup>4</sup>Let  $B: X \rightrightarrows X$ . Then  $B^{\otimes} := (-\operatorname{Id}) \circ B \circ (-\operatorname{Id})$  and  $B^{-\otimes} := (B^{-1})^{\otimes} = (B^{\otimes})^{-1}$  (see [7, Equation (10)]).

<sup>&</sup>lt;sup>5</sup>Recall that  $A : X \to X$  is cocoercive if  $(\exists \alpha > 0)$  such that  $\alpha A$  is firmly nonexpansive.

(i) *C* is paramonotone<sup>6</sup> if  $(\forall (x, u) \in \operatorname{gra} C)$   $(\forall (y, v) \in \operatorname{gra} C)$  we have

$$\begin{cases} (x,u) \in \operatorname{gra} C\\ (y,v) \in \operatorname{gra} C\\ \langle x-y,u-v \rangle = 0 \end{cases} \implies \{ (x,v), (y,u) \} \subseteq \operatorname{gra} C.$$
 (18)

(ii) C is  $3^*$  monotone<sup>7</sup> (this is also known as rectangular) if

$$(\forall x \in \operatorname{dom} C)(\forall v \in \operatorname{ran} C) \qquad \inf_{(z,w)\in\operatorname{gra} C} \langle x-z, v-w \rangle > -\infty.$$
(19)

Lemma 3.3. The following hold:

- (i) A is maximally monotone.
- (ii) A is paramonotone.
- (iii) A is 3<sup>\*</sup> monotone.

*Proof.* (i): This is [8, Example 20.27]. (ii) & (iii): Note that A = Id - (Id - A) and Id - A is firmly nonexpansive. The conclusion follows from [14, Theorem 6.1].

**Proposition 3.4.** The following hold:

- (i)  $T_{\text{FB}}$  is averaged.
- (ii)  $T_{\text{FB}}$  is asymptotically regular.
- (iii) K is a singleton.
- (iv)  $Z = \operatorname{Fix} T_{\operatorname{FB}}$ .
- (v)  $K = A(Z) = A(\text{Fix } T_{\text{FB}}).$

*Proof.* (i): Since *A* is firmly nonexpansive so is Id - A (see, e.g., [23, Lemma 2.3]). Note that  $J_B$  is firmly nonexpansive by [41, Proposition 1(c)]. It follows from [8, Remark 4.24(iii)] that Id - A and  $J_B$  are 1/2-averaged and therefore  $T = J_B(Id - A)$  is 2/3-averaged by [23, Lemma 2.2(iii)]. (ii): Combine (i) and Fact 2.3. (iii): Let  $k_1$  and  $k_2$  be in *K*. It follows from [7, Proposition 2.4] that  $(\exists z_i \in Z)$  such that  $k_i \in Az_i \cap (-Bz_i) = Az_i$ ,  $i \in \{1,2\}$ . Since *A* is single-valued, we conclude that  $k_i = Az_i$ ,  $i \in \{1,2\}$ . Using [7, Corollary 2.13] we learn that  $\langle z_1 - z_2, k_1 - k_2 \rangle = \langle z_1 - z_2, Az_1 - Az_2 \rangle = 0$ . Now combine with Lemma 3.3(ii) and use that *A* is single-valued to learn that  $k_1 = k_2$ . (iv): This follows from [8, Proposition 25.1(iv)]. (v): In view of (iii), let  $K = \{k\}$ . It follows from [7] that  $(\forall z \in Z) \ k = Az \cap (-Bz)$ , which implies, since *A* is single-valued, that k = Az; equivalently  $K = \{k\} = A(Z)$ . Now combine with (iv).

**Fact 3.5 (Baillon-Haddad).** *Let*  $f: X \to \mathbb{R}$  *be convex and differentiable. Then* 

$$\nabla f$$
 is nonexpansive  $\Leftrightarrow \nabla f$  is firmly nonexpansive. (20)

*Proof.* See [5, Corollaire 10].

<sup>&</sup>lt;sup>6</sup>For detailed discussion and examples of paramonotone operators we refer the reader to [30].

<sup>&</sup>lt;sup>7</sup>For detailed discussion and examples of 3<sup>\*</sup> monotone operators we refer the reader to [18].

**Fact 3.6.** Let  $f: X \to ]-\infty, +\infty]$  be convex, lower semicontinuous, and proper. Then the following hold:

- (i)  $\partial f$  is maximally monotone.
- (ii)  $(\partial f)^{-1} = \partial f^*$ .

*Proof.* (i): See, e.g., [43, Theorem A]. (ii): See, e.g., [43, Remark on page 216], [29, Théorème 3.1], or [8, Corollary 16.24]. ■

Suppose that *C* is a nonempty closed convex subset of *X*. It is well-known (see, e.g., [8, Example 23.4]) that

$$J_{N_C} = P_C. (21)$$

**Proposition 3.7.** Suppose that  $f: X \to \mathbb{R}$  is convex and differentiable such that  $\nabla f$  is nonexpansive and that  $g: X \to ]-\infty, +\infty]$  is convex, lower semicontinuous, and proper. Suppose that  $A = \nabla f$  and that  $B = \partial g$ . Then the following hold<sup>8,9</sup>:

- (i) Fix  $T_{\text{FB}} = \operatorname{zer}(\nabla f + \partial g) = \operatorname{argmin}(f + g)$ .
- (ii)  $T_{\text{FB}} = \text{Prox}_g(\text{Id} \nabla f).$

If in addition,  $g = \iota_V$  where V is a nonempty closed convex subset of X, then we have

(iii) 
$$T_{\text{FB}} = P_V(\text{Id} - \nabla f).$$

*Proof.* Note that dom f = X and that  $A = \nabla f$  is firmly nonexpansive by Fact 3.5. (i): The first identity is Proposition 3.4(iv) applied with (A, B) replaced by  $(\nabla f, \partial g)$ . It follows from [22, Proposition 3.2 & Corollary 3.4] that  $A + B = \nabla f + \partial g = \partial (f + g)$ . Now apply [8, Proposition 26.1]. (ii): Combine (16) and [8, Example 23.3]. (iii): Combine (ii) and (21).

**Remark 3.8.** Let  $f: X \to \mathbb{R}$  be convex and differentiable with  $1/\beta$  Lipschitz continuous gradient, where  $\beta > 0$ . Then  $\beta \nabla f$  is nonexpansive, hence firmly nonexpansive by Fact 3.5. Since  $\operatorname{argmin}(f + g) = \operatorname{argmin}(\beta f + \beta g)$ , Proposition 3.7 can be applied, with (f, g) replaced by  $(\beta f, \beta g)$ , to find a minimizer of f + g.

Suppose that<sup>10</sup> *C* is a nonempty closed convex subset of *X*. In the sequel we make use of the following useful result (see, e.g., [37, Exemple on page 286] or [8, Corollary 12.30]).

$$\nabla\left(\frac{1}{2}d_C^2\right) = \mathrm{Id} - P_C.$$
(22)

**Example 3.9 (Method of Alternating Projections (MAP) as a forward-backward iteration).** Suppose that U and V are nonempty closed convex subsets of X, that  $f = \frac{1}{2}d_U^2$  and that  $g = \iota_V$ . Suppose that  $A = \nabla f = \text{Id} - P_U$  and that  $B = \partial g = N_V$ . Then A is firmly nonexpansive and

$$T_{\text{FB}(\text{Id} - P_{U}, N_V)} = P_V P_U. \tag{23}$$

<sup>&</sup>lt;sup>8</sup>Let  $h : X \to ]-\infty, +\infty]$  be proper. The set of minimizers of h,  $\{x \in X \mid h(x) = \inf h(X)\}$ , is denoted by argmin h.

<sup>&</sup>lt;sup>9</sup>Suppose that  $g: X \to ]-\infty, +\infty]$  is convex, lower semicontinuous, and proper. Then  $\operatorname{Prox}_g$  is the *Moreau prox operator* associated with g defined by  $\operatorname{Prox}_g: X \to X: x \mapsto (\operatorname{Id} + \partial g)^{-1}(x) = \operatorname{argmin}_{u \in X} (g(y) + \frac{1}{2} ||x - y||^2).$ 

<sup>&</sup>lt;sup>10</sup>Let *C* be a nonempty closed convex subset of *X*. We use  $d_C$  to denote the *distance* from the set *C* defined by  $d_C: X \to [0, +\infty[: x \mapsto \min_{c \in C} ||x - c|| = ||x - P_C x||.$ 

*Proof.* It follows from (22) that  $\nabla f = \text{Id} - P_U$ , which is firmly nonexpansive by e.g., [48, Equation 1.7 on page 241]. Moreover (21) implies that  $J_B = J_{N_V} = P_V$ . Consequently,  $T_{\text{FB}(A,B)} = J_B \circ (\text{Id} - A) = P_V(\text{Id} - (\text{Id} - P_U)) = P_V P_U$ .

#### 4 The forward-backward operator and the normal problem

Let  $C : X \rightrightarrows X$  and let  $w \in X$ . The *inner shift* and *outer shift* of an operator C by w at  $x \in X$  are defined by

$$C_w x := C(x - w) \quad \text{and} \quad {}_w C x := -w + C x, \tag{24}$$

respectively.

Let  $w \in X$ . The *w*-perturbed problem introduced in [12] is:

$$(P_w) \text{ find } x \in X \text{ such that } 0 \in {}_wAx + B_wx = Ax + B(x - w) - w,$$
(25)

and the corresponding set of zeros is

$$Z_w := \{ x \in X \mid 0 \in 0 \in {}_w Ax + B_w x \} = \{ x \in X \mid w \in Ax + B(x - w) \}.$$
(26)

**Proposition 4.1.** *Let*  $w \in X$ *. Then* 

$$T_{FB}(_{w}A,B_{w}) = _{-w}T_{FB} = w + T_{FB},$$
 (27)

and

$$Z_w = \operatorname{Fix}(w + T_{\operatorname{FB}}). \tag{28}$$

Moreover, the following are equivalent:

(i)  $Z_w \neq \emptyset$ . (ii)  $w \in \operatorname{ran}(A + B_w)$ . (iii)  $w \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}})$ . (iv)  $w \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{DR}})$ .

*Proof.* Let  $x \in X$ . Using (16) and [8, Proposition 23.15(ii)&(iii)] we have  $T_{FB}(_wA, B_w)x = J_{B_w}(Id -_wA)x = J_B((x - (Ax - w)) - w) + w = J_B(x - Ax) + w = J_B(Id - A)x + w = w + T_{FB}x$ , which proves (27). To prove (28) apply Proposition 3.4(iv) with (A, B) replaced by  $(_wA, B_w)$  and use (27). "(i) $\Leftrightarrow$ (ii)": This follows from (26). "(i) $\Leftrightarrow$ (iii)": Indeed, using (28) we have  $Z_w \neq \emptyset \Leftrightarrow$ Fix $(w + T_{FB}) \neq \emptyset \Leftrightarrow (\exists x \in X)$  such that  $x = w + T_{FB}x \Leftrightarrow w \in \text{ran}(Id - T_{FB})$ . "(i) $\Leftrightarrow$ (iv)": This follows from [12, Proposition 3.3].

**Theorem 4.2.** We have  $^{11}$ 

(i)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{DR}}) = \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}).$ 

<sup>&</sup>lt;sup>11</sup>For convenience we shall use  $v_{\rm FB}$  and  $v_{\rm DR}$  to denote  $v_{T_{\rm FB}}$  and  $v_{T_{\rm DR}}$  respectively.

(ii)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \subseteq \operatorname{ran} A + \operatorname{ran} B$ . (iii)  $v_{\operatorname{DR}} = v_{\operatorname{FB}}$ .

*Proof.* (i): This is clear from the equivalence of (iii) and (iv) in Proposition 4.1. (ii): Combine (i) and [28, Proposition 4.1]. (iii): Indeed, using (i) and (6) we have  $v_{DR} = P_{\overline{ran}(Id - T_{DR})}0 = P_{\overline{ran}(Id - T_{FB})}0 = v_{FB}$ .

In view of Theorem 4.2(i), it is tempting to ask whether we can derive a similar conclusion for the equality of ran  $T_{FB}$  and ran  $T_{DR}$ . The next example gives a negative answer to this conjecture.

**Example 4.3** (ran  $T_{\text{DR}} \neq \text{ran } T_{\text{FB}}$ ). Suppose that A = Id. Then  $T_{\text{DR}} = \frac{1}{2} \text{Id} + J_B 0$  and  $T_{\text{FB}} \equiv J_B 0$ . Consequently,

$$X = \operatorname{ran} T_{\mathrm{DR}} \neq \operatorname{ran} T_{\mathrm{FB}} = \{J_B 0\}.$$
<sup>(29)</sup>

*Proof.* One can easily verify that  $J_A = \frac{1}{2}$  Id, hence  $R_A \equiv 0$ . Therefore,

$$T_{\rm DR} = {\rm Id} - J_A + J_B R_A = {\rm Id} - \frac{1}{2} {\rm Id} + J_B 0 = \frac{1}{2} {\rm Id} + J_B 0,$$
(30)

and

$$T_{\rm FB} = J_B(\mathrm{Id} - A) = J_B(\mathrm{Id} - \mathrm{Id}) \equiv J_B0,$$
(31)

and the conclusion readily follows.

Unlike the Douglas–Rachford operator, where we can learn about ran  $T_{DR}$  (see [11, Corollary 5.3]), we cannot obtain accurate information about the range of  $T_{FB}$  as we show next.

**Lemma 4.4.** ran  $T_{\text{FB}} \subseteq \text{dom } B$ .

*Proof.* Indeed, ran  $T_{\text{FB}} \subseteq \operatorname{ran} I_B = \operatorname{ran}(\operatorname{Id} + B)^{-1} = \operatorname{dom}(\operatorname{Id} + B) = \operatorname{dom} B$ .

The result in Lemma 4.4, cannot be improved as we illustrate now.

**Example 4.5** (ran  $T_{\text{FB}} \cong \text{dom } B$ ). Suppose that A = Id and that dom B is not a singleton. Then Example 4.3 implies that  $\{J_B 0\} = \text{ran } T_{\text{FB}} \cong \text{dom } B$ .

**Example 4.6** (ran  $T_{FB} = \text{dom } B$ ). Let C be a nonempty closed convex subset of X. Suppose that  $A \equiv 0$  and that  $B = N_C$ . Then (21) implies that  $T_{FB} = J_B = P_C$ , hence ran  $T_{FB} = C = \text{dom } B$ .

The *normal problem* (see [12, Definition 3.7]) associated with the ordered pair (A, B) is the *v*-perturbed problem where *v* is the *minimal displacement vector* defined by

$$v := v_{\rm FB} := P_{\overline{\rm ran}({\rm Id} - T_{\rm FB})}0; \tag{32}$$

and the corresponding set of *normal solutions* is  $Z_v$ .

**Corollary 4.7.**  $Z_v = \text{Fix}(v + T_{\text{FB}}).$ 

*Proof.* This follows from Proposition 4.1.

We point out that, even though the normal problem is well-defined in view of Fact 2.4, the set of normal solution may or may not be empty, as we illustrate now.

**Example 4.8** ( $Z = \emptyset$  but normal solutions exist). Let  $a^*, b^* \in X$  such that  $a^* + b^* \neq 0$ . Suppose that  $A: X \to X: x \mapsto a^*$ , and that  $B: X \to X: x \mapsto b^*$ . Then  $Z = \emptyset$ , ran $(Id - T_{FB}) = \{a^* + b^*\}$ , therefore  $v = a^* + b^* \in ran(Id - T_{FB})$  and  $Z_v = X \neq \emptyset$ .

*Proof.* We have  $J_B = (\mathrm{Id} + b^*)^{-1} = \mathrm{Id} - b^*$ , and  $T_{\mathrm{FB}} = J_B(\mathrm{Id} - A) = \mathrm{Id} - (a^* + b^*)$ . Consequently,  $\overline{\mathrm{ran}}(\mathrm{Id} - T_{\mathrm{FB}}) = \mathrm{ran}(\mathrm{Id} - T_{\mathrm{FB}}) = \{a^* + b^*\}$  and  $v = a^* + b^* \in \mathrm{ran}(\mathrm{Id} - T_{\mathrm{FB}})$ . Therefore,  $(\forall x \in X) = x - T_{\mathrm{FB}}x = a^* + b^* = v$ , which in view of Corollary 4.7, implies that  $Z_v = X$ , as claimed.

**Example 4.9** ( $Z = \emptyset$  and normal solutions do not exist). Suppose that  $X = \mathbb{R}^2$ , that  $U = \{(x,y) \in \mathbb{R}^2 \mid x > 0, y \ge 1/x\}$ , that  $V = \mathbb{R} \times \{0\}$ , that  $\beta < 0$ , that  $w = (\beta, 0) \neq (0, 0)$  and that  $f = \frac{1}{2}d_U^2$ . Set  $A = \nabla f$  and set  $B = w + N_V$ . Then  $T_{FB} = -w + P_V P_U$ , v = w,  $v \notin ran(Id - T_{FB})$  and therefore  $Z_v = \emptyset$ .

*Proof.* In view of (22) we have  $A = \text{Id} - P_U$ . Moreover (21) and [8, Proposition 23.15(ii)] implies that  $J_B = P_V(\cdot - w) = P_V - w$ , where the last identity uses that  $P_V$  is linear and that  $w \in V$ . Consequently  $T_{\text{FB}} = J_B(\text{Id} - A) = P_V(\text{Id} - (\text{Id} - P_U)) - w = P_V P_U - w$ . We claim that

$$\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = w + \operatorname{ran}(\operatorname{Id} - P_V P_U).$$
(33)

Indeed, let  $y \in X$ . Then  $y \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists x \in X)$  such that  $y = w + x - P_V P_U x \Leftrightarrow y \in w + \operatorname{ran}(\operatorname{Id} - P_V P_U)$ . It follows from Example 5.8 below that  $\overline{\operatorname{ran}}(\operatorname{Id} - P_V P_U) = \overline{(\operatorname{rec} U)^{\ominus} + (\operatorname{rec} V)^{\ominus}} = \overline{\mathbb{R}_-^2 + V^{\perp}} = \mathbb{R}_-^2 + (\{0\} \times \mathbb{R}) = \mathbb{R}_- \times \mathbb{R}$ . Using (11) applied with *S* replaced by  $\operatorname{ran}(\operatorname{Id} - P_V P_U)$  we have  $v = w + P_{\overline{\operatorname{ran}}(\operatorname{Id} - P_V P_U)}(-w) = w$ . Consequently (33) becomes  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = v + \operatorname{ran}(\operatorname{Id} - P_V P_U)$ . Furthermore, using [13, Lemma 2.2(i)]  $v \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = v + \operatorname{ran}(\operatorname{Id} - P_V P_U) \Leftrightarrow \operatorname{Fix} P_V P_U \neq \emptyset \Leftrightarrow U \cap V \neq \emptyset$ , which does not hold, hence  $Z_v = \emptyset$  by Proposition 4.1.

**Remark 4.10.** Suppose that  $A^{-1}$  is firmly nonexpansive. Then one can define the forward-backward operator for the dual pair  $(A^{-1}, B^{-0})$ . Nonetheless, the self-duality property, which is a key feature of  $T_{DR}$  (see, e.g., [7, Corollary 4.3] or [28, Lemma 3.6 on page 133]), does not hold for  $T_{FB}$  as we illustrate in Example 4.11.

**Example 4.11** ( $T_{FB}$  is not self-dual). Suppose that V is a closed linear subspace of X and let  $u \in V \setminus \{0\}$ . Suppose that  $A : X \to X : x \mapsto x - u$  and that  $B = N_V$ . Then  $A^{-1}$  is firmly nonexpansive, however

$$u \equiv T_{\mathrm{FB}(A,B)} \neq T_{\mathrm{FB}(A^{-1},B^{-\varpi})} \equiv 0.$$
(34)

*Proof.* First note that  $A^{-1} : X \to X : x \mapsto x + u$ , hence  $A^{-1}$  is firmly nonexpansive, as claimed. Since *B* is linear we learn that  $B^{-1}$  is linear and so are  $J_B$  and  $J_{B^{-1}}$  by [15, Theorem 2.1(xviii)]. By [7, Proposition 4.1(ii)] and Fact 2.1(ii) we have  $J_{B^{-0}} = J_{(B^{-1})^{\odot}} = J_{B^{-1}} = \text{Id} - J_B = \text{Id} - P_V = P_{V^{\perp}}$ . Now,  $T_{\text{FB}(A,B)} = J_B(\text{Id} - A) = P_V(\text{Id} - \text{Id} + u) = P_V u = u$ , whereas  $T_{\text{FB}(A^{-1},B^{-\odot})} = J_{B^{-\odot}}(\text{Id} - A^{-1}) = P_{V^{\perp}}(\text{Id} - \text{Id} - u) = P_{V^{\perp}}(-u) = -P_{V^{\perp}}(u) \equiv 0$ . **Remark 4.12.** Clearly the forward-backward operator is not symmetric in A and B, however, it is critical to consider the order in (16) when only A is firmly nonexpansive. If, in addition, B is firmly nonexpansive we can also define  $T_{FB(B,A)}$ .

**Corollary 4.13.** Suppose that  $B : X \to X$  is firmly nonexpansive. Then  $T_{FB(B,A)} := J_A(Id - B)$  is averaged and

$$\|v_{FB(A,B)}\| = \|v_{FB(B,A)}\|.$$
(35)

*Proof.* Combining Theorem 4.2(iii) and [12, Proposition 3.11] we have  $||v_{FB(A,B)}|| = ||v_{DR(A,B)}|| = ||v_{DR(A,B)}|| = ||v_{FB(B,A)}||$ .

## 5 The range of the displacement operator

Unless otherwise stated, in this section we work under the assumption that

*H* is a finite-dimensional Hilbert space.

The results in this section provide information on the range of the displacement map Id  $-T_{FB}$ .

**Definition 5.1** (nearly convex and nearly equal sets). Let C and D be subsets<sup>12</sup> of H.

- (i) We say that D is nearly convex<sup>13</sup> (see [40, Theorem 12.41]) if there exists a convex set subset E of H such that  $E \subseteq D \subseteq \overline{E}$ .
- (ii) We say that C and D are nearly equal<sup>14</sup> if

$$C \simeq D : \Leftrightarrow \overline{C} = \overline{D} \text{ and } \operatorname{ri} C = \operatorname{ri} D.$$
 (36)

**Fact 5.2.** Let *H* be a finite-dimensional Hilbert space. Let  $C : H \Rightarrow H$  be maximally monotone. Then dom *C* and ran *C* are nearly convex.

*Proof.* See [40, Theorem 12.41].

**Theorem 5.3.** *Let H be a finite-dimensional Hilbert space. The following hold:* 

- (i)  $\operatorname{ran}(\operatorname{Id} T_{\operatorname{FB}}) \simeq \operatorname{ran} A + \operatorname{ran} B$ .
- (ii) Suppose that A and B are affine<sup>15</sup>. Then  $ran(Id T_{FB}) = \overline{ran}(Id T_{FB}) = ran A + ran B$ .

If, in addition, A or B is surjective then we additionally have:

(iii)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = X.$ 

(iv) Fix  $T_{\text{FB}} = Z \neq \emptyset$ .

<sup>&</sup>lt;sup>12</sup>Let *C* be a subset of *H*. We use ri *C* to denote the interior of *C* with respect to the affine hull of *C*.

<sup>&</sup>lt;sup>13</sup>For detailed discussion on the algebra of nearly convex sets we refer the reader to [42, Section 3].

<sup>&</sup>lt;sup>14</sup>For detailed discussion on the properties of nearly equal and nearly convex sets we refer the reader to [15].

<sup>&</sup>lt;sup>15</sup>Let  $B : X \rightrightarrows X$ . Then B is an affine relation if gra B is an affine subspace of  $X \times X$ .

*Proof.* (i): Note that *A* is 3<sup>\*</sup> monotone (by Lemma 3.3(iii)) and dom *A* = *X*. It follows from [11, Theorem 5.2] that ran(Id  $-T_{DR}$ )  $\simeq$  (dom *A* - dom *B*)  $\cap$  (ran *A* + ran *B*). Now combine with Theorem 4.2(i) and use that dom *A* = *X*. (ii): On the one hand, ran *A* and ran *B* are closed affine subspaces of *X*, so is their sum ran *A* + ran *B*. On the other hand, since the resolvent *J<sub>B</sub>* is affine (see [15, Theorem 2.1(xix)]), so are *T<sub>FB</sub>* and Id  $-T_{FB}$ . Therefore, in view of (i), ran(Id  $-T_{FB}$ ) =  $\overline{ran}(Id - T_{FB}) = \overline{ran}A + ran B = ran A + ran B$ . (iii): Using Theorem 5.3(i) we have *X* = ri *X* = ri(ran *A* + ran *B*)  $\subseteq$  ran(Id  $-T_{FB}$ )  $\subseteq \overline{ran}(Id - T_{FB}) = \overline{ran}A + ran B = ran A + ran B =$ 

In the conclusion of Theorem 5.3(i), we cannot replace near equality by equality as we illustrate in Example 5.4.

**Example 5.4.** Suppose that  $H = \mathbb{R}^2$  and let  $f: \mathbb{R}^2 \to ]-\infty, +\infty] : (\xi_1, \xi_2) \mapsto \max \{g(\xi_1), |\xi_2|\}$ , where  $g(\xi_1) = 1 - \sqrt{\xi_1}$  if  $\xi \ge 0$  and  $g(\xi_1) = +\infty$  otherwise. Set<sup>16</sup>  $A = P_{\mathbb{R}^2_+}$  and  $B = \partial f^*$ . Then A is firmly nonexpansive and B is maximally monotone. Moreover,  $\operatorname{ran} A = \mathbb{R}^2_+$ ,  $\operatorname{ran} B = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid |\xi_2| \ge 1\}$ , hence  $\operatorname{ran} A + \operatorname{ran} B = \{(\xi_1, \xi_2) \mid \xi_1 \ge 0, \xi_2 \in \mathbb{R}\}$  but  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid \xi_2 \le \mathbb{R}\} \cup \{(0, \xi_2) \mid \xi_2 \le -1\}$ . Therefore

$$\operatorname{ri}(\operatorname{ran} A + \operatorname{ran} B) \subsetneqq \operatorname{ran}(\operatorname{Id} - T) \subsetneqq \operatorname{ran} A + \operatorname{ran} B = \operatorname{ran} A + \operatorname{ran} B.$$
(37)

*Proof.* The claim about firm nonexpansiveness of *A* follows from e.g., [48, Equation 1.6 on page 241] or [31, Section 3] and maximal monotonicity of *B* follows from Fact 3.6(i) applied to  $f^*$ . Using Fact 3.6(ii) and [42, Example on page 218] we see that dom  $\partial f = \operatorname{ran}(\partial f)^{-1} = \operatorname{ran}\partial f^* = \operatorname{ran} B = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid |\xi_2| \ge 1\}$ . Note that in view of Theorem 5.3(i) we have  $\{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} = \operatorname{ri}(\operatorname{ran} A + \operatorname{ran} B) \subseteq \operatorname{ran}(\operatorname{Id} - T) \subseteq \overline{\operatorname{ran} A + \operatorname{ran} B} = \{(\xi_1, \xi_2) \mid \xi_1 \ge 0, \xi_2 \in \mathbb{R}\}$ . Therefore we only need to check the points in  $\{(0, \beta) \mid \beta \in \mathbb{R}\}$ . To

<sup>&</sup>lt;sup>16</sup>Let  $f : X \to ]-\infty, +\infty]$  be convex, lower semicontinuous, and proper. We use  $f^*$  to denote the *convex conjugate* (*a.k.a. Fenchel conjugate*) of f, defined by  $f^* : X \to ]-\infty, +\infty] : x \mapsto \sup_{u \in X} (\langle x, u \rangle - f(x))$ .

proceed further we recall that (see [36, Example 6.5])

$$\partial f(\xi_{1},\xi_{2}) = \begin{cases} \emptyset, & \text{if } \xi_{1} < 0; \\ \emptyset, & \text{if } \xi_{1} = 0 \text{ and } |\xi_{2}| < 1; \\ \mathbb{R}_{-} \times \{1\}, & \text{if } \xi_{1} = 0 \text{ and } \xi_{2} \ge 1; \\ \mathbb{R}_{-} \times \{-1\}, & \text{if } \xi_{1} = 0 \text{ and } \xi_{2} \le -1; \\ \text{conv} \left\{ (-\frac{1}{2}\xi_{1}^{-1/2}, 0), (0, 1) \right\}, & \text{if } \xi_{2} = 1 - \sqrt{\xi_{1}} \text{ and } 0 < \xi_{1} < 1; \\ \text{conv} \left\{ (-\frac{1}{2}\xi_{1}^{-1/2}, 0), (0, -1) \right\}, & \text{if } -\xi_{2} = 1 - \sqrt{\xi_{1}} \text{ and } 0 < \xi_{1} < 1; \\ (-\frac{1}{2}\xi_{1}^{-\frac{1}{2}}, 0), & \text{if } 0 < \xi_{1} < 1 \text{ and } 1 - \sqrt{\xi_{1}} > |\xi_{2}|; \\ (0, 1), & \text{if } 0 < \xi_{1} < 1 \text{ and } \xi_{2} > 1 - \sqrt{\xi_{1}}; \\ (0, -1), & \text{if } 0 < \xi_{1} < 1 \text{ and } -\xi_{2} > 1 - \sqrt{\xi_{1}}; \\ (0, -1), & \text{if } 0 < \xi_{1} < 1 \text{ and } -\xi_{2} > 1 - \sqrt{\xi_{1}}; \\ \text{conv} \left\{ (-\frac{1}{2}, 0), (0, 1), (0, -1) \right\}, & \text{if } \xi_{1} > 1 \text{ and } \xi_{2} = 0; \\ \text{conv} \left\{ (0, 1), (0, -1) \right\}, & \text{if } \xi_{1} > 1 \text{ and } \xi_{2} = 0; \\ (0, 1), & \text{if } \xi_{1} > 1 \text{ and } \xi_{2} > 0; \\ (0, -1), & \text{if } \xi_{1} > 1 \text{ and } \xi_{2} > 0; \\ (0, -1), & \text{if } \xi_{1} > 1 \text{ and } -\xi_{2} > 0. \end{cases}$$
(38)

Let  $\beta \in \mathbb{R}$ . In view of Proposition 4.1 and Fact 3.6(ii) we have

$$(0,\beta) \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ (0,\beta) \in P_{\mathbb{R}^2_+}(\xi_1,\xi_2) + \partial f^*(\xi_1,\xi_2 - \beta)$$
(39a)

$$= P_{\mathbb{R}^2_+}(\xi_1,\xi_2) + (\partial f)^{-1}(\xi_1,\xi_2 - \beta)$$
(39b)

$$\Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) \ (0, \beta) - P_{\mathbb{R}^2_+}(\xi_1, \xi_2) \in (\partial f)^{-1}(\xi_1, \xi_2 - \beta)$$
(39c)

$$\Leftrightarrow \left(\exists (\xi_1,\xi_2) \in \mathbb{R}^2\right) (\xi_1,\xi_2-\beta) \in \partial f\left((0,\beta)-P_{\mathbb{R}^2_+}(\xi_1,\xi_2)\right).$$
(39d)

We argue by cases using (38) and (39).

*Case 1:*  $\xi_1 \geq 0$  and  $\xi_2 \geq 0$ . Then  $(0,\beta) \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ (\xi_1,\xi_2 - \beta) \in \partial f((0,\beta) - P_{\mathbb{R}^2_+}(\xi_1,\xi_2)) = \partial f(-\xi_1,\beta-\xi_2)) \Leftrightarrow [(\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ \xi_1 = 0, \xi_2 - \beta = 1 \text{ and } \beta - \xi_2 \geq 1 \text{ or } \xi_1 = 0, \xi_2 - \beta = -1 \text{ and } \beta - \xi_2 \leq -1],$  which is impossible.

*Case 2:*  $\xi_1 \leq 0$  and  $\xi_2 \leq 0$ . Then  $(0,\beta) \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ (\xi_1,\xi_2 - \beta) \in \partial f((0,\beta) - P_{\mathbb{R}^2_+}(\xi_1,\xi_2)) = \partial f(0,\beta)) \Leftrightarrow [(\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ \xi_1 \leq 0, \xi_2 - \beta = 1 \text{ and } \beta \geq 1 \text{ or } \xi_1 \leq 0, \xi_2 - \beta = -1 \text{ and } \beta \leq -1 ] \Leftrightarrow [(\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ \xi_1 \leq 0, \xi_2 = \beta + 1 \geq 2 \text{ or } \xi_1 \leq 0, \xi_2 = \beta - 1 \leq -2].$  Since  $\xi_2 \leq 0$  we conclude that  $\beta \leq -1$ .

*Case 3:*  $\xi_1 > 0$  and  $\xi_2 < 0$ . Then  $(0, \beta) \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists (\xi_1, \xi_2) \in \mathbb{R}^2) \ (\xi_1, \xi_2 - \beta) \in \partial f((0, \beta) - P_{\mathbb{R}^2_+}(\xi_1, \xi_2)) = \partial f(-\xi_1, \beta) \Rightarrow [\xi_1 > 0 \text{ and by } (38) - \xi_1 > 0]$  which is impossible.

*Case 4:*  $\xi_1 < 0$  and  $\xi_2 > 0$ . Then  $(0,\beta) \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Leftrightarrow (\exists (\xi_1,\xi_2) \in \mathbb{R}^2) \ (\xi_1,\xi_2 - \beta) \in \partial f((0,\beta) - P_{\mathbb{R}^2_+}(\xi_1,\xi_2)) = \partial f(0,\beta - \xi_2) \Leftrightarrow [\xi_1 < 0,\xi_2 - \beta = 1 \text{ and } \beta - \xi_2 \ge 1 \text{ or } \xi_1 < 0,\xi_2 - \beta = -1 \text{ and } \beta - \xi_2 \le -1]$ , which never occurs.

Altogether we conclude that  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = \{(\xi_1, \xi_2) \mid \xi_1 > 0, \xi_2 \in \mathbb{R}\} \cup \{(0, \xi_2) \mid \xi_2 \leq -1\},\$ as claimed.

Suppose that *C* and *D* are nonempty nearly convex subsets of *H*. Then [15, Proposition 2.12] implies that

$$C \simeq D \Leftrightarrow \overline{C} = \overline{D}. \tag{40}$$

**Lemma 5.5.** Let *H* be a finite-dimensional Hilbert space. Suppose that  $f: H \to ]-\infty, +\infty]$  is convex, lower semicontinuous, and proper. Then the dom  $\partial f \simeq \text{dom } f$  and ran  $\partial f \simeq \text{dom } f^*$ .

*Proof.* It follows from Fact 5.2 and Fact 3.6(i) that dom  $\partial f$  is nearly convex. Moreover, [8, Corollary 16.29] implies that dom  $\partial f = \text{dom} f$ . Therefore (40) implies that dom  $\partial f \simeq \text{dom} f$ . Using Fact 3.6(ii) we have ran  $\partial f = \text{dom}(\partial f)^{-1} = \text{dom} \partial f^*$ . Now apply the same argument to  $f^*$ .

We recall that (see [48, Theorem 3.1]) for a nonempty closed convex subset *C* of *X* the following holds<sup>17</sup>:

$$\overline{\operatorname{ran}}(\operatorname{Id} - P_{\mathcal{C}}) = (\operatorname{rec} \mathcal{C})^{\ominus}.$$
(41)

**Example 5.6.** Let *H* be a finite-dimensional Hilbert space. Suppose that *C* is a nonempty closed convex subset of *H*. Set  $f = \iota_C$  and suppose that  $A = \partial f = N_C$ . Then dom A = C and ran  $A \simeq (\operatorname{rec} C)^{\ominus}$ .

*Proof.* Clearly dom A = C. It follows from [8, Proposition 23.2(i)], Fact 2.1(ii) and (21) that ran  $A = \text{dom } A^{-1} = \text{ran} J_{A^{-1}} = \text{ran}(\text{Id} - J_A) = \text{ran}(\text{Id} - P_C)$ . In view of (41) we have  $\overline{\text{ran}}(\text{Id} - P_C) = (\text{rec } C)^{\ominus}$ . Note that  $J_{A^{-1}} = \text{Id} - P_C$  is maximally monotone by Fact 2.1(ii)&(i), therefore Fact 5.2 implies that  $\text{ran}(\text{Id} - P_C)$  is nearly convex. Now apply (40).

Suppose that  $C_1$  and  $C_2$  are nearly convex subsets of H and that  $D_1$  and  $D_2$  are subsets of H such that  $C_i \simeq D_i$  for every  $i \in \{1, 2\}$ . It follows from [15, Theorem 2.14] that

$$C_1 + C_2 \simeq D_1 + D_2.$$
 (42)

**Proposition 5.7.** Let H be a finite-dimensional Hilbert space. Suppose that  $f: H \to \mathbb{R}$  is convex and differentiable such that  $\nabla f$  is nonexpansive and that  $g: H \to ]-\infty, +\infty]$  is convex, lower semicontinuous, and proper. Suppose that  $A = \nabla f$  and that  $B = \partial g$ . Then the following hold:

(i)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \simeq \operatorname{dom} f^* + \operatorname{dom} g^*$ .

If in addition,  $g = \iota_V$  where V is a nonempty closed convex subset of H, then we have:

(ii)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \simeq \operatorname{dom} f^* + (\operatorname{rec} V)^{\ominus}$ .

*Proof.* It follows from Fact 3.5 that  $\nabla f$  is firmly nonexpansive. (i): Combine Theorem 5.3(i), Lemma 5.5 and (42). (ii): It follows from Lemma 5.5 and Example 5.6 respectively that ran  $A \simeq \text{dom } f^*$  and ran  $B \simeq (\text{rec } V)^{\ominus}$ . Now combine with Theorem 5.3(i) and (42).

<sup>&</sup>lt;sup>17</sup>Let *C* be a nonempty closed convex subset of *X*. The *recession cone* of *C* is rec  $C := \{x \in X \mid x + C \subseteq C\}$ , and the *polar cone* of *C* is  $C^{\ominus} := \{u \in X \mid \sup_{c \in C} \langle c, u \rangle \leq 0\}$ ,

**Example 5.8** (range of the displacement map of alternating projections). Let H be a finitedimensional Hilbert space. Suppose that U and V are nonempty closed convex subsets of X, that  $f = \frac{1}{2}d_U^2$ and that  $g = \iota_V$ . Suppose that  $A = \nabla f = \text{Id} - P_U$  and that  $B = \partial g = N_V$ . Then

$$\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = \operatorname{ran}(\operatorname{Id} - P_V P_U) \simeq (\operatorname{rec} U)^{\ominus} + (\operatorname{rec} V)^{\ominus}.$$
(43)

*Proof.* It follows from (41) and (40) that ran  $A = \operatorname{ran}(\operatorname{Id} - P_U) \simeq (\operatorname{rec} U)^{\ominus}$ . On the other hand Example 5.6 implies that ran  $B \simeq (\operatorname{rec} V)^{\ominus}$ . Now combine with [15, Theorem 2.12].

#### 6 Affine operators and applications

**Fact 6.1.** Let  $L: X \to X$  be linear and nonexpansive, let  $b \in X$  and suppose that  $T: X \to X: x \mapsto Lx + b$ . Let  $v_T := P_{\overline{ran}(Id - T)}0$  and let  $x \in X$ . Then

$$(\forall n \in \mathbb{N}) \quad T^n x + n v_T = (T_{-v_T})^n x = (v_T + T)^n x.$$
 (44)

*Proof.* See [6, Theorem 3.2(iv) and (v)].

**Lemma 6.2.** Let  $L: X \to X$  be linear and nonexpansive, let  $b \in X$ , suppose that  $T: X \to X: x \mapsto Lx + b$ and that  $v_T := P_{\overline{ran}(Id-T)}0 \in ran(Id-T)$ . Let  $x \in X$ . Then there exists a point  $a \in X$  such that  $v_T + b = a - La$  and  $v_T + Tx = a + L(x - a)$ . Moreover we have

$$(\forall n \in \mathbb{N}) \quad T^n x + n v_T = (T_{-v_T})^n x = (v_T + T)^n x = a + L^n (x - a)$$
(45)

and

$$\operatorname{Fix}(v_T + T) = a + \operatorname{Fix} L. \tag{46}$$

*Proof.* Note that  $v_T \in \operatorname{ran}(\operatorname{Id} - T) = \operatorname{ran}(\operatorname{Id} - L) - b \Leftrightarrow v_T + b \in \operatorname{ran}(\operatorname{Id} - L)$ . Now let  $a \in X$  be such that  $v_T + b = a - La$ . The first two identities in (45) follow from Fact 6.1. We prove the last identity in (45) by induction. The case n = 0 is obvious. Now suppose that for some  $n \in \mathbb{N}$   $(v_T + T)^n x = a + L^n(x - a)$ . Then  $(v_T + T)^{n+1}x = v_T + b + L(a + L^n(x - a)) = v_T + b + La + L^{n+1}(x - a) = a + L^{n+1}(x - a)$ . We now turn to (46). In view of (45) applied with n = 1 we have  $x \in \operatorname{Fix}(v_T + T) \Leftrightarrow x = v_T + Tx \Leftrightarrow x = a + L(x - a) \Leftrightarrow x - a \in \operatorname{Fix} L \Leftrightarrow x \in a + \operatorname{Fix} L$ , hence  $\operatorname{Fix}(v_T + T) = a + \operatorname{Fix} L$ .

**Proposition 6.3.** Let  $L: X \to X$  be linear and nonexpansive, let  $b \in X$ , suppose that  $T: X \to X: x \mapsto Lx + b$  and that  $v_T := P_{\overline{ran}(Id-T)}0 \in ran(Id-T)$ . Let  $x \in X$ . Then  $Fix(v_T + T) \neq \emptyset$ . Moreover the following are equivalent:

- (i) *L* is asymptotically regular.
- (ii)  $L^n x \to P_{\operatorname{Fix} L} x$ .
- (iii)  $T^n x + nv_T = (v_T + T)^n x = (T_{-v_T})^n x \to P_{\text{Fix}(v_T + T)} x.$
- (iv)  $T_{-v_T} = v_T + T$  is asymptotically regular.

(v)  $(T^n x + nv_T)_{n \in \mathbb{N}}$  is asymptotically regular.

*Proof.* The proof uses the same techniques as in [16]. "(i) $\Leftrightarrow$ (ii)": See [4, Proposition 4], [3, Theorem 1.1], [9, Theorem 2.2] or [8, Proposition 5.27]. "(ii) $\Rightarrow$ (iii)": Using (45) and (11) we learn that

$$T^{n}x + nv_{T} = (T_{-v_{T}})^{n}x = (v_{T} + T)^{n}x = a + L^{n}(x - a)$$
(47a)

$$\rightarrow a + P_{\text{Fix }L}(x-a) = P_{a+\text{Fix }L}x = P_{\text{Fix}(v_T+T)}x.$$
(47b)

Now combine with (46). "(iii) $\Rightarrow$ (iv)": Clear. "(iv) $\Rightarrow$ (v)": This follows from Fact 6.1. "(v) $\Rightarrow$ (i)": Using (45) we have  $L^n x - L^{n+1} x = T^n(x+a) + nv_T - (T^{n+1}(x+a) + (n+1)v_T) \rightarrow 0$ .

Let  $\mathcal{B}(X)$  denote the set of bounded linear operators on X. We have the following result.

**Proposition 6.4.** Let  $L: X \to X$  be linear and nonexpansive, let  $b \in X$ , suppose that  $T: X \to X: x \mapsto Lx + b$  and that  $v_T := P_{\overline{ran}(Id - T)}0 \in ran(Id - T)$ . Let  $x \in X$  and let  $\mu \in ]0, 1[$ . Then the following are equivalent:

- (i)  $T^n x + nv_T = (v_T + T)^n x = (T_{-v_T})^n x \to P_{\text{Fix}(v_T+T)} x \mu$ -linearly.
- (ii)  $L^n x \to P_{\text{Fix} L} x \mu$ -linearly.
- (iii)  $L^n \to P_{\text{Fix } L} \mu$ -linearly (in  $\mathcal{B}(X)$ ).

*Proof.* Note that *L* is asymptotically regular by Fact 2.3. "(i) $\Leftrightarrow$ (ii)": In view of (45), (46) and (11) we learn that  $T^n x + nv_T - P_{\text{Fix}(v_T+T)}x = (v_T + T)^n x - P_{\text{Fix}(v_T+T)}x = (T_{-v_T})^n x - P_{\text{Fix}(v_T+T)}x = a + L^n(x-a) - a - P_{\text{Fix}L}(x-a) = L^n(x-a) - P_{\text{Fix}L}(x-a)$ . "(ii) $\Leftrightarrow$ (iii)": This follows from [16, Lemma 2.6].

**Corollary 6.5.** Suppose that X is finite-dimensional. Let L:  $X \to X$  be linear, nonexpansive and asymptotically regular, let  $b \in X$ , set  $T: X \to X: x \mapsto Lx + b$  and suppose that  $v_T := P_{\overline{ran}(Id-T)}0$ . Let  $x \in X$ . Then  $v_T \in ran(Id-T)$  and

$$T^{n}x + nv_{T} = (v_{T} + T)^{n}x = (T_{-v_{T}})^{n}x \rightarrow P_{\operatorname{Fix}(v_{T}+T)}x \quad linearly.$$

$$(48)$$

*Proof.* Since *X* is finite-dimensional we learn that ran(Id - T) is a closed affine subspace of *X*, hence  $v_T \in ran(Id - T)$ . Now Proposition 6.3 implies that  $L^n x \to P_{FixL} x$ , which when combined with [16, Corollary 2.8] yields  $L^n x \to P_{FixL} x$  linearly. Now apply Proposition 6.4

**Theorem 6.6 (application to the forward-backward algorithm).** *Suppose that A and B are affine and let*  $x \in X$ *. Then the following hold:* 

(i) 
$$(T_{FB}(_{v}A, B_{v}))^{n}x = (v + T_{FB})^{n}x = ((T_{FB})_{-v})^{n}x = T_{FB}^{n}x + nv.$$
  
(ii) If  $v \in ran(Id - T_{FB})$  then  
 $(v + T_{FB})^{n}x = ((T_{FB})_{-v})^{n}x = T_{FB}^{n}x + nv \to P_{Fix(v+T)}x = P_{Z_{v}}x.$  (49)

(iii) We have the implication

$$v = 0 \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) \Rightarrow T_{\operatorname{FB}}^n x \to P_{\operatorname{Fix} T} x = P_Z x.$$
 (50)

*If, in addition, X is finite-dimensional, then we also have* 

(iv)  $v \in \operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}})$  and

$$v + T_{\text{FB}})^n x = ((T_{\text{FB}})_{-v})^n x = T_{\text{FB}}^n x + nv \to P_{\text{Fix}(v+T)} x = P_{Z_v} x \text{ linearly.}$$
(51)

(v) We have the implication

$$v = 0 \Rightarrow T_{FB}^n x \to P_{FixT} x = P_Z x \ linearly.$$
 (52)

*Proof.* Proposition 3.4(ii) implies that  $T_{\text{FB}}$  is asymptotically regular and, since  $J_B$  is affine, (see [15, Theorem 2.1(xix)]) so is  $v + T_{\text{FB}}$ . (i): The first identity follows from (27) applied with w replaced by v. Now combine with Fact 6.1. (ii): Combine Proposition 6.3 and Corollary 4.7. (iii): This is a direct consequence of (ii). (iv) & (v): Combine Corollary 6.5 with (ii) and (iii), respectively.

**Example 6.7.** Let  $L : X \to X$  be linear and firmly nonexpansive, let  $b \in X$  and suppose that U is an affine subspace of X. Suppose that  $A : X \to X : x \mapsto Lx + b$  and that  $B = N_U$ . Then the following hold<sup>18</sup>:

(i) 
$$Z_v = (v + U) \cap (L^{-1}((\operatorname{par} U)^{\perp} - b + v)).$$

*If, in addition, X is finite-dimensional then we also have:* 

(ii)  $\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = \operatorname{ran} L + (\operatorname{par} U)^{\perp} + b.$ (iii)  $v = P_{\operatorname{par} U \cap \ker L} b.$ 

*Proof.* (i): Let  $x \in X$ . Then  $x \in Z_v \Leftrightarrow 0 \in Lx + b - v + N_U(x - v) = Lx + b - v + (par U)^{\perp}$  $\Leftrightarrow [x - v \in U \text{ and } Lx \in (par U)^{\perp} - b + v] \Leftrightarrow [x \in v + U \text{ and } Lx \in (par U)^{\perp} - b + v] \Leftrightarrow x \in (v + U) \cap (L^{-1}((par U)^{\perp} - b + v)).$  (ii): Using Theorem 5.3(ii) we have

$$\operatorname{ran}(\operatorname{Id} - T_{\operatorname{FB}}) = \overline{\operatorname{ran}}(\operatorname{Id} - T_{\operatorname{FB}}) = \operatorname{ran} A + \operatorname{ran} B$$
(53a)

$$= \operatorname{ran} L + b + \operatorname{ran} N_U = \operatorname{ran} L + (\operatorname{par} U)^{\perp} + b.$$
(53b)

(iii): Using Lemma 3.3(i) we learn that *L* is (maximally) monotone. Combining (ii), (11), (53), [27, Theorem 2.19] and [8, Proposition 20.17] we have

$$v = P_{\overline{\text{ran}}(\text{Id} - T_{\text{FB}})}0 = P_{\text{ran}\,L + (\text{par}\,U)^{\perp} + b}0 = b - P_{\text{ran}\,L + (\text{par}\,U)^{\perp}}b$$
(54a)

$$= P_{(\operatorname{ran} L + (\operatorname{par} U)^{\perp})^{\perp}} b = P_{(\operatorname{ran} L)^{\perp} \cap (\operatorname{par} U)} b = P_{\operatorname{ker} L^* \cap \operatorname{par} U} b = P_{\operatorname{ker} L \cap \operatorname{par} U} b.$$
(54b)

**Example 6.8 (MAP in the affine-affine feasibility case).** Suppose that U and V are closed linear subspaces of X. Let  $w \in X$ . Suppose that  $f = \frac{1}{2}d_{w+U}^2$ , that  $g = \iota_{w+V}$ , that  $A = \nabla f$  and that  $B = \partial g$ . Then  $(\forall n \in \mathbb{N})$ 

$$(T_{\rm FB})^n = (P_{w+V}P_{w+U})^n = (P_V P_U)^n (\cdot - w) + w.$$
(55)

*Proof.* Indeed, let  $x \in X$ . It follows from Example 3.9 applied with (U, V) replaced by (w + U, w + V) and (11) that  $T_{FB} = P_{w+U}P_{w+V}x = P_{w+V}(P_U(x - w) + w) = P_V(P_U(x - w) + w - w) + w = P_VP_U(x - w) + w$ . Now (55) follows by simple induction.

<sup>&</sup>lt;sup>18</sup>Suppose that *U* is a closed affine subspace of *X*. We use par *U* to denote the parallel space of *U* defined by par U := U - U.

We now provide an application of the forward-backward algorithm that employs Pierra's product space technique introduced in [39]. For a general and more flexible framework of using the forward-backward algorithm to find a zero of the sum of more than two operators we refer the reader to the work by Combettes in [2, Section 2] and [26, Section 5].

**Proposition 6.9** (application to parallel splitting). Suppose that  $m \in \{2,3,\ldots\}$ . For every  $i \in \{1, 2, \ldots, m\}$ , let  $\alpha_i > 0$  and suppose that  $A_i : X \rightarrow X$  are  $\alpha_i$ -cocoercive. Set  $\Delta$  :=  $\{(x,...,x) \in X^m \mid x \in X\}$ , set  $\alpha = \min \{\alpha_i \mid i \in \{1,2,...,m\}\}$ , set  $\mathbf{A} = \times_{i=1}^m \alpha A_i$ , set  $\mathbf{B} = N_{\Delta}$ , set  $\mathbf{T} = T_{\mathrm{FB}(\mathbf{A},\mathbf{B})}, let \, j: X \to X^m : x \mapsto (x, x, \dots, x), and \, let \, e: X^m \to X : (x_1, x_2, \dots, x_m) \mapsto \frac{1}{m} \left( \sum_{i=1}^m x_i \right).$ Let  $\mathbf{x} \in X^m$  and suppose that  $\mathbf{v} := P_{\overline{ran}(\mathrm{Id} - \mathbf{T})} \mathbf{0} \in \mathrm{ran}(\mathrm{Id} - \mathbf{T})$ . Then the following hold:

- (i)  $\mathbf{\Delta}^{\perp} = \{(u_1, \dots, u_m) \in X^m \mid \sum_{i=1}^m u_i = 0\}.$ (ii)  $\mathbf{Z}_{\mathbf{v}} := Z_{(\mathbf{v}\mathbf{A},\mathbf{B}_{\mathbf{v}})} = (\mathbf{v} + \mathbf{\Delta}) \cap (\mathbf{A}^{-1}(\mathbf{v} + \mathbf{\Delta}^{\perp})).$ (iii)  $\mathbf{v} = 0 \Leftrightarrow \operatorname{zer}(\sum_{i=1}^m A_i) \neq \emptyset.$
- (iv) X is finite-dimensional  $\Rightarrow$  ran $(\mathrm{Id} \mathbf{T}) \simeq \Delta^{\perp} + \times_{i=1}^{m} \operatorname{ran} A_{i}$ .

*If*  $(\forall i \in \{1, 2, ..., m\})$   $A_i$  *is affine, then we additionally have:* 

- (v)  $(\mathbf{v} + \mathbf{T})^n \mathbf{x} = (\mathbf{T}_{-\mathbf{v}})^n \mathbf{x} = \mathbf{T}^n \mathbf{x} + n\mathbf{v} \rightarrow P_{\text{Fix}(\mathbf{v}+\mathbf{T})} \mathbf{x} = P_{\mathbf{Z}_{\mathbf{v}}} \mathbf{x}.$
- (vi) X is finite-dimensional  $\Rightarrow$   $(\mathbf{v} + \mathbf{T})^n \mathbf{x} \rightarrow P_{\text{Fix } \mathbf{T}} \mathbf{x} = P_{\mathbf{Z}_{\mathbf{v}}}$  linearly.
- (vii) X is finite-dimensional  $\Rightarrow$  ran $(\text{Id} \mathbf{T}) = \mathbf{\Delta}^{\perp} + \times_{i=1}^{m} \operatorname{ran} A_i$ .

*Proof.* Note that  $(\forall i \in \{1, ..., m\}) A_i$  is  $\alpha$ -cocoercive hence **A** is firmly nonexpansive. (i): This is [8, Proposition 25.5(i)]. (ii): Let  $\mathbf{z} \in X^m$ . Then  $\mathbf{z} \in \mathbf{Z}_{\mathbf{v}} \Leftrightarrow \mathbf{v} \in N_{\Delta}(\mathbf{z} - \mathbf{v}) + \mathbf{A}\mathbf{z} \Leftrightarrow [\mathbf{z} - \mathbf{v} \in \Delta \text{ and } \mathbf{z} \in \mathbf{v} \in \mathbf{A}$  $Az - v \in \Delta^{\perp}$ ]  $\Leftrightarrow [z \in v + \Delta \text{ and } z \in A^{-1}(v + \Delta^{\perp})] \Leftrightarrow z \in (v + \Delta) \cap (A^{-1}(v + \Delta^{\perp})).$  (iii): It follows from (32), Proposition 3.4(iv) applied to **A** and **B** and (i) that  $\mathbf{v} = \mathbf{0} \Leftrightarrow \text{Fix } \mathbf{T} \neq \emptyset \Leftrightarrow (\exists \mathbf{z} \in X^m)$ such that  $0 \in \mathbf{A}\mathbf{z} + N_{\Delta}\mathbf{z} = \mathbf{A}\mathbf{z} + \Delta^{\perp} \Leftrightarrow [\mathbf{z} \in \Delta \text{ and } \mathbf{A}\mathbf{z} \in \Delta^{\perp}] \Leftrightarrow [(\exists z \in X) \mathbf{z} = (z, z, \dots, z)]$ and  $\sum_{i=1}^{m} A_i z = 0 \Rightarrow z \in \operatorname{zer}(\sum_{i=1}^{m} A_i)$ . (iv): Apply Theorem 5.3(i) to **A** and **B** and note that ran  $\mathbf{A} = \times_{i=1}^{m}$  ran  $A_i$ . (v) & (vi): Apply Theorem 6.6(ii) and (iv) respectively to  $\mathbf{A}$  and  $\mathbf{B}$ . (vii): Apply Theorem 5.3(ii) to **A** and **B**. 

#### Some algorithmic consequences 7

In this section we make use of the following useful fact that is well-known in analysis.

**Fact 7.1.** Suppose that  $(a_n)_{n \in \mathbb{N}}$  is a decreasing sequence of nonnegative real numbers such that  $\sum_{n=0}^{\infty} a_n < \infty$  $+\infty$ . Then

$$na_n \to 0.$$
 (56)

Proof. See [32, Section 3.3, Theorem 1].

**Lemma 7.2.** Let  $L: X \to X$  be linear, nonexpansive and asymptotically regular, let  $b \in X$ , and suppose that  $T: X \to X: x \mapsto Lx + b$  and that  $v_T := P_{\overline{ran}(\mathrm{Id} - T)} \in \mathrm{ran}(\mathrm{Id} - T)$ . Let  $x \in X$ . Then the sequence  $(||T^n x - T^{n+1} x - v_T||)_{n \in \mathbb{N}}$  is a decreasing sequence of nonnegative real numbers that converges to 0.

*Proof.* Let  $n \in \mathbb{N}$ . It follows from Fact 6.1 that  $T^n x + nv_T = (v_T + T)^n x$ . Moreover, since *L* is nonexpansive so is  $v_T + T$ . Now

$$\|T^{n}x - T^{n+1}x - v\| = \|T^{n}x + nv_{T} - (T^{n+1}x + (n+1)v_{T})\|$$
  

$$= \|(v_{T} + T)^{n}x - (v_{T} + T)^{n+1}x\|$$
  

$$\leq \|(v_{T} + T)^{n-1}x - (v_{T} + T)^{n}x\|$$
  

$$= \|T^{n-1}x + (n-1)v_{T} - (T^{n}x + nv_{T})\|$$
  

$$= \|T^{n-1}x - T^{n}x - v_{T}\|.$$
(57)

The claim about convergence follows from Proposition 6.3.

**Theorem 7.3.** Let  $L: X \to X$  be linear, nonexpansive and asymptotically regular, let  $b \in X$ , and suppose that  $T: X \to X: x \mapsto Lx + b$  and that  $v_T := P_{\overline{ran}(Id - T)} \in ran(Id - T)$ . Let  $x \in X$  and set

$$(\forall n \in \mathbb{N}) \quad x_n := T^n x + n(T^{n^2} x - T^{n^2 + 1} x).$$
 (58)

Then  $x_n \to P_{\text{Fix}(v_T+T)}x$ .

Proof. We have

$$\|x_n - (v_T + T)^n x\| = \|T^n x + n(T^{n^2} x - T^{n^2 + 1} x) - (T^n x + nv_T)\|$$
  
=  $n\|T^{n^2} x - T^{n^2 + 1} x - v_T\| = \sqrt{n^2}\|T^{n^2} x - T^{n^2 + 1} x - v_T\| \to 0,$  (59)

where the limit follows by applying Fact 7.1 with  $a_n$  replaced by  $||T^n x - T^{n+1}x - v_T||^2$ . It follows from Proposition 6.3 that  $(v_T + T)^n x \to P_{\text{Fix}(v+T_T)}x$ , hence the conclusion follows.

**Corollary 7.4.** Suppose that A and B are affine and that  $v \in ran(Id - T_{FB})$ . Let  $x \in X$  and set

$$(\forall n \in \mathbb{N}) \quad x_n := T_{FB}^n x + n(T_{FB}^{n^2} x - T_{FB}^{n^2+1} x).$$
 (60)

Then  $x_n \to P_{\text{Fix}(v+T_{\text{FB}})}x = P_{Z_v}x$ .

Proof. Combine Proposition 3.4(i), Fact 2.3, Theorem 7.3 and Theorem 6.6(ii).

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