

# Copositivity Detection of Tensors: Theory and Algorithm

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## Abstract

A symmetric tensor is called copositive if it generates a multivariate form taking nonnegative values over the nonnegative orthant. Copositive tensors have found important applications in polynomial optimization and tensor complementarity problems. In this paper, we consider copositivity detection of tensors both from theoretical and computational points of view. After giving several necessary conditions for copositive tensors, we propose several new criteria for copositive tensors based on the representation of the multivariate form in barycentric coordinates with respect to the standard simplex and simplicial partitions. It is verified that, as the partition gets finer and finer, the concerned conditions eventually capture all strictly copositive tensors. Based on the obtained theoretical results with the help of simplicial partitions, we propose a numerical method to judge whether a tensor is copositive or not. The preliminary numerical results confirm our theoretical findings.

**Keywords:** Symmetric tensor; Strictly copositive; Positive semi-definiteness; Simplicial partition

**AMS Subject Classification(2010):** 65H17, 15A18, 90C30.

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# 1 Introduction

A symmetric tensor is called copositive if it generates a multivariate form taking nonnegative values over the nonnegative orthant [15]. Copositive tensors constitute a large class of tensors that contain nonnegative tensors and several kinds of structured tensors in the even order symmetric case such as  $M$ -tensors, diagonally dominant tensors and so on [3, 6, 9–12, 16, 17, 24]. Recently, it has been found that copositive tensors have important applications in polynomial optimization [14, 21] and the tensor complementarity problem [2, 18, 20].

Pena et al. [14] provided a general characterization for a class of polynomial optimization problems that can be formulated as a conic program over the cone of completely positive tensors or copositive tensors. As a consequence of this characterization, it follows that recent related results for quadratic problems can be further strengthened and generalized to higher order polynomial optimization problems. On the other hand, Che, Qi and Wei [2] showed that the tensor complementarity problem with a strictly copositive tensor has a nonempty and compact solution set; Song and Qi [18] proved that a real symmetric tensor is a (strictly) semi-positive if and only if it is (strictly) copositive, and Song and Qi [18, 20] obtained several results for the tensor complementarity problem with a (strictly) semi-positive tensor; and Huang and Qi [7] formulated an  $n$ -person noncooperative game as a tensor complementarity problem with the involved tensor is nonnegative. Thus, copositive tensors play an important role in the tensor complementarity problem. Now, there is a challenging problem that is how to check the copositivity of a given symmetric tensors efficiently?

Though many structured tensors are copositive, from the practical point of view, it is of great important to check directly whether a tensor is copositive or not. Several sufficient conditions or necessary and sufficient conditions for copositive tensors have been presented in [15, 19]. However, it is hard to verify numerically whether a tensor is copositive or not from these conditions. In [21], Song and Qi gave the concepts of Pareto H-eigenvalue and Pareto Z-eigenvalue for symmetric tensors. It is proved that a symmetric tensor  $\mathcal{A}$  is strictly copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of  $\mathcal{A}$  is positive, and  $\mathcal{A}$  is copositive if and only if every Pareto H-eigenvalue (Z-eigenvalue) of  $\mathcal{A}$  is nonnegative. Unfortunately, it is NP-hard the compute the minimum Pareto H-eigenvalue or Pareto Z-eigenvalue of a given symmetric tensors. In fact, for a given tensor, the problem to judge whether it is copositive or not is NP-complete, even for the matrix case [5, 13]. To the best of our knowledge, there is not any numerical detection method for copositive tensors with order greater than three. In this paper, we further give some theoretical studies on various conditions for (strictly) copositive tensors; and based on some of our theoretical findings, we propose a numerical method to judge whether a tensor is copositive or not. The algorithm we investigated can be viewed as an extension of some branch-and-bound type algorithm for

testing copositivity of symmetric matrices [1, 22, 23].

The rest of this paper is organized as follows. In Section 2, we recall some notions and basic facts about tensors and the corresponding homogeneous polynomials. Three necessary conditions for copositive tensors are given in Section 3. In Section 4, we give several criteria for (strictly) copositive tensors based on the simplicial subdivision, and an equivalent condition for a symmetric tensor that is not copositive. In Section 5, we propose a numerical detection algorithm for copositive tensors based on the results obtained in Section 4; and show that the algorithm can always capture strictly copositive tensors in finitely many iterations. The preliminary numerical results are reported in Section 6, and final remarks and some future work are given in Section 7.

## 2 Preliminaries

Throughout this paper, we denote the set consisting of all positive integers by  $\mathbb{N}$ , and always assume that  $m, n \in \mathbb{N}$ . Let  $\mathbb{R}^n$  be the  $n$  dimensional real Euclidean space and the set of all nonnegative vectors in  $\mathbb{R}^n$  be denoted by  $\mathbb{R}_+^n$ . Let  $\mathbb{R}_{++}^n$  denote the set of vectors with positive entries. Vectors are denoted by bold lowercase letters i.e.  $\mathbf{x}, \mathbf{y}, \dots$ , matrices are denoted by capital letters i.e.  $A, B, \dots$ , and tensors are written as calligraphic capitals such as  $\mathcal{A}, \mathcal{T}, \dots$ . We denote  $[n] = \{1, 2, \dots, n\}$ . The  $i$ -th unit coordinate vector in  $\mathbb{R}^n$  is denoted by  $\mathbf{e}_i$  for any  $i \in [n]$ .

A real  $m$ -th order  $n$ -dimensional tensor  $\mathcal{A} = (a_{i_1 i_2 \dots i_m})$  is a multi-array of real entries  $a_{i_1 i_2 \dots i_m}$ , where  $i_j \in [n]$  for  $j \in [m]$ . In this paper, we always assume that  $m \geq 3$  and  $n \geq 2$ . A tensor is said to be nonnegative if all its entries are nonnegative. If the entries  $a_{i_1 i_2 \dots i_m}$  are invariant under any permutation of their indices, then tensor  $\mathcal{A}$  is called a symmetric tensor. In this paper, we always consider real symmetric tensors. The identity tensor  $\mathcal{I}$  with order  $m$  and dimension  $n$  is given by  $\mathcal{I}_{i_1 \dots i_m} = 1$  if  $i_1 = \dots = i_m$  and  $\mathcal{I}_{i_1 \dots i_m} = 0$  otherwise. For any  $J \subseteq [n]$ ,  $|J|$  denotes the number of elements of  $J$ , and  $\mathcal{A}_J$  denotes a principle subtensor of  $\mathcal{A}$ .

We denote

$$S_{m,n} := \{\mathcal{A} : \mathcal{A} \text{ is an } m\text{-th order } n\text{-dimensional symmetric tensor}\}.$$

Clearly,  $S_{m,n}$  is a vector space under the addition and multiplication defined as below: for any  $t \in \mathbb{R}$ ,  $\mathcal{A} = (a_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}$  and  $\mathcal{B} = (b_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}$ ,

$$\mathcal{A} + \mathcal{B} = (a_{i_1 \dots i_m} + b_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n} \quad \text{and} \quad t\mathcal{A} = (ta_{i_1 \dots i_m})_{1 \leq i_1, \dots, i_m \leq n}.$$

For any  $\mathcal{A}, \mathcal{B} \in S_{m,n}$ , we define the inner product by  $\langle \mathcal{A}, \mathcal{B} \rangle := \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} b_{i_1 \dots i_m}$ , and the

corresponding norm by

$$\|\mathcal{A}\| = (\langle \mathcal{A}, \mathcal{A} \rangle)^{1/2} = \left( \sum_{i_1, \dots, i_m=1}^n (a_{i_1 \dots i_m})^2 \right)^{1/2}.$$

For any  $\mathbf{x} \in \mathbb{R}^n$ , we use  $x_i$  to denote its  $i$ th component; and use  $\|\mathbf{x}\|_m$  to denote the  $m$ -norm of  $\mathbf{x}$ .

For  $m$  vectors  $\mathbf{x}, \mathbf{y}, \dots, \mathbf{z} \in \mathbb{R}^n$ , we use  $\mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z}$  to denote the  $m$ -th order  $n$ -dimensional symmetric rank one tensor with

$$(\mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z})_{i_1 i_2 \dots i_m} = x_{i_1} y_{i_2} \dots z_{i_m}, \quad \forall i_1, \dots, i_m \in [n].$$

And the inner product of a symmetric tensor and the rank one tensor is given by

$$\langle \mathcal{A}, \mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z} \rangle := \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} y_{i_2} \dots z_{i_m}.$$

For  $m \in \mathbb{N}$  and  $k \in [m]$ , we denote

$$\mathcal{A}\mathbf{x}^k \mathbf{y}^{m-k} = \langle \mathcal{A}, \underbrace{\mathbf{x} \circ \dots \circ \mathbf{x}}_k \circ \underbrace{\mathbf{y} \circ \dots \circ \mathbf{y}}_{m-k} \rangle \quad \text{and} \quad \mathcal{A}\mathbf{x}^m = \langle \mathcal{A}, \underbrace{\mathbf{x} \circ \dots \circ \mathbf{x}}_m \rangle,$$

then

$$\mathcal{A}\mathbf{x}^k \mathbf{y}^{m-k} = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \dots x_{i_k} y_{i_{k+1}} \dots y_{i_m} \quad \text{and} \quad \mathcal{A}\mathbf{x}^m = \sum_{i_1, \dots, i_m=1}^n a_{i_1 \dots i_m} x_{i_1} \dots x_{i_m}. \quad (2.1)$$

For any  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in S_{m,n}$  and  $\mathbf{x} \in \mathbb{R}^n$ , we have  $\mathcal{A}\mathbf{x}^{m-1} \in \mathbb{R}^n$  with

$$(\mathcal{A}\mathbf{x}^{m-1})_i = \sum_{i_2, i_3, \dots, i_m \in [n]} a_{i i_2 \dots i_m} x_{i_2} \dots x_{i_m}, \quad \forall i \in [n].$$

It is known that an  $m$ -th order  $n$ -dimensional symmetric tensor defines uniquely an  $m$ -th degree homogeneous polynomial  $f_{\mathcal{A}}(\mathbf{x})$  on  $\mathbb{R}^n$ : for all  $\mathbf{x} = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,

$$f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m};$$

and conversely, any  $m$ -th degree homogeneous polynomial function  $f(\mathbf{x})$  on  $\mathbb{R}^n$  also corresponds uniquely a symmetric tensor. Furthermore, an even order symmetric tensor  $\mathcal{A}$  is called positive semi-definite (positive definite) if  $f_{\mathcal{A}}(\mathbf{x}) \geq 0$  ( $f_{\mathcal{A}}(\mathbf{x}) > 0$ ) for all  $\mathbf{x} \in \mathbb{R}^n$  ( $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ ).

### 3 Necessary conditions for copositivity of tensors

In this section, we first introduce the definition of copositive tensors, and then three necessary conditions are established, which are given based on binomial expansion, principle subtensor and convex combination of symmetric tensors, respectively.

**Definition 3.1** [15] Let  $\mathcal{A} \in S_{m,n}$  be given. If  $\mathcal{A}\mathbf{x}^m \geq 0$  ( $\mathcal{A}\mathbf{x}^m > 0$ ) for any  $\mathbf{x} \in \mathbb{R}_+^n$  ( $\mathbf{x} \in \mathbb{R}_+^n \setminus \{\mathbf{0}\}$ ), then  $\mathcal{A}$  is called a copositive (strictly copositive) tensor.

To move on, we first prove the following result, which looks like the binomial expansion of two real variables.

**Lemma 3.1** Let  $\mathcal{A} = (a_{i_1 i_2 \dots i_m}) \in S_{m,n}$  be given. For any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , it holds that

$$\mathcal{A}(\mathbf{x} + \mathbf{y})^m = \mathcal{A}\mathbf{x}^m + \binom{m}{1} \mathcal{A}\mathbf{x}^{m-1}\mathbf{y} + \binom{m}{2} \mathcal{A}\mathbf{x}^{m-2}\mathbf{y}^2 + \dots + \binom{m}{m} \mathcal{A}\mathbf{y}^m.$$

*Proof.* For any  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ , we know that

$$\begin{aligned} \mathcal{A}(\mathbf{x} + \mathbf{y})^m &= \langle \mathcal{A}, (\mathbf{x} + \mathbf{y}) \circ (\mathbf{x} + \mathbf{y}) \circ \dots \circ (\mathbf{x} + \mathbf{y}) \rangle \\ &= \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} (x_{i_1} + y_{i_1})(x_{i_2} + y_{i_2}) \dots (x_{i_m} + y_{i_m}). \end{aligned}$$

By using the symmetry property of  $\mathcal{A}$  and (2.1), we further obtain that

$$\begin{aligned} \mathcal{A}(\mathbf{x} + \mathbf{y})^m &= \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} x_{i_2} \dots x_{i_m} \\ &\quad + \binom{m}{1} \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} \dots x_{i_{m-1}} y_{i_m} \\ &\quad + \binom{m}{2} \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} x_{i_1} \dots x_{i_{m-2}} y_{i_{m-1}} y_{i_m} \\ &\quad + \dots \\ &\quad + \binom{m}{m} \sum_{i_1, i_2, \dots, i_m \in [n]} a_{i_1 i_2 \dots i_m} y_{i_1} y_{i_2} \dots y_{i_m} \\ &= \mathcal{A}\mathbf{x}^m + \binom{m}{1} \mathcal{A}\mathbf{x}^{m-1}\mathbf{y} + \binom{m}{2} \mathcal{A}\mathbf{x}^{m-2}\mathbf{y}^2 + \dots + \binom{m}{m} \mathcal{A}\mathbf{y}^m, \end{aligned}$$

which completes the proof.  $\square$

Now, we use Lemma 3.1 to establish a necessary condition for the copositive tensor.

**Theorem 3.1** Let  $\mathcal{A} \in S_{m,n}$  be copositive. If there is  $\mathbf{x} \in \mathbb{R}_+^n$  such that  $\mathcal{A}\mathbf{x}^m = 0$ , then  $\mathcal{A}\mathbf{x}^{m-1} \geq \mathbf{0}$ .

*Proof.* Suppose  $\mathbf{u} \in \mathbb{R}_+^n$  and  $\varepsilon > 0, \varepsilon \in \mathbb{R}$ . Since  $\mathcal{A}$  is copositive, it follows that

$$\mathcal{A}\mathbf{z}^m \geq 0, \quad \text{where } \mathbf{z} = \mathbf{x} + \varepsilon\mathbf{u}.$$

So, by Lemma 3.1, we have

$$\begin{aligned} \mathcal{A}\mathbf{z}^m &= \mathcal{A}(\mathbf{x} + \varepsilon\mathbf{u})^m \\ &= \mathcal{A}\mathbf{x}^m + \binom{m}{1} \varepsilon \mathcal{A}\mathbf{x}^{m-1}\mathbf{u} + \binom{m}{2} \varepsilon^2 \mathcal{A}\mathbf{x}^{m-2}\mathbf{u}^2 + \dots + \binom{m}{m} \varepsilon^m \mathcal{A}\mathbf{u}^m. \end{aligned} \tag{3.1}$$

Now, assume that there is an index  $i \in [n]$  such that  $(\mathcal{A}\mathbf{x}^{m-1})_i < 0$ . Take  $\mathbf{u} = \mathbf{e}_i$  in (3.1), we obtain that

$$\begin{aligned} \mathcal{A}\mathbf{z}^m &= \mathcal{A}(\mathbf{x} + \varepsilon\mathbf{e}_i)^m \\ &= \mathcal{A}\mathbf{x}^m + \binom{m}{1}\varepsilon\mathcal{A}\mathbf{x}^{m-1}\mathbf{e}_i + \cdots + \binom{m}{m}\varepsilon^m\mathcal{A}\mathbf{e}_i^m \\ &= \binom{m}{1}\varepsilon(\mathcal{A}\mathbf{x}^{m-1})_i + \binom{m}{2}\varepsilon^2\mathcal{A}\mathbf{x}^{m-2}\mathbf{e}_i^2 + \cdots + \binom{m}{m}\varepsilon^m\mathcal{A}\mathbf{e}_i^m \\ &< 0 \end{aligned}$$

holds for any sufficiently small  $\varepsilon > 0$ , which contradicts the fact that  $\mathcal{A}$  is copositive; and hence, the desired result follows.  $\square$

Next, we establish a necessary condition for the copositive tensor by using the concept of principle subtensor.

**Theorem 3.2** *Let  $\mathcal{A} \in S_{m,n}$  be copositive. Then, for any  $J \subseteq [n]$ , the system  $\mathcal{A}_J\mathbf{x}^{m-1} \geq \mathbf{0}$  admits a nonzero solution  $\mathbf{x} \in \mathbb{R}_+^{|J|}$ .*

*Proof.* We prove it by contradiction. If there is  $J \subseteq [n]$  such that

$$\mathcal{A}_J\mathbf{x}^{m-1} < \mathbf{0} \text{ for all nonzero } \mathbf{x} \in \mathbb{R}_+^{|J|}, \quad (3.2)$$

then, we may define a vector  $\mathbf{y} \in \mathbb{R}^n$  with  $y_i = \frac{x_i}{\|\mathbf{x}\|_m}$  when  $i \in J$ ; and  $y_i = 0$  otherwise. By a direct computation, we obtain  $\mathcal{A}\mathbf{y}^m = \frac{\mathcal{A}_J\mathbf{x}^m}{\|\mathbf{x}\|_m^m} < 0$ , which contradicts that  $\mathcal{A}$  is copositive.  $\square$

To end this section, we establish a necessary condition for the copositivity of a convex combination of two symmetric tensors.

**Theorem 3.3** *Let  $\mathcal{A}, \mathcal{B} \in S_{m,n}$  be given. Suppose that there exists  $t \in [0, 1]$  such that  $(1-t)\mathcal{A} + t\mathcal{B}$  is copositive, then  $\max\{\mathcal{A}\mathbf{u}^m + \mathcal{A}\mathbf{v}^m, \mathcal{B}\mathbf{u}^m + \mathcal{B}\mathbf{v}^m\} \geq 0$  for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$ .*

*Proof.* Since  $(1-t)\mathcal{A} + t\mathcal{B}$  is copositive, it follows that

$$(1-t)\mathcal{A}\mathbf{u}^m + t\mathcal{B}\mathbf{u}^m \geq 0, \quad (1-t)\mathcal{A}\mathbf{v}^m + t\mathcal{B}\mathbf{v}^m \geq 0$$

for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$ . Adding these two inequalities, one has that

$$\max\{\mathcal{A}\mathbf{u}^m + \mathcal{A}\mathbf{v}^m, \mathcal{B}\mathbf{u}^m + \mathcal{B}\mathbf{v}^m\} \geq (1-t)(\mathcal{A}\mathbf{u}^m + \mathcal{A}\mathbf{v}^m) + t(\mathcal{B}\mathbf{u}^m + \mathcal{B}\mathbf{v}^m) \geq 0$$

holds for all  $\mathbf{u}, \mathbf{v} \in \mathbb{R}_+^n$ . This completes the proof.  $\square$

When  $m = 2$ , Crouzeix, Martínéz-Legaz, and Seeger proved the conclusions of Theorem 3.3 is sufficient and necessary in the matrix case [4].

## 4 Detection criteria based on simplicial partition

In this section, several sufficient conditions or necessary conditions of copositive tensors are characterized based on some simplices. First of all, we show an useful result obtained by Song and Qi [19].

**Lemma 4.1** *Let  $\mathcal{A} \in S_{m,n}$  be given and  $\|\cdot\|$  denote any norm on  $\mathbb{R}^n$ . Then, we have*

- (i)  $\mathcal{A}$  is copositive if and only if  $\mathcal{A}\mathbf{x}^m \geq 0$  for all  $\mathbf{x} \in \mathbb{R}_+^n$  with  $\|\mathbf{x}\| = 1$ ;
- (ii)  $\mathcal{A}$  is strictly copositive if and only if  $\mathcal{A}\mathbf{x}^m > 0$  for all  $\mathbf{x} \in \mathbb{R}_+^n$  with  $\|\mathbf{x}\| = 1$ .

It is well known that the set  $S_0 = \{\mathbf{x} \in \mathbb{R}_+^n \mid \|\mathbf{x}\|_1 = 1\}$  is the so-called standard simplex with vertices  $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ . So, it follows from Lemma 4.1 that the copositivity of tensor  $\mathcal{A} \in S_{m,n}$  can be translated to check

$$f(\mathbf{x}) = \mathcal{A}\mathbf{x}^m \geq 0, \quad \forall \mathbf{x} \in S_0.$$

Thus, our main goal in this section is to search for conditions that can guarantee the homogeneous polynomial  $f(\mathbf{x})$  to be nonnegative on a simplex. A simple way to describe a polynomial with respect to a simplex is to use barycentric coordinates, which gives a convenient verifiable sufficient condition for a tensor to be copositive on a simplex. This approach has been much used for convex surface fitting in computer aided geometric design [8] and the copositivity detection of matrices [1, 22, 23].

**Lemma 4.2** *Let  $S_1 = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be a simplex. If*

$$\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \cdots \circ \mathbf{u}_{i_m} \rangle \geq 0 \quad (\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \cdots \circ \mathbf{u}_{i_m} \rangle > 0) \quad (4.1)$$

*for all  $i_1, i_2, \dots, i_m \in [n]$ , then  $\mathcal{A}\mathbf{x}^m \geq 0$  ( $\mathcal{A}\mathbf{x}^m > 0$  respectively) for all  $\mathbf{x} \in S_1$ .*

*Proof.* For any  $\mathbf{x} \in S_1$ , we have that

$$\mathbf{x} = \lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i \geq 0, \quad \forall i \in [n]. \quad (4.2)$$

So, it holds that

$$\begin{aligned} \mathcal{A}\mathbf{x}^m &= \langle \mathcal{A}, (\lambda_1 \mathbf{u}_1 + \cdots + \lambda_n \mathbf{u}_n)^m \rangle \\ &= \sum_{j_1, j_2, \dots, j_m \in [n]} \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_m} \langle \mathcal{A}, \mathbf{u}_{j_1} \circ \mathbf{u}_{j_2} \cdots \circ \mathbf{u}_{j_m} \rangle. \end{aligned}$$

By (4.1) and (4.2), we further obtain that

$$\mathcal{A}\mathbf{x}^m \geq 0 \quad (\text{or } \mathcal{A}\mathbf{x}^m > 0),$$

which completes the proof. □

If we apply Lemma 4.2 to the standard simplex  $S_0 = \{\mathbf{x} \in \mathbb{R}_+^n \mid \|\mathbf{x}\|_1 = 1\}$ , it shows that

$$\langle \mathcal{A}, \mathbf{e}_{i_1} \circ \mathbf{e}_{i_2} \circ \cdots \circ \mathbf{e}_{i_m} \rangle = a_{i_1 i_2 \cdots i_m} \geq 0, \quad \forall i_1, i_2, \dots, i_m \in [n].$$

This means that all nonnegative tensors are copositive tensors.

Let  $S, S_1, S_2, \dots, S_r$  be finite simplices in  $\mathbb{R}^n$ . The set  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is called a simplicial partition of  $S$  if it satisfies that

$$S = \bigcup_{i=1}^r S_i \quad \text{and} \quad \text{int}S_i \cap \text{int}S_j = \emptyset \quad \text{for any } i, j \in [r] \text{ with } i \neq j,$$

where  $\text{int}S$  denotes the interior of  $S$ .

**Theorem 4.1** *Let  $\mathcal{A} \in S_{m,n}$  be given. Suppose  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is a simplicial partition of simplex  $S_0 = \{\mathbf{x} \in \mathbb{R}_+^n \mid \|\mathbf{x}\|_1 = 1\}$ ; and the vertices of simplex  $S_k$  are denoted by  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  for any  $k \in [r]$ .*

*Then, the following results hold:*

- (i) *if  $\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \cdots \circ \mathbf{u}_{i_m}^k \rangle \geq 0$  for all  $k \in [r], i_j \in [n], j \in [m]$ , then  $\mathcal{A}$  is copositive;*
- (ii) *if  $\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \cdots \circ \mathbf{u}_{i_m}^k \rangle > 0$  for all  $k \in [r], i_j \in [n], j \in [m]$ , then  $\mathcal{A}$  is strictly copositive.*

*Proof.* By Lemma 4.1, it suffices to prove that

$$\mathcal{A}\mathbf{x}^m \geq 0 \quad \text{for all } \mathbf{x} \in S_0.$$

For any  $\mathbf{x} \in S_0$ , since  $\tilde{S}$  is a simplicial partition of  $S_0$ , it follows that there is an index  $k \in [r]$  such that  $\mathbf{x} \in S_k \subseteq \tilde{S}$ . By assumptions and Lemma 4.2, the desired results follow.  $\square$

It is easy to see that a simplex  $S$  is determined by its vertices, which can be further represented by a matrix  $V_S$  whose columns are vertices of the simplex. It is obvious that  $V_S$  is nonsingular and unique up to a permutation of its columns. So, analogue to Theorem 4.1, we have the following results.

**Theorem 4.2** *Let  $\mathcal{A} \in S_{m,n}$  be given. Suppose  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  is a simplicial partition of simplex  $S_0 = \{\mathbf{x} \in \mathbb{R}_+^n \mid \|\mathbf{x}\|_1 = 1\}$ ; and the vertices of simplex  $S_k$  are denoted by  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  for any  $k \in [r]$ .*

*Let  $V_{S_k} = (\mathbf{u}_1^k \ \mathbf{u}_2^k \ \cdots \ \mathbf{u}_n^k)$  be the matrix corresponding to simplex  $S_k$  for any  $k \in [r]$ . Then, the following results hold:*

- (i) *if  $V_{S_k}^T \mathcal{A} V_{S_k}$  is copositive for all  $k \in [r]$ , then  $\mathcal{A}$  is copositive;*
- (ii) *if  $V_{S_k}^T \mathcal{A} V_{S_k}$  is strictly copositive for all  $k \in [r]$ , then  $\mathcal{A}$  is strictly copositive.*

*Proof.* It is sufficient to prove (i), since (ii) can be verified similarly. For any  $k \in [r]$  and  $i_1, i_2, \dots, i_m \in [n]$ , we have that

$$\begin{aligned}
(V_{S_k}^T \mathcal{A} V_{S_k})_{i_1 i_2 \dots i_m} &= \sum_{j_1, j_2, \dots, j_m \in [n]} (V_{S_k}^T)_{i_1 j_1} a_{j_1 j_2 \dots j_m} (V_{S_k})_{j_2 i_2} \dots (V_{S_k})_{j_m i_m} \\
&= \sum_{j_1, j_2, \dots, j_m \in [n]} (V_{S_k})_{j_1 i_1} a_{j_1 j_2 \dots j_m} (V_{S_k})_{j_2 i_2} \dots (V_{S_k})_{j_m i_m} \\
&= \sum_{j_1, j_2, \dots, j_m \in [n]} a_{j_1 j_2 \dots j_m} (\mathbf{u}_{i_1}^k)_{j_1} (\mathbf{u}_{i_2}^k)_{j_2} \dots (\mathbf{u}_{i_m}^k)_{j_m} \\
&= \langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle.
\end{aligned} \tag{4.3}$$

For any  $\mathbf{x} \in S_0$ , it follows that  $\mathbf{x} \in S_k$  for some  $k \in [r]$  such that

$$\mathbf{x} = \lambda_1 \mathbf{u}_1^k + \lambda_2 \mathbf{u}_2^k + \dots + \lambda_n \mathbf{u}_n^k, \quad \sum_{i=1}^n \lambda_i = 1, \quad \lambda_i \geq 0, \quad \forall i \in [n].$$

Thus,

$$\begin{aligned}
\mathcal{A} \mathbf{x}^m &= \langle \mathcal{A}, (\lambda_1 \mathbf{u}_1^k + \lambda_2 \mathbf{u}_2^k + \dots + \lambda_n \mathbf{u}_n^k)^m \rangle \\
&= \sum_{i_1, i_2, \dots, i_m \in [n]} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle \\
&= \sum_{i_1, i_2, \dots, i_m \in [n]} (V_{S_k}^T \mathcal{A} V_{S_k})_{i_1 i_2 \dots i_m} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \\
&= (V_{S_k}^T \mathcal{A} V_{S_k}) \lambda^m,
\end{aligned}$$

where  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}_+^n$ , and the third equality is obtained from (4.3). By conditions and  $\lambda \in \mathbb{R}_+^n$ , it holds that  $\mathcal{A} \mathbf{x}^m \geq 0$  holds for any  $\mathbf{x} \in S_0$ ; and hence, the desired results follow.  $\square$

To show the simplicial partition is fine enough, we will give a necessary condition for strictly copositivity of tensor. For the standard simplex  $S_0$  with a simplicial partition  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$ , the vertices of simplex  $S_k$  are denoted by  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  for any  $k \in [r]$ . Let  $d(\tilde{S})$  denote the maximum diameter of a simplex in  $\tilde{S}$ :

$$d(\tilde{S}) = \max_{k \in [r]} \max_{i, j \in [n]} \|\mathbf{u}_i^k - \mathbf{u}_j^k\|_2.$$

**Theorem 4.3** *Let  $\mathcal{A} \in S_{m,n}$  be a strictly copositive tensor. Then, there exists  $\varepsilon > 0$  such that for all finite simplicial partitions  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  of  $S_0$  with  $d(\tilde{S}) < \varepsilon$ , it follows that*

$$\langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle > 0$$

for all  $k \in [r], i_j \in [n], j \in [m]$ , where  $\mathbf{u}_1^k, \mathbf{u}_2^k, \dots, \mathbf{u}_n^k$  are vertices of the simplex  $S_k$ .

*Proof.* First of all, we define the following function:

$$f(\mathbf{w}) = \langle \mathcal{A}, \underbrace{\mathbf{x} \circ \mathbf{y} \circ \dots \circ \mathbf{z}}_m \rangle, \quad \forall \mathbf{w} := (\mathbf{x}, \mathbf{y}, \dots, \mathbf{z}) \in \underbrace{S_0 \times S_0 \times \dots \times S_0}_m \in \mathbb{R}^{mn}.$$

The strictly copositivity of  $\mathcal{A}$  implies that  $f(\mathbf{w}_\mathbf{x}) > 0$  for any  $\mathbf{w}_\mathbf{x} := (\mathbf{x}, \mathbf{x}, \dots, \mathbf{x}) \in S_0 \times S_0 \times \dots \times S_0$ . By continuity, it follows that, for any  $\mathbf{x} \in S_0$ , there exists  $\varepsilon_\mathbf{x} > 0$  such that

$$f(\mathbf{w}) > 0 \text{ for all } \mathbf{w} \text{ satisfying } \|\mathbf{w} - \mathbf{w}_\mathbf{x}\|_2 \leq \varepsilon_\mathbf{x}.$$

Let  $\varepsilon = \min_{\mathbf{x} \in S_0} \varepsilon_\mathbf{x} > 0$ . Then, it follows from uniform continuity of  $f$  that

$$f(\mathbf{w}) > 0 \text{ for all } \mathbf{w} \text{ satisfying } \|\mathbf{w} - \mathbf{w}_\mathbf{x}\|_2 \leq \varepsilon \text{ for all } \mathbf{x} \in S_0. \quad (4.4)$$

For any simplicial partition  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  of  $S_0$  with  $d(\tilde{S}) < \frac{1}{m}\varepsilon$ , it holds that

$$\|\mathbf{u}_i^k - \mathbf{u}_j^k\|_2 \leq \frac{1}{m}\varepsilon, \quad \forall k \in [r], \forall i, j \in [n].$$

Moreover, for any  $\mathbf{x} \in S_k \subseteq \tilde{S}$ , it follows that, for any  $i \in [n]$ ,

$$\|\mathbf{u}_i^k - \mathbf{x}\|_2 \leq \|\mathbf{u}_i^k - \mathbf{u}_j^k\|_2 \leq d(\tilde{S}) < \frac{1}{m}\varepsilon \text{ for some } j \in [n],$$

which implies that

$$\|(\mathbf{u}_{i_1}^k, \mathbf{u}_{i_2}^k, \dots, \mathbf{u}_{i_m}^k) - \mathbf{w}_\mathbf{x}\|_2 \leq \varepsilon, \quad \forall i_j \in [n], \forall j \in [m].$$

Combining this with (4.4), we obtain that

$$f(\mathbf{u}_{i_1}^k, \mathbf{u}_{i_2}^k, \dots, \mathbf{u}_{i_m}^k) = \langle \mathcal{A}, \mathbf{u}_{i_1}^k \circ \mathbf{u}_{i_2}^k \circ \dots \circ \mathbf{u}_{i_m}^k \rangle > 0$$

holds for all  $k \in [r], i_j \in [n], j \in [m]$ ; and hence, the desired results follow.  $\square$

The following lemma gives a detection criterion for the case of a tensor being not copositive.

**Theorem 4.4** *Let  $\mathcal{A} \in S_{m,n}$  be given. Then, the following two assertions are equivalent.*

(i) *Tensor  $\mathcal{A}$  is not copositive.*

(ii) *There exists  $\varepsilon > 0$  such that, for all simplicial partition  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  of  $S_0$  with  $d(\tilde{S}) < \varepsilon$ , there are at least one  $k \in [r]$  and one  $i \in [n]$  satisfying  $\mathcal{A}(\mathbf{u}_i^k)^m < 0$ .*

*Proof.* It is obvious that (ii)  $\Rightarrow$  (i). To prove the converse, we assume that  $\mathcal{A}$  is not copositive. Then, there is  $\mathbf{x} \in S_0$  such that  $\mathcal{A}\mathbf{x}^m < 0$ . By continuity, there exists  $\varepsilon > 0$  such that

$$\mathcal{A}\mathbf{y}^m < 0 \text{ for all } \mathbf{y} \text{ satisfying } \|\mathbf{y} - \mathbf{x}\|_2 < \varepsilon. \quad (4.5)$$

For any simplicial partition  $\tilde{S} = \{S_1, S_2, \dots, S_r\}$  of  $S_0$  with  $d(\tilde{S}) < \varepsilon$ , there is at least one  $k \in [r]$  such that  $\mathbf{x} \in S_k$  with  $\|\mathbf{u}_i^k - \mathbf{x}\|_2 \leq d(\tilde{S}) < \varepsilon$ . Thus, it follows from (4.5) that  $\mathcal{A}(\mathbf{u}_i^k)^m < 0$ , which implies that the desired results follow.  $\square$

## 5 Detection algorithm based on simplicial partition

Based on the results obtained in the last section, we can develop an algorithm to verify whether a tensor is copositive or not, which is stated as follows.

---

**Algorithm 5.1. Test whether a given symmetric tensor is copositive or not**

---

**Input:**  $\mathcal{A} \in S_{m,n}$

Set  $\tilde{S} := \{S_0\}$ , where  $S_0 = \text{conv}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is the standard simplex

while  $\tilde{S} \neq \emptyset$  do

choose  $S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \in \tilde{S}$

if there exists  $i \in [n]$  such that  $\mathcal{A}\mathbf{u}_i^m < 0$ , then

return “ $\mathcal{A}$  is not copositive”

else if  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \geq 0$  for all  $i_1, i_2, \dots, i_m \in [n]$ , then

$\tilde{S} = \tilde{S} \setminus \{S\}$

else

simplicial partition  $S = S_1 \cup S_2$ ; and set  $\tilde{S} := \tilde{S} \setminus \{S\} \cup \{S_1, S_2\}$

end if

end while

return “ $\mathcal{A}$  is copositive.”

**Output:** “ $\mathcal{A}$  is copositive” or “ $\mathcal{A}$  is not copositive”.

---

From Theorems 4.3 and 4.4, it is easy to see that the following result holds.

**Theorem 5.1** *In Algorithm 5.1, if the input symmetric tensor  $\mathcal{A}$  is strictly copositive or  $\mathcal{A}$  is not copositive, then the method will terminate in finitely many iterations.*

Thus, it is clear that Algorithm 5.1 can capture all strictly copositive tensors and non-copositive tensors. Unfortunately, when  $\mathcal{A}$  is copositive but not strictly copositive, it is possible that the partition procedure of the algorithm leads to  $d(\tilde{S}) \rightarrow 0$ ; and in this case, the algorithm dose not stop in general. This case also exists for the matrix detecting process [1, 22]. The reason for this is the following result.

**Proposition 5.1** *Suppose  $\mathcal{A} \in S_{m,n}$  is copositive. Let  $S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be a simplex with  $\mathcal{A}\mathbf{u}_i^m > 0$  for all  $i \in [n]$ . If there exists  $\mathbf{x} \in S \setminus \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  such that  $\mathcal{A}\mathbf{x}^m = 0$ , then there are  $i_1, i_2, \dots, i_m \in [n]$  such that  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle < 0$ .*

*Proof.* We prove this proposition by contradiction. Assume that

$$\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \geq 0$$

for all  $i_1, i_2, \dots, i_m \in [n]$ . By conditions, there is  $\mathbf{x} \in S \setminus \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  such that  $\mathcal{A}\mathbf{x}^m = 0$ . It follows that there exist  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$  such that  $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{u}_i$  and  $\sum_{i=1}^n \lambda_i = 1$ . Thus,

$$\begin{aligned} \mathcal{A}\mathbf{x}^m &= \langle \mathcal{A}, (\lambda_1 \mathbf{u}_1 + \dots + \lambda_n \mathbf{u}_n)^m \rangle \\ &= \sum_{i_1, i_2, \dots, i_m \in [n]} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \\ &\geq \sum_{i=1}^n \lambda_i \mathcal{A}\mathbf{u}_i^m > 0, \end{aligned}$$

which contradicts the result  $\mathcal{A}\mathbf{x}^m = 0$ . Therefore, the desired results follow.  $\square$

To get rid of the termination problem with the given tensor  $\mathcal{A}$  being copositive but not strictly copositive, we can first try to check the copositivity of  $\mathcal{A}$  by Algorithm 5.1. If it terminates in finitely many iterations, then we get a correct answer; and if not, we can consider a relaxation form of  $\mathcal{A}$ , i.e.,  $\mathcal{B} = \mathcal{A} + \sigma\mathcal{E}$ , where  $\mathcal{E}$  is the tensor of all ones and  $\sigma > 0$  is a small tolerance.

**Theorem 5.2** *Let  $\mathcal{A} \in S_{m,n}$  be given. Then,  $\mathcal{A}$  is copositive if and only if  $\mathcal{B} = \mathcal{A} + \sigma\mathcal{E}$  is strictly copositive for any  $\sigma > 0$ .*

*Proof.* It is obvious that the necessary condition holds. For the sufficient statement, since

$$\mathcal{B}\mathbf{x}^m = (\mathcal{A} + \sigma\mathcal{E})\mathbf{x}^m = \mathcal{A}\mathbf{x}^m + \sigma > 0,$$

for all  $\mathbf{x} \in S_0$  and  $\sigma > 0$ , by letting  $\sigma \rightarrow 0$ , we can obtain the desired result.  $\square$

From Theorems 5.1 and 5.2, the following conclusion holds.

**Corollary 5.1** *If the given tensor  $\mathcal{A}$  is copositive but not strictly copositive, by replacing  $\mathcal{A}$  by  $\mathcal{B} = \mathcal{A} + \sigma\mathcal{E}$  for some  $\sigma > 0$ , then Algorithm 5.1 terminates in finitely many iterations.*

For  $\sigma > 0$ , we call the symmetric tensor  $\mathcal{A}$  a  $\sigma$ -copositive tensor with respect to simplex  $S_0$ , if  $\mathcal{A}\mathbf{x}^m \geq -\sigma$  for all  $\mathbf{x} \in S_0$ . And we have the following conclusion.

**Proposition 5.2** *Suppose  $\mathcal{A} \in S_{m,n}$  is given. Let  $S = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$  be a simplex and  $\sigma > 0$ . If  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \geq -\sigma$  for all  $i_1, i_2, \dots, i_m \in [n]$ , then  $\mathcal{A}\mathbf{x}^m \geq -\sigma$  for any  $\mathbf{x} \in S$ .*

*Proof.* For any  $\mathbf{x} \in S$ , there exist  $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$  such that  $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{u}_i$  and  $\sum_{i=1}^n \lambda_i = 1$ . Thus,

$$\begin{aligned} \mathcal{A}\mathbf{x}^m &= \langle \mathcal{A}, (\lambda_1 \mathbf{u}_1 + \dots + \lambda_n \mathbf{u}_n)^m \rangle \\ &= \sum_{i_1, i_2, \dots, i_m \in [n]} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_m} \langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \\ &\geq -\sigma, \end{aligned}$$

which implies that the desired result holds.  $\square$

## 6 Numerical examples

In order to implement Algorithm 5.1, we specify Algorithm 5.1 as follows.

---

**Algorithm 6.1. Test whether a given symmetric tensor is copositive or not**

---

**Input:**  $\mathcal{A} \in S_{m,n}$

Set  $\tilde{S} := \{S_1\}$ , where  $S_1 = \text{conv}\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  is the standard simplex

Set  $k := 1$

while  $k \neq 0$  do

  set  $S := S_k = \text{conv}\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\} \in \tilde{S}$

  if there exists  $i \in [n]$  such that  $\mathcal{A}\mathbf{u}_i^m < 0$ , then

    return “ $\mathcal{A}$  is not copositive”

  else if  $\langle \mathcal{A}, \mathbf{u}_{i_1} \circ \mathbf{u}_{i_2} \circ \dots \circ \mathbf{u}_{i_m} \rangle \geq 0$  for all  $i_1, i_2, \dots, i_m \in [n]$ , then

    set  $\tilde{S} := \tilde{S} \setminus \{S_k\}$  and  $k := k - 1$

  else

    set

$S_k := \text{conv}\{\mathbf{u}_1, \dots, \mathbf{u}_{p-1}, \mathbf{v}, \mathbf{u}_{p+1}, \dots, \mathbf{u}_n\};$

$S_{k+1} := \text{conv}\{\mathbf{u}_1, \dots, \mathbf{u}_{q-1}, \mathbf{v}, \mathbf{u}_{q+1}, \dots, \mathbf{u}_n\},$

      where  $\mathbf{v} = \frac{\mathbf{u}_p + \mathbf{u}_q}{2}$ ,  $[p, q] = \arg \max_{i,j \in [n]} \|\mathbf{u}_i - \mathbf{u}_j\|_2$  and  $p < q$ .

    set  $\tilde{S} := \tilde{S} \setminus \{S\} \cup \{S_k, S_{k+1}\}$  and  $k := k + 1$

  end if

end while

return “ $\mathcal{A}$  is copositive.”

**Output:** “ $\mathcal{A}$  is copositive” or “ $\mathcal{A}$  is not copositive”.

---

In this section, we use this specified version of Algorithm 5.1 to detect whether a tensor is copositive or not. All experiments are finished in Matlab2014b on a Philips desktop computer with Intel(R) Core(TM)2 Duo CPU E8500 @ 3.16GHz 3.17 GHz and 4 GB of RAM. We detect several classes of tensors from three aspects, which are given in the following three parts, respectively.

**Part 1.** Suppose that  $\mathcal{B} \in S_{m,n}$  is a nonnegative tensor and  $\rho(\mathcal{B})$  denotes its spectral radius. Let  $\mathcal{I} \in S_{m,n}$  denote the identity tensor. Then, by Definition 3.1 and Theorem 3.12 of [24], we have the following results: The tensor  $\eta\mathcal{I} - \mathcal{B}$  is copositive if and only if  $\eta \geq \rho(\mathcal{B})$ ; and the tensor  $\eta\mathcal{I} - \mathcal{B}$  is strictly copositive if and only if  $\eta > \rho(\mathcal{B})$ . Based on these results, we construct several tensors for testing. We first test the following specific tensors.

**Example 6.1** Suppose that  $\mathcal{A} \in S_{3,3}$  (or  $\mathcal{A} \in S_{4,4}$ ) is given by

$$\mathcal{A} = \eta \mathcal{I} - \mathcal{B}, \tag{6.1}$$

where  $\mathcal{B} \in S_{3,3}$  (or  $\mathcal{B} \in S_{4,4}$ ) is a tensor of ones and  $\eta$  is specified in our numerical results.

The numerical results are given in Table 1, where “ $\rho$ ” denotes the spectral radius of the tested tensor, “IT” denotes the number of iterations, “CPU(s)” denotes the CPU time in seconds, and “Result” denotes the output result in which “No” denotes the output result that the tested tensor is not copositive and “Yes” denotes the output result that the tested tensor is copositive.

Table 1: The numerical results of the problem in Example 6.1

$m$	$n$	$\rho$	$\eta$	IT	CPU(s)	Result
3	3	9	1	2	0.078	No
			8.99	43	0.437	No
			9	> 100		
			9.01	59	0.593	Yes
			19	11	0.172	Yes
4	4	64	10	14	0.499	No
			64	63	2.32	Yes
			74	63	2.14	Yes

Next, we test some randomly generated tensors with the form being same as the one by (6.1).

**Example 6.2** Suppose that  $\mathcal{A} \in S_{m,n}$  is given by (6.1), where  $\mathcal{B} \in S_{m,n}$  is randomly generated with all its elements are in the interval  $(0, 1)$ .

In our experiments, we use the higher order power method to compute the spectral radius  $\rho$  of every tensor  $\mathcal{B}$ .  $m$ ,  $n$  and  $\eta$  are specified in our numerical results. For the same  $m$  and  $n$ , we generate randomly every tested problem 10 times, the numerical results are shown in Table 2, where “MinIT” and “MaxIT” denote the minimal number and the maximal number of iterations among ten times for every tested problem, respectively, “MinCPU(s)” and “MaxCPU(s)” denote the smallest and the largest CPU times in second among ten times for every tested problem, respectively, “Nyes” denotes the number of the output result that the tested tensor is copositive, and “Nno” denotes the number of the output result that the tested tensor is not copositive. The same notations are also used in Table 6.3.

Table 2: The numerical results of the problem in Example 6.2

$m$	$n$	$\eta$	MinIT	MaxIT	MinCPU(s)	MinCPU(s)	Nyes	Nno
3	3	$\rho - 1$	6	25	0.0624	0.234		10
		$\rho + 1$	19	19	0.187	0.187	10	
		$\rho + 10$	11	11	0.109	0.125	10	
3	4	$\rho - 1$	21	65	0.39	1.19		10
		$\rho + 1$	63	75	1.14	1.36	10	
		$\rho + 10$	49	53	0.905	0.983	10	
4	3	$\rho - 1$	17	17	0.234	0.312		10
		$\rho + 1$	27	31	0.39	0.452	10	
		$\rho + 10$	19	19	0.265	0.296	10	
4	4	$\rho - 1$	21	25	0.686	0.827		10
		$\rho + 1$	65	91	2.11	3.15	10	
		$\rho + 10$	63	63	2.09	2.22	10	
6	3	$\rho - 1$	20	28	0.562	0.811		10
		$\rho + 1$	43	47	1.25	1.36	10	
		$\rho + 10$	27	27	0.764	0.796	10	

**Part 2.** It is obvious that any nonnegative tensor is copositive. By Corollary 6.1 of [19], we also know that for any  $\mathcal{A} \in S_{m,n}$ , if  $\mathcal{A}$  is (strictly) copositive, then  $(a_{ii\dots i} > 0) \ a_{ii\dots i} \geq 0$  for all  $i \in [n]$ . Based on these results, we consider to detect the tensors given in the following example.

**Example 6.3** (i) Consider the tensor  $\mathcal{A} \in S_{m,n}$  which is randomly generated with all its elements are in the interval  $(0, 1)$ ; (ii) we set  $\mathcal{B} := \mathcal{A}$  and  $b_{11\dots 1} = -1$ .

In our experiments, for the same  $m$  and  $n$ , we generate randomly every tested problem 10 times, the numerical results are shown in Table 3.

**Part 3.** It is well known that there is a one-to-one relationship between the homogeneous polynomial and the symmetric tensor. In this part, we consider several tensors which come from several famous homogeneous polynomials. For convenience, we use the following notation: for any integers  $i_1, i_2, \dots, i_m$ , we use  $\pi(i_1 i_2 \dots i_m)$  to denote a permutation of  $i_1 i_2 \dots i_m$ , and  $S_{\pi(i_1 i_2 \dots i_m)}$  to denote the set of all these permutations.

Table 3: The numerical results of the problem in Example 6.3

$m$	$n$	Tensor	MinIT	MaxIT	MinCPU(s)	MinCPU(s)	Nyes	Nno
3	3	$\mathcal{A}$	1	1	0.0624	0.0936	10	
		$\mathcal{B}$	1	1	0.0624	0.078		10
3	4	$\mathcal{A}$	1	1	0.0624	0.078	10	
		$\mathcal{B}$	1	1	0.0624	0.078		10
4	3	$\mathcal{A}$	1	1	0.078	0.078	10	
		$\mathcal{B}$	1	1	0.0624	0.078		10
4	4	$\mathcal{A}$	1	1	0.0936	0.125	10	
		$\mathcal{B}$	1	1	0.0624	0.078		10
6	3	$\mathcal{A}$	1	1	0.0936	0.125	10	
		$\mathcal{B}$	1	1	0.0936	0.109		10

**Example 6.4** Suppose that  $\mathcal{A} \in S_{6,3}$  is given by

$$\begin{cases} \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(111122)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(112222)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 1, \\ a_{333333} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(112233)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -3, \end{cases}$$

The corresponding polynomial of the tensor  $\mathcal{A}$  given in this example is

$$f(x, y, z) = x^4 y^2 + x^2 y^4 + z^6 - 3x^2 y^2 z^2.$$

This is the famous Motzkin polynomial, which is non-negative but not a sum of squares; and hence, the tensor  $\mathcal{A}$  is copositive. It is easy to see that the tensor  $\mathcal{A}$  is not strictly copositive. We use Algorithm 5.1 to test this tensor, the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor  $\mathcal{A} + \sigma \mathcal{E}$  with  $\sigma > 0$ , however, the algorithm can correctly detect the copositivity of the tensor; and we list several cases as follows.

- When  $\sigma = 0.01$ , the algorithm can correctly detect the copositivity of the tensor with 11 iterations in 0.406 seconds;
- when  $\sigma = 0.001$ , the algorithm can correctly detect the copositivity of the tensor with 27 iterations in 0.874 seconds; and

- when  $\sigma = 0.0001$ , the algorithm can correctly detect the copositivity of the tensor with 71 iterations in 2.25 seconds.

**Example 6.5** Suppose that  $\mathcal{A} \in S_{6,3}$  is given by

$$\left\{ \begin{array}{l} a_{111111} = 1, \quad a_{222222} = 1, \quad a_{333333} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(111122)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(112222)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(111133)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(113333)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(222233)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(223333)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(112233)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 3, \end{array} \right.$$

The corresponding polynomial of the tensor  $\mathcal{A}$  given in this example is

$$f(x, y, z) = x^6 + y^6 + z^6 - x^4 y^2 - x^2 y^4 - x^4 z^2 - x^2 z^4 - y^4 z^2 - y^2 z^4 + 3x^2 y^2 z^2.$$

This is the famous Robinson polynomial, which is non-negative but not a sum of squares; and hence, the tensor  $\mathcal{A}$  is copositive. It is easy to see that the tensor  $\mathcal{A}$  is not strictly copositive. We use Algorithm 5.1 to test this tensor, the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor  $\mathcal{A} + \sigma \mathcal{E}$  with  $\sigma > 0$ , however, the algorithm can correctly detect the copositivity of the tensor; and we list several cases as follows.

- When  $\sigma = 0.01$ , the algorithm can correctly detect the copositivity of the tensor with 11 iterations in 0.406 seconds;
- when  $\sigma = 0.001$ , the algorithm can correctly detect the copositivity of the tensor with 27 iterations in 0.842 seconds; and
- when  $\sigma = 0.0001$ , the algorithm can correctly detect the copositivity of the tensor with 83 iterations in 2.5 seconds.

**Example 6.6** Suppose that  $\mathcal{A} \in S_{6,3}$  is given by

$$\left\{ \begin{array}{l} \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(111122)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(222233)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(333311)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = 1, \\ \sum_{i_1 i_2 i_3 i_4 i_5 i_6 \in S_{\pi(112233)}} a_{i_1 i_2 i_3 i_4 i_5 i_6} = -3, \end{array} \right.$$

The corresponding polynomial of the tensor  $\mathcal{A}$  given in this example is

$$f(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2.$$

This is the famous Choi-Lam polynomial, which is non-negative but not a sum of squares; and hence, the tensor  $\mathcal{A}$  is copositive. It is easy to see that the tensor  $\mathcal{A}$  is not strictly copositive. We use Algorithm 5.1 to test this tensor, the algorithm does not terminate within 100 iterations. We use Algorithm 5.1 to test the tensor  $\mathcal{A} + \sigma\mathcal{E}$  with  $\sigma > 0$ , however, the algorithm can correctly detect the copositivity of the tensor; and we list several cases as follows.

- When  $\sigma = 0.01$ , the algorithm can correctly detect the copositivity of the tensor with 5 iterations in 0.218 seconds;
- when  $\sigma = 0.001$ , the algorithm can correctly detect the copositivity of the tensor with 27 iterations in 0.858 seconds; and
- when  $\sigma = 0.0001$ , the algorithm can correctly detect the copositivity of the tensor with 41 iterations in 1.29 seconds.

From the numerical results given in Part 1-Part 3, we can see that Algorithm 6.1 is effective for the problems we tested.

## 7 Conclusions

In this paper, we proposed new criteria to judge whether a tensor is (strictly) copositive or not, including three necessary conditions which are given based on binomial expansion, principle subtensor and convex combination, respectively; and several necessary conditions or sufficient conditions which are investigated by taking advantage of the simplicial partition. These theoretical results can be viewed as extensions of those obtained in the case of matrix. Moreover, by the obtained criteria based on the simplicial partition, we proposed a detection algorithm for the copositive tensor. The preliminary numerical results demonstrate that the proposed algorithm is effective.

## References

- [1] S. Bundfuss, M. Dür, *Algorithmic copositivity detection by simplicial partition*, Linear Algebra and its Applications 428(7) (2008) 1511-1523.

- [2] M. Che, L. Qi and Y. Wei, *Positive definite tensors to nonlinear complementarity problems*, Journal of Optimization Theory and Applications 168 (2016) 475-487.
- [3] H. Chen, L. Qi, *Positive definiteness and semi-definiteness of even order symmetric Cauchy tensors*, Journal of Industrial and Management Optimization 11 (2015) 1263-1274.
- [4] J.P. Crouzeix, J.E. Martinez-Legaz, A. Seeger, *An alternative theorem for quadratic forms and extensions*, Linear algebra and its applications 215 (1995) 121-134.
- [5] P.J.C. Dickinson, L. Gijben, *On the computational complexity of membership problems for the completely positive cone and its dual*, Computational Optimization and Applications 57 (2014) 403-415.
- [6] W. Ding, Z. Luo, L. Qi, *P-Tensors,  $P_0$ -Tensors, and Tensor Complementarity Problem*, arXiv preprint arXiv:1507.06731 (2015).
- [7] Z.H. Huang, L. Qi, *Formulating an n-person noncooperative game as a tensor complementarity problem*, arXiv preprint arXiv:1602.03280v1
- [8] B. Jüttler, *Arbitrarily weak linear convexity conditions formultivariate poynomials* Studia Scientiarum Mathematicarum Hungarica 36(1-2) (2000) 165-184.
- [9] M.R. Kannan, N. Shaked-Monderer, A. Berman, *Some properties of strong H-tensors and general H-tensors*, Linear Algebra and its Applications 476 (2015) 42-55.
- [10] C. Li, Y. Li, *Double B tensors and quasi-double B tensors*, Linear Algebra and its Applications 466 (2015) 343-356.
- [11] C. Li, L. Qi, Y. Li, *MB-tensors and  $MB_0$ -tensors*, Linear Algebra Appl. 484 (2015) 141-153.
- [12] C. Li, F. Wang, J. Zhao, Y. Zhu, Y. Li, *Criteria for the positive definiteness of real supersymmetric tensors*, Journal of Computational and Applied Mathematics 255 (2014) 1-14.
- [13] K.G. Murty, S.N. Kabadi, *Some NP-complete problems in quadratic and nonlinear programming*, Mathematical programming 39(2) (1987) 117-129.
- [14] J. Pena, J.C. Vera, L.F. Zuluaga, *Completely positive reformulations for polynomial optimization*, Mathematical Programming 151(2) (2014) 405-431.
- [15] L. Qi, *Symmetric nonnegative tensors and copositive tensors*, Linear Algebra and its Applications 439 (2013) 228-238.

- [16] L. Qi,  *$H^+$ -eigenvalues of Laplacian and signless Laplacian tensors*, Communications in Mathematical Sciences 12 (2014) 1045-1064.
- [17] L. Qi, Y. Song, *An even order symmetric B tensor is positive definite*, Linear Algebra and its Applications 457 (2014) 303-312.
- [18] Y. Song, L. Qi, *Tensor complementarity problem and semi-positive tensors*, Journal of Optimization Theory and Applications DOI 10.1007/s10957-015-0800-2.
- [19] Y. Song, L. Qi, *Necessary and sufficient conditions for copositive tensors*, Linear and Multilinear Algebra 63(1) (2015) 120-131.
- [20] Y. Song, L. Qi, *On strictly semi-positive tensors*, arXiv:1509.01327 (2015).
- [21] Y. Song, L. Qi, *Eigenvalue analysis of constrained minimization problem for homogeneous polynomials*, Journal of Global Optimization 64 (2016) 563-575.
- [22] J. Sponsel, S. Bundfuss, M. Dür, *An improved algorithm to test copositivity*, Journal of Global Optimization 52(3) (2012) 537-551.
- [23] J. Xu, Y. Yao, *An algorithm for determining copositive matrices*, Linear Algebra and its Applications 435(11) (2011) 2784-2792.
- [24] L. Zhang, L. Qi, G. Zhou, *M-tensors and some applications*, SIAM Journal on Matrix Analysis and Applications 35 (2014) 437-452.