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published in
Journal of Optimization Theory and Applications
2019

DOI (link to publisher)
10.1007/s10957-018-1259-8

document version
Publisher's PDF, also known as Version of record

document license
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citation for published version (APA)
Wang, W., Sun, H., van den Brink, R., & Xu, G. (2019). The Family of Ideal Values for Cooperative Games. *Journal of Optimization Theory and Applications*, *180*(3), 1065–1086. https://doi.org/10.1007/s10957-018-1259-8

J Optim Theory Appl (2019) 180:1065–1086
https://doi.org/10.1007/s10957-018-1259-8

The Family of Ideal Values for Cooperative Games

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Received: 9 January 2018 / Accepted: 23 February 2018 / Published online: 9 March 2018

Abstract In view of the nature of pursuing profit, a selfish coefficient function is employed to describe the degrees of selfishness of players in different coalitions, which is the desired rate of return to the worth of coalitions. This function brings in the concept of individual expected reward to every player. Built on different selfish coefficient functions, the family of ideal values can be obtained by minimizing deviations from the individual expected rewards. Then, we show the relationships between the family of ideal values and two other classical families of values: the procedural values and the least square values. For any selfish coefficient function, the corresponding ideal value is characterized by efficiency, linearity, an equal-expectation player property and a nullifying player punishment property, and also interpreted by a dynamic process. As two dual cases in the family of ideal values, the center of gravity of imputation set value and the equal allocation of nonseparable costs value are raised from new axiomatic angles.

Communicated by Irinel Chiril Dragan.

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Keywords Game theory · m-Individual expected reward · The family of ideal values · Dynamic process · CIS and EANS values

Mathematics Subject Classification 91A12

1 Introduction

In the theory of cooperative games with transferable utility, the Shapley value [1] is the most eminent (single-valued) solution. It assigns to every player its expected marginal contribution assuming that all possible orders of entrance of the players occur with equal probability. For every player, the Banzhaf value [2] assumes that every coalition without this player has equal probability to be the coalition, that is present when the player enters. Under this assumption, it gives every player its expected marginal contribution. Both values determine the payoff distribution depending on the marginal contributions of the players. Deegan and Packel [3] switch perspectives and determine the payoff for a player by considering the worths of coalitions the player belongs to. They put forward the Deegan–Packel (DP) value, which provides for every player the sum of the per capita worth of each coalition the player belongs to.

The DP value is not efficient. Even though the DP value opens up a new perspective, it ignores the possibility of coalition formation and the selfishness of the players. The social selfish coefficient is established by Wang et al. [4] to offer a new interpretation for the egalitarian Shapley value with an underlying procedure of sharing marginal contributions to coalitions formed by players joining in random order. To pursue more profit, the players assemble to form 'the grand' coalition. When players join a coalition, it is appropriate for them to ask a part of payoff from the coalition. The DP value divides the worth equally among the players in the coalition. We assume that every player wants a specific share of the worth of every coalition it belongs to. A so-called selfish coefficient function is used to describe the players' selfishness in different coalitions, i.e., the shares they request from every coalitions worth. The *individual expected reward* is the player's expected payoff over all coalitions the player may take part in, assuming these coalitions occur with equal probability.

Given a game, we are usually interested to know how the fruits of cooperation are shared among the players. In other words, we are looking for an allocation rule, satisfying a list of requirements, the axioms, that attribute payments to players in the game. One basic requirement is that all players together have and can only distribute the worth of the grand coalition consisting of all players. In consideration of this requirement, assigning to every player its individual expected reward is usually unattainable.

Yet another approach to allocate payoffs is the basis of the nucleolus (Schmeidler [5]) and prenucleolus (Sobolev [6]) which are both the outcome of a lexicographic minimization procedure over the excess vector that can be associated with any coalition. Ruiz et al. [7–9] introduce optimality theory to allocation in cooperative games. In order to look for an allocation in which all the excesses are similar, according to an egalitarian philosophy, Ruiz et al. [7] put forward the least square prenucleolus and the least square nucleolus by choosing the payoff vector which minimizes the variance of the excesses of the coalitions. Subsequently, Ruiz et al. [8,9] extend the definition to the family of least square values by minimizing the weighted variance and to the family of individually rational least square values with adding the individual rationality constraint. Different from considering the excess vector of coalitions, Nguyen [10] considers the core allocation that is closest to the Shapley value, as the most fair core allocation. In the underlying paper, the optimality problem, minimizing the deviations from the individual expected rewards, will be the main pathway to define new allocation methods, resulting in what we call *the family of ideal values*, by choosing the allocations that satisfy this optimization theory principle for different selfish coefficient functions.

For any efficient, symmetric and linear value, Ruiz et al. [8] give a special convey to characterize its payoff vector with a certain sequence of coefficients. Driessen [11] presents another equivalent formula, which reveals the explicit relationship between the Shapley value and any efficient, symmetric, and linear value. Assuming that the players arrive in the grand coalition in a random order, Malawski [12] introduces a new notion of “procedural” value for cooperative TU games by redefining the distributive method of the marginal contribution of every player. To further understand the family of ideal values, our work shows a new equivalent statement for efficient, symmetric and linear values. The family of least square values and the “procedural” values are both special subsets of the family of ideal values.

There are several approaches to justify a value for TU games. Two approaches are axiomatization and providing a dynamic process. An axiomatization gives a set of axioms that are satisfied by only one solution. For any selfish function m, the *m-equal-expectation player property* and the *nullifying player m-punishment property* are used to axiomatically characterize the *m-ideal value*. A dynamic process for a value leads the players to that value, starting from an arbitrary efficient payoff vector. Hwang et al. [13] propose a dynamic process leading to the Shapley value based on a modified version of Hamiache’s notion of an associated game. Later, Hwang et al. [14] adopt excess functions to propose a dynamic process for the efficient Banzhaf–Owen index. Following the steps of Hwang, we offer a dynamic process for the family of ideal values with respect to a new *complaint function*.

After providing general results on axiomatization and a dynamic process for the ideal values, we look more close at two special ideal values. The *CIS value*, defined by Driessen and Funaki [15], assigns to every player its individual worth and distributes the remainder of the worth of the grand coalition equally among all players. The *EANS value*, introduced by Moulin [16], is the dual of the CIS value. Using a reduced game consistency, van den Brink and Funaki [17] provide characterizations for a class of equal surplus sharing solutions including the CIS value and EANS value. Though Hamiache [18] initially proposes the associated consistency with respect to a specific associated game, Hwang [19,20] and Xu et al. [21,22] apply the associated consistency to the two values by modifying the construction of associated game. Xu et al. [23] also provide a bidding mechanism as the noncooperative interpretation to the CIS value. The underlying work will provide characterizations that are based on the individual expected reward for the CIS value and the EANS value.

The paper is organized as follows. Section 2 recalls some preliminaries on cooperative game theory. Section 3 gives the definition of the family of ideal values and compares it with two other classical families of values: the procedural values and the least square values. Section 4 introduces the axiomatization and dynamic process to characterize the ideal values. Section 5 focuses on the CIS value and the EANS value. Section 6 concludes and develops some suggestions for future research.

2 Preliminaries: Values for Cooperative Games

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is the finite set of n *players* and $v : 2^N \to \mathbb{R}$ is the *characteristic function* assigning to each *coalition* $S \in 2^N\backslash\{\emptyset\}$ the *worth* $v(S)$, with the convention that $v(\emptyset) = 0$. For each coalition S, the real number $v(S)$ represents the reward that coalition S can guarantee by itself without the cooperation of the other players. The size of the player set S is denoted by s. We denote by $\mathcal{G}^N$ the game space consisting of all these TU games with player set N.

In this context, any $x \in \mathbb{R}^N$ will be called a *payoff vector*, and for any coalition S, $x(S) = \sum_{i\in S} x_i$. A payoff vector x is said to be *efficient* or a *preimputation* if $x(N) = v(N)$. The set of preimputations of a game $\langle N, v\rangle$ is denoted $I(N, v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$. Formally, a *value* on $\mathcal{G}^N$ is a function ϕ that assigns a payoff vector $\phi(N, v) = (\phi_i(N, v))_{i\in N} \in \mathbb{R}^N$ to every game $\langle N, v\rangle \in \mathcal{G}^N$. The value $\phi_i(N, v)$ of player i represents an assessment by i of his or her gains for participating in the game $\langle N, v\rangle$.

The *Shapley value* [1] is the solution that assigns to every player in any game its expected marginal contribution assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability,

$$\mathrm{Sh}_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{(n-s)!(s-1)!}{n!}\big(v(S) - v(S\backslash\{i\})\big), \quad \text{for all } i \in N.$$

The *Banzhaf value* [2] assigns to every player in any game its expected marginal contribution assuming that every coalition not containing this player, is equally likely to occur,

$$\mathrm{Ba}_i(N, v) = \frac{1}{2^{n-1}} \sum_{S\subseteq N, S\ni i} \big(v(S) - v(S\backslash\{i\})\big), \quad \text{for all } i \in N.$$

As an alternative to the player's marginal contributions to coalitions, the assessment of player's gains can also be determined by the worths of the coalitions they belong to. The *Deegan–Packel* (*DP*)*-value* [3] assumes that all coalitions are equally likely to form, and players in a coalition divide the payoff (or the loss) equally. For any game $\langle N, v\rangle \in \mathcal{G}^N$,

$$\mathrm{DP}_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{v(S)}{s}, \quad \text{for all } i \in N.$$

For any game $\langle N, v\rangle \in \mathcal{G}^N$, two players $i, j \in N$ are *symmetric* if, for every coalition $S \subseteq N\backslash\{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$. A game $\langle N, v\rangle \in \mathcal{G}^N$ is *inessential*, if for all $S \subseteq N$, it holds that $v(S) = \sum_{i\in S} v(\{i\})$. Denote by $\mathcal{I}^N$ the linear space of all inessential games with player set N. A game $\langle N, v\rangle \in \mathcal{G}^N$ is *monotonic*, if for all $T \subseteq S \subseteq N$, it holds that $v(T) \leq v(S)$. Let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. We give the following axioms for a value ϕ,

- *Efficiency* For any game $\langle N, v\rangle \in \mathcal{G}^N$, $\sum_{i\in N} \phi_i(N, v) = v(N)$.
- *Symmetry* (*or, equal treatment property*) For any game $\langle N, v\rangle \in \mathcal{G}^N$, if players $i, j \in N$ are symmetric, then $\phi_i(N, v) = \phi_j(N, v)$.
- *Linearity* For any game $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and $a, b \in \mathbb{R}$, $\phi(N, av + bw) = a\phi(N, v)+b\phi(N, w)$, where $av+bw$ is given by $(av+bw)(S) = av(S)+bw(S)$, for all $S \subseteq N$.
- *Inessential game property* For any inessential game $\langle N, v\rangle \in \mathcal{I}^N$, the value satisfies $\phi_i(N, v) = v(\{i\})$ for all $i \in N$.
- *Weak monotonicity* For any monotonic game $\langle N, v\rangle \in \mathcal{G}^N$, the value satisfies $\phi_i(N, v) \geq 0$, for all $i \in N$.
- *Coalitional monotonicity* For any game $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$ and for every coalition $T \subseteq N$, if $v(T) > w(T)$ and $v(S) = w(S)$ for every $S \neq T$, then $\phi_i(N, v) \geq \phi_i(N, w)$ for $i \in T$.

For any efficient, symmetric and linear value, Ruiz et al. [8] propose an universal formula with respect to a sequence of coefficients.

Proposition 2.1 [8] *A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, symmetry and linearity if and only if there exists $p_s \in \mathbb{R}$, $s = 1, 2, \ldots, n-1$, such that for any game $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$,*

$$\phi_i(N, v) = \frac{1}{n}v(N) + \sum_{S\subsetneq N, S\ni i} \frac{p_s}{s}v(S) - \sum_{S\subsetneq N, S\not\ni i} \frac{p_s}{n-s}v(S). \tag{1}$$

On account of the universal formula of efficient, symmetric and linear values provided by Ruiz et al. [8], Malawski [12] lists the conditions that a value satisfies efficiency, symmetry, linearity and coalitional monotonicity and that a value satisfies efficiency, symmetry, linearity and weak monotonicity.

Lemma 2.1 [12] (*i*) *A linear efficient value having the equal treatment property is coalitionally monotonic if and only if, for every $t < n$, $p_t \geq 0$.*

(*ii*) *If a linear efficient value on $\mathcal{G}^N$ with the equal treatment property is weakly monotonic, then for every $t = 1, 2, \ldots, n-1$, the coefficients p_t satisfy*

(*a*) $\binom{n}{t} p_t \leq 1$;

(*b*) $\forall u = 1, 2, \ldots, t,\ \sum_{s=u}^{t} \binom{n}{s} p_s \geq -1$.

Malawski [12] introduces a new notion of a "procedural" value, which is determined by an underlying procedure of sharing marginal contributions to coalitions formed by players joining in random order. A *procedure* r is a family of nonnegative coefficients $((r_{k,j})_{j=1}^{k})_{k=1}^{n}$ such that $\sum_{j=1}^{k} r_{k,j} = 1$, $\forall k$. The coefficient $r_{k,j}$ describes the share of the player who is at place j in the order in the marginal contribution of the player who is at place k. For any game $\langle N, v\rangle \in \mathcal{G}^N$ and all players $i \in N$, the corresponding *procedural value* is

$$\psi_i^r(N, v) = \sum_{\pi\in\Pi} \sum_{j\in N_{\pi,i}} \frac{r_{\pi(j),\pi(i)} m_{j,\pi}(v)}{n!},$$

where Π is the set of all permutations of the set N. For any player $j \in N$ and any permutation $\pi \in \Pi$, $H_{\pi,j} = \{i : \pi(i) \leq \pi(j)\}$ and $N_{\pi,j} = \{i : \pi(i) \geq \pi(j)\}$. Then, $m_{j,\pi}(v)$ is the marginal contribution of player j to coalition $H_{\pi,j}$, i.e., $m_{j,\pi}(v) = v(H_{\pi,j}) - v(H_{\pi,j}\backslash\{j\})$.

Theorem 2.1 [12] *A value on $\mathcal{G}^N$ is procedural if and only if it satisfies efficiency, linearity, the equal treatment property, weak monotonicity and coalitional monotonicity.*

Based on the excess vector, Ruiz et al. [7] select the unique payoff vector which minimizes the variance of the excesses of the coalitions. Assuming different weights for different coalitions, they [8] introduce the family of least square values by minimizing the weighted variance of the excesses. For any coalitional weights function $w : \{0, 1, 2, \ldots n\} \to \mathbb{R}$ and any game $\langle N, v\rangle \in \mathcal{G}^N$, the corresponding *least square value LS^w* assigns the solution of the following minimization problem,

$$\text{Minimize}_{x\in\mathbb{R}^N} \sum_{S\subseteq N} w(s)[v(S) - x(S)]^2 \quad \text{s.t.} \sum_{i\in N} x_i = v(N). \tag{2}$$

The corresponding least square value, being the solution of the minimization problem (2), is given by

$$\mathrm{LS}_i^w(N, v) = \frac{v(N)}{n} + \frac{1}{n\alpha}\left[\sum_{S:i\in S}(n-s)w(s)v(S) - \sum_{S:i\notin S} s w(s)v(S)\right],$$

where $\alpha = \sum_{s=1}^{n-1} w(s)\binom{n-2}{s-1}$.

They also provide an axiomatic characterization for the least square family.

Proposition 2.2 [9] *A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, linearity, symmetry, coalitional monotonicity and the inessential game property if and only if it belongs to the family of least square values.*

3 The Family of Ideal Values

3.1 Definition

As mentioned in the introduction, the DP value offers an interesting alternative to the Shapley and Banzhaf values, focusing on the worths of coalitions a player belongs to, instead of marginal contributions of a player. Especially in situations where players do not focus on their individual marginal contributions but more on what they can earn by cooperating with other players, the DP value seems an attractive value. However, in our opinion, the DP value misses two important points. The first is that it emphasizes the equal possibility of the coalitions to form, ignoring that coalitions are build sequentially. The second is that it assumes that players in a coalition divide the full worth of that coalition equally. Together, this implies that the sum of all coalitional worths are allocated, which might not be feasible.[1]

Every player, who is motivated by profit to cooperate and join a coalition, may want selfishly to get part of the formed coalition's worth. We employ a function of coalitional selfish coefficients to describe the individual selfish degree in the coalition. A *selfish coefficient function* on N is a weights' map $m : 2^N \backslash \{\emptyset\} \rightarrow \mathbb{R}$ that associates with every nonempty $S \subseteq N$ a real number $m(S)$, which identifies the selfish degree of players in this coalition. It means that every player wants to get $m(S)v(S)$ from the cooperation within coalition S. Without loss of generality, we restrict our attention to nonnegative selfish coefficient functions, namely such that $m(S) \geq 0$ for all $S \subseteq N$. Further, we assume the selfish coefficient function to be symmetric assigning the same selfish coefficient to coalitions of the same size, i.e., $m(S) = m(s)$ for all $S \subseteq N$.

Based on a selfish coefficient function m, assuming that the probability that player i participates to every coalition $S \ni i$, is equal, the *m-individual expected reward* of player $i \in N$ in game $\langle N, v\rangle \in \mathcal{G}^N$ is defined as

$$E_i^m(N, v) = \frac{1}{2^{n-1}} \sum_{S \subseteq N, S \ni i} m(s)v(S).$$

We try to select the payoff vector in the preimputation set that makes every player closer to their expected reward. Formally, consider the following problem for any game $\langle N, v\rangle \in \mathcal{G}^N$,

$$\begin{aligned} \text{Problem } X : \text{Minimize}_{x \in \mathbb{R}^N} & \sum_{i \in N} \sum_{S \subseteq N, S \ni i} [m(s)v(S) - x_i]^2 \\ & \text{s.t.} \sum_{i \in N} x_i = v(N). \end{aligned} \tag{3}$$

Notice the difference with the minimization problem in (2), where the minimum is taken over coalitional payoffs instead of individual rewards.

Theorem 3.1 *Given any selfish coefficient function m, for $\langle N, v\rangle \in \mathcal{G}^N$, Problem X has a unique solution x^m that is given by*

$$x_i^m = \sum_{S \subseteq N, S \ni i} \frac{m(s)}{2^{n-1}} v(S) + \frac{1}{n}\left[v(N) - \sum_{j \in N} \sum_{S \subseteq N, S \ni j} \frac{m(s)}{2^{n-1}} v(S)\right], \quad i \in N. \tag{4}$$

[1] The Shapley value allocates the *dividends* of every coalition equally over the players in the coalition, and since the sum of the dividends over all coalitions equals the worth of the grand coalition, the Shapley value is efficient.

Proof The objective function and the feasible set of Problem X are both convex. Hence, there is only one optimal solution if it exists. It is necessary and sufficient to verify the Lagrange conditions so as to find the optimal solution. The Lagrangian of Problem X is

$$L(x, \lambda) = \sum_{i \in N} \sum_{S \subseteq N, S \ni i} [m(s)v(S) - x_i]^2 + \lambda \left[\sum_{i \in N} x_i - v(N) \right].$$

Then, the derivative with respect to x_i, $i \in N$ of $L(x, \lambda)$ is the following

$$L_{x_i}(x, \lambda) = -2 \sum_{S \subseteq N, S \ni i} [m(s)v(S) - x_i] + \lambda = 0.$$

Obviously, the derivative with respect to λ gives the efficiency constraint

$$L_\lambda(x, \lambda) = \sum_{i \in N} x_i - v(N) = 0.$$

A simple calculation solves this linear system and shows that the unique point x^m satisfying these conditions is given by (4). □

The solutions (4) to the maximization problem X form, what we call, the family of *ideal values*. Notice that, using the individual expected rewards $E_i^m(N, v)$, these solutions can be written as in the following definition.

Definition 3.1 For any selfish coefficient function m, value $\mathrm{IV}^m : \mathcal{G}^N \to \mathbb{R}^N$ which for any game $\langle N, v \rangle \in \mathcal{G}^N$ assigns the payoff vector

$$\mathrm{IV}_i^m(N, v) = E_i^m(N, v) + \frac{1}{n} \left[v(N) - \sum_{j \in N} E_j^m(N, v) \right] \quad \text{for every } i \in N,$$

is called an ideal value.

So, for any given selfish coefficient function m, the corresponding ideal value distributes the m-individual expected reward to every player, and then the remainder of the worth of the grand coalition N is equally distributed over all players. This gives the solution of Problem X. Next, we explore the relation of ideal values with the least square values and procedural values.

3.2 Relationships with Procedural and Least Square Values

Obviously, all ideal values are efficient, symmetric and linear. Aiming to facilitate research of the family of ideal values, we develop the further relationship between any ideal value and any efficient, symmetric and linear value by relating the selfish coefficients $m(s)$ to the coefficients p_s in Proposition 2.1.

Proposition 3.1 *A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, symmetry and linearity if and only if there exists $m_s \in \mathbb{R}$, $s = 1, 2, \ldots, n-1$ such that for any game $\langle N, v\rangle \in \mathcal{G}^N$ and $i \in N$,*

$$\phi_i(N, v) = \sum_{S \subsetneq N, S \ni i} m_s v(S) + \frac{1}{n}\left[v(N) - \sum_{j \in N}\sum_{S \subsetneq N, S \ni j} m_s v(S)\right]. \tag{5}$$

Proof The right-hand side of (5) can be rewritten as

$$\begin{aligned}
&\sum_{S \subsetneq N, S \ni i} m_s v(S) + \frac{1}{n}\left[v(N) - \sum_{j \in N}\sum_{S \subsetneq N, S \ni j} m_s v(S)\right] \\
&\quad = \frac{1}{n}v(N) + \sum_{S \subsetneq N, S \ni i} m_s v(S) - \frac{1}{n}\sum_{S \subsetneq N} s m_s v(S) \\
&\quad = \frac{1}{n}v(N) + \sum_{S \subsetneq N, S \ni i} \left(1 - \frac{s}{n}\right) m_s v(S) - \sum_{S \subsetneq N, S \not\ni i} \frac{s}{n} m_s v(S) \\
&\quad = \frac{1}{n}v(N) + \sum_{S \subsetneq N, S \ni i} \left(\frac{n-s}{n}\right) m_s v(S) - \sum_{S \subsetneq N, S \not\ni i} \frac{s}{n} m_s v(S)
\end{aligned}$$

By straightforward computations, it then follows that the expression on the right-hand side of (5) agrees with the one on the right-hand side of (1) by choosing $m_s = \frac{n}{s(n-s)} p_s$ for all $s = 1, 2, \ldots, n-1$. □

For any game $\langle N, v\rangle \in \mathcal{G}^N$ and for any selfish coefficient function $m : 2^N \backslash \{\emptyset\} \to \mathbb{R}$, taking $m_s = \frac{m(s)}{2^{n-1}}$, we can get the ideal value $\mathrm{IV}^m(N, v)$. Especially coefficients m_s obtained from ideal values satisfy $m_s \geq 0$. Moreover, the relationship between $m(s)$ and p_s is $m(s) = \frac{n2^{n-1}}{s(n-s)} p_s$.

Notice that the value of $m(n)$ does not have any influence on the ideal value, so from now on we put away the requirement on $m(n)$.

From the expression $m(s) = 2^{n-1} m_s = \frac{n2^{n-1}}{s(n-s)} p_s$ from the proof above, it is clear that $p_s \geq 0$ if and only if $m(s) \geq 0$ for all $s = 1, 2, \ldots, n-1$. Then, using the nonnegativity of the selfish coefficient function, with Lemma 2.1(i) we obtain an axiomatic characterization for the family of ideal values.

Theorem 3.2 *A value $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, symmetry, linearity, and coalitional monotonicity if and only if it belongs to the family of ideal values.*

This result strongly motivates the family of ideal values as being the coalitional monotonic values among the ESL (efficient, symmetric, linear) values.

Combining Theorem 3.2 with Theorem 2.1, the family of ideal values has the following connection with the procedural values.

Corollary 3.1 *A value on $\mathcal{G}^N$ belonging to the family of ideal values is procedural if and only if it satisfies weak monotonicity.*

Combining with Lemma 2.1(ii), we get the conditions on the selfish coefficient functions to obtain ideal values that are procedural.

Proposition 3.2 *For any given selfish coefficient function m, if the ideal value $IV^m : \mathcal{G}^N \to \mathbb{R}^N$ is procedural, then for every $s = 1, 2, \ldots, n-1$, the coefficients $m(s)$ satisfy $\binom{n}{s}\frac{s(n-s)}{n2^{n-1}}m(s) \leq 1$.*

Proof The equation can be deduced easily from condition (a) in Lemma 2.1(ii) and the relation between $m(s)$ and p_s, $s = 1, 2 \ldots, n-1$. Condition (b) follows directly from the selfish coefficients being nonnegative. □

From Theorem 3.2, using Proposition 2.2, we also obtain the connection between the family of ideal values and the family of least square values.

Corollary 3.2 *A value on $\mathcal{G}^N$, belonging to the family of ideal values is a least square value if and only if it satisfies the inessential game property.*

Next, we want an explicit condition on $m(s)$ for an ideal value to be a least square value. For that, we first derive the explicit condition on the coefficients p_s.

Lemma 3.1 *An efficient, symmetric and linear value satisfies the inessential game property, if and only if, $\sum_{s=1}^{n-1}\binom{n}{s}p_s = n-1$.*

Proof Consider the unanimity game $\langle N, u_T\rangle$ which is defined as: for each $S \subseteq N$, $u_T(S) = 1$ if $S \supseteq T$, and $u_T(S) = 0$ if $S \not\supseteq T$. The ordered collection of unanimity games $(\langle N, u_{\{1\}}\rangle, \langle N, u_{\{2\}}\rangle, \ldots, \langle N, u_{\{n\}}\rangle)$ forms a basis for $\mathcal{I}^N$. So any inessential game $\langle N, v\rangle \in \mathcal{I}^N$ can be written as $v(S) = \sum_{j\in N} v(\{j\})u_j(S)$, for all $S \subseteq N$.

Let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be a value that satisfies efficiency, symmetry and linearity. Following the definition of the inessential game property, the value ϕ owning the inessential game property is equivalent to that, for $i \in N$, $\phi_i(v) = v(\{i\})$ if $\langle N, v\rangle \in \mathcal{I}^N$, i.e.,

$$\phi_i(v) = \phi_i\left(\sum_{j\in N} v(\{j\})u_j\right) = \sum_{j\in N} v(\{j\})\phi_i(u_j) = v(\{i\}).$$

It is also equivalent to

$$\begin{aligned}\phi_i(u_i) &= u_i(\{i\}) = 1;\\ \phi_i(u_j) &= u_j(\{i\}) = 0, \quad j \neq i.\end{aligned}$$

By Proposition 2.1, the equivalent condition can be inferred as

$$\begin{aligned}\phi_i(N, u_i) &= \frac{1}{n}u_i(N) + \sum_{S\subsetneq N, S\ni i}\frac{p_s}{s}u_i(S) - \sum_{S\subsetneq N, S\not\ni i}\frac{p_s}{n-s}u_i(S)\\ &= \frac{1}{n} + \sum_{S\subsetneq N, S\ni i}\frac{p_s}{s} = 1.\end{aligned}$$

So, $n \sum_{S \subsetneq N, S \ni i} \frac{p_s}{s} = \sum_{s=1}^{n-1} n \binom{n-1}{s-1} \frac{p_s}{s} = \sum_{s=1}^{n-1} \binom{n}{s} p_s = n - 1$.

And for any $j \in N$, $j \neq i$,

$$\begin{aligned}
\phi_i(N, u_j) &= \frac{1}{n} u_j(N) + \sum_{S \subsetneq N, S \ni i} \frac{p_s}{s} u_j(S) - \sum_{S \subsetneq N, S \not\ni i} \frac{p_s}{n-s} u_j(S) \\
&= \frac{1}{n} + \sum_{S \subsetneq N, S \ni \{i,j\}} \frac{p_s}{s} - \sum_{S \subseteq N \setminus \{i\}, S \ni j} \frac{p_s}{n-s} \\
&= \frac{1}{n} + \sum_{s=2}^{n-1} \binom{n-2}{s-2} \frac{p_s}{s} - \sum_{s=1}^{n-1} \binom{n-2}{s-1} \frac{p_s}{n-s} \\
&= \frac{1}{n} - \sum_{s=1}^{n-1} \binom{n}{s} \frac{p_s}{n(n-1)} = 0.
\end{aligned}$$

This also indicates that $\sum_{s=1}^{n-1} \binom{n}{s} p_s = n - 1$. □

From this we obtain the conditions on the selfish coefficient functions $m(s)$.

Proposition 3.3 *For any given selfish coefficient function m, the ideal value* $IV^m : \mathcal{G}^N \to \mathbb{R}^N$ *is a least square value, if and only if, the coefficients* $m(s)$ *satisfy* $\sum_{s=1}^{n-1} \binom{n}{s} \frac{s(n-s)}{n 2^{n-1}} m(s) = n - 1$.

In this subsection, we described the relationship between the family of ideal values and two important families of values from the literature: the procedural values and the least square values. In the next section we provide two characterizations of specific values within this family.

4 Characterization of the Ideal Values

There are several approaches to justify values for TU games. Two of these approaches are axiomatization and providing a dynamic process.

4.1 Axiomatization

For any game $\langle N, v\rangle \in \mathcal{G}^N$ and for any selfish coefficient function m, two players $i, j \in N$ are *m-equal-expectation players* if their individual expected reward is equal, i.e., $E_i^m(N, v) = E_j^m(N, v)$. Player $i \in N$ is a *nullifying player* if, $v(S) = 0$ for all coalition $S \subseteq N$ with $i \in S$. Given any selfish coefficient function m, let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. We consider the following properties.

- *m-Equal-expectation player property* For every game $\langle N, v\rangle \in \mathcal{G}^N$, if players $i, j \in N$ are m-equal-expectation player, then $\phi_i(N, v) = \phi_j(N, v)$.
- *Nullifying player m-punishment property* For every game $\langle N, v\rangle \in \mathcal{G}^N$, if player $i \in N$ is a nullifying player, then $\phi_i(N, v) = -\frac{1}{n} \sum_{j \in N} E_j^m(N, v)$.

The m-equal-expectation player property points out that players should get the same payoff, if their individual expected rewards are equal. This makes sense if the players take their individual expected reward as basis for their claim on the payoff.

The nullifying player m-punishment property determines the payoff for nullifying players. If a player is a nullifying player, then every coalition he belongs to, specifically the grand coalition, will gain zero. If the coalition without this player earns a positive worth, then the nullifying player has a negative impact on the worth of this coalition. In that case it seems appropriate to punish the nullifying player. The nullifying player m-punishment property puts this punishment for a nullifying player equal to the average of all players' individual expected rewards.

This punishment can be motivated as follows. Although this paper considers classes of games on a fixed player set N, suppose that a nullifying player i leaves the game. The resulting game is the projection $\langle N\backslash\{i\}, v_{-i}\rangle$ given by $v_{-i}(S) = v(S)$ for all $S \subseteq N\backslash\{i\}$. Assuming that the selfish coefficients $m(s),\ s = 1, \ldots, n-1$, do not change, the total gain for the other players of i leaving the game is

$$\begin{aligned}
&\sum_{j\in N\backslash\{i\}} \left[E_j^m(N\backslash\{i\}, v_{-i}) - E_j^m(N, v)\right] \\
&\quad = \sum_{j\in N\backslash\{i\}} \left[\frac{1}{2^{n-2}} \sum_{S\subseteq N\backslash\{i\}} m(s)v(S) - \frac{1}{2^{n-1}} \sum_{S\subseteq N} m(s)v(S)\right] \\
&\quad = \sum_{j\in N\backslash\{i\}} \left[\frac{2}{2^{n-1}} \sum_{S\subseteq N} m(s)v(S) - \frac{1}{2^{n-1}} \sum_{S\subseteq N} m(s)v(S)\right] \\
&\quad = \sum_{j\in N\backslash\{i\}} \frac{1}{2^{n-1}} \sum_{S\subseteq N} m(s)v(S) = \sum_{j\in N\backslash\{i\}} E_j^m(N, v) = \sum_{j\in N} E_j^m(N, v),
\end{aligned}$$

where the second and fifth equality follow since $v(S) = 0$ if $i \in S$. So, the nullifying player pays an equal share in the total loss resulting from its presence.

Remark 4.1 Let $\phi : \mathcal{G}^N \to \mathbb{R}^N$ be a value. For any $\langle N, v\rangle \in \mathcal{G}^N$, given any selfish coefficient function m, the value $\phi(N, v)$ satisfies the m-equal-expectation player property since $E_i^m(N, v) = E_j^m(N, v)$ if i and j are symmetric players in $\langle N, v\rangle$. It implies that $\phi(N, v)$ satisfies symmetry.

With efficiency and linearity, these axioms characterize the corresponding ideal value.

Theorem 4.1 *For any given selfish coefficient function m, the ideal value $IV^m : \mathcal{G}^N \to \mathbb{R}^N$ is the unique value which satisfies efficiency, linearity, the m-equal-expectation player property and the nullifying player m-punishment property.*

Proof For any given selfish coefficient function m, it is obvious that the ideal value $IV^m : \mathcal{G}^N \to \mathbb{R}^N$ satisfies efficiency, linearity, the m-equal-expectation player property and the nullifying player m-punishment property.

It remains to prove the uniqueness part. For any given selfish coefficient function m, suppose that $\phi^m : \mathcal{G}^N \to \mathbb{R}^N$ is a value with the four mentioned properties. For any $T \subseteq N$ and $T \neq \emptyset$, consider the standard game $\langle N, b_T \rangle$ defined as: for each $S \subseteq N$,

$$b_T(S) = \begin{cases} 1, & S = T; \\ 0, & \text{otherwise.} \end{cases}$$

Let $T \subseteq N, T \neq \emptyset$. Given any player $i \in N \backslash T$, we have $b_T(S) = 0$ for all $i \in S \subseteq N$, so $E_i^m(N, b_T) = 0$. Now discussing player $i \in T$, it is apparent that $b_T(T) = 1$ and $b_T(S) = 0$ for all $i \in S \subseteq N$, $S \neq T$. This yields (i) $E_i^m(N, b_T) = \frac{m(t)}{2^{n-1}}$ for all $i \in T$, (ii) all players in coalition T are m-equal-expectation players, and (iii) $\sum_{j \in N} E_j^m(N, b_T) = \sum_{j \in T} E_j^m(N, b_T) = \frac{tm(t)}{2^{n-1}}$.

Since any player $i \in N \backslash T$ is a nullifying player, by the nullifying player m-punishment property, we have

$$\phi_i^m(N, b_T) = -\frac{1}{n} \sum_{j \in N} E_j^m(N, b_T) = -\frac{tm(t)}{n2^{n-1}} \quad \text{for all } i \in N \backslash T.$$

According to efficiency,

$$\sum_{i \in T} \phi_i^m(N, b_T) = b_T(N) - \sum_{i \in N \backslash T} \phi_i^m(N, b_T) = b_T(N) + \frac{(n-t)tm(t)}{n2^{n-1}}.$$

Because of the m-equal-expectation player property, for any player $i \in T$,

$$\phi_i^m(N, b_T) = \frac{b_T(N)}{t} + \frac{(n-t)m(t)}{n2^{n-1}}.$$

Summarizing,

$$\phi_i^m(N, b_T) = \begin{cases} \frac{b_T(N)}{t} + \frac{(n-t)m(t)}{n2^{n-1}}, & i \in T, \\ -\frac{tm(t)}{n2^{n-1}}, & i \in N \backslash T. \end{cases}$$

We conclude that $\phi^m(N, b_T)$ is unique for any $\emptyset \neq T \subseteq N$. Recall that the game set $\{\langle N, b_T \rangle \in \mathcal{G}^N : \emptyset \neq T \subseteq N\}$ forms a basis of the linear space $\mathcal{G}^N$. Together with linearity of $\phi^m(N, v)$, this implies that $\phi^m(N, v)$ is unique for any $\langle N, v \rangle \in \mathcal{G}^N$. So, if $\phi^m(N, v)$ exists, it can only be the ideal value IV^m. □

4.2 Dynamic Process

In a characterization by a dynamic process, it is shown how, starting from any efficient payoff vector, such a process can lead to an ideal value. In our dynamic process, the main basis is a complaint function based on the selfish coefficient.

For any game $\langle N, v \rangle \in \mathcal{G}^N$ and payoff vector $x \in I(N, v)$, the *excess* of the coalition S with respect to the vector x in the game $\langle N, v \rangle$ is defined to be $e(S, v, x) = v(S) - x(S)$, i.e., it is the difference between the worth of the coalition and the total payoff assigned to the players in this coalition. For every selfish coefficient function m, each player in coalition S wants to take the payoff $m(s)v(S)$. So, the *complaint* of player i in coalition S with respect to m is the real number $e_i^m(S, v, x) = m(s)v(S) - x_i$.

Theorem 4.2 *Let $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in I(N, v)$. For any selfish coefficient function m, we have*

$$\sum_{S\subseteq N\backslash\{i,j\}} e_i^m(S\cup\{i\}, v, 2x) = \sum_{S\subseteq N\backslash\{i,j\}} e_j^m(S\cup\{j\}, v, 2x) \quad \forall\, i, j \in N$$
$$\Longleftrightarrow x = \mathrm{IV}^m(N, v).$$

Proof Let $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in I(N, v)$. For any selfish coefficient function m, and $i, j \in N$,

$$\begin{aligned}
&\sum_{S\subseteq N\backslash\{i,j\}} e_i^m(S\cup\{i\}, v, 2x) = \sum_{S\subseteq N\backslash\{i,j\}} e_j^m(S\cup\{j\}, v, 2x)\\
&\quad\Longleftrightarrow \sum_{S\subseteq N\backslash\{i,j\}} [m(s+1)v(S\cup\{i\}) - 2x_i] = \sum_{S\subseteq N\backslash\{i,j\}} \left[m(s+1)v(S\cup\{j\}) - 2x_j\right]\\
&\quad\Longleftrightarrow \sum_{S\subseteq N\backslash\{i,j\}} 2(x_i - x_j) = \sum_{S\subseteq N\backslash\{i,j\}} m(s+1)[v(S\cup\{i\}) - v(S\cup\{j\})]\\
&\quad\Longleftrightarrow x_i - x_j = \sum_{S\subseteq N\backslash\{i,j\}} \frac{m(s+1)}{2^{n-1}}[v(S\cup\{i\}) - v(S\cup\{j\})].
\end{aligned} \tag{6}$$

On the other hand, by the definitions of $\mathrm{IV}^m(N, v)$,

$$\begin{aligned}
&\mathrm{IV}_i^m(N, v) - \mathrm{IV}_j^m(N, v)\\
&\quad= \sum_{S\subseteq N, S\ni i} \frac{1}{2^{n-1}} m(s)v(S) - \sum_{S\subseteq N, S\ni j} \frac{1}{2^{n-1}} m(s)v(S)\\
&\quad= \left[\sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-1}} m(s+1)v(S\cup\{i\}) + \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-1}} m(s+2)v(S\cup\{i, j\})\right]\\
&\quad- \left[\sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-1}} m(s+1)v(S\cup\{j\}) + \sum_{S\subseteq N\backslash\{i,j\}} \frac{1}{2^{n-1}} m(s+2)v(S\cup\{i, j\})\right]\\
&\quad= \sum_{S\subseteq N\backslash\{i,j\}} \frac{m(s+1)}{2^{n-1}} [v(S\cup\{i\}) - v(S\cup\{j\})].
\end{aligned} \tag{7}$$

By Eqs. (6) and (7), $x_i - x_j = \mathrm{IV}_i^m(N, v) - \mathrm{IV}_j^m(N, v)$ for all $i, j \in N$. Hence,

$$\sum_{j\in N}(x_i - x_j) = \sum_{j\in N}\left[\mathrm{IV}_i^m(N, v) - IV_j^m(N, v)\right].$$

That is, $nx_i - \sum_{j\in N} x_j = n\mathrm{IV}_i^m(N, v) - \sum_{j\in N} \mathrm{IV}_j^m(N, v)$. Because of $x \in I(N, v)$ and efficiency of $\mathrm{IV}^m(N, v)$, $nx_i - v(N) = n\mathrm{IV}_i^m(N, v) - v(N)$. So, $x = \mathrm{IV}^m(N, v)$. □

Notice that $e_i^m(S\cup\{i\}, v, 2x)$ in Theorem 4.2 is the complaint of player i in coalition $S \cup \{i\}$ with respect to the payoff vector $2x$. Although it is not immediately clear why to consider twice the payoff vector, notice that the equation on the left side of the equivalence in Theorem 4.2 can be written, for all $i, j \in N$, as

$$\sum_{S\subseteq N\setminus\{i,j\}} \left(e_i^m(S \cup \{i\}, v, x) - x_i\right) = \sum_{S\subseteq N\setminus\{i,j\}} \left(e_j^m(S \cup \{j\}, v, x) - x_j\right)$$

which is equivalent to

$$x_i - x_j = \frac{1}{2^{n-2}} \sum_{S\subseteq N\setminus\{i,j\}} \left(e_i^m(S \cup \{i\}, v, x) - e_j^m(S \cup \{j\}, v, x)\right) \quad \forall\, i, j \in N$$

Defining the complaint of player i against player j as the difference between the average complaint of i in all coalitions that contain player i and do not contain player j (and vice versa), this can be seen as some kind of *balanced mutual complaint* property stating that the difference in average complaint of i against j and the average complaint of j against i is equal to the difference in their payoffs. In this way, the ideal value IV^m is the unique efficient value satisfying the balanced mutual complaint property.

Next, we adopt complaint functions to introduce a dynamic process that leads the players to the ideal value.

Let $\langle N, v\rangle \in \mathcal{G}^N$ and $x \in I(N, v)$. For any selfish coefficient function m, we define the *m-correction function* $f^m : I(N, v) \to \mathbb{R}^N$ as follows: for all $i \in N$,

$$\begin{aligned} f_i^m(x) &= x_i + \lambda \sum_{j\in N\setminus\{i\}} \sum_{T\subseteq N\setminus\{i,j\}} \left[e_i^m(T \cup \{i\}, v, 2x) - e_j^m(T \cup \{j\}, v, 2x)\right] \\ &= x_i + \lambda \sum_{j\in N\setminus\{i\}} \sum_{T\subseteq N\setminus\{i,j\}} \left[(e_i^m(T \cup \{i\}, v, x) - e_j^m(T \cup \{j\}, v, x)) - (x_i - x_j)\right]. \end{aligned}$$

where λ belongs to $(0, 1)$.

$\sum_{j\in N\setminus\{i\}} \sum_{T\subseteq N\setminus\{i,j\}} \left[(e_i^m(T \cup \{i\}, v, x) - e_j^m(T \cup \{j\}, v, x)) - (x_i - x_j)\right]$ is a correction on the current payoff assignment. The correction is based on the differences in payoffs and mutual complaints. The m-correction function reflects the assumption that player i does not ask for full correction (when $\lambda = 1$) but only a fraction λ of it.

The following lemma shows that the correction function is well defined, i.e., if $x \in I(N, v)$, then $f^m(x) \in I(N, v)$. This lemma plays a key role to prove the necessary convergence results.

Lemma 4.1 *Let $\langle N, v\rangle \in \mathcal{G}^N$ with $n \geq 3$ and $x \in I(N, v)$. For any selfish coefficient function m, and for all $i \in N$,*

$$\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}\left[e_i^m(T\cup\{i\},v,2x)-e_j^m(T\cup\{j\},v,2x)\right]\right\}$$
$$=n2^{n-1}(\mathrm{IV}_i^m(N,v)-x_i)$$

and

$$\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}\left[e_i^m(T\cup\{i\},v,2x)-e_j^m(T\cup\{j\},v,2x)\right]\right\}=0.$$

Proof Let $\langle N,v\rangle\in\mathcal{G}^N$ and $x\in I(N,v)$. For any selfish coefficient function m, $i,j\in N$,

$$\begin{aligned}
&\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}\left[e_i^m(T\cup\{i\},v,2x)-e_j^m(T\cup\{j\},v,2x)\right]\right\}\\
&=\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}[m(s+1)v(S\cup\{i\})-2x_i\right.\\
&\qquad\left.-m(s+1)v(S\cup\{j\})+2x_j]\vphantom{\sum_{T}}\right\}\\
&=\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}m(s+1)[v(S\cup\{i\})-v(S\cup\{j\})]-2^{n-1}(x_i-x_j)\right\}\\
&\overset{\text{Eq (7)}}{=}\sum_{j\in N\setminus\{i\}}2^{n-1}\left[\mathrm{IV}_i^m(N,v)-IV_j^m(N,v)-x_i+x_j\right]\\
&=n2^{n-1}(IV_i^m(N,v)-x_i),
\end{aligned}$$

where the last equality follows from x and $\mathrm{IV}^m(N,v)$ both belonging to $I(N,v)$. Moreover,

$$\sum_{i\in N}\sum_{j\in N\setminus\{i\}}\left\{\sum_{T\subseteq N\setminus\{i,j\}}\left[e_i^m(T\cup\{i\},v,2x)-e_j^m(T\cup\{j\},v,2x)\right]\right\}$$
$$=\sum_{i\in N}n2^{n-1}(\mathrm{IV}_i^m(N,v)-x_i)=n2^{n-1}(v(N)-v(N))=0.$$

This completes the proof. □

Let $\langle N,v\rangle\in\mathcal{G}^N$ and $x\in I(N,v)$. For any selfish coefficient function m, we define the dynamic sequence $\{x_{f^m}^q\}_{q=1}^{\infty}$ with respect to the correction function f^m, for all $q\in\mathbb{N}$, by

$$x^0_{f^m} = x,\ x^1_{f^m} = f^m(x^0_{f^m}),\ x^2_{f^m} = f^m(x^1_{f^m}), \ldots,\ x^q_{f^m} = f^m(x^{q-1}_{f^m}).$$

For 'small enough' values of λ, this dynamic process converges to the corresponding ideal value.

Theorem 4.3 *Let $\langle N, v\rangle \in \mathcal{G}^N$. For any selfish coefficient function m, and $0 < \lambda < \frac{1}{n2^{n-2}}$, $\{x^q_{f^m}\}_{q=1}^{\infty}$ converges geometrically to $IV^m(N, v)$ for each $x \in I(N, v)$.*

Proof Let $\langle N, v\rangle \in \mathcal{G}^N$, $x \in I(N, v)$, and take any selfish coefficient function m. By definition of f^m and Lemma 4.1, for $i \in N$,

$$\begin{aligned} &f^m_i(x) - x_i \\ &= \lambda \sum_{j\in N\setminus\{i\}} \left\{ \sum_{T\subseteq N\setminus\{i,j\}} \left[e^m_i(T \cup \{i\}, v, 2x) - e^m_j(T \cup \{j\}, v, 2x) \right] \right\} \\ &= n2^{n-1}\lambda(\mathrm{IV}^m_i(N, v) - x_i). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{IV}^m_i(N, v) - f^m_i(x) &= \mathrm{IV}^m_i(N, v) - x_i + x_i - f^m_i(x) \\ &= (1 - n2^{n-1}\lambda)(\mathrm{IV}^m_i(N, v) - x_i). \end{aligned}$$

For all $q \in \mathbb{N}$,

$$\mathrm{IV}^m(N, v) - x^q_{f^m} = (1 - n2^{n-1}\lambda)^q(\mathrm{IV}^m(N, v) - x).$$

If $0 < \lambda < \frac{1}{n2^{n-2}}$, then $-1 < 1 - n2^{n-1}\lambda < 1$ and $\{x^q_{f^m}\}_{q=1}^{\infty}$ converges geometrically to $\mathrm{IV}^m(N, v)$. □

5 Two Special Cases: The CIS Value and the EANS Value

The *center of gravity of imputation set value (CIS value)*, introduced by Driessen and Funaki [15], is a solution on $\mathcal{G}^N$, which associates with each game $\langle N, v\rangle$ and all players $i \in N$,

$$\mathrm{CIS}_i(N, v) = v(\{i\}) + \frac{1}{n}\left[v(N) - \sum_{j\in N} v(\{j\}) \right].$$

The CIS value assigns to every player its individual worth, and distributes the remainder of the worth of the grand coalition N equally among all players.

The *equal allocation of nonseparable cost value (EANS value)* introduced by Moulin [16] is given as

$$\mathrm{EANS}_i(N, v) = \mathrm{SC}_i(N, v) + \frac{1}{n}\left[v(N) - \sum_{j\in N} \mathrm{SC}_j(N, v)\right],$$

where $\mathrm{SC}_j(N, v) = v(N) - v(N\backslash\{j\})$ is the separable cost and the EANS value refers to all players sharing the nonseparable cost $v(N) - \sum_{j\in N} \mathrm{SC}_j(N, v)$ equally.

For any game $\langle N, v\rangle \in \mathcal{G}^N$, its *dual game* $\langle N, v^D\rangle$ is $v^D(S) = v(N) - v(N\backslash S)$ for all $S \subseteq N$. Obviously, $\mathrm{EANS}(N, v) = \mathrm{CIS}(N, v^D)$ for all $\langle N, v\rangle \in \mathcal{G}^N$ since by the definition of dual game, $\mathrm{SC}_j(N, v) = v^D(j)$ for all $j \in N$. So, the CIS value and the EANS value are dual to each other. Furthermore, it is easy to show that the CIS value is an ideal value by taking $m(1) = 2^{n-1}$ and $m(s) = 0$, $s = 2, 3, \ldots, n-1$, so is the EANS value by taking $m(n-1) = 2^{n-1}$ and $m(s) = 0$, $s = 1, 2 \ldots, n-2$.

Consistency, including reduced consistency and associated consistency, has been used to characterize the CIS value [17,21,22] and the EANS value [17,19–22]. Xu et al. [23] also provide a noncooperative interpretation of the α-CIS value, the extension of CIS value, by a bidding mechanism. We apply Theorem 4.1 to the specific selfish coefficient functions of the CIS and EANS values.

Taking as selfish coefficient function $\bar{m}$, where $\bar{m}(1) = 2^{n-1}$ and $\bar{m}(s) = 0$, $s = 2, 3, \ldots, n-1$, for any game $\langle N, v\rangle \in \mathcal{G}^N$, $E_i^{\bar{m}}(N, v) = v(\{i\})$, $i \in N$. Using this selfish coefficient function in Theorem 4.1 characterizes the CIS value. In that case, we can replace the nullifying player m-punishment property by the inessential game property.

Theorem 5.1 *For any game $\langle N, v\rangle \in \mathcal{G}^N$, the CIS value is the unique value that satisfies efficiency, linearity, the inessential game property and the $\bar{m}$-equal-expectation player property.*

Proof It can be easily checked that the CIS value satisfies efficiency, linearity, the inessential game property and the $\bar{m}$-equal-expectation player property. It remains to prove the uniqueness.

Suppose that a solution $\phi : \mathcal{G}^N \to \mathbb{R}^N$ satisfies these four properties. For any game $\langle N, v\rangle \in \mathcal{G}^N$, define $v^0(S) := v(S) - \sum_{j\in S} v(\{j\})$, $S \subseteq N$. Then, $\forall i, j \in N$, $0 = v^0(i) = E_i^{\bar{m}}(N, v^0) = E_j^{\bar{m}}(N, v^0) = v^0(j) = 0$. Because of the $\bar{m}$-equal-expectation player property, we have $\phi_i(N, v^0) = \phi_j(N, v^0)$. So based on the efficiency, for any $i \in N$,

$$\phi_i(N, v^0) = \frac{1}{n} v^0(N) = \frac{1}{n}\left[v(N) - \sum_{j\in N} v(\{j\})\right].$$

Let $w := v - v^0$, it is obvious that $\langle N, w\rangle$ is an inessential game. According to the inessential game property, we have $\phi_i(N, w) = w(\{i\}) = v(\{i\})$.

Because $v = w + v^0$, with linearity it follows that

$$\phi_i(N, v) = \phi_i(N, w) + \phi_i(N, v^0) = v(\{i\}) + \frac{1}{n}\left[v(N) - \sum_{j \in N} v(\{j\})\right] = \mathrm{CIS}_i(N, v).$$

This completes the proof. □

Notice that in the proof of Theorem 5.1, we used only part of the linearity axiom. In fact, in the axiomatization, we can replace linearity by the weaker *additivity* axiom.

– *Additivity* For any game $\langle N, v\rangle, \langle N, w\rangle \in \mathcal{G}^N$, $\phi(N, v+w) = \phi(N, v) + \phi(N, w)$, where $v + w$ is given by $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

With the appropriate selfish coefficient function, we can also obtain an axiomatization of the EANS value as a corollary from Theorem 4.1. However, we can also take the dual axiom of the $\bar{m}$-equal-expectation player property.

– *Dual $\bar{m}$-equal-expectation player property* For any game $\langle N, v\rangle \in \mathcal{G}^N$ and $i, j \in N$, if $E_i^{\bar{m}}(N, v^D) = E_j^{\bar{m}}(N, v^D)$, then $\phi_i(N, v) = \phi_j(N, v)$.

Theorem 5.2 *For any game $\langle N, v\rangle \in \mathcal{G}^N$, the EANS value is the unique value that satisfies efficiency, additivity, the inessential game property and the dual $\bar{m}$-equal-expectation player property.*

Proof This proof is similar to the proof of Theorem 5.1 except the following.

For any game $\langle N, v\rangle \in \mathcal{G}^N$, define $v^0(S) := v(S) - \sum_{j \in S} \mathrm{SC}_j(N, v)$, $S \subseteq N$, where $\mathrm{SC}_j(N, v) = v(N) - v(N \backslash \{j\})$. It is easy to verify that $\forall i, j \in N$, $E_i^{\bar{m}}(N, (v^0)^D) = E_j^{\bar{m}}(N, (v^0)^D)$. Then, imitating the proof of Theorem 5.1, we can complete this proof. □

Motivated by the duality of the CIS value and the EANS value, we build the relationship of selfish coefficient functions of dual values in the family of ideal values as follows.

Proposition 5.1 *Let $\langle N, v\rangle \in \mathcal{G}^N$. For any two selfish coefficient functions m and m^*, the ideal values, $IV^m(N, v)$ and $IV^{m^*}(N, v)$, are dual if the selfish coefficient functions satisfy $m^*(n - s) = m(s)$, $s = 1, 2, \ldots, n - 1$.*

Proof Let $\langle N, v\rangle \in \mathcal{G}^N$. For any two selfish coefficient functions m and m^*, the ideal values, $\mathrm{IV}^m(N, v)$ and $\mathrm{IV}^{m^*}(N, v)$, are dual if and only if, $\mathrm{IV}^m(N, v) = \mathrm{IV}^{m^*}(N, v^D)$. So we need to prove that $m^*(n - s) = m(s)$, $s = 1, 2, \ldots, n - 1$ implies that $\mathrm{IV}^m(N, v) = \mathrm{IV}^{m^*}(N, v^D)$.

According to Proposition 3.1, it is easy to get that, for $i \in N$,

$$\begin{aligned}
&\mathrm{IV}_i^m(N, v) \\
&\quad = \sum_{S \subsetneq N, S \ni i} \tfrac{1}{2^{n-1}} m(s) v(S) + \tfrac{1}{n}\left[v(N) - \sum_{j \in N} \sum_{S \subsetneq N, S \ni j} \tfrac{1}{2^{n-1}} m(s) v(S)\right] \\
&\quad = \tfrac{1}{n} v(N) + \sum_{S \subsetneq N, S \ni i} \tfrac{n-s}{n2^{n-1}} m(s) v(S) - \sum_{S \subsetneq N, S \not\ni i} \tfrac{s}{n2^{n-1}} m(s) v(S).
\end{aligned} \tag{8}$$

Similarly,

$$
\begin{aligned}
&\mathrm{IV}_i^{m^*}(N, v^D) \\
&= \sum_{S \subsetneq N, S \ni i} \tfrac{1}{2^{n-1}} m^*(s) v^D(S) + \tfrac{1}{n}\left[v^D(N) - \sum_{j \in N} \sum_{S \subsetneq N, S \ni j} \tfrac{1}{2^{n-1}} m^*(s) v^D(S) \right] \\
&= \tfrac{1}{n} v(N) + \sum_{S \subsetneq N, S \ni i} \tfrac{n-s}{n 2^{n-1}} m^*(s) v^D(S) - \sum_{S \subsetneq N, S \not\ni i} \tfrac{s}{n 2^{n-1}} m^*(s) v^D(S) \\
&= \tfrac{1}{n} v(N) + \sum_{S \subsetneq N, S \ni i} \tfrac{n-s}{n 2^{n-1}} m^*(s) [v(N) - v(N \backslash S)] \\
&\quad - \sum_{S \subsetneq N, S \not\ni i} \tfrac{s}{n 2^{n-1}} m^*(s) [v(N) - v(N \backslash S)] \\
&= \tfrac{1}{n} v(N) - \sum_{S \subsetneq N, S \ni i} \tfrac{n-s}{n 2^{n-1}} m^*(s) v(N \backslash S) + \sum_{S \subsetneq N, S \not\ni i} \tfrac{s}{n 2^{n-1}} m^*(s) v(N \backslash S) \\
&= \tfrac{1}{n} v(N) + \sum_{S \subsetneq N, S \ni i} \tfrac{n-s}{n 2^{n-1}} m^*(n-s) v(S) - \sum_{S \subsetneq N, S \not\ni i} \tfrac{s}{n 2^{n-1}} m^*(n-s) v(S).
\end{aligned}
\tag{9}
$$

By comparing Eq. (8) with Eq. (9), we can get that $m^*(n-s) = m(s)$, $s = 1, 2, \ldots, n-1$, implies that $\mathrm{IV}^m(N, v) = \mathrm{IV}^{m^*}(N, v^D)$. □

As a corollary, we obtain that the family of ideal values is self-dual.

6 Conclusions

In this paper, we gave two types of characterization of ideal values for cooperative TU games: an axiomatization and a dynamic process. Ideal values are based on the idea that players expect to receive a certain part, determined by a selfish coefficient function, from the worths of the coalitions they belong to. Since it is not usually feasible to respect all players individual expected rewards, the values need to be normalized. We compared the ideal values with three other classes from the literature and saw that (i) they are exactly the coalitional monotonic ESL values, (ii) contain the class of procedural values being the weakly monotonic ideal values, and (iii) contain the least square values being the ideal values satisfying the inessential game property.

Future research on ideal values will be done on, for example, strategic implementation. Also, we will consider more general selfish coefficient functions. In this paper, we assumed the selfish coefficient function to be symmetric meaning that the share the players in a coalition expect to receive from the coalition's worth only depends on the size of the coalition. In reality, individual players might have different expectations about their share in the worths of coalitions, and it is interesting to see what results are still valid (in original or modified form) for these more general selfish coefficient functions. Also, the impact of different degrees of selfishness on the above-mentioned strategic implementation will be studied.

Since the family of ideal values contains the procedural values, it also contains the egalitarian Shapley values [24,25], and it is worthwhile to investigate whether within the family of ideal values there are ways to bring egalitarianism into TU game solutions.

Finally, certain specific ideal values might be worth investigating in more detail. In this paper, we already considered the CIS and EANS values. Another interesting ideal value might be based on the DP value, where the selfish coefficient function and corresponding individual expected rewards are simply taken as every player expecting a fraction $\frac{1}{s}$ of the worth of coalition S.

Acknowledgements This research has been supported by the National Natural Science Foundation of China (Grant Nos. 71571143 and 71671140), the China Scholarship Council (Grant No. 201706290181).

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