

Nonconvex Weak Sharp Minima on Riemannian Manifolds

M. M. Karkhaneei · N. Mahdavi-Amiri

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Abstract We are to establish necessary conditions (of the primal and dual types) for the set of weak sharp minima of a nonconvex optimization problem on a Riemannian manifold. Here, we are to provide a generalization of some characterizations of weak sharp minima for convex problems on Riemannian manifold introduced by Li et al. (SIAM J. Optim., 21 (2011), pp. 1523–1560) for nonconvex problems. We use the theory of the Fréchet and limiting subdifferentials on Riemannian manifold to give the necessary conditions of the dual type. We also consider a theory of contingent directional derivative and a notion of contingent cone on Riemannian manifold to give the necessary conditions of the primal type. Several definitions have been provided for the contingent cone on Riemannian manifold. We show that these definitions, with some modifications, are equivalent. We establish a lemma about the local behavior of a distance function. Using the lemma, we express the Fréchet subdifferential (contingent directional derivative) of a distance function on a Riemannian manifold in terms of normal cones (contingent cones), to establish the necessary conditions. As an application, we show how one can use weak sharp minima property to model a Cheeger type constant of a graph as an optimization problem on a Stiefel manifold.

Keywords weak sharp minima · Riemannian manifolds · distance functions · nonconvex functions · generalized differentiation · graph clustering

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Mohammad Mahdi Karkhaneei
Sharif University of Technology
Tehran, Iran
karkhaneei@gmail.com

Nezam Mahdavi-Amiri
Sharif University of Technology
Tehran, Iran
nezamm@sharif.edu

1 Introduction

In recent decades, extensive research has been carried out on optimization on manifolds. These studies scatter in various contexts such as convex [1], smooth [2–4], nonsmooth [5], and over special manifolds such as sphere [6], Stiefel manifolds [7], etc. Particularly, special attention have been devoted to nonconvex analysis and optimization on Riemannian manifolds. Several problems in machine learning, pattern recognition and computer vision, can be modeled as nonconvex optimization problems on Riemannian manifolds [8–12]. Moreover, various numerical procedures for solving nonconvex optimization problems on Riemannian manifolds have been designed, such as line search and trust region methods along with Newton-like methods in smooth cases (see [2, 3, 6, 13, 14]), and subgradient decent, gradient sampling, and proximal algorithms in nonsmooth cases (see [15, 16]).

Distance functions appear in various optimization methods, such as proximal point methods, penalty methods, etc. Generalized differential properties of distance functions play remarkable roles in variational analysis, optimization, and their applications. The authors of [17, 18] investigated properties of generalized derivatives of distance functions in linear space setting. Properties of distance functions on a Riemannian manifold are not trivially obtained by generalization of the corresponding properties in the linear space setting. Li et al. [19] gave a relation for the subdifferential of a convex distance function in term of a normal cone to the corresponding set in a Riemannian manifold setting with the non-positive sectional curvature. They used a comparison result for geodesic triangles in a Riemannian manifold with the non-positive sectional curvature as a global property. Subdifferentials and normal cones are local notions, and we do not require any global condition on a Riemannian manifold for investigating local properties of a distance function. Here, we establish a lemma about local behavior of a distance function on a manifold (called local distance lemma). Using this lemma, we express the Fréchet subdifferential (contingent directional derivative) of a distance function on a Riemannian manifold in terms of normal cones (contingent cones).

Then, we investigate the weak sharp minima notion for nonconvex optimization problems on Riemannian manifolds. We provide generalization of some characterizations of weak sharp minima of [19] to the nonconvex optimization problems on a Riemannian manifold modeled in a Hilbert space. To the best of our knowledge, our work is the first study concerning the notion of weak sharp minima for nonconvex optimization problems on Riemannian manifolds.

The concept of sharp minima was introduced by Polyak [20] in the case of finite-dimensional Euclidean space for the sensitivity analysis of optimization problems and convergence analysis of some numerical algorithms. Next, Ferris [21] extended the notion of weak sharp minima to the situation where the optimization problem has multiple solutions. The concept was expanded by many authors for convex and nonconvex optimization problems over finite and infinite dimensional linear spaces (see [22–28]).

The primary goal of our work here is to present some primal and dual necessary conditions for the set of weak sharp minima of a nonconvex optimization problem on Riemannian manifold. The key ingredient of deriving the necessary conditions is the representation of a generalized derivative of a distance function in terms of a cone. We will use Fréchet and limiting subdifferentials along with Fréchet and limiting normal cones on Riemannian manifold to state some necessary conditions of dual type for the set of weak sharp minima of a nonconvex optimization problem. To state some necessary conditions of primal type for the set of weak sharp minima of a nonconvex optimization problem, we use a contingent directional derivative and a contingent cone on Riemannian manifold. The contingent directional derivative, which we introduce, is closely related to the Fréchet subdifferential. Several definitions have been provided for the contingent cone on Riemannian manifold; see [29] and [30]. We will show that these definitions, with some modifications, are equivalent.

The remainder of our work is organized as follows. In Section 2, some needed preliminaries on linear spaces and metric spaces are given. Also, we recall some fundamental notions of variational analysis in linear space setting. In Section 3, we first introduce some basic notions of Riemannian manifolds. Then, definitions and properties of the (limiting) Fréchet subdifferential and (limiting) normal cone on Riemannian manifold are provided. Also, we introduce contingent directional derivative on Riemannian manifold and recall some definitions of contingent cones on Riemannian manifold and show that these definitions, with some modifications, are equivalent. In Section 4, we establish a local distance lemma and use it to attain a formula for the Fréchet subdifferential (the directional derivative) of a distance function a Riemannian manifold in terms of the normal cone (the continent cone). In Section 5, we establish some necessary conditions for weak sharp minima in the nonconvex case on Riemannian manifold. In Section 6, we give an application of the results of the previous sections. In Section 7, we provide our concluding remarks.

2 Linear and Metric Spaces

We provide some definitions and symbols required from variational analysis and topology; the reader is referred to [31] for more details. We denote \mathbb{R} as extended real numbers $\mathbb{R} \cup \{\infty\}$. Suppose that E is a Banach space. Then, E^* , \mathbb{B}_E , and I_E respectively denote dual space, closed unit ball, and identity function on E . Denote by $\text{conv } A$ and $\text{cone } A$, respectively, the convex hull of and the cone of a subset $A \subset E$. Let (M, d) be a metric space and $A \subset M$. Recall that $\text{cl } A$ stands for the closure of A , and the distance function for subset A is defined by $\text{dist}(p; A) := \inf_{u \in M} d(p, u)$, for all $p \in M$. The closed ball with center p and radius $r > 0$ is denoted by $\mathbb{B}(p, r) := \{q \in M \mid d(p, q) \leq r\}$. Moreover, if $f : M \rightarrow \mathbb{R}$ is a function on M , then we denote $\text{dom}(f) := \{p \in M : f(p) < \infty\}$ and $\text{ker}(f) := \{p \in M : f(p) = 0\}$. We say that f is proper, if $\text{dom}(f) \neq \emptyset$. Also, we say that f is lower semicontinuous (l.s.c.) at $p \in M$, if

$f(p) \leq \liminf_{u \rightarrow p} f(u)$. We say that f is locally Lipschitz at $p \in M$ with rate r , if there exists a neighbor of p , say U , such that $|f(u) - f(p)| \leq rd(u, p)$, for all $u \in U$. Furthermore, we say that f is a Lipschitzian function with rate r (around $q \in M$), if the earlier inequality holds for all $p, u \in M$ (for all p and u in a neighborhood of q). Let $\Omega \subset M$. The indicator function of Ω is defined as $\delta_\Omega(u) := 0$ for $u \in \Omega$, and $\delta_\Omega(u) := \infty$ for $u \notin \Omega$. Now, we recall definition of weak sharp minima on a metric space.

Definition 2.1 (weak sharp minima) Let $f: M \rightarrow \bar{\mathbb{R}}$ be a proper function on a metric space M and $S \subset M$. We say that

1. $p \in \Omega$ (where $\Omega := \operatorname{argmin}_S f$) is a local weak sharp minimizer for the problem $\min_{u \in S} f(u)$ with modulus $\alpha > 0$, if there is $\epsilon > 0$ such that for all $u \in S \cap \mathbb{B}(p, \epsilon)$, we have

$$f(u) \geq f(p) + \alpha \operatorname{dist}(u; \Omega). \quad (1)$$

2. $\Omega := \operatorname{argmin}_S f$ is the set of (global) weak sharp minima for the problem $\min_{u \in S} f(u)$ with the modulus $\alpha > 0$, if (1) holds for all $p \in \Omega$ and $u \in S$.

In the sequel, we recall definitions of some basic notions of variational analysis in linear space setting. Suppose that $f: E \rightarrow \bar{\mathbb{R}}$ is a function on a Banach space E and is finite at $p \in E$. The Fréchet subdifferential of f at p is defined to be

$$\hat{\partial}f(p) := \left\{ x^* \in E^* \left| \liminf_{u \rightarrow p} \frac{f(u) - f(p) - \langle x^*, u - p \rangle}{\|u - p\|} \geq 0 \right. \right\}.$$

The limiting subdifferential of f at p is the set $\partial f(p)$ of all $x^* \in E^*$ such that there is a sequence of covectors $x_i^* \in \hat{\partial}f(p_i)$ such that $f(p_i) \rightarrow f(p)$, and $\lim x_i^* = x^*$. The contingent directional derivative of $f: E \rightarrow \bar{\mathbb{R}}$ at p in the direction $v \in E$ is defined as follows:

$$f^-(p; v) := \liminf_{w \rightarrow v, t \downarrow 0} \frac{f(p + tw) - f(p)}{t}. \quad (2)$$

Suppose that Ω is a subset of the Banach space E and $p \in \operatorname{cl} \Omega$. The Fréchet normal cone of Ω at p is defined to be

$$\hat{N}_\Omega(p) := \left\{ x^* \in E^* \left| \limsup_{u \xrightarrow{\Omega} p} \frac{\langle x^*, u - p \rangle}{\|u - p\|} \leq 0 \right. \right\}.$$

The limiting normal cone of Ω at p is the set $N_\Omega(p)$ of limits of sequences $\{x_i^*\} \subset E^*$ for which there is a sequence $\{p_i\} \subset \Omega$ converging to p such that $x_i^* \in \hat{N}_\Omega(p_i)$, for all i . The contingent cone of Ω at p is defined as

$$\hat{T}_\Omega(p) := \{v \in E : \exists v_i \in E, t_i \downarrow 0 \text{ such that } v_i \rightarrow v, p + t_i v_i \in \Omega\}.$$

Proposition 2.1 *For a subset Ω of E , with E being a Banach space, its distance function $\text{dist}(\cdot; \Omega)$, and $p \in \text{cl } \Omega$, one has*

$$\hat{\partial} \text{dist}(p; \Omega) = \hat{N}_\Omega(p) \cap \mathbb{B}_{E^*}. \quad (3)$$

The following theorem relates directional derivative of the distance function with contingent cone.

Proposition 2.2 ([17]) *Suppose that E is a Banach space and $p \in \Omega \subset E$. Then, for all $v \in E$, we have*

$$d_\Omega^-(p; v) \leq \text{dist}(v; \hat{T}_\Omega(p)), \quad (4)$$

where $d_\Omega^-(p; v)$ is the contingent directional derivative of $\text{dist}(\cdot; \Omega)$ at p in direction v . If it is further assumed that E is finite dimensional, then equality holds in (4).

3 Variational Analysis on Riemannian Manifolds

Here, we recall some definitions and results about variational analysis on Riemannian manifolds which will be useful later on; see, e.g., [30, 32] for more details. We will be dealing with functions defined on Riemannian manifolds (either finite or infinite-dimensional). A Riemannian manifold (\mathcal{M}, g) is a C^∞ smooth manifold \mathcal{M} modeled on some Hilbert space \mathbb{H} (possibly infinite-dimensional), such that for every $p \in \mathcal{M}$ we are given a scalar product $g(p) = g_p := \langle \cdot, \cdot \rangle_p$ on the tangent space $T_p \mathcal{M} \simeq \mathbb{H}$ so that $\|x\|_p = \langle x, x \rangle_p^{1/2}$ defines an equivalent norm on $T_p \mathcal{M}$ for each $p \in \mathcal{M}$ and in such a way that the mapping $p \in \mathcal{M} \rightarrow g_p \in \mathcal{S}^2(\mathcal{M})$ is a C^∞ section of the bundle of symmetric bilinear forms. For each $p \in \mathcal{M}$, the metric g induces a natural isomorphism between $T_p \mathcal{M}$ and $T_p^* \mathcal{M}$. So, we define the norm on $T_p^* \mathcal{M}$ as $\|v^*\|_p^2 = g_p(v, v)$. If a function $f : \mathcal{M} \rightarrow \mathbb{R}$ is (Fréchet) differentiable at $p \in \mathcal{M}$, then norm of the differential $df(p) \in T_p^* \mathcal{M}$ at the point p is defined by $\|df(p)\|_p := \sup\{df(p)(v) | v \in T_p \mathcal{M}, \|v\|_p \leq 1\}$. Given two points $p, q \in \mathcal{M}$, the Riemannian distance from p to q is denoted by $d_{\mathcal{M}}(p, q)$. Throughout our work here, \mathcal{M} is a Riemannian manifold modeled on a Hilbert space.

Since Fréchet/limiting subdifferential and Fréchet/limiting normal cone are local notions in linear space setting, we can define these notions on Riemannian manifold, by using a local chart. These definitions are independent of the chosen chart; see, e.g., [30, 32, 33] for more details. Here, we define these concepts on Riemannian manifold by using exponential charts. Suppose that $f : \mathcal{M} \rightarrow \bar{\mathbb{R}}$ is a function on a Riemannian manifold \mathcal{M} and is finite at $p \in \mathcal{M}$. The Fréchet subdifferential and the limiting subdifferential of f at p , respectively, are defined to be $\hat{\partial}_{\mathcal{M}} f(p) := \hat{\partial}(f \circ \exp_p)(0)$ and $\partial f(p) := \partial(f \circ \exp_p)(0)$, where $\exp_p : U \rightarrow \mathcal{M}$ is the exponential map of \mathcal{M} defined on U , which is a sufficiently small neighborhood of 0 in $T_p \mathcal{M}$. Similarly, the contingent directional derivative of $f : \mathcal{M} \rightarrow \bar{\mathbb{R}}$ at p in the direction $v \in T_p \mathcal{M}$ is defined

to be $f^-(p; v) := (f \circ \exp_p)^-(0; v)$. Let $\Omega \subset \mathcal{M}$ and $p \in \text{cl } \Omega$. The Fréchet normal cone and the limiting normal cone of Ω at p , respectively, are defined to be $\hat{N}_\Omega^\mathcal{M}(p) := \hat{N}_{\exp_p^{-1} \Omega_*}^\mathcal{M}(0)$ and $N_\Omega^\mathcal{M}(p) := N_{\exp_p^{-1} \Omega_*}^\mathcal{M}(0)$, where Ω_* is the intersection of Ω with an arbitrarily small neighborhood of p on which \exp_p^{-1} is defined. Recall that the Fréchet normal cone of Ω at point p is equal to the Fréchet subdifferential of the indicator function of Ω at p , that is,

$$\hat{N}_\Omega^\mathcal{M}(p) = \hat{\partial}_\mathcal{M} \delta_\Omega(p). \quad (5)$$

Similarly, for the limiting subdifferential and normal cone, we have $N_\Omega^\mathcal{M}(p) = \partial_\mathcal{M} \delta_\Omega(p)$. Suppose that \mathcal{M} is a submanifold of a Euclidean space E , $\Omega \subset \mathcal{M}$ and $p \in \text{cl } \Omega$. Then, from the definition, we have

$$\hat{N}_\Omega^\mathcal{M}(p) = \hat{N}_\Omega^E(p) \cap T_p^* \mathcal{M}. \quad (6)$$

The following proposition states that the Fréchet subdifferential on Riemannian manifold has the homotone property. The proof is directly obtained using the definition.

Proposition 3.1 (homotone property of subdifferential) *Consider functions $f, g: \mathcal{M} \rightarrow \mathbb{R}$ on a Riemannian manifold \mathcal{M} . Suppose that f and g are finite at $p \in \mathcal{M}$, $g \leq f$ and $f(p) = g(p)$. Then, we have $\hat{\partial}_\mathcal{M} g(p) \subset \hat{\partial}_\mathcal{M} f(p)$.*

3.1 Contingent Cone

In the sequel, we define contingent cone on a Riemannian manifold. Similarly, one can define contingent cone on a Riemannian manifold by means of the exponential function and the corresponding definition in linear space setting. But, at first, for a general subset Ω of a Riemannian manifold \mathcal{M} and a point $p \in \text{cl } \Omega$, the notion of contingent cone (the Bouligand tangent) was introduced by Ledyev and Zhu [30, Definition 3.8], as all tangent vectors $v \in T_p \mathcal{M}$ so that

there exist a sequence $t_i \downarrow 0$ and $v_i \in T_p \mathcal{M}$ such that $v_i \rightarrow v$ and $c_{v_i}(t_i) \in \Omega$, (7)

where c_{v_i} is an integral curve on \mathcal{M} with $c_{v_i}(0) = p$ and $c'_{v_i}(0) = v_i$. Li et al. [19, Remark 3.6] mentioned that this definition is incomplete and required some additional restrictions on the sequence $\{c_{v_i}\}$, which were in fact used by Ledyev and Zhu [30, Proposition 3.9]; they are

$$\lim_i c_{v_i}(t_i) = p \text{ and } \lim_i c'_{v_i}(t_i) = v. \quad (8)$$

But, it seems that these additional conditions are still insufficient. We should add more conditions, until the conclusions in [19, Remark 3.6] and [30, Proposition 3.9] hold true. That condition is,

$$\text{uniform convergence of } c'_{v_i} \text{ to } c'_v, \quad (9)$$

in the sense that for every scalar function $g \in C^1(\mathcal{M})$, $d(g \circ c_{v_i})$ uniformly converges to $d(g \circ c_v)$ on a neighborhood of 0. With these additional conditions, we can show that the definition of contingent cone as given in [30, Definition 3.8] reduces to the following definition (see Remark 3.1 below).

Definition 3.1 (contingent cone) Suppose that Ω is a subset of the Riemannian manifold \mathcal{M} and $p \in \text{cl } \Omega$. The contingent cone of Ω at p is defined as

$$\hat{T}_\Omega^\mathcal{M}(p) := \{v \in T_p\mathcal{M} : \exists v_i \in T_p\mathcal{M}, t_i \downarrow 0 \text{ such that } v_i \rightarrow v, \exp_p(t_i v_i) \in \Omega\}.$$

It is worthwhile to mention that Definition 3.1 has been used by Hosseini and Pouryayevali [29].

Remark 3.1 Here, we show that Definition 3.1 is equivalent to [30, Definition 3.8] when we add additional restrictions (8) and (9) on a sequence $\{c_{v_i}\}$ for a vector $v_i \in T_p\mathcal{M}$ therein. Our proof makes use of a technique of [19, Proposition 3.5]. Let $T_\Omega(p)$ be the set of all tangent vectors $v \in T_p\mathcal{M}$ for which the conditions (7), (8) and (9) hold true. We want to show that $\hat{T}_\Omega^\mathcal{M}(p) = T_\Omega(p)$. Obviously, $\hat{T}_\Omega^\mathcal{M}(p) \subset T_\Omega(p)$. We will show the reverse of the inclusion. Let $v \in T_\Omega(p)$. By (7), there exist a sequence $t_i \downarrow 0$ and $v_i \in T_p\mathcal{M}$ such that $v_i \rightarrow v$ and $c_{v_i}(t_i) \in \Omega$. Denote $w_i := \exp_p^{-1} c_{v_i}(t_i)$ for sufficiently small t_i . Then, we show that

$$v = \lim_{t_i \rightarrow 0} \frac{w_i}{t_i}, \quad (10)$$

which clearly shows that $v \in \hat{T}_\Omega^\mathcal{M}(p)$. To establish (10), take $f \in C^1(\mathcal{M})$ and, by its smoothness, obtain $f(c_{v_i}(t_i)) = f(\exp_p w_i) = f(p) + \langle df(p), w_i \rangle + o(\|w_i\|)$, which in turn implies

$$\langle df(p), v \rangle = \lim_{t_i \rightarrow 0+} \frac{f(c_{v_i}(t_i)) - f(p)}{t_i} = \lim_{t_i \rightarrow 0+} \langle df(p), \frac{w_i}{t_i} \rangle + \lim_{t_i \rightarrow 0+} \frac{o(\|w_i\|)}{t_i}. \quad (11)$$

Since c'_{v_i} uniformly converges to c'_v , for a constant $L > 0$, we have

$$\|w_i\| = d(c_{v_i}(t_i), p) \leq \int_0^{t_i} \|c'_{v_i}(t)\| dt \leq Lt_i.$$

We get that $\frac{\|w_i\|}{t_i}$ is bounded as $t_i \rightarrow 0+$. The latter, together with $\lim_i w_i = 0$ and (11), implies that (10) holds, because $f \in C^1(\mathcal{M})$ was chosen arbitrarily. Thus, the proof is complete.

4 Generalized Derivatives of a Distance Function

Generalized differential properties of distance functions play remarkable roles in variational analysis, optimization, and their applications. Generalized derivatives of distance function have fundamental roles in the analysis of optimization algorithms, such as Proximal point methods in both linear space setting

and Riemannian manifold. The authors of [17, 18] investigated properties of generalized derivatives of distance functions in linear space setting. Properties of distance functions on a Riemannian manifold are not trivially obtained by generalization of the corresponding properties in linear spaces setting.

The following statement shows a relation between distance of two points on a Riemannian manifold and distance of image of two points under a chart.

Proposition 4.1 ([34]) *For any point $p \in \mathcal{M}$ and chart (U', ψ) around p there exist a $U \subseteq U'$ and a constant $C \geq 1$ such that for all $p, x \in U$, we have*

$$\frac{1}{C} \|\psi(p) - \psi(x)\| \leq d(p, x) \leq C \|\psi(p) - \psi(x)\|. \quad (12)$$

Now, Suppose that $\Omega \subset \mathcal{M}$ and $p \in \text{cl } \Omega$. For every $r > 0$, denote $\Omega_r := \Omega \cap \mathbb{B}(p, r)$. In Proposition 4.1, by setting $h = \exp_p^{-1}$ and taking supremum over $x \in \Omega_r$, one gets¹

$$\frac{1}{C} \text{dist}(\exp_p^{-1}(u); \exp_p^{-1}(\Omega_r)) \leq d(u; \Omega_r) \leq C \text{dist}(\exp_p^{-1}(u); \exp_p^{-1}(\Omega_r)), \quad (13)$$

which is an local estimate of a distance function on a Riemannian manifold in terms of normal coordinates. By these estimates, one can obtain some properties of a distance function on a Riemannian manifold from corresponding results in linear space setting. For example, by using homotone property of the Fréchet subdifferential and a well-known result in linear space setting (Proposition 2.1), we get

$$\frac{1}{C} \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}} \leq \hat{\partial}_{\mathcal{M}} \text{dist}(p; \Omega) \leq C \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}}. \quad (14)$$

Li et al. [19] established (14) with $C = 1$ for a convex subset of a Riemannian manifold with the non-positive sectional curvature. In their proof, there were two fundamental points: first, since the authors used the convex calculus on manifolds, it was necessary to assume that the manifolds were Hadamard manifolds and Ω was a geodesic convex set so that the distance function $\text{dist}(\cdot; \Omega)$ be a convex function; Second, they used a comparison result for geodesic triangles in Hadamard manifold, which was a global property. Since we are to use the nonsmooth calculus on manifolds, we do not need the distance function to be convex. On the other hand, since the Fréchet subdifferential and normal cone are local notions, we can prove (14) with $C = 1$ for an arbitrary subset of an arbitrary Riemannian manifold with a better local estimate of a distance function than (13).

Lemma 4.1 (local distance lemma) *Let Ω be a subset of \mathcal{M} and $p \in \text{cl } \Omega$. For sufficiently small $r > 0$, denote $\Omega_r := \Omega \cap \mathbb{B}(p, r)$. For each $u \in \mathbb{B}(p, r)$, we have*

$$\frac{\text{dist}(\exp_p^{-1} u; \exp_p^{-1} \Omega_r)}{1 + |o(r)|} \leq \text{dist}(u; \Omega_r) \leq (1 + |o(r)|) \text{dist}(\exp_p^{-1} u; \exp_p^{-1} \Omega_r).$$

¹ See: <https://mathoverflow.net/q/301064>.

Proof Let r be sufficiently small such that \exp_p^{-1} is defined on $\mathbb{B}(p, r)$. It is enough to show that for every $u_1, u_2 \in \mathbb{B}(p, r)$, we have

$$(1 - |o(r)|)d(v_1, v_2) \leq d(u_1, u_2) \leq (1 + |o(r)|)d(v_1, v_2), \quad (15)$$

where $v_1 := \exp_p^{-1} u_1$ and $v_2 = \exp_p^{-1} u_2$. Suppose that $\gamma(t) = v_1 + t(v_2 - v_1)$, $0 \leq t \leq 1$, is the straight line segment joining v_1 and v_2 . Denote $c(t) := \exp_p \gamma(t)$, $0 \leq t \leq 1$. Let $f = \exp_p^*(g)$ be the pull back of the metric g of \mathcal{M} in $\exp_p^{-1} \mathbb{B}(p, r)$. We have $\dot{c}(t) = (D \exp_p)_{\gamma(t)}(\dot{\gamma}(t))$ and

$$d(u_1, u_2) \leq \int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))} dt = \int_0^1 \sqrt{f_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

According to [35, Theorem 5.5], we have the following Taylor's expansion of f_v at 0: $f_v = g_p + q_v + h_v$, for $|v| \rightarrow 0$, where $q_v(w_1, w_2) := \frac{1}{3} R_p(v, w_1, v, w_2)$ is a symmetric bilinear function obtained from the Riemann curvature R_p of \mathcal{M} and h_v is a bilinear function on $T_v T_p \mathcal{M}$ whose norm is of order $O(|v|^3)$. So, we have

$$\begin{aligned} d(u_1, u_2) &\leq \int_0^1 \sqrt{g_p(\dot{\gamma}, \dot{\gamma}) + q_{\gamma(t)}(\dot{\gamma}, \dot{\gamma}) + h_{\gamma(t)}(\dot{\gamma}, \dot{\gamma})} dt \\ &= \int_0^1 \sqrt{1 + \frac{1}{3} \left(\frac{R_p(\gamma, \dot{\gamma}, \gamma, \dot{\gamma})}{g_p(\dot{\gamma}, \dot{\gamma}) g_p(\gamma, \gamma)} \right) g_p(\gamma, \gamma) + \frac{h_{\gamma(t)}(\dot{\gamma}, \dot{\gamma})}{g_p(\dot{\gamma}, \dot{\gamma})} g_p(\gamma, \gamma)} dt \\ &\leq \left(1 + \frac{\|R_p\|}{6} r^2 + O(r^3)\right) d(v_1, v_2), \end{aligned}$$

where $\|R_p\|$ is the supremum of $R_p(w_1, w_2, w_3, w_4)$ over all $w_1, w_2, w_3, w_4 \in T_p \mathcal{M}$ with norms equal to 1. Note that since R_p is a continuous multilinear function, $\|R_p\|$ is finite. So, the second inequality of (15) is established. To establish the first inequality of (15), we consider an arbitrary curve c joining u_1 and u_2 in $\mathbb{B}(p, r)$. Then, using a similar technique as above, one can show that the length of c is at least $(1 - \frac{\|R_p\|}{6} r^2 + O(r^3))d(v_1, v_2)$. So, by taking the infimum, the desired inequality is at hand. Therefore, the proof is complete. \square

Remark 4.1 The proof of Lemma 4.1 gives a tighter estimate of $o(r)$. Indeed, we have $\text{dist}(u; \Omega_r) = (1 \pm \frac{\|R_p\|}{6} r^2 + O(r^3)) \text{dist}(\exp_p^{-1} u; \exp_p^{-1} \Omega_r)$.

Now, we can prove (14) with $C = 1$ in a general setting, which plays an essential role for stating some necessary conditions of dual type for the set of weak sharp minima of a nonconvex optimization problem in Section 5.

Note that results of this type, relating subdifferential of the distance function and normals to the corresponding set, are known for various subdifferentials in general nonconvex settings of Banach spaces and are of great importance for many aspects of variational analysis; e.g., see [31, 36].

Theorem 4.1 *With Ω a subset of \mathcal{M} , for the distance function $\text{dist}(\cdot; \Omega)$, and $p \in \text{cl } \Omega$, one has*

$$\hat{\partial}_{\mathcal{M}} \text{dist}(p; \Omega) = \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}}. \quad (16)$$

Proof By local distance lemma (Lemma 4.1), for sufficiently small values of r , we have $\text{dist}(u; \Omega) = \text{dist}(u; \Omega_r) \leq (1 + |o(r)|) \text{dist}(\exp_p^{-1} u; \exp_p^{-1} \Omega_r)$, for each $u \in \mathbb{B}(0, r/2)$. Now, the homotone property of subdifferential and the similar property in linear space setting (Proposition 2.1) imply that

$$\begin{aligned} \hat{\partial}_{\mathcal{M}} \text{dist}(p; \Omega) &\leq (1 + |o(r)|) \hat{\partial}_{\mathcal{M}} \text{dist}(0; \exp_p^{-1} \Omega_r) \\ &= (1 + |o(r)|) \hat{N}_{\exp_p^{-1} \Omega_r}^{\mathcal{M}}(0) \cap \mathbb{B}_{T_p^* \mathcal{M}} \\ &= (1 + |o(r)|) \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}}. \end{aligned}$$

By $r \rightarrow 0$, we have $\hat{\partial}_{\mathcal{M}} \text{dist}(p; \Omega) \leq \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}}$. The reverse of the inequality can be established similarly, to complete the proof. \square

Immediately, we have the following corollary from Theorem 4.1.

Corollary 4.1 *With the notation of Theorem 4.1, we have*

$$\hat{N}_{\Omega}^{\mathcal{M}}(p) = \text{cone } \hat{\partial}_{\mathcal{M}} \text{dist}(p; \Omega).$$

The following theorem relates directional derivative of the distance function with contingent cone. It immediately follows from a corresponding property in linear space setting by using the local distance lemma. We use this property to obtain a necessary condition for weak sharp minima. This proposition follows immediately from a similar result in linear spaces [17, Theorem 4] by using the mentioned local distance lemma above.

Theorem 4.2 *Suppose that \mathcal{M} is a Riemannian manifold and $p \in \Omega \subset \mathcal{M}$. Then, for all $v \in T_p \mathcal{M}$, we have*

$$d_{\Omega}^{-}(p; v) \leq \text{dist}(v; \hat{T}_{\Omega}^{\mathcal{M}}(p)), \quad (17)$$

where $d_{\Omega}^{-}(p; v)$ is the directional derivative of $\text{dist}(\cdot; \Omega)$ at p in direction v . If it is further assumed that \mathcal{M} is finite dimensional, then equality holds in (17).

Proof We have

$$\frac{\text{dist}(\exp_p tw; \Omega) - \text{dist}(p; \Omega)}{t} \leq (1 + |o(r)|) \frac{\text{dist}(tw; \exp_p^{-1} \Omega_r) - 0}{t}.$$

By taking \liminf as $t \downarrow 0$ and $w \rightarrow v$, and by using a similar property in linear space setting (Proposition 2.2), we have $d_{\Omega}^{-}(p; v) \leq \text{dist}(v; \hat{T}_{\exp_p^{-1} \Omega_r}^{\mathcal{M}}(0)) = \text{dist}(v; \hat{T}_{\Omega}^{\mathcal{M}}(p))$. Similarly, one can prove that if it is further assumed that \mathcal{M} is finite dimensional, then equality holds in (17). \square

The following corollary is now at hand.

Corollary 4.2 *With the notation of Theorem 4.2, if $p \in \Omega \subset \mathcal{M}$, then*

$$\hat{T}_\Omega^\mathcal{M}(p) \subset \{v : d_\Omega^-(p; v) \leq 0\},$$

and equality holds when \mathcal{M} is finite dimensional.

Remark 4.2 To prove Theorem 4.1 and Theorem 4.2, we do not need the underlying Riemannian manifold to be a Hadamard manifold. But, for the development of some optimization techniques on Riemannian manifolds, such as variational inequalities [37] and equilibrium problems [38], properties of Hadamard manifolds are indeed essential; see Kristály [38, Remark 5.1].

5 Nonconvex Weak Sharp Minima on a Riemannian Manifold

Here, we give some necessary conditions of the primal type and the dual type for the set of weak sharp minima of an optimization problem on a (possibly infinite dimensional) Riemannian manifold. Formerly, Li et al. [19] provided some characterizations of weak sharp minima in the case of convex problems on finite dimensional Riemannian manifold.

Note that $p \in \Omega$ (where $\Omega := \operatorname{argmin}_S f$) being a local weak sharp minimizer for the problem $\min_{u \in S} f(u)$ with modulus $\alpha > 0$, is equivalent to $p \in \Omega$ being a local minimizer of the following perturbed problem:

$$\min_{u \in S} (f(u) - \alpha \operatorname{dist}(u; \Omega)). \quad (18)$$

Similarly, Ω being the set of weak sharp minima for the problem $\min_{u \in S} f(u)$ with the modulus $\alpha > 0$, is equivalent to $p \in \Omega$ being a global minimizer of the perturbed problem (18). So, the set of weak sharp minima of an optimization problem is equivalent to the set of minimizers of a difference optimization problem.

Remark 5.1 The properties of weak sharp minima on Riemannian manifold could not trivially be advocated as the properties of weak sharp minima on a linear space setting. Indeed, if we substitute u by $\exp_p w$ in (18), the objective function is converted to $(f \circ \exp_p)(w) - \alpha \operatorname{dist}(\exp_p w; \Omega)$. But, the local minimum of the converted problem is not trivially related to the weak sharp minima of an optimization problem on a linear space.

The homotone property of Fréchet subdifferential (Proposition 3.1) admits a necessary optimality condition for a local minimum of a function on a Riemannian manifold.

Proposition 5.1 *Let $f: \mathcal{M} \rightarrow \bar{\mathbb{R}}$ be a function on a Riemannian manifold \mathcal{M} . Suppose that the value of f is finite at $p \in \mathcal{M}$. If p is a local minimizer of f , then $0 \in \hat{\partial}_\mathcal{M} f(p)$.*

Next, we state a simple rule about the Fréchet subdifferential of sum of two functions, which is directly deduced from the definition.

Proposition 5.2 *Consider functions $f_1, f_2: \mathcal{M} \rightarrow \mathbb{R}$ on a Riemannian manifold \mathcal{M} . Suppose $p \in \mathcal{M}$ and $f_1(p), f_2(p) < \infty$. Then, we have*

$$\hat{\partial}_{\mathcal{M}} f_1(p) + \hat{\partial}_{\mathcal{M}} f_2(p) \subset \hat{\partial}_{\mathcal{M}}(f_1 + f_2)(p). \quad (19)$$

Moreover, if one of the f_i is Fréchet differentiable, then we have equality in (19).

Now, we state some necessary conditions of the primal type and the dual type for a local weak sharp minimizer of an unconstrained optimization problem.

Theorem 5.1 (necessary conditions for a local weak sharp minimizer of an unconstrained problem on a Riemannian manifold) *Let Ω be the solution set for problem (14). Suppose that $S := \mathcal{M}$ and $p \in \Omega := \operatorname{argmin}_S f$ is a local weak sharp minimizer for the problem $\min_{u \in S} f(u)$ with modulus $\alpha > 0$. Then, the followings holds.*

(i) *We have*

$$\alpha \mathbb{B}_{T_p^* \mathcal{M}} \cap \hat{N}_{\Omega}^{\mathcal{M}}(p) \subset \hat{\partial}_{\mathcal{M}} f(p). \quad (20)$$

(ii) *For all $v \in T_p \mathcal{M}$, we have*

$$f^-(p; v) \geq \alpha \operatorname{dist}(v; \hat{T}_{\Omega}^{\mathcal{M}}(p)). \quad (21)$$

Proof By definition, there exists $\epsilon > 0$ such that

$$f(u) \geq f(p) + \alpha \operatorname{dist}(u; \Omega), \quad \forall u \in \mathbb{B}(p, \epsilon).$$

Since the Fréchet subdifferential has homotone property (Proposition 3.1), we have $\hat{\partial}_{\mathcal{M}} \alpha \operatorname{dist}(p; \Omega) \subset \hat{\partial}_{\mathcal{M}} f(p)$. Theorem 4.1 implies $\alpha \hat{N}_{\Omega}^{\mathcal{M}}(p) \cap \mathbb{B}_{T_p^* \mathcal{M}} \subset \hat{\partial}_{\mathcal{M}} f(p)$. So, (i) is proved. Next, we prove (ii). Let $v \in T_p \mathcal{M}$. The hypothesis guarantees that for all w sufficiently close to v and for all sufficiently small $t > 0$, we have $f(\exp_p tw) - f(p) \geq \alpha \operatorname{dist}(\exp_p tw; \Omega)$, which implies

$$\frac{f(\exp_p tw) - f(p)}{t} \geq \alpha \frac{\operatorname{dist}(\exp_p tw; \Omega) - \operatorname{dist}(p; \Omega)}{t}.$$

By taking \liminf of both sides of the latter inequality, as $w \rightarrow v$ and $t \downarrow 0$, and applying Theorem 4.2, (ii) is obtained. \square

Since every element of the set of global weak sharp minima is a local weak sharp minimizer, we immediately have the following corollary.

Corollary 5.1 (necessary conditions for unconstrained weak sharp minima on a Riemannian manifold) *Suppose that $S := \mathcal{M}$ and $\Omega := \operatorname{argmin}_S f$ is the set of weak sharp minima for the problem $\min_S f$ with modulus $\alpha > 0$. Then,*

1. *for every $p \in \Omega$, we have the inclusion (20).*
2. *for every $p \in \Omega$ and $v \in T_p \mathcal{M}$, we have the inequality (21).*

Remark 5.2 Similar to Ward [27], with some modifications of the definition of contingent directional derivative on Riemannian manifold, one can state some necessary conditions for weak sharp minima of higher orders.

Remark 5.3 Similar to the approach of Studniarski and Ward [26] in linear spaces, one can state some sufficient conditions for weak sharp minima on Riemannian manifold based on a generalization of the contingent directional derivative.

5.1 Constrained Weak Sharp Minima on Riemannian Manifolds

In the sequel, we give some necessary conditions for the weak sharp minima of a constrained optimization problem on a Riemannian manifold. As said at the beginning of this section, the set of weak sharp minima is equivalent to the set of minimizers of a difference optimization problem (18). So, at first, we state some necessary conditions for minimizers of a difference optimization problem, i.e., an optimization problem whose cost function is given in a difference form. To present these necessary conditions, we use the Fréchet and limiting subdifferentials on a Riemannian manifold. Consider the difference optimization problem with the geometric constraint:

$$\min_{u \in S} f(u), \quad (22)$$

where $f: \mathcal{M} \rightarrow \bar{\mathbb{R}}$ may be represented as $f = f_1 - f_2$. With the help of indicator function on the set S , constrained optimization problem (22) can be rewritten in an unconstrained form:

$$\min_{u \in \mathcal{M}} f(u) + \delta_S(u). \quad (23)$$

To present some necessary optimality conditions for the unconstrained optimization problem, we need to decompose the subdifferential of the objective function of problem (23). We say that f is Fréchet decomposable at $p \in S$ on S , when

$$(f + \delta_S)(p) \subset \hat{\partial}_{\mathcal{M}} f(p) + \hat{N}_S^{\mathcal{M}}(p). \quad (24)$$

Note that this definition is a Riemannian manifold counterpart of the one used by Mordukhovich et al. [24] in the linear space setting. If f is Fréchet differentiable at p , then Proposition 5.2 implies that f is Fréchet decomposable at $p \in S$ on S . Moreover, by [19, Proposition 4.3], for a convex function f with $\text{dom } f$ having nonempty interior and a nonempty convex set S such that $S \cap \text{dom } f$ is convex, we have that f is Fréchet decomposable on S at every point in $\text{int}(\text{dom } f) \cap S$. According to the attractive form of the calculus for the limiting subdifferential of a Lipschitzian function, decompose condition (24) for the limiting subdifferential is at hand. In the first assertion of the following theorem, we impose the decomposition assumption for the Fréchet subdifferential on a Riemannian manifold, while the second part is justified without

this assumption via the limiting subdifferential on a Riemannian manifold and its attractive forms in the Lipschitzian case, i.e., if $f_1: \mathcal{M} \rightarrow \mathbb{R}$ is Lipschitzian around $p \in \mathcal{M}$ and $f_2: \mathcal{M} \rightarrow \mathbb{R}$ is an l.s.c. function and is finite at p , then

$$\partial_{\mathcal{M}}(f_1 + f_2)(p) \subset \partial_{\mathcal{M}}f_1(p) + \partial_{\mathcal{M}}f_2(p). \quad (25)$$

Theorem 5.2 (necessary conditions for difference problems with geometric constraints) *Suppose that p is a local solution of (22), and f is represented as $f = f_1 - f_2$, where $f_i: \mathcal{M} \rightarrow \mathbb{R}$ is finite at p . Then,*

(i) *If f_1 is Fréchet decomposable at $p \in S$ on S , then we have*

$$\hat{\partial}_{\mathcal{M}}f_2(p) \subset \hat{\partial}_{\mathcal{M}}f_1(p) + \hat{N}_S^{\mathcal{M}}(p). \quad (26)$$

Particularly, when $f \equiv -f_2$, we have $\hat{\partial}_{\mathcal{M}}f_2(p) \subset \hat{N}_S^{\mathcal{M}}(p)$.

(ii) *If f_1 is Lipschitzian around p and S is closed, then we have*

$$\hat{\partial}_{\mathcal{M}}f_2(p) \subset \partial_{\mathcal{M}}f_1(p) + N_S^{\mathcal{M}}(p).$$

Proof Problem (22) can be written as the following unconstrained difference problem: $\min_{x \in \mathcal{M}} [f_1(x) + \delta_S(x) - f_2(x)]$. By Corollary 5.1, we have $0 \in \hat{\partial}_{\mathcal{M}}(f_1(x) + \delta_S(x) - f_2(x))(p)$. Proposition 5.2 implies

$$\hat{\partial}_{\mathcal{M}}f_2(p) \subset \hat{\partial}_{\mathcal{M}}(f_1 + \delta_S)(p). \quad (27)$$

If f_2 is Fréchet decomposable at $p \in S$ on S , then we have $\hat{\partial}_{\mathcal{M}}f_2(p) \subset \hat{\partial}_{\mathcal{M}}f_1(p) + \hat{N}_S^{\mathcal{M}}(p)$. So, (i) is proved. Now, we prove part (ii). Since S is assumed to be closed, δ_S is l.s.c and the inclusion (27) with the sum rule for the limiting subdifferential (25) imply

$$\begin{aligned} \hat{\partial}_{\mathcal{M}}f_2(p) &\subset \hat{\partial}_{\mathcal{M}}(f_1 + \delta_S)(p) \subset \partial_{\mathcal{M}}(f_1 + \delta_S)(p) \\ &\subset \partial_{\mathcal{M}}f_1(p) + \partial_{\mathcal{M}}\delta_S(p) = \partial_{\mathcal{M}}f_1(p) + N_S^{\mathcal{M}}(p), \end{aligned}$$

and the proof of (ii) is complete. \square

In the following, as a corollary of Theorem 5.2, we can state some necessary conditions for the set of weak sharp minima of a constrained optimization problem. The result can be established using Theorem 5.2 by reducing weak sharp minima to minimizers of a constrained difference optimization problem.

Corollary 5.2 (necessary conditions for weak sharp minima under geometric constraints on Riemannian manifolds) *Suppose $\Omega := \operatorname{argmin}_S f$ is the set of weak sharp minima for the problem $\min_S f$ with modulus $\alpha > 0$. Then, we have*

1. *If $p \in \Omega$ and f is Fréchet decomposable on S at the point p , then*

$$\alpha \mathbb{B}_{T_p^* \mathcal{M}} \cap \hat{N}_\Omega^{\mathcal{M}}(p) \subset \hat{\partial}_{\mathcal{M}}f(p) + \hat{N}_S^{\mathcal{M}}(p).$$

2. *If f is Lipschitzian around $p \in S$ and S is closed, then we have*

$$\alpha \mathbb{B}_{T_p^* \mathcal{M}} \cap \hat{N}_\Omega^{\mathcal{M}}(p) \subset \partial_{\mathcal{M}}f(p) + N_S^{\mathcal{M}}(p).$$

6 Application

Here, we give an application of the nonconvex weak sharp minima on Riemannian manifolds. We will show how the notion of weak sharp minima and the results of the previous sections can be used to model a discrete optimization problem as an unconstrained optimization problem on a Stiefel manifold. Recall that for integers $0 < k \leq n$, the Stiefel manifold $St(n, k)$ is defined as the set of all matrices $U \in \mathbb{R}^{n \times k}$, with $U^T U = I_k$, where U^T denotes the transpose of U and I_k denotes the $k \times k$ identity matrix. For each $P \in St(n, k)$, we have

$$T_P St(n, k) = \{X \in \mathbb{R}^{n \times k} \mid X^T P + P^T X = 0\},$$

and we endow $T_P St(n, k)$ with the induced metric of the Euclidean space $\mathbb{R}^{n \times k}$: $\langle X, Y \rangle := \text{tr}(X^T Y)$, for $X, Y \in \mathbb{R}^{n \times k}$, also we identify $T_P^* St(n, k)$ with $T_P St(n, k)$, by the natural isomorphism induced by this Riemannian metric. For more details on the properties of the Stiefel manifold, see [3]. For each matrix $A = [a_{ij}]$, denote $\|A\|_\beta := (\sum_{i,j} |a_{ij}|^\beta)^{1/\beta}$ and $A^- := [\max\{-a_{ij}, 0\}]$.

Example 6.1 Let G be a graph with vertices $V(G) := \{v_1, \dots, v_n\}$ and Edges $E(G)$. Fix integer $k > 0$. Denote $D_k(G) := \{(A_1, \dots, A_k) : \emptyset \neq A_i \subset V(G), A_i \cap A_j = \emptyset, \text{ for all } i \neq j\}$ as the set of all k sub-partitions of $V(G)$. Define ∂A , for a subset A of $V(G)$, as the set of all edges of G whose only one endpoint belongs to A . Now, consider the following Cheeger type constant of G :

$$\gamma_k(G) := \min \sum_{i=1}^k \frac{|\partial A_i|}{\sqrt{|A_i|}}, \quad (28)$$

where the minimum runs over all $(A_1, \dots, A_k) \in D_k(G)$. The constant $\gamma_k(G)$ indicates how well G can be partitioned into k clusters. For more information on the Cheeger constant of graphs and its applications to clustering, see [39–41]. For each $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$, define $\|\nabla \alpha\|_1 := \sum_{\{v_i, v_j\} \in E(G)} |\alpha_i - \alpha_j|$. By a technique similar to that of Rothaus [42], one can show² that the discrete minimization problem (28), is equivalent to the following continuous optimization problem on a manifold with non-negativity constraints:

$$\min \sum_{i=1}^k \|\nabla u_i\|_1, \quad (29)$$

where the minimum runs over all $U := (u_1, \dots, u_k) \in St(n, k)$ with non-negative entries (which hereafter is denoted by $St_+(n, k)$). The non-negativity constraints provide a combinatorial nature to the feasible set of the problem (29). But, since the cost function of (29) is Lipschitzian on $St(n, k)$ with a rate C , the penalization lemma (see [31, Proposition 1.121]) implies that the

² The key point is to show that for each vector $u := (u_1, \dots, u_n) \in \mathbb{R}^n$ with $\|u\|_2 = 1$, there is a non-empty subset $A \subset \{i \mid u_i \neq 0\}$ such that $\frac{|\partial A|}{\sqrt{|A|}} \leq \|\nabla u\|_1$.

problem (29) is equivalent to the following unconstrained optimization problem on a Stiefel manifold:

$$\min_{U \in St(n,k)} \sum_{i=1}^k \|\nabla u_i\|_1 + C \operatorname{dist}(U; St_+(n, k)). \quad (30)$$

Although computing the distance term in (30) is not a simple task, a simple observation shows that one can replace the distance term with any upper estimate $h(U)$ which is zero if and only if the distance term is zero, or equivalently, with a function $h(U)$ such that 0 is the minimum value of h , the set of minimizers of h is $St_+(n, k)$, and for every $U \in St(n, k)$,

$$\operatorname{dist}(U; St_+(n, k)) \leq h(U). \quad (31)$$

In other words, $St_+(n, k)$ should be the set of weak sharp minima of the problem $\min_{St(n,k)} h(U)$ with 0 as the optimal value. Since the distance of a matrix $U \in St(n, k)$ from the non-negative matrices in the Euclidean space $\mathbb{R}^{n \times k}$ is a function of the negative part of U , it seems that a natural candidate for $h(U)$ is a function of U^- . So, we investigate the following question:

Is $St_+(n, k)$ the set of weak sharp minima for the problem of minimizing $f_\beta(U) := \|U^-\|_\beta^\beta$ over the Stiefel manifold $St(n, k)$, for some $\beta > 0$?

According to Corollary 5.1, a necessary condition for provision of an affirmative response to this question is that there exists $\alpha > 0$ such that, for every $P \in St_+(n, k)$,

$$\alpha \mathbb{B}_{T_P St(n,k)} \cap \hat{N}_{St_+(n,k)}^{St(n,k)}(P) \subset \hat{\partial}_{St(n,k)} f_\beta(P). \quad (32)$$

Fix $P := [p_{ij}] \in St_+(n, k)$ and, for simplicity, suppose that only the first t rows of P are non-zero (for some $t \geq 0$). For each $n \times k$ matrix A , Denote \tilde{A} by the matrix whose rows are the first t rows of A and \hat{A} by the matrix whose rows are the last $n - t$ rows of A . According to (6), we have $X \in \hat{N}_{St_+(n,k)}^{St(n,k)}(P)$, if and only if $X \in T_P St(n, k)$ and $\limsup \langle X, U - P \rangle / \|U - P\| \leq 0$, where the \limsup runs over all U approaching P , in $St_+(n, k)$. By approaching P , from some suitable curves in $St_+(n, k)$, one can see

$$\hat{N}_{St_+(n,k)}^{St(n,k)}(P) = \{X \in T_P St(n, k) \mid \hat{X} \leq 0, \tilde{X} \circ \tilde{P} = O_{t \times k}\}, \quad (33)$$

where \circ is the entry-wise product, $O_{t \times k}$ is the $t \times k$ zero matrix, and $\hat{X} \leq 0$ means that \hat{X} is a non-positive matrix. Next, we will investigate when the necessary condition (32) holds. First, note that if $\beta > 1$, then f_β is smooth. So, the subdifferential $\hat{\partial}_{St(n,k)} f_\beta(P)$ has only one element. But, there are some $P \in St_+(n, k)$ such that the left hand of (32) have infinitely many elements. So, for $\beta > 1$, the necessary condition (32) does not hold.

Next, we will show that the necessary condition (32) holds, for each $\beta < 1$, with $\alpha = 1$. Consider an arbitrary element $X \in \mathbb{B}_{T_P St(n,k)} \cap \hat{N}_{St_+(n,k)}^{St(n,k)}(P)$. We will show $X \in \hat{\partial}_{St(n,k)} f_\beta(P)$. Define the smooth function $\bar{g}(U) = \langle X, \exp_P^{-1} U \rangle$

on a sufficiently small neighborhood of P . For simplicity, denote $dU = \exp_P^{-1} U$. Note that since $X \in \mathbb{B}_{T_P St(n,k)}$, we have $\text{tr}(X^T X) \leq 1$. So, the absolute value of the entries of \widehat{X} are at most equal to 1. Moreover, $\widehat{X} \leq 0$. Thus, $\langle \widehat{X}, \widehat{dU} \rangle \leq \|\widehat{dU}^-\|_1$. Therefore, by the Cauchy-Schwarz inequality, we have

$$\bar{g}(U) = \langle \widetilde{X}, \widetilde{dU} \circ \widetilde{I_P} \rangle + \langle \widehat{X}, \widehat{dU} \rangle \leq \|\widetilde{dU} \circ \widetilde{I_P}\|_2 + \|\widehat{dU}^-\|_1, \quad (34)$$

where I_P is the $n \times k$ matrix whose (i, j) th entry is 1 if $p_{ij} = 0$ and is 0 otherwise. One can see that the function $\|\cdot\|_1$ is a norm on $S := \{V \in \mathbb{R}^{t \times k} | \widetilde{P}^T V + V^T \widetilde{P} = 0\}$. Indeed, the function $\|\cdot\|_1$ satisfies the triangle inequality, since for each real a, b we have $(a+b)^- \leq a^- + b^-$. So, it is enough to show that $V = 0$ whenever $V \in S$ and $\|V^-\|_1 = 0$. This follows from the non-negativity of the entries of P and that P has no zero row. Now, since $dU \in T_P St(n, k)$, we have $\widetilde{dU} \in S$ and $\widetilde{dU} \circ \widetilde{I_P} \in S$. By the equivalence of norms on a finite dimensional linear space, we have $\|\widetilde{dU} \circ \widetilde{I_P}\|_2 \leq C_P \|\widetilde{dU} \circ \widetilde{I_P}^-\|_1$, for some $C_P > 1$. Note that $\widehat{dU} = \widetilde{dU} \circ \widehat{I_P}$. So, from (34), we have

$$\bar{g}(U) \leq C_P (\|\widetilde{dU} \circ \widetilde{I_P}^-\|_1 + \|\widehat{dU} \circ \widehat{I_P}^-\|_1) = C_P \|(dU \circ I_P)^-\|_1. \quad (35)$$

Now, consider the function $g(U) := \bar{g}(U) + C_P \|(U \circ I_P)^-\|_1 - C_P \|(dU \circ I_P)^-\|_1$. Since $\|(U \circ I_P)^-\|_1 - \|(dU \circ I_P)^-\|_1$ is of order $o(\|U - P\|)$ (note that $dU = (U - P) + o(\|U - P\|)$), the function $\|\cdot\|_1$ satisfies the triangle inequality, and $P \circ I_P = O_{n \times k}$, we have $dg(P) = d\bar{g}(P) = X$. Therefore, as (35), we have

$$\begin{aligned} g(U) &\leq C_P \|(U \circ I_P)^-\|_1 \leq C_P \|U^-\|_1 = \|U^-\|_1^\beta (C_P \|U^-\|_1^{1-\beta}) \\ &\leq \|U^-\|_1^\beta \leq \|U^-\|_\beta^\beta, \end{aligned} \quad (36)$$

on a sufficiently small neighborhood of P , in which $C_P \|U^-\|_1^{1-\beta} \leq 1$ (the last inequality of (36) follows from the fact that for each $a, b \geq 0$ and $0 < \beta < 1$, we have $(a+b)^\beta \leq a^\beta + b^\beta$). Thus, $g \leq f_\beta$ on a neighborhood of P . Since $g(P) = f_\beta(P)$ and $dg(P) = X$, as the homotone property of the Fréchet subdifferential, we have $X \in \hat{\partial}_{St(n,k)} f_\beta(P)$. \square

7 Concluding Remarks

We presented a lemma and used it to derive some local properties of a distance function on a Riemannian manifold using the corresponding properties in linear space. In this regard, we established a relation between the Fréchet subdifferential (directional derivative) of a distance function and a normal cone (contingent cone) on a Riemannian manifold. Then, we established some primal and dual necessary conditions for the set of weak sharp minima of non-convex optimization problems on Riemannian manifold. As an application, we showed how the notion of weak sharp minima and our stated results can be used to model a Cheeger type constant of a graph as an unconstrained optimization problem on a Stiefel manifold.

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