# Representation of weak solutions of convex 

## Hamilton-Jacobi-Bellman equations on infinite horizon

## Vincenzo Basco


#### Abstract

In the present paper it is provided a representation result for the weak solutions of a class of evolutionary Hamilton-Jacobi-Bellman equations on infinite horizon, with Hamiltonians measurable in time and fiber convex. Such Hamiltonians are associated to a - faithful - representation namely involving two functions measurable in time and locally Lipschitz in the state and control. Our results concern to recover a representation of convex Hamiltonians under a relaxed assumption on the Fenchel transform of the Hamiltonian with respect the fiber. We apply them to investigate uniqueness of weak solutions vanishing at infinity of a class of time dependent Hamilton-Jacobi-Bellman equations, regarded as an appropriate value function of an infinite horizon


[^0]control problem under state constraints, assuming a viability condition on the domain of the aforementioned Fenchel transform.

Keywords Hamilton-Jacobi-Bellman equations • Weak solutions • Infinite horizon • State constraints • Representation of Hamiltonians. Mathematics Subject Classification (2000) 70H20 • 49L25 • 49J24 . 35E10.

## 1 Introduction

In this paper we address the Hamilton-Jacobi-Bellman (HJB) equation on infinite horizon

$$
\begin{equation*}
-V_{t}+H\left(t, x,-D_{x} V\right)=0 \quad \text { on }(0,+\infty) \times \mathscr{O}, \tag{1}
\end{equation*}
$$

where $H: \mathbb{R}^{+} \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ is the Hamiltonian, $\mathscr{O} \subset \mathbb{R}^{n}$ is an open subset, and $D_{x}$ stands for the gradient with respect to the space variable. The notion of weak - or viscosity - solution to a first-order partial differential equation to study stationary and evolutionary HJB equations is due to Crandall, Evans, and Lions [7.16. Using superdifferentials and subdifferentials of continuous functions, they obtained existence and uniqueness results in the class of continuous functions for Cauchy problems associated to HJB equations, when the Hamiltonian is continuous, by means of concept of sub/super-solution. Barles and Souganidis [3,22 extended the existence results to a large class of continuous Hamiltonians. As matter of fact, such notion of solution turns out to be quite unsatisfactory for HJB equations arising in control theory and the
calculus of variations - cfr. [23] for further discussions. In fact, the value function, that is a viscosity solution of HJB equation, loses the differentiability property - even in the absence of state constraints - whenever multiple optimal solutions are present at the same initial datum or when additional state constraints arise. Further, the definition of solution can be stated equivalently in terms of 'normals' to the epigraph and the hypograph of the solution. But, when the dynamics is only measurable in time such equivalence may fail to be true.

Nevertheless, the study of uniqueness of weak solutions can be carried out by using the definition of solution from [9]. In order to deal with Hamiltonians which are measurable in time, Ishii 11 proposed a new notion of weak solution in the class of continuous functions, proving the existence and uniqueness in the stationary case, and, for the evolutionary case, on $(0,+\infty) \times \mathbb{R}^{n}$. The continuously differentiable test functions needed to define such solutions are more complex, involving in addition some integrable mappings. This yields an existence result for weak solutions. Since uniqueness results for viscosity solutions of the Bellman equation

$$
-V_{t}+\sup _{u \in \mathbb{R}^{m}}\left\{\left\langle f(t, x, u),-D_{x} V\right\rangle-\ell(t, x, u)\right\}=0 \quad \text { on }(0,+\infty) \times \mathscr{O},
$$

assert that the weak solutions are represented as the value function of the control problem associated to the couple $(f, \ell)$ where $f$ is the dynamics and $\ell$ the Lagrangian, one may ask the possibility for the viscosity solution of the HJB in (1) to be represented as the value function of an appropriate optimal control control problem under state constraints. In the compact time case,
this viewpoint was investigated by Ishii in [12] for the convex case, providing Hölder continuous representation, and in 13 for Hamiltonian non necessarily convex. In this latter work, the lagrangian $\ell$ is merely continuous and the space of control is infinite dimensional. On the other hand, in [18] the author construct a faithful representation, Lipschitz continuous in the state and control. Frankowska and Sadrakyan [10] investigated faithful representations of Hamiltonians that are measurable in time, giving sharp results on the Lipschitz constants of faithful representations, and the stability of the faithful representation, key property to show convergences results of value functions. However, in [17] the author, under some weaker assumptions and assuming the boundedness from above of the Fenchel transform $H^{*}(t, x,$.$) on its domain,$ constructed a faithful representation $(f, \ell)$ showing, for the finite horizon setting, the equivalence between the calculus of variation problem and the optimal control problem associated to the representation $(f, \ell)$ for the free-constrained setting $\mathscr{O}=\mathbb{R}^{n}$.

Unfortunately, when addressing state constrained problems, i.e. $\mathscr{O} \neq \mathbb{R}^{n}$, the usual assumptions on the Hamiltonian may be insufficient to derive existence and uniqueness results for the HJB equations, even for finite horizon problem. In the framework of control problems, Soner [21] proposed a controllability assumption - called inward pointing condition - to investigate the continuity of the value function and the uniqueness of viscosity solutions of an autonomous control problem. However, such a property cannot be used for sets with nonsmooth boundaries and boundedness assumptions on $\mathscr{O}$ may be quite
restrictive for many applied models. To allow for nonsmooth boundaries, Ishii and Koike [15] generalized Soner's condition in the setting of infinite horizon problems and continuous solutions - cfr. also [5]. To deal with discontinuous solutions, Ishii [14] introduced the concept of lower and upper semicontinuous envelopes of a function, proving that the upper semicontinuous envelope of the value function of an optimal control problem is the largest upper semicontinuous subsolution and its lower semicontinuous envelope is the smallest lower semi-continuous supersolution. This approach, however, does not ensure the uniqueness of - weak - solutions of the HJB equation. On the other hand, the upper semicontinuous envelope does not have any meaning in optimal control theory while dealing with minimization problems - the lower semicontinuous envelope determines the value function of the relaxed problem. Barron, Frankowska, and Jensen 44 8] developed a different concept of solutions for the HJB equation associated to constraint-free Mayer optimal control problems, with a discontinuous cost. In this approach only subdifferentials are involved. While investigating infinite horizon problems, in the early work [6], the merely measurable case, it became clear that, in order to get uniqueness, it is convenient to replace subdifferentials by normals to the epigraph of solutions. Such 'geometric' definition of solution avoids using test functions and allows to have a unified approach to both the continuous and measurable case.

The contribution of the present paper is to give a representation and an uniqueness theorem for weak solutions - in the sense of definition given in Section [5- of non-autonomous HJB equations (1), with Hamiltonians time
measurable and convex in the fiber. We prove, assuming the relaxed assumption of local boundedness from above of $H^{*}(t, x,$.$) on its domain, a faithful$ representation result - cfr. Section 3 - of convex Hamiltonians in order to recover, under a backward viability assumption on the domain of the Fenchel transform $H^{*}(t, x,$.$) , the uniqueness of weak solutions in the class of vanishing$ functions at infinity

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \sup _{x \in \operatorname{dom} V(t, .)}|V(t, x)|=0 . \tag{2}
\end{equation*}
$$

The outline of this paper is as follows. In Section 2 we recall some basic concept and result in non-smooth analysis. The Sections 3 and 4 are devoted to the parametrization of set-valued maps and the representation of convex Hamiltonians, respectively. In Section 5 we state the main result of this paper, showing an uniqueness theorem for weak solution of HJB on infinite horizon with vanishing condition at infinity (2) and a representation result of such - weak - solutions as the value function of an appropriate infinite horizon optimal control problem under state constraints.

Notations: $\mathbb{R}^{+},|$.$| , and \langle.,$.$\rangle stands for the set of all non-negative real$ numbers, the Euclidean norm, and the scalar product, respectively. Let $E \subset \mathbb{R}^{k}$ be a subset and $x \in \mathbb{R}^{k}$. The Euclidean distance between $x$ and $E$ and the closed ball in $\mathbb{R}^{k}$ of radius $r>0$ and centered at $x$ are denoted, respectively, by $d(x, E)$ and $B(x, r)(\mathbb{B}:=B(0,1))$. cl $E$, int $E$, bdr $E, E^{c}$, and co $E$ stands, respectively, for the closure, the interior, the boundary, the complement, and the convex hull of $E(\overline{\operatorname{co}} E:=\operatorname{cl} \operatorname{co} E)$. We put $\|E\|:=\sup _{k \in E}|k| \in \mathbb{R}^{+} \cup$ $\{+\infty\} . \mathscr{C}^{m}$ stands for the family of all non-empty closed convex subsets in
$\mathbb{R}^{m}$ and we write $J \in \mathscr{C}_{b}^{m}$ if $J$ is bounded and $J \in \mathscr{C}^{m} . \mu$ denotes the Lebesgue measure.

## 2 Preliminaries on non-smooth analysis

The negative polar cone of a non-empty subset $C \subset \mathbb{R}^{k}$ is the set defined by

$$
C^{-}:=\left\{p \in \mathbb{R}^{k} \mid\langle p, c\rangle \leq 0 \text { for all } c \in C\right\} .
$$

The positive polar cone of $C$ is the set defined by $C^{+}:=-C^{-}$. Let $D \subset \mathbb{R}^{n}$ be non-empty and $x \in \operatorname{cl} D$. The contingent cone to $D$ at $x$ is the set defined by

$$
T_{D}(x):=\left\{v \in \mathbb{R}^{n} \left\lvert\, \liminf _{h \rightarrow 0+} \frac{d_{D}(x+h v)}{h}=0\right.\right\}
$$

The limiting normal cone to $D$ at $x$, written $N_{D}(x)$, is the closed set of all $p \in \mathbb{R}^{n}$ such that $\liminf _{y \rightarrow_{D} x} d_{T_{D}(y)^{-}}(p)=0$ and it is known that $N_{D}(x)^{-} \subset$ $T_{D}(x)$, provided $D$ is closed. If $D$ is convex, then $N_{D}(x)$ is also called normal cone and holds (cfr. [23])

$$
\begin{equation*}
p \in N_{D}(x) \Longleftrightarrow\langle p, y-x\rangle \leq 0 \quad \forall y \in D \tag{3}
\end{equation*}
$$

We denote for any $r>0$

$$
N_{D}^{r}(x):=\left\{p \in \mathbb{R}^{n} \mid p \in \overline{\operatorname{co}} N_{D}(y), y \in(\text { bdr } D) \cap B(x, r),|p|=1\right\}
$$

Now, assume that $D$ is closed. A vector $p \in \mathbb{R}^{n}$ is called a proximal normal to $D$ at $x$ if there exists $\lambda>0$ such that $d_{D}(x+\lambda p)=\lambda|p|$, i.e.,

$$
\begin{equation*}
\operatorname{int} B(x+\lambda p, \lambda|p|) \subset D^{c} \tag{4}
\end{equation*}
$$

We note that, since $D$ is closed, for any $x \in D$ the set of all proximal normals is non-empty and it reduces to the singleton $\{0\}$ at any interior point of $D$.

Let $\varphi: \mathbb{R}^{k} \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ be an extended real function. We write dom $\varphi$, epi $\varphi$, hypo $\varphi$, and graph $\varphi$ for the domain, the epigraph, the hypograph, and the graph of $\varphi$, respectively. We recall that $\varphi$ is said to be measurable if $\varphi^{-1}(\{+\infty\}), \varphi^{-1}(\{-\infty\})$, and $\varphi^{-1}(I)$ are measurable for any Borel subset $I \subset \mathbb{R}$. The function $\varphi$ is said to be proper if $\operatorname{dom} \varphi \neq \emptyset$ and $\varphi$ never attain $-\infty$ values. We reacall that the contingent epiderivative/hypoderivative, in direction $u \in \mathbb{R}^{k}$, of $\varphi$ at $x \in \operatorname{dom} \varphi$ are, respectively, defined by

$$
\begin{aligned}
& D_{\uparrow} \varphi(x)(u):=\liminf _{h \rightarrow 0+, u^{\prime} \rightarrow u} \frac{\varphi\left(x+h u^{\prime}\right)-\varphi(x)}{h} \\
& D_{\downarrow} \varphi(x)(u):=\limsup _{h \rightarrow 0+, u^{\prime} \rightarrow u} \frac{\varphi\left(x+h u^{\prime}\right)-\varphi(x)}{h}
\end{aligned}
$$

The Fréchet subdifferential/superdifferential of $\varphi$ at $x \in \operatorname{dom} \varphi$ are, respectively, defined by

$$
\begin{aligned}
& \partial_{-} \varphi(x):=\left\{p \in \mathbb{R}^{k} \left\lvert\, \liminf _{y \rightarrow x} \frac{\varphi(y)-\varphi(x)-\langle p, y-x\rangle}{|y-x|} \geq 0\right.\right\} \\
& \partial_{+} \varphi(x):=\left\{p \in \mathbb{R}^{k} \left\lvert\, \limsup _{y \rightarrow x} \frac{\varphi(y)-\varphi(x)-\langle p, y-x\rangle}{|y-x|} \leq 0\right.\right\} .
\end{aligned}
$$

The following result is well known (cfr. [19, Theo. 1] and [20, Prop. 8.12]).

Lemma 2.1 ( $\mathbf{1 9}, \mathbf{2 0}$ ) Assume that $\varphi$ is lower semicontinuous and convex.

Then, for any $x \in \operatorname{dom} \varphi$ :
(i) $\partial_{+} \varphi(x)=\partial \varphi(x):=\left\{p \in \mathbb{R}^{k} \mid \varphi(y) \geq \varphi(x)+\langle p, y-x\rangle\right.$ for all $\left.y \in \mathbb{R}^{k}\right\}$ and it is called subdifferential (in the sense of convex analysis) of $\varphi$ at $x$;
(ii) for any $(p, 0) \in N_{\mathrm{epi}} \varphi(x, \varphi(x))$ there exist two sequences $x_{i} \in \operatorname{dom} \varphi$ and $\left(p_{i}, q_{i}\right) \in N_{\text {epi }} \varphi\left(x_{i}, \varphi\left(x_{i}\right)\right)$ such that $q_{i}<0$ for all $i \in \mathbb{N}$ and $\left(x_{i}, \varphi\left(x_{i}\right)\right) \rightarrow(x, \varphi(x)),\left(p_{i}, q_{i}\right) \rightarrow(p, 0)$.

The Fenchel transform (or conjugate) of $\varphi, \operatorname{written} \varphi^{*}$, is the extended real function $\varphi^{*}: \mathbb{R}^{k} \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ defined by $\varphi^{*}(v):=\sup _{p \in \mathbb{R}^{k}}\{\langle v, p\rangle-\varphi(p)\}$. The following results are known (cfr. [20, Theo. 11.1, 11.3]).

Lemma 2.2 ([20]) Assume that $\varphi$ is proper, lower semicontinuous, and convex.

Then $\varphi^{*}$ is a proper lower semicontinuous convex function, $\left(\varphi^{*}\right)^{*}=\varphi$, dom $\varphi^{*}$ is convex, and for all $p, v \in \mathbb{R}^{k}$ it holds that $p \in \partial \varphi^{*}(v) \Longleftrightarrow v \in$ $\partial \varphi(p) \Longleftrightarrow \varphi(p)+\varphi^{*}(v)=\langle v, p\rangle$.

## 3 Parametrization of set-valued maps

We recall that the extended Hausdorff distance between $J, K \in \mathscr{C}^{m}$ is defined by

$$
\mathrm{d}(J, K):=\max \left\{\sup _{x \in K} d(x, J), \sup _{x \in J} d(x, K)\right\} \in \mathbb{R} \cup\{+\infty\}
$$

Notice that $\operatorname{dl}(J, K)<+\infty$ for any $J, K \in \mathscr{C}_{b}^{m}$. Next we state a result on Lipschitz parametrization of convex sets (cfr. [1, Chapter 9]).

Lemma 3.1 ([]]) Let $P: \mathbb{R}^{m} \times \mathscr{C}^{m} \rightsquigarrow \mathscr{C}^{m}$ be the projection map defined by $P(u, J):=J \cap B(u, 2 d(u, J))$.

Then $\operatorname{dl}(P(u, J), P(v, K)) \leq 5(d l(J, K)+|u-v|)$ for all $J, K \in \mathscr{C}^{m}$ and all $u, v \in \mathbb{R}^{m}$.

In the following, we consider the map $S_{m}: \mathscr{C}_{b}^{m} \rightarrow \mathbb{R}^{m}$ defined by

$$
S_{m}(J):=\frac{1}{\mu(\mathbb{B})} \int_{\mathbb{B}} \operatorname{pr}\left(\partial \sigma_{J}(p)\right) \mu(d p),
$$

where: for any $K \in \mathscr{C}^{m}, \operatorname{pr}(K)$ stands for the projection of $0 \in \mathbb{R}^{m}$ onto $K \in \mathscr{C}^{m}$, i.e., the element in $K$ with minimal norm; for any $J \in \mathscr{C}_{b}^{m}, \sigma_{J}($. denotes the support function of $J$, that is $\sigma_{J}(p):=\max _{q \in J}\langle p, q\rangle$. The function $S_{m}($.$) is called Steiner map (or Steiner selection) and the following result is$ well known (cfr. [1, Theorem 9.4.1]).

Lemma 3.2 ([1]) The function $S_{m}($.$) is m$-Lipschitz continuous on $\mathscr{C}_{b}^{m}$ with respect the Hausdorff distance and satisfies

$$
\begin{equation*}
S_{m}(J) \in J \quad \forall J \in \mathscr{C}_{b}^{m} \tag{5}
\end{equation*}
$$

Remark 3.1 We notice that (5) follows immediately from the properties of Fenchel transform and the definition of subdifferential. Indeed, fix $J \in \mathscr{C}_{b}^{m}$ and let $p \in \mathbb{B}$. Define $\psi()=.\psi_{p}():.=\sigma_{J}(.+p)$. The function $\psi$ is proper convex. From Lemma 2.17(i), it follows that

$$
\begin{equation*}
\partial \sigma_{J}(p)=\partial \psi(0)=\arg \min \psi^{*} \tag{6}
\end{equation*}
$$

So, $\psi^{*}(q)=\sup _{y \in \mathbb{R}^{m}}\left\{\langle y, q\rangle-\sigma_{J}(y+p)\right\}=-\langle p, q\rangle$, if $q \in J$, and $+\infty$ otherwise. From (6), we have $\partial \sigma_{J}(p)=\arg \max _{q \in J}\langle p, q\rangle$, and, by arbitrariness of $p$, we conclude $\operatorname{pr}\left(\partial \sigma_{J}(p)\right) \in J$ for all $p \in \mathbb{B}$. Since $\frac{1}{\mu(\mathbb{B})} \int_{\mathbb{B}} J \mu(d p)=J$, we get (5).

Next, we state the main result of this section on parametrization of convex sets following the main ideas of those discussed in [1, Chapter 9] (cfr. also the literature therein) and providing sharper conditions.

Theorem 3.1 Let $I$ be a closed interval of $\mathbb{R}^{+}$and $Q: I \times \mathbb{R}^{k} \rightsquigarrow \mathbb{R}^{m}$ be a set-valued map such that $Q(t, x) \in \mathscr{C}^{m}$ for all $(t, x) \in I \times \mathbb{R}^{k}, Q(., x)$ is measurable for all $x \in \mathbb{R}^{k}$, and:
(a) for all $t \in I$ and any $r>0$ there exists $c_{r}(t)>0$ satisfying $Q(t, x) \subset$ $Q(t, y)+c_{r}(t)|x-y| \mathbb{B}$ for all $x, y \in B(0, r)$.

Then, for any set-valued map $\delta: I \times \mathbb{R}^{k} \rightsquigarrow \mathbb{R}^{m}$, with non-empty closed values and $\delta(., x)$ measurable for all $x \in \mathbb{R}^{k}$, there exist two functions $\phi$ : $I \times \mathbb{R}^{k} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ and $\eta: I \times \mathbb{R}^{k} \rightarrow(0,+\infty)$ satisfying

$$
\eta(t, x)= \begin{cases}\|\delta(t, x)\| & \text { if }\|\delta(t, x)\|>0  \tag{7}\\ 1 & \text { otherwise }\end{cases}
$$

and:
(i) $\phi(., x, u)$ and $\eta(., x)$ are measurable for all $x \in \mathbb{R}^{k}, u \in \mathbb{R}^{m}$;
(ii) for any $t \in I$ and any $r>0$

$$
\begin{aligned}
& |\phi(t, x, u)-\phi(t, y, v)| \leq 5 m\left(c_{r}(t)|x-y|+|\eta(t, x) u-\eta(t, y) v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{m}
\end{aligned}
$$

(iii) $\phi(t, x, \mathbb{B}) \subset Q(t, x)$ for all $(t, x) \in I \times \mathbb{R}^{k}$;
(iv) if $\delta(t, x) \neq\{0\}$ and it is bounded, then $Q(t, x) \cap \delta(t, x) \subset \phi(t, x, \mathbb{B})$.

In particular, if $\delta(.,.) \equiv \mathbb{R}^{m}$, then

$$
\begin{equation*}
Q(t, x)=\phi\left(t, x, \mathbb{R}^{m}\right) \quad \forall(t, x) \in I \times \mathbb{R}^{k} \tag{8}
\end{equation*}
$$

Proof Assume first that $\delta: I \times \mathbb{R}^{k} \rightsquigarrow \mathbb{R}^{m}$ is the constant set-valued map $\delta(.,.) \equiv \mathbb{R}^{m}$. Next, we prove (i)-(iii). Notice that, from our assumptions, $\mathbb{1}$, Theorem 8.2.3], and since the intersection of measurable set-valued maps is measurable, we have that

$$
\begin{equation*}
\forall(x, u) \in \mathbb{R}^{k} \times \mathbb{R}^{m}, \quad t \rightsquigarrow P(u, Q(t, x)) \text { is measurable, } \tag{9}
\end{equation*}
$$

where $P(.,$.$) is the projection map defined in Lemma 3.1. Fix r>0$. From assumption (a) and applying Lemma 3.1, for all $t \in I$ and all $x, y \in B(0, r)$

$$
\begin{align*}
\mathrm{d}(P(u, Q(t, x)), P(v, Q(t, y))) & \leq 5(\mathbb{d}(Q(t, x), Q(t, y))+|u-v|)  \tag{10}\\
& \leq 5\left(c_{r}(t)|x-y|+|u-v|\right) .
\end{align*}
$$

Now, consider the function $\phi: I \times \mathbb{R}^{k} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ defined by

$$
\phi(t, x, u):=S_{m} \circ P(u, Q(t, x))
$$

Let $t \in I, x \in \mathbb{R}^{k}$, and $u \in \mathbb{R}^{m}$. By (5) immediately follows that $\phi(t, x, u) \in$ $Q(t, x)$. In particular, (iii) holds. Moreover, let $w \in Q(t, x)$. Since $Q(t, x) \cap$ $B(w, 2 d(w, Q(t, x)))=\{w\}$, then $\phi(t, x, w)=S_{m} \circ P(w, Q(t, x))=w$. So, (8) is proved. From the $m$-Lipschitz continuity of $S_{m}($.$) and (10), it follows that$ for all $t \in I$,

$$
\begin{align*}
|\phi(t, x, u)-\phi(t, y, v)| & \leq m \mathrm{dl}(P(u, Q(t, x)), P(v, Q(t, y)))  \tag{11}\\
& \leq 5 m\left(c_{r}(t)|x-y|+|u-v|\right)
\end{align*}
$$

for all $x, y \in B(0, r)$ and all $u, v \in \mathbb{R}^{m}$. Hence, recalling (9), (11), and the continuity of $S_{k}($.$) , (i) and (ii) follows.$

Now, consider a set-valued map $\delta: I \times \mathbb{R}^{k} \rightsquigarrow \mathbb{R}^{m}$ with non-empty closed values and $\delta(., x)$ measurable for all $x \in \mathbb{R}^{k}$. From [1, Theorem 8.2.11] and since $\delta(., x)$ is measurable for any $x \in \mathbb{R}^{k}$, we have that the map $t \rightarrow\|\delta(t, x)\| \in$ $\mathbb{R} \cup\{ \pm \infty\}$ is measurable for any $x \in \mathbb{R}^{k}$. Define, for any $x \in \mathbb{R}^{k}$, the measurable set $\Lambda(x):=\{t \in I \mid\|\delta(t, x)\| \in\{0,+\infty\}\}$ and denote for all $(t, x) \in I \times \mathbb{R}^{k}$

$$
\eta(t, x):=\chi_{\Lambda(x)}(t)+\chi_{\Lambda(x)^{c}}(t) \cdot\|\delta(t, x)\|,
$$

where $\chi_{V}$ is the function that takes values 1 on the set $V$ and 0 elsewhere. Then, (7) holds and, from our assumptions, the map $t \mapsto \eta(t, x)$ is measurable for all $x \in \mathbb{R}^{k}$. Consider the map $\phi: I \times \mathbb{R}^{k} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ defined by

$$
\phi(t, x, u):=S_{m} \circ P(\eta(t, x) u, Q(t, x)) .
$$

Arguing in the same way as above, the statements (i), (ii), and (iii) holds. Next we show (iv). Assume that $\delta(t, x)$ is bounded and $\delta(t, x) \neq\{0\}$. From the definition of $\eta(t, x)$, we have that $\eta(t, x)=\|\delta(t, x)\|>0$. Now, if $Q(t, x) \cap \delta(t, x)=$ $\emptyset$, then (iv) holds. Otherwise, let $w \in Q(t, x) \cap \delta(t, x)$. Then, there exists $\hat{\delta} \in[0,\|\delta(t, x)\|]$ and $|\hat{w}|=1$ satisfying $w=\hat{w} \hat{\delta}$. We have $w=\left(\hat{w} \frac{\hat{\delta}}{\eta(t, x)}\right) \eta(t, x)$. Since $\left|\hat{w} \frac{\hat{\delta}}{\eta(t, x)}\right| \leq 1$, it follows $\phi\left(t, x, \hat{w} \frac{\hat{\delta}}{\eta(t, x)}\right)=w$. Thus, (iv) holds.

Remark 3.2 Let $(t, x) \in I \times \mathbb{R}^{n}$. From the proof of Theorem 3.1, it follows that $\mathrm{dl}(Q(t, x) \cap \delta(t, x), \phi(t, x, \mathbb{B})) \leq 10 m\|\delta(t, x)\|$. Indeed, for any $\gamma \in Q(t, x) \cap$ $\delta(t, x)$ and any $\theta \in \phi(t, x, \mathbb{B})$, we have

$$
\begin{aligned}
|\gamma-\theta| & =\left|\gamma-S_{m} \circ P(\|\delta(t, x)\| u, Q(t, x))\right| \\
& =\left|S_{m} \circ P(\gamma, Q(t, x))-S_{m} \circ P(\|\delta(t, x)\| u, Q(t, x))\right| \\
& \leq 5 m|\gamma-\|\delta(t, x)\| u| \leq 10 m\|\delta(t, x)\| .
\end{aligned}
$$

From Theorem 3.1 we get the following corollary:

Corollary 3.1 Assume the assumptions of Theorem 3.1 and that for all $x \in$ $\mathbb{R}^{k}$ there exists $r(., x): I \rightarrow(0,+\infty)$ measurable such that $Q(t, x) \subset r(t, x) \mathbb{B}$ for all $(t, x) \in I \times \mathbb{R}^{k}$.

Then there exists a function $\phi: I \times \mathbb{R}^{k} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}$ such that

$$
Q(t, x)=\phi(t, x, \mathbb{B}) \quad \forall(t, x) \in I \times \mathbb{R}^{k}
$$

and:
(i) $\phi(., x, u)$ is measurable for all $x \in \mathbb{R}^{k}, u \in \mathbb{R}^{m}$;
(ii) for any $t \in I$ and any $r>0$

$$
\begin{aligned}
& |\phi(t, x, u)-\phi(t, y, v)| \leq 5 m\left(c_{r}(t)|x-y|+|r(t, x) u-r(t, y) v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{m} .
\end{aligned}
$$

Proof All the conclusions follows from Theorem 3.1 by choosing $\delta(.,)=$. $r(.,.) \mathbb{B}$.

## 4 Representation of convex Hamiltonians

Let $I$ be a closed interval of $\mathbb{R}^{+}$and $H: I \times \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}$ be a function such that $t \mapsto H(t, x, p)$ is measurable for any $x, p \in \mathbb{R}^{n}$. We consider the following conditions on $H$ :
H.1.1. $p \mapsto H(t, x, p)$ is convex for all $t \in I$ and $x \in \mathbb{R}^{n}$;
H.1.2. for all $r>0$ there exists $C_{r}: I \rightarrow \mathbb{R}^{+}$measurable such that

$$
|H(t, x, p)-H(t, y, p)| \leq C_{r}(t)(1+|p|)|x-y|
$$

for all $t \in I, x, y \in B(0, r)$, and $p \in \mathbb{R}^{n}$;
H.1.3. there exists $\tilde{c}: I \rightarrow \mathbb{R}^{+}$measurable such that

$$
|H(t, x, p)-H(t, x, q)| \leq \tilde{c}(t)(1+|x|)|p-q|
$$

for all $t \in I$ and $x, p, q \in \mathbb{R}^{n}$.
In the following, for any $(t, x) \in I \times \mathbb{R}^{n}$, we denote by $H^{*}(t, x,):. \mathbb{R}^{n} \rightarrow$ $\mathbb{R} \cup\{ \pm \infty\}$ the Fenchel transform of the function $H(t, x,$.$) , we define the set-$
valued maps $D: I \times \mathbb{R}^{n} \rightsquigarrow \mathbb{R}^{n}, E p: I \times \mathbb{R}^{n} \rightsquigarrow \mathbb{R}^{n+1}$, and $G r: I \times \mathbb{R}^{n} \rightsquigarrow \mathbb{R}^{n+1}$
respectively by

$$
\begin{aligned}
& D(t, x):=\operatorname{dom} H^{*}(t, x, .) \\
& E p(t, x):=\operatorname{epi} H^{*}(t, x, .) \\
& G r(t, x):=\operatorname{graph} H^{*}(t, x, .),
\end{aligned}
$$

and we put $\gamma(t, x):=\max \left\{0, \sup _{q \in D(t, x)} H^{*}(t, x, q)\right\} \in \mathbb{R}^{+} \cup\{+\infty\}$.
We also consider the following condition on the Hamiltonian:
H.1.4. $\forall(t, x) \in I \times \mathbb{R}^{n}, \forall \bar{q} \in \operatorname{cl} D(t, x), \exists \varepsilon>0$ :

$$
\sup _{q \in D(t, x) \cap B(\bar{q}, \varepsilon)} H^{*}(t, x, q)<\infty .
$$

Theorem 4.1 (Representation) Assume H.1.1-2 and H.1.4.
Then there exists a function $\phi: I \times \mathbb{R}^{n} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$,

$$
\phi(t, x, u)=:(f(t, x, u), \ell(t, x, u)) \in \mathbb{R}^{n} \times \mathbb{R},
$$

satisfying:
(i) $f(., x, u)$ and $\ell(., x, u)$ are measurable for all $x \in \mathbb{R}^{n}$ and $u \in \mathbb{R}^{n+1}$;
(ii) for all $t \in I$ and $x, p \in \mathbb{R}^{n}$,

$$
H(t, x, p)=\sup _{u \in \mathbb{R}^{n+1}}\{\langle p, f(t, x, u)\rangle-\ell(t, x, u)\}
$$

(iii) for all $t \in I$ and $r>0$,

$$
\begin{aligned}
& |f(t, x, u)-f(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|u-v|\right) \\
& |\ell(t, x, u)-\ell(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|u-v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{n+1}
\end{aligned}
$$

If, in addition, condition H.1.3 holds, then the statements (ii)-(iii) are replaced by the following:
(ii)' for all $t \in I$ and $x, p \in \mathbb{R}^{n}$,

$$
H(t, x, p)=\sup _{u \in \mathbb{B}}\{\langle p, f(t, x, u)\rangle-\ell(t, x, u)\}
$$

(iii)' for all $t \in I$ and $r>0$,

$$
\begin{aligned}
& |f(t, x, u)-f(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|\eta(t, x) u-\eta(t, y) v|\right) \\
& |\ell(t, x, u)-\ell(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|\eta(t, x) u-\eta(t, y) v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{n+1}
\end{aligned}
$$

$$
\text { where } \eta(t, .):=\tilde{c}(t)(1+|.|)+\gamma(t, .)+|H(t, ., 0)| ;
$$

and moreover:
$(\mathrm{iv})^{\prime} D(t, x)=f(t, x, \mathbb{B})$ for all $t \in I$ and $x \in \mathbb{R}^{n}$;
$(\mathrm{v})^{\prime} G r(t, x) \subset \phi(t, x, \mathbb{B})$ for all $t \in I$ and $x \in \mathbb{R}^{n}$.

Before to give a proof of Theorem 4.1, we show some intermediate results.

Lemma 4.1 Assume H.1.1-2 and let $(t, x) \in I \times \mathbb{R}^{n}$.
Then:
(i) $D(t, x)$ is non-empty and convex.

Moreover, if, in addition, the condition H.1.3 holds, then:
(ii) $D(t, x) \subset \tilde{c}(t)(1+|x|) \mathbb{B}$.

Proof The first claim follows immediately from Lemma 2.2 The proof of statement (ii) follows in the same way as in [10].

Lemma 4.2 Assume H.1.1-2 and H.1.4.
Then:
(i) $D(t, x) \in \mathscr{C}^{n}$;
(ii) for all $t \in I$ and $r>0$,

$$
D(t, x) \subset D(t, y)+C_{r}(t)|x-y| \mathbb{B} \quad \forall x, y \in B(0, r)
$$

Proof Let $(t, x) \in I \times \mathbb{R}^{n}$. Next we show that $D(t, x)$ is closed. Consider a sequence $q_{i} \in D(t, x)$ converging to $\tilde{q} \in \mathbb{R}^{n}$. Since $\tilde{q} \in \operatorname{cl} D(t, x)$, from assumption H.1.4, there exist $\varepsilon>0$ and $M>0$ such that $\left|H^{*}\left(t, x, q_{i}\right)\right| \leq M$ for all $q_{i} \in B(\tilde{q}, \varepsilon)$. Hence, since the Fenchel transform $q \mapsto H^{*}(t, x, q)$ is lower semicontinuous (cfr. Lemma 2.2), $M \geq \liminf _{i \rightarrow+\infty} H^{*}\left(t, x, q_{i}\right) \geq H^{*}(t, x, \tilde{q})$. So, $\tilde{q} \in D(t, x)$, and recalling Lemma 4.1 the assertion (i) is proved.

Now, to show (ii), suppose by contradiction that there exist $t \in I, r>0$, $x, y \in B(0, r), w \in D(t, x)$, and $\eta>C_{r}(t)$ such that

$$
D(t, y) \cap B(w, \eta|x-y|)=\emptyset
$$

We divide the proof into three steps.
Step 1: Applying Lemma 4.1 and (i), the set $D(t, x)$ is closed and convex. Let $\bar{q} \in D(t, y)$ be the projection of $w$ onto $D(t, y)$ and put $z:=(w-\bar{q}) /|w-q|$. We have that $z$ is a proximal normal to $D(t, y)$ at $\bar{q}$, i.e., there exists $\bar{\lambda}>\eta|x-y|$ such that $d_{D(t, y)}(\bar{q}+\bar{\lambda} z)=\bar{\lambda}$. Consider the hyperplane $\left\{\xi \in \mathbb{R}^{n} \mid\langle z, \xi\rangle=\right.$ $\langle z, \bar{q}\rangle\}$. Since $D(t, y)$ is convex and $z$ is a proximal normal, we have that $D(t, y) \subset\left\{\xi \in \mathbb{R}^{n} \mid\langle z, \xi\rangle \leq\langle z, \bar{q}\rangle\right\}$. Moreover, from (4), $B(w, \eta|x-y|) \subset$ int $B(\bar{q}+\bar{\lambda} z, \bar{\lambda}) \subset D(t, y)^{c}$, and we get

$$
\begin{equation*}
\langle z, q\rangle \leq\langle z, \bar{q}\rangle<\langle z, w+\eta| x-y|h\rangle \quad \forall q \in D(t, y), \forall h \in \mathbb{B} . \tag{12}
\end{equation*}
$$

Notice that, applying [23, Proposition 4.2.9], $z \in N_{D(t, y)}(\bar{q})$. Hence, using (3), we have $(z, 0) \in N_{\text {epi } H^{*}(t, y, .)}\left(\bar{q}, H^{*}(t, y, \bar{q})\right)$.

Step 2: From Step 1 and applying Lemma2.1f(ii), consider two sequences $w_{i} \in \operatorname{dom} H^{*}(t, y,$.$) and \left(p_{i}, q_{i}\right) \in N_{\text {epi } H^{*}(t, y, .)}\left(w_{i}, H^{*}\left(t, y, w_{i}\right)\right)$, with $q_{i}<0$, satisfying

$$
\begin{equation*}
\left(p_{i}, q_{i}\right) \rightarrow(z, 0), \quad\left(w_{i}, H^{*}\left(t, y, w_{i}\right)\right) \rightarrow\left(\bar{q}, H^{*}(t, y, \bar{q})\right) \tag{13}
\end{equation*}
$$

So, $\left(p_{i} /\left|q_{i}\right|,-1\right) \in N_{\text {epi } H^{*}(t, y, .)}\left(w_{i}, H^{*}\left(t, y, w_{i}\right)\right)$ for all $i \in \mathbb{N}$. We conclude that $p_{i} /\left|q_{i}\right| \in \partial H^{*}(t, y,).\left(w_{i}\right)$ for all $i \in \mathbb{N}$. Thus, from Lemma 2.2,

$$
\begin{equation*}
H\left(t, y, p_{i} /\left|q_{i}\right|\right)+H^{*}\left(t, y, w_{i}\right)=\left\langle w_{i}, p_{i} /\right| q_{i}| \rangle \quad \forall i \in \mathbb{N} . \tag{14}
\end{equation*}
$$

Step 3: Using (13) and (12) with $h=-z /|z|$, we can assume that $\left\langle w_{i}, p_{i}\right\rangle<$ $\left\langle w, p_{i}\right\rangle-\eta|x-y|$ for all large $i \in \mathbb{N}$. Hence, from assumption H.1.2 and recalling that $w \in D(t, x)$, we get for all large $i \in \mathbb{N}$

$$
\begin{aligned}
& \left\langle w_{i}, p_{i}\right\rangle-\left|q_{i}\right| H\left(t, y, p_{i} /\left|q_{i}\right|\right) \\
& <\left\langle w, p_{i}\right\rangle-\eta|x-y|-\left|q_{i}\right| H\left(t, x, p_{i} /\left|q_{i}\right|\right)+\left(\left|q_{i}\right|+\left|p_{i}\right|\right) C_{R}(t)|x-y| \\
& =\left|q_{i}\right|\left(\left\langle w, p_{i} /\right| q_{i}| \rangle-H\left(t, x, p_{i} /\left|q_{i}\right|\right)\right)+\left(\left(\left|q_{i}\right|+\left|p_{i}\right|\right) C_{r}(t)-\eta\right)|x-y| \\
& \leq\left|q_{i}\right| H^{*}(t, x, w)+\left(\left(\left|q_{i}\right|+\left|p_{i}\right|\right) C_{r}(t)-\eta\right)|x-y| .
\end{aligned}
$$

So, by (14), for all large $i \in \mathbb{N}$

$$
\begin{equation*}
\left|q_{i}\right| H^{*}\left(t, y, w_{i}\right)<\left|q_{i}\right| H^{*}(t, x, w)+\left(\left(\left|q_{i}\right|+\left|p_{i}\right|\right) C_{r}(t)-\eta\right)|x-y| . \tag{15}
\end{equation*}
$$

From assumption H.1.4, the lower semicontinuity of $H^{*}(t, y,$.$) , and since w_{i} \rightarrow$ $\bar{q}$, the sequence $\left\{H^{*}\left(t, y, w_{i}\right)\right\}_{i \in \mathbb{N}}$ is bounded. Then, using again (13), and passing to the lower limit as $i \rightarrow+\infty$ in (15) we get $0 \leq\left(|z| C_{r}-\eta\right)|x-y|$. Since $|z|=1,0 \leq C_{r}(t)-\eta$, and a contradiction follows.

Lemma 4.3 Assume H.1.1-2 and H.1.4.
Then:
(i) $t \rightsquigarrow E p(t, x)$ is measurable for all $x \in \mathbb{R}^{n}$;
(ii) $E p(t, x) \in \mathscr{C}^{n+1}$ for all $t \in I$ and $x \in \mathbb{R}^{n}$;
(iii) for all $t \in I$ and $r>0$,

$$
E p(t, x) \subset E p(t, y)+C_{r}(t)|x-y|(\mathbb{B} \times[-1,1]) \quad \forall x, y \in B(0, r)
$$

Proof We notice that, from H.1.1-2, the lower semicontinuity and the convexity of Fenchel transform (cfr. Lemma 2.2), it follows immediately that, for any $t \in I$ and $x \in \mathbb{R}^{n}$, the set-valued map $s \rightsquigarrow E p(s, x)$ is measurable and $E p(t, x)$ is non-empty, closed, and convex. So, (i) and (ii) holds.

Now, we show (iii). Fix $x, y \in \mathbb{R}^{n}, t \in I$, and consider $(q, \lambda) \in E p(t, x)$. Without loss of generality we may assume that $C_{r}(t)|x-y| \neq 0$. We claim the following: there exists $w \in D(t, y)$ satisfying $\left(w, q+C_{r}(t)|x-y|\right) \in E p(t, y)$. Indeed, from assumption H.1.2, we have that $H(t, x, p) \leq H(t, y, p)+C_{r}(t)(1+$ $|p|)|x-y|$ for all $p \in \mathbb{R}^{n}$, and, from the definition of Fenchel transform,

$$
\begin{equation*}
\left(H(t, y, .)+C_{r}(t)(1+|.|)|x-y|\right)^{*}(\tilde{q}) \leq H^{*}(t, x, \tilde{q}) \quad \forall \tilde{q} \in \mathbb{R}^{n} \tag{16}
\end{equation*}
$$

Now, define $h():.=-C_{r}(t)|x-y|$ on $B\left(0, C_{r}(t)|x-y|\right)$ and $+\infty$ elsewhere. Notice that $h($.$) is a proper lower semicontinuous convex function and h^{*}()=$. $C_{r}(t)(1+||).|x-y|$. Notice that, since dom $H(t, y,)=.\mathbb{R}^{n}$ and from assumption H.1.1, applying Lemma 2.2 we get $\left(H^{*}(t, y, .)\right)^{*}=H(t, y,$.$) . So, for all z \in \mathbb{R}^{n}$

$$
\begin{aligned}
& \left(\inf _{q_{1} \in \mathbb{R}^{k}} H^{*}\left(t, y, q_{1}\right)+h\left(.-q_{1}\right)\right)^{*}(z) \\
& :=\sup _{q_{2} \in \mathbb{R}^{k}}\left\{\left\langle q_{2}, z\right\rangle-\inf _{q_{1} \in \mathbb{R}^{k}}\left\{H^{*}\left(t, y, q_{1}\right)+h\left(q_{2}-q_{1}\right)\right\}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& =\sup _{q_{2} \in \mathbb{R}^{k}}\left\{\left\langle q_{2}, z\right\rangle+\sup _{q_{1} \in \mathbb{R}^{k}}\left\{-H^{*}\left(t, y, q_{1}\right)-h\left(q_{2}-q_{1}\right)\right\}\right\} \\
& =\sup _{q_{2}, q_{1} \in \mathbb{R}^{k}}\left\{\left\langle q_{2}, z\right\rangle-H^{*}\left(t, y, q_{1}\right)-h\left(q_{2}-q_{1}\right)\right\} \\
& =\sup _{q_{1} \in \mathbb{R}^{k}}\left\{\left\langle q_{1}, z\right\rangle-H^{*}\left(t, y, q_{1}\right)+\sup _{q_{2} \in \mathbb{R}^{k}}\left\{\left\langle q_{2}-q_{1}, z\right\rangle-h\left(q_{2}-q_{1}\right)\right\}\right\} \\
& =H(t, y, z)+C_{r}(t)(1+|z|)|x-y| .
\end{aligned}
$$

Since dom $h=B\left(0, C_{r}(t)|x-y|\right)$ and the function

$$
z \mapsto\left(\inf _{q_{1} \in \mathbb{R}^{k}} H^{*}\left(t, y, q_{1}\right)+h\left(z-q_{1}\right)\right)
$$

is proper lower semicontinuous and convex, passing to the Fenchel transform and using Lemma 2.2 we deduce

$$
\begin{aligned}
& \inf _{q_{1} \in \mathbb{R}^{k}} H^{*}\left(t, y, q_{1}\right)+h\left(q-q_{1}\right) \\
& =\inf _{q_{1} \in B\left(q, C_{r}(t)|x-y|\right)} H^{*}\left(t, y, q_{1}\right)-C_{r}(t)|x-y| \\
& =\left(H(t, y, .)+C_{r}(t)(1+|.|)|x-y|\right)^{*}(q) .
\end{aligned}
$$

Thus, from (16), there exists $w \in B\left(q, C_{r}(t)|x-y|\right)$ satisfying

$$
H^{*}(t, y, w)-C_{r}(t)|x-y| \leq H^{*}(t, x, q)
$$

Hence, the claim follows. Now, applying Lemma4.2(ii), $|q-w| \leq C_{r}(t)|x-y|$ because $q \in D(t, x)$ and $w \in D(t, y)$. Finally, since

$$
(q, \lambda)=\left(w, \lambda+C_{r}(t)|x-y|\right)+C_{r}(t)|x-y|\left(\frac{q-w}{C_{r}(t)|x-y|},-1\right),
$$

the statement (iii) follows by the arbitrariness of $(q, \lambda) \in E p(t, x)$.

Proposition 4.1 Assume H.1.1-2 and H.1.4.
Then there exists a function $\phi: I \times \mathbb{R}^{n} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$,

$$
\phi(t, x, u)=:(f(t, x, u), \ell(t, x, u)) \in \mathbb{R}^{n} \times \mathbb{R},
$$

satisfying

$$
\begin{equation*}
E p(t, x)=\phi\left(t, x, \mathbb{R}^{n+1}\right) \quad \forall(t, x) \in I \times \mathbb{R}^{n} \tag{17}
\end{equation*}
$$

and:
(i) $\phi(., x, u)$ is measurable for all $x \in \mathbb{R}^{n}$ and $u \in \mathbb{R}^{n+1}$;
(ii) for all $t \in I$ and $r>0$

$$
\begin{aligned}
& |\phi(t, x, u)-\phi(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|u-v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{n+1}
\end{aligned}
$$

If, in addition, H.1.3 holds, then

$$
G r(t, x) \subset \phi(t, x, \mathbb{B}) \quad \forall(t, x) \in I \times \mathbb{R}^{n},
$$

the statement (ii) is replaced by the following:
(ii)' for all $t \in I$ and $r>0$

$$
\begin{aligned}
& |\phi(t, x, u)-\phi(t, y, v)| \leq 5(n+1)\left(C_{r}(t)|x-y|+|\eta(t, x) u-\eta(t, y) v|\right) \\
& \forall x, y \in B(0, r), \forall u, v \in \mathbb{R}^{n+1}
\end{aligned}
$$

$$
\text { where } \eta(t, .):=\left(\tilde{c}(t)+C_{r}(t)\right)(1+|.|)+|H(t, 0,0)|+\gamma(t, .) ;
$$

and moreover:
(iii) $D(t, x)=f(t, x, \mathbb{B})$ for all $t \in I$ and $x \in \mathbb{R}^{n}$.

Proof Let $t \in I$ and $r>0$. Notice that, if H.1.3 holds, then $\gamma(t, x)<+\infty$ for any $x \in B(0, r)$. Furthermore, for all $x \in B(0, r)$ and $v \in D(t, x)$

$$
-H(t, x, 0) \leq H^{*}(t, x, v) \leq \gamma(t, x)
$$

We get $\left|\left(v, H^{*}(t, x, v)\right)\right| \leq \tilde{c}(t)(1+|x|)+\gamma(t, x)+|H(t, x, 0)| \leq \tilde{c}(t)(1+|x|)+$ $C_{r}(t)(1+|x|)+|H(t, 0,0)|+\gamma(t, x)$ for all $x \in B(0, r)$. So

$$
\begin{equation*}
\|G r(t, x)\| \leq\left(\tilde{c}(t)+C_{r}(t)\right)(1+|x|)+|H(t, 0,0)|+\gamma(t, x) \tag{18}
\end{equation*}
$$

for all $x \in B(0, r)$. Hence, the conclusions follows immediately from Theorem 3.1 and Lemma 4.3, with $Q(t, x)=E p(t, x)$ and $\delta(t, x)=G r(t, x)$.

Next, we give a proof of Theorem 4.1.
Proof of Theorem 4.1 Consider the function $\phi=(f, \ell)$ of Proposition 4.1. The statements (i) and (iii) follows from Proposition 4.1. Next we show (ii).

Fix $t \in I, x \in \mathbb{R}^{n}$, and $p \in \mathbb{R}^{n}$. Recalling that $\left(H^{*}(t, x, .)\right)^{*}=H(t, x,$.$) ,$ from (17) it follows that for any $u \in \mathbb{R}^{n+1}$ the pair $(f(t, x, u), \ell(t, x, u))$ lays in $E p(t, x)$, i.e.,

$$
\begin{equation*}
H^{*}(t, x, f(t, x, u)) \leq \ell(t, x, u) . \tag{19}
\end{equation*}
$$

So, for any $u \in \mathbb{R}^{n+1}$

$$
\begin{aligned}
& \langle p, f(t, x, u)\rangle-\ell(t, x, u) \\
& \leq\langle p, f(t, x, u)\rangle-H^{*}(t, x, f(t, x, u)) \\
& \leq \sup _{v \in \mathbb{R}^{n+1}}\left\{\langle p, v\rangle-H^{*}(t, x, v)\right\}=H(t, x, p)
\end{aligned}
$$

Then, by arbitrariness of $u \in \mathbb{R}^{n+1}$, we get

$$
\sup _{u \in \mathbb{R}^{n+1}}\{\langle p, f(t, x, u)\rangle-\ell(t, x, u)\} \leq H(t, x, p)
$$

On the other hand, let $v \in D(t, x)$. Since $\left(v, H^{*}(t, x, v)\right) \in E p(t, x)$, from (17), there exists $w \in \mathbb{R}^{n+1}$ such that $\left(v, H^{*}(t, x, v)\right)=(f(t, x, w), \ell(t, x, w))$. So, $\langle p, v\rangle-H^{*}(t, x, v)=\langle p, f(t, x, w)\rangle-l(t, x, w) \leq \sup _{u \in \mathbb{R}^{n+1}}\{\langle p, f(t, x, u)\rangle-$ $\ell(t, x, u)\}$. Hence,

$$
\begin{aligned}
H(t, x, p) & =\sup _{v \in D(t, x)}\left\{\langle p, v\rangle-H^{*}(t, x, v)\right\} \\
& \leq \sup _{u \in \mathbb{R}^{n+1}}\{\langle p, f(t, x, u)\rangle-\ell(t, x, u)\} .
\end{aligned}
$$

The last statements, assuming that H.1.3 holds, can be obtained with the same arguments as above using Proposition 4.1

Remark 4.1 Let $t \in I$ and $x \in \mathbb{R}^{n}$.
(a) Assumption H.1.4 is weaker than (H4) in [10, p. 33] and [18, p. 869].
(b) The condition in H.1.4 is equivalent to require that $H^{*}(t, x,$.$) is locally$ bounded on its domain. If $H(t, x,$.$) is globally Lipschitz, then one can$ show that H.1.4 is equivalent to assume that $H^{*}(t, x,$.$) is bounded on its$ domain.
(c) We would like to underline that, if $H(t, x,$.$) is convex, then, from Lemma$ 4.1, the domain $D(t, x)$ of the Fenchel transform $H^{*}(t, x,$.$) turn out to be$ bounded under the global Lipschitz assumption H.1.3. In particular, we notice that the Lipschitz condition imply the sublinear growth of $H(t, x,$.$) . On$ the other hand, when $H(t, x,$.$) is merely locally Lipschitz continuous, then$ $D(t, x)=\mathbb{R}^{n}$ if and only if $H(t, x,$.$) is coercive, i.e., \lim _{|p| \rightarrow+\infty} \frac{H(t, x, p)}{|p|}=$ $+\infty$ (cfr. [20, Theorem 11.8]).
(d) Let $r>0$ such that $x \in B(0, r)$. From Remark 3.2, (18), and (19) we get for all $u \in \mathbb{B}$
$\ell(t, x, u)-H^{*}(t, x, f(t, x, u)) \leq 10 m\left(\tilde{c}(t)+C_{r}(t)\right)(1+|x|)+|H(t, 0,0)|+$ $\gamma(t, x)$.
(e) It is necessary to point out that, although not in the main interest of this paper, by using the same arguments that those proposed in (10] (cfr. [18), the representation of Theorem 4.1 is faithful.

## 5 Hamilton-Jacobi-Bellman equations on infinite horizon

For any $a \in \mathbb{R}, b \in \mathbb{R} \cup\{+\infty\}$, with $a<b$, and $A \subset \mathbb{R}^{k}$ we take the following notation:

- $\quad \mathscr{L}^{1}(a, b ; A)$ denotes the normed space of all $A$-valued Lebesgue integrable functions on $[a, b)$ (we write $w \in \mathscr{L}_{\text {loc }}^{1}(a, b ; A)$ if $w \in \mathscr{L}^{1}(c, d ; A)$ for any $[c, d] \subset[a, b))^{1}$;
- $\quad \mathscr{W}_{l o c}^{1,1}(a, b ; A)$ is the normed space of all $A$-valued locally absolutely continuous functions on cl $[a, b)$;
- $\mathscr{L}_{\text {loc }}$ is the set of all $f \in \mathscr{L}_{\text {loc }}^{1}\left(0,+\infty ; \mathbb{R}^{+}\right)$such that $\lim _{\sigma \rightarrow 0} \sup \left\{\int_{J} f(\tau) d \tau \mid\right.$ $\left.J \subset \mathbb{R}^{+}, \mu(J) \leq \sigma\right\}=0$.

In this section we consider a closed non-empty subset $\Omega \subset \mathbb{R}^{n}$.
5.1 Weak solutions

We denote by H.2.1,3,4 the assumptions in H.1.1,3,4, and by H.2.2 the assumption H.1.2 with $C_{r}(.) \leq C($.$) for any r>0$ and a suitable $C \in \mathscr{L}_{\text {loc }}$. We consider the further two conditions on the Hamiltonian:
H.2.5. there exist $\tilde{\varphi} \in \mathscr{L}_{\text {loc }}^{1}\left(0,+\infty ; \mathbb{R}^{+}\right)$and $\varphi \in \mathscr{L}^{1}(0,+\infty ; \mathbb{R})$ such that for a.e. $t \geq 0$, for all $x \in \mathbb{R}^{n}$, and all $q \in D(t, x)$

$$
\varphi(t) \leq H^{*}(t, x, q) \leq \tilde{\varphi}(t)(1+|x|)
$$

[^1]H.2.6. there exists $\psi \in \mathscr{L}_{\text {loc }}$ such that for a.e. $t \geq 0$, for all $x \in \operatorname{bdr} \Omega$, and
$$
\text { all } q \in D(t, x)
$$
$$
|q|+\left|H^{*}(t, x, q)\right| \leq \psi(t)
$$

Definition 5.1 A lower semicontinuous function $V: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ is called a weak - or viscosity - solution of the HJB equation (1) if there exists a set $C \subset(0,+\infty)$, with $\mu((0,+\infty \backslash C))=0$, such that

$$
\begin{aligned}
& -p_{t}+H\left(t, x,-p_{x}\right) \geq 0 \\
& \forall\left(p_{t}, p_{x}\right) \in \partial_{-} V(t, x), \forall(t, x) \in \operatorname{dom} V \cap(C \times \operatorname{bdr} \Omega),
\end{aligned}
$$

and

$$
\begin{aligned}
& -p_{t}+H\left(t, x,-p_{x}\right)=0 \\
& \forall\left(p_{t}, p_{x}\right) \in \partial_{-} V(t, x), \forall(t, x) \in \operatorname{dom} V \cap(C \times \operatorname{int} \Omega)
\end{aligned}
$$

For all $t \in \mathbb{R}^{+}, x \in \mathbb{R}^{n}$, and $u \in \mathbb{R}^{n+1}$ we denote by $(f(t, x, u), \ell(t, x, u)):=$ $\phi(t, x, u)$ the representation of the Hamiltonian given by Theorem 4.1, and by $\mathscr{U}_{\Omega}(t, x)$ the (possibly empty) set of all trajectory-control pairs $(\xi, u)$ : $[t,+\infty) \rightarrow \mathbb{R}^{n} \times \mathbb{R}^{n+1}$ such that $u($.$) is measurable and$

$$
\left\{\begin{array}{l}
\xi^{\prime}(s)=f(s, \xi(s), u(s)) \quad s \in[t,+\infty) \text { a.e. }  \tag{20}\\
\xi(t)=x \\
u(.) \subset \mathbb{B}, \quad \xi(.) \subset \Omega
\end{array}\right.
$$

The value function $v: \mathbb{R}^{+} \times \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{ \pm \infty\}$ associated to the representation $(f, \ell)$ is defined by

$$
v(t, x):=\inf \left\{\int_{t}^{+\infty} \ell(s, \xi(s), u(s)) d s \mid(\xi(.), u(.)) \in \mathscr{U}_{\Omega}(t, x)\right\}
$$

where $v(t, x)=+\infty$ if $\mathscr{U}_{\Omega}(t, x)=\emptyset$, by convention. In the following we consider the outward pointing condition (briefly O.P.C.):
O.P.C. there exist $\eta, r, M>0$ such that
for a.e. $t>0, \forall y \in \partial \Omega+\eta \mathbb{B}, \forall q \in D(t, y)$, satisfying $\inf _{n \in N_{\Omega}^{\eta}(y)}\langle n, q\rangle \leq$

0 ,
there exists $w \in D(t, y) \cap B(q, M)$ such that $\inf _{n \in N_{\Omega}^{\eta}(y)}\{\langle n, w\rangle,\langle n, w-$ $q\rangle\} \geq r$.

Theorem 5.1 Assume H.2.1-6 and O.P.C.

Let $V: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function satisfying the vanishing condition at infinity (2).

Then the following statements are equivalent:
(i) $V=v$;
(ii) $V$ is weak solution of the HJB equation (1).

The proof of Theorem 5.1 is given in Section 5.3.

## Remark 5.1

(a) We would like to underline that condition O.P.C. is helpful to construct feasible trajectories for infinite horizon control problems. More precisely, it provides uniform neighboring feasible trajectories results (cfr. 5), on any compact interval $[0, T] \subset \mathbb{R}^{+}$, for the dynamics $F(s, x)=-f(T-s, x, \mathbb{B})$. Such results basically says that any absolutely continuous trajectory $\xi($. starting from a point in $\Omega$ and solving the differential inclusion $\xi(.) \in$ $F(., \xi()$.$) can be approximated by a sequence of trajectories which remain$ in the interior of the state constraints $\Omega$.
(b) For existence results of the HJB in (1) in the free-constraints case we refer to [2] and the literature therein. However, for infinite horizon optimal control problems under state constraints, O.P.C. does not ensure the existence of feasible trajectories. Existence results under state constraints are investigate in [6] in the case of Lagrangian with discount factor $e^{-\lambda t}$ and a suitable inward pointing condition.
(c) Assumption H.2.6 can be skipped for finite horizon settings in order to recover neighbouring feasible trajectory results (cfr. Remark5.1-(b) above). However, for infinity horizon control problem, such condition is sufficient to ensure uniform neighbouring feasible trajectory theorems (cfr. [5]).
5.2 Locally absolutely continuity of epigraph

For all $t \in \mathbb{R}^{+}, x \in \mathbb{R}^{n}$, and $u \in \mathbb{R}^{n+1}$ we put $L(t, x, u):=H^{*}(t, x, f(t, x, u))$.

## Lemma 5.1 Assume H.2.1-5.

Then:
(i) for all $x \in \mathbb{R}^{n}$ the mappings $f(., x,$.$) and \ell(., x,$.$) are Lebesgue-Borel$ measurable and there exists $\phi \in \mathscr{L}^{1}(0,+\infty ; \mathbb{R})$ such that $\ell(t, x, u) \geq$ $\phi(t)$ for a.e. $t \geq 0$ and all $x \in \mathbb{R}^{n}$ and $u \in \mathbb{R}^{n+1} ;$
(ii) there exists $c \in \mathscr{L}_{\text {loc }}^{1}\left(0,+\infty ; \mathbb{R}^{+}\right)$such that $|f(t, x, u)|+|\ell(t, x, u)| \leq$ $c(t)(1+|x|)$ for a.e. $t \geq 0$ and for all $x \in \mathbb{R}^{n}, u \in \mathbb{B} ;$
(iii) for a.e. $t \geq 0$ and all $x \in \mathbb{R}^{n}$, the set-valued map $\mathbb{R}^{n} \ni y \rightsquigarrow\{(f(t, y, u)$, $\ell(t, y, u)) \mid u \in \mathbb{B}\}$ is continuous with closed images;
(iv) there exists $k \in \mathscr{L}_{\text {loc }}$ such that $|f(t, x, u)-f(t, y, u)|+|\ell(t, x, u)-\ell(t, y, u)| \leq$ $k(t)|x-y|$ for a.e. $t \geq 0$ and for all $x, y \in \mathbb{R}^{n}, u \in \mathbb{B}$.

Proof All the conclusions follows from our assumptions and Theorem 4.1.

Next, we recall the definition of locally absolutely continuity for set-valued maps.

Definition 5.2 A set-valued map $S: \mathbb{R}^{+} \rightsquigarrow \mathbb{R}^{k}$ is called locally absolutely continuous (briefly l.a.c.) if it takes non-empty closed images and every $\varepsilon>0$, any $[t, T] \subset \mathbb{R}^{+}$, and any compact subset $K \subset \mathbb{R}^{k}$, there exists $\delta>0$ such that for any finite partition $t \leq t_{1}<\tau_{1} \leq t_{2}<\tau_{2} \leq \ldots \leq t_{m}<\tau_{m} \leq T$ of $[t, T]$ satisfying $\sum_{i=1}^{m}\left(\tau_{i}-t_{i}\right)<\delta$ holds

$$
\sum_{i=1}^{m} \max \left\{e\left(S\left(t_{i}\right), S\left(\tau_{i}\right) \cap K\right), e\left(S\left(\tau_{i}\right), S\left(t_{i}\right) \cap K\right)\right\}<\varepsilon,
$$

where $e\left(E, E^{\prime}\right):=\inf \left\{r>0: E^{\prime} \subset E+r \mathbb{B}\right\}$ for all $E, E^{\prime} \subset \mathbb{R}^{k}(\inf \emptyset:=$ $+\infty$, by convention).

Proposition 5.1 Assume H.2.1-6 and O.P.C. Denote by $W: \mathbb{R}^{+} \times \mathbb{R}^{n} \rightarrow$ $\mathbb{R} \cup\{+\infty\}$ the value function of the following infinite horizon control problem under state constraints: minimize $\int_{t}^{+\infty} L(s, \xi(s), u(s)) d s$ over all $(\xi(),. u().) \in$ $\mathscr{U}_{\Omega}(t, x)$ such that $\xi(t)=x$, where $(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}$ is the initial datum.

Then:
(i) $W(t, x)=\inf \left\{\int_{t}^{+\infty} H^{*}\left(s, \xi(s), \xi^{\prime}(s)\right) d s \mid \xi \in \mathscr{W}_{l o c}^{1,1}\left(t,+\infty ; \mathbb{R}^{n}\right), \xi(t)=\right.$ $x, \xi(.) \subset \Omega\}$, for any $(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}$ such that $\mathscr{U}_{\Omega}(t, x) \neq \emptyset ;$
(ii) $W$ and $v$ are lower semicontinuous and $t \rightsquigarrow \operatorname{epi} W(t,$.$) is l.a.c.;$
(iii) there exists a set $C^{\prime} \subset(0,+\infty)$, with $\left.\mu\left((0,+\infty) \backslash C^{\prime}\right)\right)=0$, such that for any $(t, x) \in \operatorname{dom} v \cap\left(C^{\prime} \times \operatorname{int} \Omega\right)$

$$
\forall u \in \mathbb{B}, \quad D_{\uparrow} v(t, x)(-1,-f(t, x, u)) \leq \ell(t, x, u)
$$

(iv) there exists a set $C^{\prime \prime} \subset(0,+\infty)$, with $\left.\mu\left((0,+\infty) \backslash C^{\prime \prime}\right)\right)=0$, such that for any $(t, x) \in \operatorname{dom} v \cap\left(C^{\prime \prime} \times \operatorname{int} \Omega\right)$

$$
\forall u \in \mathbb{B}, \quad-\ell(t, x, u) \leq D_{\downarrow} v(t, x)(1, f(t, x, u)) .
$$

Proof The statement (i) is a known fact (cfr. [2]), and for the lower semicontinuity of $W$ and $v$ and the locally absolutely continuity of the epigraph of the value function $W$, under the viability condition O.P.C., we refer to [5, 6]. So, (ii) holds.

Let us define for all $t \in \mathbb{R}^{+}$and all $x \in \mathbb{R}^{n}$

$$
\left.\begin{array}{rl}
G(t, x):=\{(f(t, x, u),-\ell(t, x, u)-r) \mid & u
\end{array}\right)
$$

Next, we prove (iii). Let $j \in \mathbb{N}^{+}$. Recalling Lemma 5.1, we apply [9, Theorem 2.9] to the set-valued map $[1 / j, j] \times \mathbb{R}^{n} \times \mathbb{R} \ni(s, \xi, \beta) \rightsquigarrow-G(j-s, \xi) \in \mathbb{R}^{n} \times \mathbb{R}$ and the measurable selection theorem: there exists a subset $C_{j}^{\prime} \subset[1 / j, j]$, with $\mu\left(C_{j}^{\prime}\right)=0$, such that for any $\left(t_{0}, x_{0}\right) \in\left((1 / j, j] \backslash C_{j}^{\prime}\right) \times \operatorname{int} \Omega$ and any $u_{0} \in \mathbb{B}$ we
can find $t_{1} \in\left[1 / j, t_{0}\right)$ and a trajectory-control pair $((\xi, \beta),(u, r))($.$) satisfying$

$$
\begin{cases}(\xi, \beta)^{\prime}(t)=(f(t, \xi(t), u(t)),-\ell(t, \xi(t), u(t))-r(t)) & t \in\left[t_{1}, t_{0}\right] \text { a.e. } \\ (u, r)(t) \in \mathbb{B} \times[0, c(t)(1+|\xi(t)|)-\ell(t, \xi(t), u(t))] & t \in\left[t_{1}, t_{0}\right] \text { a.e. } \\ (\xi, \beta)\left(t_{0}\right)=\left(x_{0}, 0\right) & \\ (\xi, \beta)^{\prime}\left(t_{0}\right)=\left(f\left(t_{0}, x_{0}, u_{0}\right),-\ell\left(t_{0}, x_{0}, u_{0}\right)\right) & \end{cases}
$$

and $\xi\left(\left[t_{1}, t_{0}\right]\right) \subset \Omega$. Hence, if $\left(t_{0}, x_{0}\right) \in \operatorname{dom} v$, by the dynamic programming principle it follows that $\frac{v(s, \xi(s))-v\left(t_{0}, x_{0}\right)}{t_{0}-s} \leq \frac{1}{t_{0}-s}\left(\beta(s)-\beta\left(t_{0}\right)\right)$ for all $s \in\left[t_{1}, t_{0}\right]$. Passing to the lower limit as $s \rightarrow t_{0}-$ and using the lower semicontinuity of $v$, we conclude $D_{\uparrow} v\left(t_{0}, x_{0}\right)\left(-1,-f\left(t_{0}, x_{0}, u_{0}\right)\right) \leq \ell\left(t_{0}, x_{0}, u_{0}\right)$.

Since $u_{0} \in \mathbb{B}$ is arbitrary, the statement (iii) follows with $C^{\prime}=(0,+\infty) \backslash \cup_{j \in \mathbb{N}}$ $C_{j}^{\prime}$. The statement (iv) holds as well arguing in a similar way.

Remark 5.2 Notice that, from [6, Proposition 4.4] and Proposition 5.1] under the assumptions H.2.1-6 and O.P.C., the set-valued map $t \rightsquigarrow \operatorname{epi} v(t,$.$) is l.a.c.$ even though $v$ may be discontinuous.
5.2. Viability of hypograph. Next lemma provides a viability result of the hypograph of weak solutions.

Lemma 5.2 Assume H.2.1-5. Let $V: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R} \cup\{+\infty\}$ be such that

$$
t \rightsquigarrow\{(x, \lambda) \in \Omega \times \mathbb{R} \mid \lambda \leq V(t, x)<+\infty\} \text { is l.a.c. }
$$

If there exists a set $E^{\prime} \subset(0,+\infty)$, with $\mu\left((0,+\infty) \backslash E^{\prime}\right)=0$, such that

$$
\begin{align*}
& -p_{t}+\sup _{u \in \mathbb{B}}\left\{\left\langle f(t, x, u),-p_{x}\right\rangle+q \ell(t, x, u)\right\} \leq 0  \tag{22}\\
& \forall\left(p_{t}, p_{x}, q\right) \in T_{\mathrm{hypo}} V(t, x, V(t, x))^{+}, \forall(t, x) \in \operatorname{dom} V \cap\left(E^{\prime} \times \operatorname{int} \Omega\right),
\end{align*}
$$

then for all $0<\tau_{0}<\tau_{1}$ and any feasible trajectory-control pair $(\xi(),. u()$.$) on$ $I=\left[\tau_{0}, \tau_{1}\right]$, with $\xi\left(\left[\tau_{0}, \tau_{1}\right]\right) \subset \operatorname{int} \Omega$ and $\left(\tau_{0}, \xi\left(\tau_{0}\right)\right) \in \operatorname{dom} V$, we have

$$
\left(\xi(t), V\left(\tau_{0}, \xi\left(\tau_{0}\right)\right)-\int_{\tau_{0}}^{t} \ell(s, \xi(s), u(s)) d s\right) \in \operatorname{hypo} V(t, .) \quad \forall t \in\left[\tau_{0}, \tau_{1}\right]
$$

Proof Notice that, by the separation theorem, (22) is equivalent to $\{1\} \times$ $G(t, x) \subset \overline{\operatorname{co}} T_{\text {hypo } V}(t, x, \beta)$ for all $\beta \leq V(t, x)$ and all $(t, x) \in\left(E^{\prime} \times \operatorname{int} \Omega\right) \cap$ dom $V$, where we defined

Let $0<\tau_{0}<\tau_{1}$ and put $Q(s):=$ hypo $V(s,$.$) for any s \in\left[\tau_{0}, \tau_{1}\right]$. We have

$$
\begin{equation*}
(1, f(s, x, u),-\ell(s, x, u)) \in \overline{\operatorname{co}} T_{\operatorname{graph} Q}(s, x, \beta) \tag{23}
\end{equation*}
$$

for a.e. $s \in\left[\tau_{0}, \tau_{1}\right]$, any $(x, \beta) \in Q(s) \cap(\operatorname{int} \Omega \times \mathbb{R})$, and any $u \in \mathbb{B}$. Consider a trajectory-control pair $(\xi(),. u()$.$) solving (20) on I=\left[\tau_{0}, \tau_{1}\right]$, with $\xi\left(\left[\tau_{0}, \tau_{1}\right]\right) \subset$ int $\Omega$ and $\left(\tau_{0}, \xi\left(\tau_{0}\right)\right) \in \operatorname{dom} V$. We claim that $d_{Q(s)}((\xi(s), u(s)))=0$ for all $s \in I$, where $w($.$) is the unique solutions of$

$$
w^{\prime}(t)=-\ell(t, \xi(t), u(t)) \text { for a.e. } t \in\left[\tau_{0}, \tau_{1}\right], \quad w\left(\tau_{0}\right)=V\left(\tau_{0}, \xi\left(\tau_{0}\right)\right)
$$

Putting $g(s)=d_{Q(s)}((\xi(s), w(s)))$, applying [9, Lemma 4.8] and Lemma5.1to the single-valued map $s \rightsquigarrow\{(f(s, \xi(s), u(s)),-\ell(s, \xi(s), u(s)))\}$, we have that $g($.$) is absolutely continuous. Let for any s \in I$ the pair $(p(s), r(s)) \in Q(s)$ be such that

$$
g(s)=|(\xi(s), w(s))-(p(s), r(s))| .
$$

We claim that $g(s)=0$ for all $s \in\left(\tau_{0}, \tau_{1}\right]$. Suppose, by contradiction, that we can find $T \in\left(\tau_{0}, \tau_{1}\right]$ with $g(T)>0$. Denoting $t^{*}=\sup \left\{t \in\left[\tau_{0}, T\right]: g(t)=0\right\}$, let $\varepsilon>0$ be such that $p(s) \in \operatorname{int} \Omega$ and $g(s)>0$ for any $s \in\left(t^{*}, t^{*}+\right.$ $\varepsilon]$. Consider $s \in\left(t^{*}, t^{*}+\varepsilon\right)$ where $g(),. \xi($.$) , and w($.$) are differentiable,$
with $\xi^{\prime}(s)=f(s, \xi(s), u(s))$ and $w^{\prime}(s)=\ell(s, \xi(s), u(s))$. Consider $(\theta, \lambda) \in$ $T_{\text {graph } Q}(s, p(s), r(s))$ and $\theta_{i} \rightarrow \theta, \lambda_{i} \rightarrow \lambda, h_{i} \rightarrow 0+\operatorname{satisfy}(p(s), r(s))+h_{i} \lambda_{i} \in$ $Q\left(s+h_{i} \theta_{i}\right) \quad$ for all $i \in \mathbb{N}$. Then, setting $q=(\xi(s), w(s))$ and $\bar{q}=(p(s), r(s))$, we get

$$
g\left(s+h_{i} \theta_{i}\right)-g(s) \leq\left|\left(\xi\left(s+h_{i} \theta_{i}\right), w\left(s+h_{i} \theta_{i}\right)\right)-\bar{q}-h_{i} \lambda_{i}\right|-|q-\bar{q}| .
$$

Dividing this inequality by $h_{i}$, passing to the limit as $i \rightarrow+\infty$, and putting $\zeta:=\frac{q-\bar{q}}{|q-\bar{q}|}$, we have

$$
\begin{equation*}
g^{\prime}(s) \theta \leq\langle\zeta,(f(s, \xi(s), u(s)),-\ell(s, \xi(s), u(s))) \theta-\lambda\rangle . \tag{24}
\end{equation*}
$$

Since (24) holds for any $(\theta, \lambda) \in T_{\text {graph } Q}(s, p(s), r(s))$, taking convex combinations of elements in $T_{\operatorname{graph} Q}(s, p(s), r(s))$ we conclude that (24) holds for all $(\theta, \lambda) \in \overline{\operatorname{co}} T_{\text {graph } Q}(s, p(s), r(s))$. By (23), the inequality (24) holds true for $\theta=1$ and $\lambda=(f(s, p(s), u(s)),-\ell(s, p(s), u(s)))$. Therefore, from Lemma 5.17(iv),

$$
g^{\prime}(s) \leq k(s)|\xi(s)-p(s)| \leq k(s) g(s) .
$$

From the Gronwall lemma we conclude that $g()=$.0 on $\left[t^{*}, t^{*}+\varepsilon\right]$, and a contradiction follows.
5.3. Proof of Theorem 5.1. In this section we provide a proof of Theorem 5.1.

Proposition 5.2 Assume H.2.1-6 and O.P.C. Let $V: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R} \cup\{+\infty\}$ be a lower semicontinuous function, satisfying the vanishing condition at infinity (2), such that $\operatorname{dom} v(t,.) \subset \operatorname{dom} V(t,.) \neq \emptyset$ for all large $t>0$ and

$$
\begin{equation*}
t \rightsquigarrow\{(x, \lambda) \in \Omega \times \mathbb{R} \mid \lambda \leq V(t, x)<+\infty\} \text { is l.a.c. } \tag{25}
\end{equation*}
$$

Then the following statements are equivalent:
(i) $V=v$;
(ii) $t \rightsquigarrow \operatorname{epi} V(t,$.$) is l.a.c. and there exists E \subset(0,+\infty)$, with $\mu((0,+\infty) \backslash E)=$ 0, such that:
(ii.a) $-p_{t}+\sup _{u \in \mathbb{B}}\left\{\left\langle f(t, x, u),-p_{x}\right\rangle+q \ell(t, x, u)\right\} \geq 0$

$$
\forall\left(p_{t}, p_{x}, q\right) \in T_{\mathrm{epi} V}(t, x, V(t, x))^{-}, \forall(t, x) \in \operatorname{dom} V \cap(E \times \Omega)
$$

(ii.b) $-p_{t}+\sup _{u \in \mathbb{B}}\left\{\left\langle f(t, x, u),-p_{x}\right\rangle+q \ell(t, x, u)\right\} \leq 0$

$$
\forall\left(p_{t}, p_{x}, q\right) \in T_{\operatorname{hypo} V}(t, x, V(t, x))^{+}, \forall(t, x) \in \operatorname{dom} V \cap(E \times \operatorname{int} \Omega)
$$

Proof Notice that, by the definition of locally absolutely continuous set-valued map, the hypograph of $V(t,$.$) restricted to dom V(t,$.$) is closed. To show the$ equivalence between statements (i) and (ii), we use the following claim: for any $(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}$ with $\mathscr{U}_{\Omega}(t, x) \neq \emptyset, v(t, x)$ is equal to the following infimum
$(\mathrm{CV})\left\{\begin{array}{l}\inf \int_{t}^{+\infty} H^{*}\left(s, \xi(s), \xi^{\prime}(s)\right) d s \text { over all } \xi \in \mathscr{W}_{\text {loc }}^{1,1}\left(t,+\infty ; \mathbb{R}^{n}\right) \\ \text { such that } \xi(t)=x \text { and } \xi(.) \subset \Omega .\end{array}\right.$
Indeed, let $(t, x) \in \mathbb{R}^{+} \times \mathbb{R}^{n}$ such that $\mathscr{U}_{\Omega}(t, x) \neq \emptyset$ and denote by $\alpha(t, x) \in$ $\mathbb{R} \cup\{ \pm \infty\}$ the infimum in (CV) above. From assumption H.2.5 we have that $\alpha(t, x) \neq-\infty$. If $\alpha(t, x)=+\infty$ then $\alpha(t, x) \geq v(t, x)$. Assume $\alpha(t, x) \in \mathbb{R}$. Fix $\varepsilon>0$ and consider $\xi \in \mathscr{W}_{l o c}^{1,1}\left(t,+\infty ; \mathbb{R}^{n}\right)$ with $\xi(t)=x$ and $\xi(.) \subset \Omega$ satisfying $\int_{t}^{+\infty} H^{*}\left(s, \xi(s), \xi^{\prime}(s)\right) d s<\alpha(t, x)+\varepsilon$. We have that $\left(\xi^{\prime}(s), u^{\prime}(s)\right) \in$ graph $H^{*}(s, \xi(s),$.$) for a.e. s \geq t$, where we put $u(s):=\int_{t}^{s} H^{*}\left(\tau, \xi(\tau), \xi^{\prime}(\tau)\right) d \tau$ for all $s \geq t$. Applying now the representation Theorem4.17(v) ${ }^{\prime}$ and the measurable selection theorem, we have that there exists a measurable function $w:[t,+\infty) \rightarrow \mathbb{B}$ such that $\left(\xi^{\prime}(s), u^{\prime}(s)\right)=(f(s, \xi(s), w(s)), \ell(s, \xi(s), w(s)))$ for
a.e. $s \geq t$. We get
$\int_{t}^{+\infty} H^{*}\left(s, \xi(s), \xi^{\prime}(s)\right) d s=\int_{t}^{+\infty} u^{\prime}(s) d s=\int_{t}^{+\infty} \ell(s, \xi(s), w(s)) d s \geq v(t, x)$. So, $\alpha(t, x)+\varepsilon>v(t, x)$. Since $\varepsilon$ is arbitrary, we deduce that $\alpha(t, x) \geq v(t, x)$. Arguing in analogous way as above and using (19), we get $\alpha(t, x) \leq v(t, x)$.

Next, we show $(\mathrm{i}) \Longleftrightarrow$ (ii). Assume (i). Applying Lemma 5.1 and the claim, we have that $t \rightsquigarrow$ epi $v(t,$.$) is l.a.c. for any x \in \Omega$. Now, from Proposition 5.1f(iv) and the claim, we can find a subset $C \subset(0,+\infty)$, with $\mu(C)=$ 0 , such that for any $(t, x) \in((0,+\infty) \backslash C) \times \operatorname{int} \Omega$ we have $-\ell(t, x, u) \leq$ $D_{\downarrow} v(t, x)(1, f(t, x, u))$ for all $u \in \mathbb{B}$. Hence, from [1 Proposition 6.1.4], we get

$$
(1, f(t, x, u),-\ell(t, x, u)) \in T_{\text {hypo } v}(t, x, v(t, x))
$$

for any $u \in \mathbb{B}$. Then we get

$$
\begin{aligned}
& -p_{t}+\sup _{u \in \mathbb{B}}\left\langle f(t, x, u),-p_{x}\right\rangle+q \ell(t, x, u) \leq 0 \\
& \forall\left(p_{t}, p_{x}, q\right) \in T_{\text {hypo } v}(t, x, v(t, x))^{+} .
\end{aligned}
$$

Hence, statement (ii.b) holds. Using a similar argument and applying Lemma 5.1 and [6, Theorem 3.3], we get (ii.a). Thus, (ii) follows.

Now, assume (ii). From condition (21) and [6, Theorem 3.3] and its proof, it is just sufficient to show the following: there exists $C \subset(0,+\infty)$, with $\mu((0,+\infty) \backslash C)=0$, such that

$$
\begin{align*}
& \forall(t, x) \in \operatorname{dom} V \cap(C \times \operatorname{int} \Omega), \forall u \in \mathbb{B},  \tag{26}\\
& D_{\uparrow} V(t, x)(-1,-f(t, x, u)) \leq \ell(t, x, u) .
\end{align*}
$$

Recalling the definition of $G(.,$.$) given in (21), applying Lemma 5.1$ and 9 , Theorem 2.9] to the set-valued maps $[0, j] \times \mathbb{R}^{n} \times \mathbb{R} \ni(s, \xi, \beta) \rightsquigarrow-G(j-s, \xi) \in$
$\mathbb{R}^{n} \times \mathbb{R}$ where $j \in \mathbb{N}$, and from the measurable selection theorem, we can find a family of subsets $C_{j}^{\prime} \subset(0, j)$, with $\mu\left(C_{j}^{\prime}\right)=0$ for all $j \in \mathbb{N}$, such that for any $\left(t_{0}, x_{0}\right) \in\left((0,+\infty) \backslash \cup_{j \in \mathbb{N}} C_{j}^{\prime}\right) \times \operatorname{int} \Omega$ and any $u_{0} \in \mathbb{B}$, there exists $t_{1} \in\left(0, t_{0}\right)$ and a trajectory-control pair $((\xi, \beta),(u, r))($.$) satisfying$

$$
\begin{cases}(\xi, \beta)^{\prime}(t)=(f(t, \xi(t), u(t)),-\ell(t, \xi(t), u(t))-r(t)) & t \in\left[t_{1}, t_{0}\right] \text { a.e. } \\ (u, r)(t) \in \mathbb{B} \times[0, c(t)(1+|\xi(t)|)-\ell(t, \xi(t), u(t))] & t \in\left[t_{1}, t_{0}\right] \text { a.e. } \\ \xi\left(\left[t_{1}, t_{0}\right]\right) \subset \operatorname{int} \Omega\end{cases}
$$

with initial condition and final velocity

$$
(\xi, \beta)\left(t_{0}\right)=\left(x_{0}, 0\right), \quad(\xi, \beta)^{\prime}\left(t_{0}\right)=\left(f\left(t_{0}, x_{0}, u_{0}\right),-\ell\left(t_{0}, x_{0}, u_{0}\right)\right)
$$

Hence, applying Lemma 5.2 and taking a sequence $s_{i} \in\left(t_{1}, t_{0}\right)$ with $s_{i} \rightarrow t_{0}-$, we get $V\left(s_{i}, \xi\left(s_{i}\right)\right)-\int_{s_{i}}^{t_{0}} \ell(s, \xi(s), u(s)) d s \leq V\left(t_{0}, x\left(t_{0}\right)\right)$ for all $i \in \mathbb{N}$. So,

$$
V\left(s_{i}, \xi\left(s_{i}\right)\right)-V\left(t_{0}, x_{0}\right) \leq \int_{s_{i}}^{t_{0}} \ell(s, \xi(s), u(s)) d s \leq \beta\left(s_{i}\right) \quad \forall i \in \mathbb{N}
$$

Dividing by $t_{0}-s_{i}$ and passing to the lower limit as $i \rightarrow \infty$, we get (26) with $C=(0,+\infty) \backslash \cup_{j \in \mathbb{N}} C_{j}^{\prime}$, and the proof is complete.

Proof of Theorem 5.1 Let $V: \mathbb{R}^{+} \times \Omega \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function and $(t, x) \in \mathbb{R}^{+} \times \Omega$. Notice that, from the locally Lipschitz continuity of $V$, the following set-valued maps $t \rightsquigarrow$ epi $V(t,$.$) and t \rightsquigarrow$ hypo $V(t,$. are locally absolutely continuous and, since $\partial_{-} V(t, x), \partial_{+} V(t, x)$ are nonempty closed sets, it is straightforward to see that $\cup_{\lambda \geq 0} \lambda\left(\partial_{-} V(t, x),-1\right)$, $\cup_{\lambda \geq 0} \lambda\left(\partial_{+} V(t, x),-1\right)$ are closed too. Now, we claim that:

$$
\begin{equation*}
\cup_{\lambda \geq 0} \lambda\left(\partial_{-} V(t, x),-1\right)=T_{\mathrm{epi} V}(t, x, V(t, x))^{-} \tag{27}
\end{equation*}
$$

Indeed, from the following well known relation (cfr. [1, Chapter 6.4])

$$
\begin{equation*}
\zeta \in \partial_{-} V(t, x) \Longleftrightarrow(\zeta,-1) \in T_{\mathrm{epi} V}(t, x, V(t, x))^{-} \tag{28}
\end{equation*}
$$

it follows that $\cup_{\lambda \geq 0} \lambda\left(\partial_{-} V(t, x),-1\right) \subset T_{\text {epi } V}(t, x, V(t, x))^{-}$. On the other hand, let $(\zeta, q) \in T_{\text {epi } V}(t, x, V(t, x))^{-}$. Since $(0, \delta) \in T_{\text {epi } V}(t, x, V(t, x))^{-}$for all $\delta \geq 0$, we have $q \leq 0$. If $q<0,(\zeta /|q|,-1) \in T_{\text {epi } V}(t, x, V(t, x))^{-}$and, applying (28), $\frac{\zeta}{|q|} \in \partial_{-} V(t, x)$. So, $\left.(\zeta, q) \in \cup_{\lambda \geq 0} \lambda\left(\partial_{-} V(t, x),-1\right)\right)$. If $q=0$, consider $\bar{\zeta} \in \partial_{-} V(t, x)$. Then $(\bar{\zeta},-1) \in T_{\text {epi } V}(t, x, V(t, x))^{-}$, and, from the convexity of the polar cone, $(r \bar{\zeta}+(1-r) \zeta,-r) \in T_{\text {epi } V}(t, x, V(t, x))^{-}$for all $0<r<1$. Arguing as above, we conclude that $(r \bar{\zeta}+(1-r) \zeta,-r) \in$ $\cup_{\lambda \geq 0} \lambda\left(\partial_{+} V(t, x),-1\right)$, and the claim (27) follows. Using the same argument as above, we have also

$$
\begin{equation*}
\cup_{\lambda \geq 0} \lambda\left(\partial_{+} V(t, x),-1\right)=T_{\text {hypo } V}(t, x, V(t, x))^{+} . \tag{29}
\end{equation*}
$$

Finally, from (27), (29), Proposition 5.2 and the representation Theorem 4.1. the conclusion follows.

## References

1. J.-P. Aubin and H. Frankowska. Set-valued analysis. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2009.
2. M. Bardi and I. Capuzzo-Dolcetta. Optimal control and viscosity solutions of Hamilton-Jacobi-Bellman equations. Systems \& Control: Foundations \& Applications. Birkhäuser Boston, Inc., Boston, MA, 1997.
3. G. Barles. Existence results for first-order Hamilton-Jacobi equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 1(5):325-340, 1984.
4. E. N. Barron and R. Jensen. Optimal control and semicontinuous viscosity solutions. Proc. Amer. Math. Soc., 113(2):397-402, 1991.
5. V. Basco and H. Frankowska. Lipschitz continuity of the value function for the infinite horizon optimal control problem under state constraints. In F. Alabau-Boussouira et al., editor, Trends in Control Theory and Partial Differential Equations, volume 32 of Springer INdAM Series, pages $15-52$. Springer International Publishing, 1985.
6. V. Basco and H. Frankowska. Hamilton-Jacobi-Bellman equations for infinite horizon control problems under state constraints with time-measurable data. NoDEA-Nonlinear Differential Equations and Applications, 26(1):7, 2019.
7. M. G. Crandall, L. C. Evans, and P.-L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. Trans. Amer. Math. Soc., 282(2):487-502, 1984.
8. H. Frankowska. Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations. SIAM J. Control Optim., 31(1):257-272, 1993.
9. H. Frankowska, S. Plaskacz, and T. Rzeżuchowski. Measurable viability theorems and the Hamilton-Jacobi-Bellman equation. J. Differential Equations, 116(2):265-305, 1995.
10. H. Frankowska and H. Sedrakyan. Stable representation of convex hamiltonians. Nonlinear Analysis: Theory, Methods \& Applications, 100:30-42, 2014.
11. H. Ishii. Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets. Bull. Fac. Sci. Engrg. Chuo Univ., 28:33-77, 1985.
12. H. Ishii. On representation of solutions of Hamilton-Jacobi equations with convex hamiltonians. In K. Masuda and M. Mimura, editors, Recent Topics in Nonlinear PDE II, volume 128 of North-Holland Mathematics Studies, pages $15-52$. North-Holland, 1985.
13. H. Ishii. Representation of solutions of Hamilton-Jacobi equations. Nonlinear Analysis: Theory, Methods $\mathcal{E}$ Applications, 12(2):121-146, 1988.
14. H. Ishii. Perron's method for monotone systems of second-order elliptic partial differential equations. Differential Integral Equations, 5(1):1-24, 1992.
15. H. Ishii and S. Koike. A new formulation of state constraint problems for first-order PDEs. SIAM J. Control Optim., 34(2):554-571, 1996.
16. P.-L. Lions. Generalized solutions of Hamilton-Jacobi equations, volume 69 of Research Notes in Mathematics. Pitman, 1982.
17. A. Misztela. Representation of Hamilton-Jacobi equation in optimal control theory with compact control set. SIAM Journal on Control and Optimization, 57(1):53-77, 2019.
18. F. Rampazzo. Faithful representations for convex Hamilton-Jacobi equations. SIAM J. Control Optim., 44(3):867-884, 2005.
19. R. T. Rockafellar. Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization. Math. Oper. Res., 6(3):424-436, 1981.
20. R. T. Rockafellar and R. J. B. Wets. Variational analysis, volume 317 of Grundlehren der Mathematischen Wissenschaften. Springer-Verlag, Berlin, 1998.
21. H. M. Soner. Optimal control problems with state-space constraints I. SIAM J. Control Optim., 24:552-562, 1986.
22. P. E. Souganidis. Existence of viscosity solutions of Hamilton-Jacobi equations. J. Differential Equations, 56(3):345-390, 1985.
23. R. B. Vinter. Optimal Control. Birkhäuser, Boston, MA, 2000.

[^0]:    Vincenzo Basco
    Department of Electrical \& Electronic Engineering
    The University of Melbourne
    Victoria 3010

    Australia

[^1]:    ${ }^{1}$ If $w \in \mathscr{L}_{\text {loc }}^{1}(a,+\infty ; \mathbb{R})$, we denote $\int_{a}^{\infty} w(s) d s:=\lim _{b \rightarrow \infty} \int_{a}^{b} w(s) d s$, provided this limit exists.

