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Introducing Identity

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Abstract
The best-known syntactic account of the logical constants is *inferentialism* . Following Wittgenstein's thought that meaning is use, inferentialists argue that meanings of expressions are given by introduction and elimination rules. This is especially plausible for the logical constants, where standard presentations divide inference rules in just this way. But not just any rules will do, as we've learnt from Prior's famous example of *tonk*, and the usual extra constraint is *harmony*. Where does this leave identity? It's usually taken as a logical constant but it doesn't seem harmonious: standardly, the introduction rule (reflexivity) only concerns a subset of the formulas canvassed by the elimination rule (Leibniz's law). In response, Read [5, 8] and Klev [3] amend the standard approach. We argue that both attempts fail, in part because of a misconception regarding inferentialism and identity that we aim to identify and clear up.

Keywords Identity · Logical constants · Inferentialism · Meaning-is-use · Proof-theoretic semantics

The best-known syntactic account of the logical constants is *inferentialism*. Following Wittgenstein's thought that meaning is use, inferentialists argue that meanings of expressions are given by introduction and elimination rules. This is especially plausible for the logical constants, where standard presentations divide inference rules in just this way. But not just any rules will do, as we've learnt from *tonk*, and the usual extra constraint is *harmony*. Roughly, the introduction and elimination rules should be balanced in strength.

Where does this leave *identity*? It's usually taken as a logical constant, but it doesn't seem harmonious: standardly, its introduction rule (reflexivity) only concerns a subset

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of the formulas canvassed by the elimination rule (Leibniz's law). In response, Read [5, 8] and Klev [3] amend the standard approach, either directly amending the rules (Read) or changing the background against which they function (Klev).

We argue that both attempts fail, in part because of a misconception regarding inferentialism and identity that we aim to identify and to clear up.

In short: Read may and Klev does take the disharmony of the standard presentation to lie in the impossibility of *justifying* the elimination rule in terms of the introduction rule. The elimination rule, on this presentation, is *too strong* relative to the introduction rule. And they set out alternative representations on which the elimination rule *is* justifiable relative to the introduction rule.

But that is the wrong target. As we argue at §1, the problem was not that the elimination rule was too *strong* but that it was too *weak*. There is no problem with *justifying* the standard elimination rule (Leibniz's Law), given the standard introduction rule (reflexivity). The problem is that the standard introduction rule licenses *more* inferences from an identity $a=b$ than the standard elimination rule allows. Leibniz's Law does not exploit the full strength of any logical constant $\#$ for which the only introduction rule is that $a\#a$ is always assertible. *That* is the problem for inferentialism about identity.

§2 argues that Read's definition is inferentially inert relative to the traditional one given first-order predicate logic without identity. Against that background, Read's rules make identity behave just as under the traditional rules. Clearly then, they cannot fix the disharmony of identity, on Read's construal of the problem or on ours. §2 defends this understanding of the situation against Read's own objections.

As we see in §3, Klev's definition of identity *does* have an inferential effect given his framework of definitional identity. Relative to that his justification of the elimination rule is cogent. But that doesn't help with harmony because it doesn't address what causes the problem.

So neither Read nor Klev has offered harmonious introduction and elimination rules. This hardly proves that none *can* be found. §4 briefly suggests why a harmonious theory of identity may be unavailable in principle. If so, identity is not a logical constant after all, not by inferentialist lights. We cannot settle this here.

1 The Standard Rules

1.1 Harmony

Calling introduction and elimination rules for a constant *harmonious* means that the elimination rules 'draw no more and no less from an assertion than the introduction rules warrant'.[1] Thus harmony makes two demands, one the converse of the other[2]:

> (H1) The elimination rules (#*E*) for a logical constant # ('sharp') license us to draw a #-free conclusion *Q* from a #-involving assertion *P* together with any

[1] [5]: 115.

[2] When judging harmony, we are only concerned with the interaction between #-free and #-involving sentences.

side-premises, *only if* we could already have inferred Q from those side-premises together with *any* #-free ground from which the introduction rules (#I) licensed us in inferring P in the first place.

(H2) The elimination rules (#E) for a logical constant # license us to draw a #-free conclusion Q from a #-involving assertion P together with any side-premises, *if* we could already have inferred Q from those side-premises together with *any* #-free ground from which the introduction rules (#I) licensed us in inferring P in the first place.

Consider e.g. the standard rules for conjunction:

$$(\&\mathrm{I})\ \frac{A\quad B}{A\&B.}$$

$$(\&\mathrm{E1})\ \frac{A\&B}{A}$$

$$(\&\mathrm{E2})\ \frac{A\&B}{B}$$

First, suppose we can infer Q from $A \,\&\, B$ using the elimination rules and given side premises. Since (&E1) and (&E2) are the only elimination rules for conjunction, this means that we can infer Q from $\{A, B\}$ given those side premises. Since (&I) is the only introduction rule, we can infer Q from the side-premises together with any of the grounds for $A \,\&\, B$ that are licensed by the introduction rules.[3] So the rules satisfy (H1).

Second, suppose we can infer a conjunction-free Q from *any* grounds for $A \,\&\, B$ given some side premises. Since (&I) is the only introduction rule for conjunction, this means that we can infer Q from $\{A, B\}$ together with those side-premises. Then since by (&E1) and (&E2) we can infer both A and B from $A \,\&\, B$, we can infer Q from $A \,\&\, B$ together with those side premises.[4] So the rules satisfy (H2).

The rules satisfy *both* conditions. First, the elimination rules license drawing Q from $A \,\&\, B$ *only* when we could have already drawn Q from any of the grounds for $A \,\&\, B$. Second, the elimination rules license drawing Q from $A \,\&\, B$ *whenever* we could have already drawn Q from any of the grounds for $A \,\&\, B$. So these rules are harmonious. This reasoning is classical, and we assume classical logic throughout the paper. The assumption is unproblematic, however: the cases that we consider will not essentially rely on classical rules that e.g. the intuitionist or relevance logician would reject.

[3] We assume that inference tracks a derivability relation that satisfies dilution (adding premises doesn't invalidate a derivation).

[4] We assume that inference tracks a derivability relation that satisfies the cut or transitivity condition.

1.2 Harmony and Identity

But consider now the usual schematic rules for identity:

$$\text{(Refl)}\quad a = a$$

$$\text{(LL)}\ \frac{a = b \quad \alpha_x^a}{\alpha_x^b}$$

In (Refl) and (LL), the instances of a and b are any singular terms. In (Refl), 'α_x^a denotes the result of replacing every free occurrence of "x" by "a" if "x" is free for "a" in α, and if not, the result of replacing every free occurrence of "x" by "a" in a well-defined bound alphabetic variant of α in which "x" is free for "a"', and similarly for α_x^b.[5]

(Refl) and (LL) are harmonious if and only if (LL) draws (H1) no more and (H2) no less from an assertion than (Refl) warrants. That is: (LL) licenses us to draw an = −free conclusion Q from an = −involving assertion P *if and only if* we could already have inferred Q from any of the = −free grounds from which (Refl) licensed us in inferring P in the first place.

(Refl) and (LL) are *not* harmonious. They meet (H1) i.e. the left-to-right direction of the foregoing biconditional, but not (H2) i.e. not the right-to-left direction.

Regarding (H1), consider first a 'homonymous' identity $a = a$, where the *same* term appears on both sides. Clearly (LL) licenses no non-trivial inferences from $a = a$: given side-premise α_x^a we can only infer that α_x^a itself. And clearly, any such conclusion α_x^a *could* already have been inferred from any of the grounds that (Refl) provides for $a = a$ together with the side-premise, because the conclusion *is* the side-premise.[6]

Next consider a 'heteronymous' $a = b$, where *different* terms appear on either side. There is *no* application of the introduction rule (Refl) that terminates in a heteronymous $a = b$. So there are *no* grounds on which that rule licenses $a = b$. Therefore trivially, *any* such grounds for $a = b$, together with α_x^a, justify α_x^b. (LL), in other words, never permits the drawing of conclusions from $a = b$ and side premises that was not already warranted by the grounds for $a = b$ together with those side premises.

But every identity statement is either homonymous or heteronymous. So (Refl) and (LL) satisfy (H1). (LL) licenses us to draw an =-free conclusion Q from an =-involving assertion P *only if* we could already have inferred Q from any of the =-free grounds from which (Refl) licensed us in inferring P in the first place.

But they do *not* satisfy (H2). Let a and b be different terms, and α_x^a a sentence, of the identity-free language L into which identity is being introduced. Then there are no grounds from which (Refl) lets us infer $a = b$, because a and b are different terms, and (Refl) only lets us infer instances of $x = x$. Now let Q be an arbitrary sentence of L. Then vacuously, for *any* of the grounds from which (Refl) licenses us in inferring $a = b$, we

[5] [8]: 8 n. 18.

[6] We assume that inference tracks a derivability relation that satisfies reflexivity (everything is derivable from itself) and dilution.

could have inferred Q directly from α^a_x together with those grounds – because there *are* no such grounds. So (LL) should let us infer *any* Q from α^a_x and $a = b$.[7] But it does not.

So the elimination rule unduly restricts what we may infer from $a = b$. In the terminology (see e.g. [10]), the rules are *E-weakly* disharmonious. The elimination rule lets one infer less from $a = b$ than might have already been inferred from any of the grounds on which the introduction rule licensed $a = b$ as a conclusion. This stands in contrast with much of the literature on identity and harmony. Griffiths [2], for example, assumes that the disharmony here must be E-strong, given that Refl only concerns a subclass of the formulas canvassed by LL. But when we take the wording of harmony seriously, and are not misled by appearances, we see that Refl and LL are in fact E-*weakly* disharmonious.

2 Read's Rules

2.1 Read's Introduction Rule

Read's amended introduction rule is supposed to harmonize with his elimination rule by explicitly specifying grounds from which we can infer a heteronymous $a = b$. Read's proposed rules are[8]:

$$(=I')\quad \frac{\begin{array}{cc}[Fa] & [Fb]\\ \vdots & \vdots\\ Fb & Fa\end{array}}{a = b}$$

$$(\mathrm{Congr})\quad \frac{a = b \quad Fa}{Fb}$$

$$(\mathrm{Congr}')\quad \frac{a = b \quad Fb}{Fa}$$

[7] More formally: let meta-linguistic variables Q and Σ range over sentences of a language L and over sets of sentences and derivations in L respectively. (H2) requires that:

$$\forall Q\left(\forall \Sigma\left(\Sigma \vdash_{\mathrm{L,Refl}} a = b \rightarrow \Sigma \cup \left\{\alpha^a_x\right\} \vdash_L Q\right) \rightarrow \left\{a = b, \alpha^a_x\right\} \vdash_{\mathrm{L,LL}} Q\right)$$

$\Sigma \vdash_{\mathrm{L,\ Refl}} a = b$ is false for every Σ. So $\forall \Sigma\left(\Sigma \vdash_{\mathrm{L,Refl}} a = b \rightarrow \Sigma \cup \left\{\alpha^a_x\right\} \vdash_L s\right)$ is true for every s. The stated condition implies $\left\{a = b, \alpha^a_x\right\} \vdash_{\mathrm{L,Congr}} s$ for every s.

[8] [8]: 416.

In (=I′), F is a predicate variable ranging over monadic predicate letters that do not occur in any side-premises.[9]

Are (=I′), (Congr) and (Congr′) harmonious? It looks like it. (H1) demands:

(i) (Congr) lets us infer from $a = b$ and the specified side-premise Fa only what we could have inferred from Fa together with any of the grounds for $a = b$ that were licensed by (=I′).

(ii) (Congr′) lets us infer from $a = b$ and the specified side-premise Fb only what we could have inferred from Fb together with any of the grounds for $a = b$ that were licensed by (=I′).

And (H2) demands:

(iii) (Congr) and (Congr′) together let us infer from $a = b$ and a side-premise everything that we could already have inferred directly from that side-premise together with any grounds for $a = b$ given by (=I′).

Starting with (i) and (ii): something is a ground from which we could have inferred $a = b$ using (=I′) only if it includes derivations, in the host language, of Fb from Fa, and of Fa from Fb. Given Fa as a side-premise, we *could* have inferred Fb directly from these grounds; given Fb as a side-premise, we could have inferred Fa directly from these grounds. More specifically: (Congr) lets us infer from $a = b$ and Fa only what we could have inferred from Fa together with the grounds for $a = b$ that were licensed by (=I′); (Congr′) lets us infer from $a = b$ and Fb only what we could have inferred from Fb together with any of the grounds for $a = b$ that were licensed by (=I′). So Read's rules apparently satisfy (H1).

As for (iii), suppose we start with $a = b$. Then obviously we can produce a derivation of Fb from the assumption [Fa] by a single application of (Congr). Similarly we can produce a derivation of Fa from the assumption [Fb] by means of (Congr′). So taken together, (Congr) and (Congr′) license production of any of the grounds from which (=I′) licenses the inference of $a = b$. It *looks* therefore as if (Congr) and (Congr′) also jointly license the derivation of anything that was already derivable from any grounds of the original language from which (=I′) licenses the inference of $a = b$. So the rules appear to satisfy (H2).

Read's rules therefore seem harmonious. And if the possibility of explanation by harmonious rules suffices for logicality, then they also seem to prove identity a logical constant.

2.2 Griffiths on Harmony and Identity

But as Griffiths shows, Read's rules are exactly as strong as the original rules (Refl) and (LL) when the host language is first-order classical predicate logic without identity. The

[9] The second-order variables in Read's rules don't take us beyond the resources of first-order logic. Only their first-order instances appear in proofs. If the rules do take us beyond first-order resources, then this is trouble for Read since he certainly intends them to be first-order.

reason is that in that setting, (=I′) licenses $a = b$ in just the same circumstances as does (Refl).

Informally, this is because in classical first-order predicate logic plus (=I′), there is *no* way to derive Fb from [Fa] when a and b are different terms, for *any* monadic F in which identity does not occur and that does not itself occur in any side-premise. And if F is a predicate in which identity *does* occur, say $b = x$, then derivation of Fb from [Fa] is possible using Read's introduction rules, but *also* possible using the original rule (Refl). On the other hand, Fb is trivially derivable from [Fa] when a and b are the *same* term, for *every* monadic F. So (=I′) licenses inference of $a = a$ from any set of premises. But then so does (Refl); so again replacing (Refl) with (=I′) has no effect.[10]

Read's (=I′), (Congr) and (Congr′) therefore *fail* (H2) for the same reason that (Refl) and (LL) do. Condition (iii) says that (Congr) and (Congr′) together let us infer from $a = b$ and a side-premise everything we could have inferred directly from that side-premise together with any of the grounds for $a = b$ specified by (=I′). But if a and b are distinct terms then there are *no* such grounds in classical logic without identity, since such a ground must include a derivation of Fb from [Fa], and as Griffiths shows there are no such derivations. For any Q, it is therefore (vacuously) true that it can be inferred from any of the grounds from which (=I′) licenses $a = b$. So harmony demands that (Congr) and (Congr′) license inference of any such Q from $a = b$; but they do not.[11]

It may have obscured this point that Read occasionally writes as though the problem with the traditional rules is that (Refl) is too weak to *justify* (LL) i.e. that the traditional rules violate (H1). For instance, in an early paper he writes that '[t]he problem with the standard rules for "=" is that (Refl) seems *too weak to justify* [LL]'.[12] Similarly, Klev writes:

> The cause of the felt disharmony is the difference between the form of the conclusion of [Refl], viz. $t = t$, and the form of the major premiss of [LL] viz. $t = u$. This difference in form makes it unclear how one should go about justifying [LL] on the basis of [Refl].[13]

If *that* were the problem, then indeed Read's definition of identity solves it, because (=I′), (Congr) and (Congr′) satisfy (H1). But it can't have been the problem, because (Refl) and (LL) *also* satisfy (H1). The real problem is that the traditional rules violate the *other* condition, (H2). And by establishing that Read's rules are inferentially

[10] See further [2].

[11] More formally, (H1) requires:

$$\forall s\left(\forall\Sigma\left(\Sigma\vdash_{L,=I'} a = b \rightarrow \Sigma\cup\left\{\alpha_x^a\right\}\vdash_L s\right)\rightarrow\left\{a = b, \alpha_x^a\right\}\vdash_{\mathrm{L,Congr,Congr'}} s\right)$$

But since $\Sigma\vdash_{L,=I'} a = b$ is false for every Σ, $\forall\Sigma\left(\Sigma\vdash_{L,=I'} a = b\rightarrow\Sigma\cup\left\{\alpha_x^a\right\}\vdash_L s\right)$ is true for every s. The stated condition therefore implies $\left\{a = b, \alpha_x^a\right\}\vdash_{L,\mathrm{Congr,Congr'}} s$ for every s.

[12] [5]: 115 (our emphasis).

[13] [3]: 869.

equivalent to the traditional ones, Griffiths's argument implies that they fall short of harmony in the same way.

2.3 Read's Modal Analogy

Read has responded that although his rules are equivalent to the standard ones, there is more to harmony than balance of inferential power between the I- and E-rules. Harmony also requires that the meaning of the expression be given entirely by its I-rules. Read argues that taking this *I-rules first* condition seriously makes his identity rules superior to the standard ones. This notion of harmony Read calls *general elimination* (ge) harmony.

This understanding of harmony has the surprising upshot that two inferentially equivalent pairs of inference rules can differ with respect to harmony:

> on the general-elimination account of harmony, inferentially equivalent rules need not be equally harmonious. Whether a set of I- and E-rules is ge-harmonious is not simply a matter of what can be inferred; it is also required that the full meaning is expressed by the I-rules, and that the E-rules are generated in accordance with the meaning by the ge-procedure ([8]: 7)

Read claims that this phenomenon is not surprising and that there are mundane examples of it.

His central example is the possibility operator of S4 in the Curry-Fitch-Prawitz (CFP) presentation:

$$(\Diamond\mathrm{I}) \quad \frac{\alpha}{\Diamond\alpha}$$

$$(\Diamond\mathrm{E}) \quad \frac{\Diamond\alpha \quad \begin{array}{c}[\alpha]\\ \vdots \\ \gamma\end{array}}{\gamma}$$

In ($\Diamond$E), every undischarged assumption on which γ depends, other than α, is *modal* (has the form $\Box\beta$ or $\neg\Diamond\beta$ or $\bot$) and γ is *co-modal* (has the form $\Diamond\beta$ or $\neg\Box\beta$ or $\neg\bot$). These rules can be strengthened for S5 or weakened for K and T.

Read [6] notes that these rules can be proven *normalizable*. That is: (i) any *local peak,* where a connective is introduced and immediately eliminated, can be *levelled* to achieve a proof in which it does not occur; and (ii) any proof in which that connective is introduced and subsequently eliminated can be rewritten as a proof in which it only features in local peaks.

Despite the CFP rules' normalizability, he argues, they are not in ge-harmony since the whole meaning is not contained in the I-rules. Focus on the ($\diamond$I) rule: $\diamond\alpha$ is justified when we have established α. Harmony dictates that the grounds for assertion of $\diamond\alpha$ should be what follows from an application of ($\diamond$E), but this is not the case. The grounds for assertion of $\diamond\alpha$ is α but we don't want α to follow from $\diamond\alpha$.

Read explains the problem as follows:

> The failure of harmony in the case of the CFP rules for possibility is one of I-weak disharmony, as dubbed by Steinberger [10]: that is, the CFP ($\diamond$I) rule is too weak to justify the restrictions that need to be placed on the ($\diamond$E) rules to ensure that between them they characterise possibility. ([8]: 11)

For this reason, he endorses[14] alternative rules for the modal operators in a *labelled* deductive system[15]:

$$(\diamond\mathrm{I}^*)\ \frac{\alpha_j \quad i<j}{\diamond\alpha_i} \qquad (\diamond\mathrm{E}^*)\ \frac{\alpha_i \qquad \begin{array}{c}[\alpha_j,\ i<j]\\ \vdots\\ \gamma_k\end{array}}{\gamma_k}$$

In ($\diamond$E*), k is any index, $i \neq j$, $j \neq k$ and $i \neq k$, and j must not occur in any undischarged assumption on which γ_k depends other than (α_j, $i<j$).

For these rules to characterise S4, the accessibility relation expressed by < must be governed by the following rules:

$$(\mathrm{T})\ \frac{\begin{array}{c}(i<i)\\ \vdots\\ \alpha_j\end{array}}{\alpha_j} \qquad (\mathrm{B})\ \frac{i<j \qquad \begin{array}{c}[j<i]\\ \vdots\\ \alpha_k\end{array}}{\alpha_k} \qquad (\mathrm{S4})\ \frac{i<j \quad j<k \qquad \begin{array}{c}[i<k]\\ \vdots\\ \alpha_l\end{array}}{\alpha_l}$$

Unlike the CFP rules, these rules *are* in ge-harmony since they are normalizable *and* the whole meaning is contained in the I-rule. Read draws an analogy between (Refl) and (= I') on the one hand, and the standard CFP ($\diamond$I) and his ($\diamond$I*) rule on the other. They are analogous in the following sense: Read's (= I') and ($\diamond$I*) are in ge-harmony with their corresponding E-rules – (Congr), (Congr′) and ($\diamond$E*), respectively – whereas the standard (Refl) and ($\diamond$I) are not, despite both pairs of modal rules equivalently characterising S4 and both pairs of identity rules equivalently characterising identity.

[14] Read attributes the labelled system to Rahman and Keiff [4], Simpson [9], and Viganò [11].
[15] The labels are purely syntactic to keep track of derivations. They are not names of worlds or anything similarly semantic.

2.4 The Modal Disanalogy

Here's the state of play. The standard identity rules – (Refl) and (LL) – are not harmonious. Read offered new rules – (=I'), (Congr) and (Congr′) – which are supposedly harmonious. Griffiths showed that the two sets of rules are equivalent, which should worry the inferentialist, since harmony should be sensitive only to inferential power. The debate therefore turns on:

> (Equiv) If two pairs of I- and E-rules are inferentially equivalent, then they are either both harmonious or both disharmonious.

Read argues that (Equiv) is false and that this isn't surprising or new: the CFP rules for possibility in S4 are inferentially equivalent to his labelled rules for the same, but only the latter are harmonious. And he claims that identity is analogous: the standard rules are inferentially equivalent to his, but only the latter are harmonious.

But the modal and identity cases are importantly disanalogous. First, the CFP modal rules are in I-weak disharmony whereas the standard identity rules are (as we saw) in E-weak disharmony. Second, as Read [8] states, 'the CFP ($\diamond$I) rule is too weak to justify the restrictions that need to be placed on the ($\diamond$E) rule'. But his preferred ($\diamond$I*) *is* in ge-harmony with its E-rule. So there is a clear sense in which Read's ($\diamond$I*) is stronger than the CFP ($\diamond$I) rule.

To call one rule R_1 *stronger* than another R_2 usually means that, with respect to some background set of deductive rules D, everything provable with $D \cup \{R_2\}$ is provable with $D \cup \{R_1\}$ and at least one sentence S is provable with $D \cup \{R_1\}$ but not with $D \cup \{R_2\}$. We must tread carefully here: this cannot be quite what we mean in Read's modal case. There, the background set D changes when we move from the CFP to the labelled system. But there is a clear intuitive sense in which Read's rule is stronger: his ($\diamond$I*) allows the derivation of $\diamond\alpha_i$ from both the actual truth of α at world i *and* from the truth of α at world j, accessible from i. Whereas the CFP ($\diamond$I) rule only allows the derivation of $\diamond\alpha$ from the actual truth of α. The situation is not analogous with identity: Read's (=I') and the old (Refl) are exactly as strong as one another with respect to the classical rules, in that neither is stronger than the other, in the straightforward sense of strength.

As Read [6] notes:

> if the [CFP] rules were in harmony, ($\diamond$I) would give the whole meaning of '$\diamond$'. But even in reflexive logics, α is only one ground for asserting $\diamond\alpha$. $\diamond\alpha$ can be true without α also being true.

He ([8]: 7) adds that the

> CFP I-rule is too weak, inferring '$\diamond\alpha$' from α itself. If that were the sole ground for asserting $\diamond\alpha$, then modalities would collapse and '$\diamond$' would just mean 'true', not 'possibly true'.

Crucially, as Read notes, the CFP ($\diamond$I) rule is too weak and his labelled ($\diamond$I*) is stronger. In other words, consider the following principle:

> (Equiv*) If two sets of #-introduction rules (#*I*) and (#*I*′) are inferentially equivalent, and if two sets of #-elimination rules (#*E*) and (#*E*′) are inferentially equivalent, then (#*I*) and (#*E*) are in harmony if and only if (#*I*′) and (#*E*′) are in harmony.

We have argued that ($\diamond$I*) is stronger than ($\diamond$I), hence Read's modal example is not a counterexample to (Equiv*) – the antecedent is false – even though it *is* a counterexample to (Equiv).

But in the case of identity we *do* have a counterexample to (Equiv*). Here the old and new *introduction* rules are equivalent, and similarly for the *elimination* rules, but the old rules are not in harmony, whereas the new ones are. Here is a crucial disanalogy between the identity case and the modal case: the former violates (Equiv*) whereas the latter does not.

2.5 Inferentialist Meaning

Read may now admit that the analogy fails but add that the identity case itself shows the falsity of (Equiv*): perhaps, contrary to his claims, the falsity of (Equiv*) is news. But the inferentialist should not *want* to reject (Equiv*), on the usual understanding of 'inferentially equivalent'. To deny (Equiv*) is to deny that meaning supervenes on inferential power. To see why, we must briefly discuss the philosophical motivation for inferentialism.

In his ([7]: 558), Read motivates inferentialism like this:

> Traditionally, semantics has been denotational and representational, consisting in a homomorphic valuation from expressions to some range of objects. The approach risks ontological explosion, first in hypostatizing denotations for empty names, predicates, conjunctions, prepositions and so on … then in seeking values for false propositions in the form of non-actual states of affairs.

The worry, generally, is that *externalism* about meaning – roughly, taking the meaning of an expression to fall out of its relation to the world – faces well-known problems. One response is *internalism*, which, roughly, takes the meaning of an expression to be given not by its relation to the world but by its relation to other expressions. Inferentialism is a kind of internalism that tries to give meanings of expressions by their *inferential* relations to other expressions. As Read [7] puts it:

> Inferentialism, in contrast [with denotational semantics], is ontologically neutral. Expressions are meaningful if there are rules governing their use, in particular, logical expressions are given meaning by their introduction-rules, specifying the

grounds for assertion of propositions containing them, and elimination-rules drawing inferences from those assertions.

Logical constants are obvious candidates for inferentialist treatment. *Logical inferentialism* says that inferentialism is true of the logical constants. *Global inferentialism* says that inferentialism is true of every expression. Here we focus on logical inferentialism; of course if it is false then so is global inferentialism.

Recall: ge-harmony requires: (i) balance of inferential powers between I- and E-rules; and (ii) meaning being given entirely by the I-rules. For ($\diamond$I) and ($\diamond$I*), we noted a sense in which the latter is stronger and so can plausibly be said to capture more of the meaning of S4 possibility. We also saw that that doesn't hold for (Refl) and (=*I*′), which are equally strong. So if Read is right that (=*I*′) captures more of the meaning of identity, that meaning must be *non-inferential*.

After all, if (=*I*′) really does improve on (Refl), then it must be superior on either (i) balance of inferential power with its elimination rules or (ii) what it captures of the meaning of identity. (=*I*′) can't beat (Refl) on (i), as they are equivalent, so it must be superior on (ii). So (=*I*′) must capture more of the meaning of identity than (Refl). But, given inferential equivalence, anything extra captured by (=*I*′) must be non-inferential. In short, (=*I*′) is superior to (Refl) with respect to ge-harmony only if the meaning of identity is partly non-inferential. So Read *is* committed to non-supervenience of meaning on inferential power.

2.6 Inferentialism and Non-inferential Meaning

But *should* inferentialism allow that meaning does not supervene on inferential power? There may be parts of language where inferentialists could admit non-inferential meaning, but the logical fragment is not among them. The logical constants were meant to be paradigms of inferentialist treatment. In the quote from Read at §2.5, avoiding a *referent* for the conjunction functor was an explicit motivation. And mathematical objects like functions are the sorts of object that may raise ontological worries. So it seems that *logical* inferentialists ought to resist accepting non-inferential meaning, on pain of undermining the motivation for their view.[16]

Read does not explicitly admit non-inferential meaning, though we find hints:

> [T]he meaning of '=' is given by the rules for its assertion; and although Refl and =*I*′ (and = I[17]) are equivalent in that they each only permit assertions of self-identity, 'a = a', how they do so is different. The meaning of '=' is given not by

[16] Read may now insist on a *presentational* component to harmony, and that presentation of the introduction rule as (=*I*′) is superior to presentation of it as (Refl). But the logical inferentialist should resist. As Steinberger [10] clearly explains, inferentialism is motivated by *use-theoretic* internalism about meaning. Logical constants get their meaning from how they are used inferentially. The formalism – the rules of inference – seeks to capture these meaning-conferring aspects of use. But *use* is what confers meaning on the logical constants, not the formalism itself; we must not confuse the formal medium with the inferential message.

[17] This being Read's original proposal for = −introduction [5].

> what identity statements can be asserted, but by the grounds for those assertions.[18]

The thought seems to be yes, (Refl) and (=*I*′) license 'heteronymous' identities $a = b$ under the same conditions, but they license homonymous 'self-identity' claims $a = a$ under different conditions. In particular, (Refl) treats $a = a$ as assertible on no premises, whereas (=*I*′) treats $a = a$ as assertible only given a derivation of Fa from Fa, for arbitrary F. But really those are the same *inferential* conditions: on the one hand, we can always assert $a = a$; on the other, we can always derive Fa from Fa.[19] If there is an important difference between the cases, it must be because (=*I*′) captures some non-inferential aspect of the meaning of identity (not *what* but *how* we infer). This again looks at odds with inferentialism.

If inferentialism *does* acknowledge that identity has some non-inferential meaning, it is unclear that it should favour Read's rule. Does (=*I*′) capture its meaning better than (Refl)? It is hard to settle. If anything, it seems more likely that we learn the logical truth of $a = a$ in something like this way: we realise that self-identity claims hold without exception, take $a = a$ to be a logical truth, and formulate (Refl). The alternative is that we derive Fa from Fa for arbitrary F and conclude $a = a$. It seems unlikely that such trivial derivations are required to realise the logical truth of 'self-identities'. But whichever story we prefer, we only find ourselves comparing them if we have already bought into non-inferential meaning. This, we contend, the logical inferentialist should not do.

2.7 Equivalence Properties of Identity

Read [8] offers one more argument:

> (Refl) allows assertions of self-identity, of the reflexivity of identity. But it adds nothing about other properties of identity, e.g., symmetry or transitivity. In contrast, (=*I*′) commits one to the symmetry of identity (and to its transitivity too). The reason is that the premises of (=*I*′) do not occur in any particular order. So if we are justified in asserting $a = b$, by means of derivations of Fb from Fa and of Fa from Fb, then we are also justified in asserting $b = a$, by derivations of Fa from Fb and of Fb from Fa.

The thought seems to be that (=*I*′) somehow tells us more about identity than (Refl) does.

But (Refl) *does* tell us something about symmetry and transitivity. Suppose we start in classical predicate logic without identity, from which starting point it is true that '(Refl) and (=*I*′) are equivalent in that they each only permit assertions of self-identity $a = a$'. Now suppose somebody proposes to add = to

[18] [8]: 10.

[19] This of course assumes reflexivity. Again, we are assuming classical logic but e.g. Fjellstad [1] has developed irreflexive logic. He does not endorse such a logic but shows that we should give up reflexivity if we want to give a semantics for 'tonk'.

our language, by means of the explanation that (Refl) gives its *whole* meaning. We claim that we can now establish symmetry and transitivity of identity.

For symmetry, we must show (i) that $a = a$ is inferable from $a = a$; and (ii) that $b = a$ is inferable from $a = b$, where a and b are different terms. (i) follows by reflexivity of derivability. (ii): if (Refl) exhausts the meaning of identity, then there are no grounds from which we can infer $a = b$ other than those from which we can infer $\perp$.[20] Hence it is vacuously true that any grounds from which we can infer $a = b$ are grounds from which we can infer $b = a$. Hence, if we are justified in asserting $a = b$ then we are justified in asserting $b = a$.

For transitivity, we need to show – again assuming that (Refl) exhausts the meaning of identity – that for distinct *a, b* and *c*: (i) $a = a$ is inferable from $a = a$ and $a = a$; (ii) $a = b$ is inferable from $a = a$ and $a = b$; (iii) $a = a$ is inferable from $a = b$ and $b = a$; (iv) $a = b$ is inferable from $a = b$ and $b = b$; (v) $a = c$ is inferable from $a = b$ and $b = c$. (i), (ii) and (iv) follow by reflexivity and dilution properties of derivability, and (iii) by (Refl) and dilution. That leaves (v), but again, if (Refl) exhausts the meaning of identity, then the only grounds for $a = b$ and $b = c$ are grounds for $\perp$. So vacuously, any grounds from which we are justified in asserting $a = b$ and $b = c$ are grounds from which we can infer $a = c$. So, if we are justified in asserting $a = b$ and $b = c$, then we are trivially justified in asserting $a = c$. So, if (Refl) exhausts the meaning of '=', then identity is transitive. Read is wrong, therefore, that his rules settle more about identity than do the traditional rules.

Further, given any background language in which (Refl) and ($=I'$) are equivalent in terms of what they allow us to prove, the form of Read's argument in the above quote proves too much. The argument was that $b = a$ is derivable from $a = b$ on the grounds that 'if we are justified in asserting a=b, by means of derivations of *Fb* from *Fa* and of *Fa* from *Fb*, then we are also justified in asserting b=a, by derivations of *Fa* from *Fb* and of *Fb* from *Fa*.' But the point of Griffiths's argument is that there are *no* derivations of *Fb* from *Fa* and of *Fa* from *Fb*; the conditional just quoted is therefore vacuously true. So we can just as well say, for any *Q*: if we are justified in asserting $a = b$, by means of derivations from *Fb* from *Fa* and *Fa* from *Fb*, then we are also justified in asserting *Q*, by derivations of *Fb* from *Fa* and *Fa* from *Fb*. Therefore Q is derivable from a = b. This result is clearly too strong, so Read's argument has gone wrong.

In short: if Read denies (Equiv*), then he must deny that meaning supervenes on inferential power. But inferentialism shouldn't admit this. Moreover, Read's argument is presented as applying within a language in which (Refl) and ($=I'$) are inferentially equivalent, i.e. classical predicate logic without identity. So it is hardly surprising that the standard rules and Read's rules are equal – and equally deficient – with respect to harmony.

[20] This threatens to make anything inferable from $a = b$, because anything is inferable from the classical grounds for $\perp$. But it still holds that taking (Refl) to give the whole meaning of = *does* tell us *at least* that identity is symmetric. And the threat of explosion is also present when we take ($=I'$) to give its whole meaning. So the point remains that ($=I'$) says no more (or less) about identity than does (Refl).

3 Klev's Rules

When the axioms of a theory define new expressions (e.g. '1') in old terms (e.g. '$s(0)$'), that theory also licenses identities between the new terms and the old ('$1 = s(0)$'). These identities are always heteronymous. So theoretical definitions are among our grounds for heteronymous identities. This insight is at the centre of Klev's recent discussion. His basic claim is that one can justify the *standard* rules for identity – (Refl) and (LL) – given a relation of *definitional identity* relative to some given background theory (say, a first-order theory of arithmetic).

3.1 Definitional Identity

More precisely, Klev explains definitional identity, written ≡, via several axioms, axiom schemata and rules *relative to a background theory T*. We assume some formal way to identify the definitions of that theory: let us write $D(a, b, T)$ to mean that T defines a to mean the same as b. a and b might belong to any category – they might be terms *or* formulae – provided they both belong to the same one. Here are Klev's axioms:

$$(\equiv_1) \quad a \equiv a$$

$$(\equiv_2) \quad \frac{a \equiv b}{b \equiv a}$$

$$(\equiv_3) \quad \frac{a \equiv b \quad b \equiv c}{a \equiv c}$$

$$(\equiv_4) \quad a \equiv b \text{ if } D(a, b, T)$$

$$(\equiv_5) \quad a \equiv b \text{ if } b \text{ arises from } a \text{ by a renaming of its bound variables.}$$

$$(\equiv_6) \quad \frac{a \equiv b \quad c \equiv c'}{a \equiv b \;\; [c'/c]_!}$$

$$(\equiv_7) \quad \frac{a \equiv b}{a[t/x] \equiv b[t/x]}$$

$$(\equiv_8) \quad \frac{A \quad a \equiv b}{A[b/a]_!}$$

In $(\equiv_6)$, $b[c'/c]_!$ denotes any formula that results from replacing *any number* of occurrences of c in b by b'. Similarly for $(\equiv_8)$.

In $(\equiv_7)$, x is a first-order variable, t a term, and $a[t/x]$ the result of replacing x by t for *all* its occurrences in a. For instance, if T is a theory of arithmetic that licenses the definitional identity $x + 0 \equiv 0 + x$, $(\equiv_7)$ lets us write e.g. $1 + 0 \equiv 0 + 1$, $2 + 0 \equiv 0 + 2\ldots$ (provided that 1, 2… are all terms of our language). So $a[t/x]$ in Klev's notation is the same as a^t_x in Read's notation (see §1.2).

In $(\equiv_8)$, A is an arbitrary formula and $(\equiv_8)$ permits substitutions of definitionally identical expressions into A. For instance, if we are given $3 \equiv (2 + 1)$ and $3 + 3 > 1 + 2$ then $(\equiv_8)$ permits the derivation of $3 + (2 + 1) > 1 + 2$. Since $(\equiv_8)$ is the only rule that mentions formulas other than definitional identities, it functions as what Klev calls a 'bridge principle' connecting the 'pure' theory of definitional identity (relative to T), with the rest of the language.[21]

3.2 Justification for (LL)

Klev argues that we can now *justify* the *standard* elimination rule for identity, (LL):

$$(\text{LL}) \ \frac{a = b \quad A[a]}{A[b]}$$

(Klev writes $A[a]$ and $A[b]$ respectively for Read's α^a_x and α^b_x).

Saying that we can *justify* (LL) means that we can derive its conclusion from the grounds on which the introduction rule licensed the identity statement that is its major premise. Definitional identity is relevant because the rules for $\equiv$, especially $(\equiv_8)$, determine the grounds on which the identity statement was derivable in the first place.

To justify (LL) we need to show that any grounds from which the introduction rule for identity lets us derive $a = b$, together with $A[a]$, warrant derivation of $A[b]$. For Read, this is where the traditional rules go wrong: (Refl) identifies *no* grounds from which we may derive a heteronymous identity like $a = b$.

[21] [3]: 876.

[22] Klev uses Martin-Löf's elimination rule, which licenses derivation of $A(t, u)$ from $A(x, x)$ and $t = u$. This is equivalent to (LL) given the rules for implication and the universal quantifier ([3]: 868–9). Klev's procedure applies equally to both.

It is also where Klev's definitional identities come in. We can't derive $a = b$ using (Refl) alone, but we *can* derive it from (Refl) given definitional identities. Klev says that such derivations take this canonical form:

$$(= -\text{CanDer})\ \dfrac{\dfrac{a' = a' \qquad \begin{matrix}\mathcal{D}_1\\ a{\equiv}a'\end{matrix}}{a = a'}(\equiv_8) \qquad \begin{matrix}\mathcal{D}_2\\ a'{\equiv}b\end{matrix}}{a = b}(\equiv_8)$$

In (=-CanDer), we derive the homonymous $a' = a'$ from (Refl) and then use substitutions permitted by ($\equiv_8$) and definitional identities $a \equiv a'$ and $a' \equiv b$ respectively to derive the heteronymous $a = a'$ and $a = b$.

Given (=-CanDer) we use the resources of this derivation – specifically $\mathcal{D}_1$ and $\mathcal{D}_2$ – to justify what $a = b$ justifies i.e. the step from $A[a]$ to $A[b]$ in (LL). We do this with one application of ($\equiv_3$) and one application of the bridge principle ($\equiv_8$):

$$(\text{LLJ})\ \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ a{\equiv}a'\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ a'{\equiv}b\end{matrix}}{a{\equiv}b}(\equiv_3) \qquad A[a]}{A[b]}(\equiv_8)$$

This shows that any grounds for $a = b$ in a system involving $\equiv$ also justify everything that (LL) lets us infer from $a = b$. So given *traditional* rules for identity, the apparatus of definitional identity can indeed justify the elimination rule in terms of the introduction rule.

3.3 E-Weak and E-Strong Disharmony

In one respect Klev's approach *does* improve on Read's. As Griffiths showed (§2.2), Read's rules supply no new grounds for deriving heteronymous identities. So if the problem with the traditional approach was that it supplied no grounds from which heteronymous identities are derivable, then Read's approach faces the same problem.

Klev's approach doesn't face *that* problem. As we just saw, the traditional rules *do* allow derivation of heteronymous identities given definitional identity relative to a background theory T. We also saw that this apparatus justifies the traditional elimination rule (LL) without any modification either to it or to the traditional introduction rule (Refl).

But that shows only that Klev is hitting a target at which neither he nor Read should have been aiming in the first place. As (apparently) with Read, Klev's guiding thought throughout is to *justify* the identity-elimination rule i.e. show that it satisfies the first harmony condition (H1) (see §1.1). That is: show that the conclusion of such a rule is *always* warranted by its minor premises together with any grounds on which one might have asserted the identity statement in the first place. Klev succeeds at that. But it was unnecessary to try, because the traditional identity-elimination rule (LL) is *already* justified, without $\equiv$; indeed the justification is trivial.

As we saw at §1.2, it goes like this. *No* application of the traditional introduction rules terminates in a heteronymous $a = b$. So there are *no* grounds from which that rule permits inference of $a = b$. So trivially, *any* such grounds for $a = b$, together with $A[a]$, justify $A[b]$. (LL), in other words, never permits the drawing of conclusions from $a = b$ and side premises that was not already warranted by the grounds for $a = b$ together with those premises. This justification of the elimination rule was available all along. If that were the problem with identity, we should never have needed to invoke definitional identity in the first place.

The real problem with the traditional rules was not that the traditional E-rule was too *strong* to be justifiable by any grounds mentioned in the traditional I-rule. The problem was that the E-rule was too *weak* to exploit all those grounds. When a and b are distinct there are *no* grounds from which (Refl) licenses $a = b$. So trivially we could have inferred *any* Q from any $A[a]$ and these grounds, as there *are* no such grounds. So (LL) *should* license the inference of arbitrary Q from $A[a]$ and $a = b$, but of course it doesn't.

Is E-weak disharmony also a problem for the non-traditional approaches that we considered in this paper? We already saw that Read's rules cannot help with E-weak disharmony any more than with E-strong disharmony, for as Griffiths's argument showed, Read's rules warrant the same inferential transitions as the traditional rules. What about Klev's approach? We argue now that although Klev's system is inferentially more powerful than the traditional one, it still leaves the rules E-weakly disharmonious.

3.4 Klev's System Is E-Weakly Disharmonious

For Klev, a canonical derivation of $a = b$ takes the form (=-CanDer). We can use the resources in (=-CanDer) to derive the definitional identity $a \equiv b$: as with (LLJ) this follows from the transitivity principle ($\equiv_3$):

$$(\text{Def}-\text{Der})\ \frac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2 \\ a \equiv a' & a' \equiv b\end{matrix}}{a \equiv b}(\equiv_3)$$

So any grounds that justify $a = b$ in Klev's system also justify $a \equiv b$. So unless Klevian identity is E-weakly disharmonious, it must be possible in that system to infer $a \equiv b$ from $a = b$.

On the face of it this *does* look possible:

$$(\text{Id}-\text{Der})\quad \frac{\dfrac{}{a \equiv a}(\equiv_1) \quad a = b}{a \equiv b}(LL)$$

In this derivation $a \equiv a$ is an instance of ($\equiv_1$); $a \equiv b$ is supposed to follow by (LL). But in Klev's system (Id-Der) is *not* a legitimate application of (LL).

The reason is as follows: in Klev's system, the *only* way that definitional identities enter into derivations involving formulas other than definitional identities is via the substitution of definitional equivalents licensed by the bridge principle ($\equiv_8$). To dramatize this separation between definitional identity and the rest of his system, Klev writes that we can imagine the derivation of a definitional identity being written on a separate paper from any derivation in which it plays a role, and invoked only to justify the substitutions of definitional equivalents into a formula that was derived without $\equiv$, this sequence of substitutions being the end of the derivation.[23]

[23] [3]: 878.

But clearly, no such procedure could justify (Id-Der), because (Id-Der) does *not* rewrite $a = b$ (or anything else) via substitution of definitional equivalents. So in Klev's system $a \equiv b$ is *not* derivable from $a = b$.

Indeed it is hardly surprising that you can't get $a \equiv b$ from $a = b$ in Klev's system: if you could, $a = b$ and $a \equiv b$ would be notational variants of one another. More importantly, it would compromise the topic-neutral character of identity. Recall that $\equiv$ is not topic-neutral but is tied to some specific background theory T (say, a theory of arithmetic) whose special vocabulary (say, numerals and arithmetical operators) is what the theory defines. But to claim that a and b are numerically identical (e.g. 'Hesperus = Phosphorus') is to make a claim about a, b and nothing else: the identity-sign itself introduces no new subject-matter. That identity claim *should* not commit me to their, or their names', being *definitionally equivalent* according to the background theory. Claims of numerical identity should have *no* consequences for the notational substitutions that this or that background theory sets up.[24] So it is no surprise that, as Klev himself notes, there are theories (like first-order arithmetic) relative to which definitional identity cuts strictly finer than identity itself.[25]

But if $a \equiv b$ is not derivable from $a = b$ then even in a Klevian system the rules for identity still fail (H2) i.e. they are still E-weakly disharmonious. For those rules to be E-weakly harmonious, (LL) should license an $=$−free conclusion Q from an $=$−involving P *if* we could already have inferred Q from any of the $=$−free grounds from which (Refl) licensed P in the first place. In particular, (LL) should license $a \equiv b$ from $a = b$ *if* we could already have inferred $a \equiv b$ from any of the $=$−free grounds from which (Refl) licensed $a = b$ in the first place. But (Refl) and (LL) fail this condition. For as (Def-Der) shows, we *could* already have inferred $a \equiv b$ from any of the $=$−free grounds, namely those involved in (=-CanDer) from which (Refl) licensed $a = b$ in the first place. But as we argued here, (LL) does *not* license us to conclude $a \equiv b$ from $a = b$.

Identity in Klev's system is therefore E-weakly disharmonious.[26]

4 Conclusion

We considered two inferentialist attempts to establish harmony of identity; both failed. Both seem to assume that (LL) permits us to infer *more* from a heteronymous $a = b$ than is justified given that the introduction rule is (Refl). We argued that the real trouble is

[24] This argument parallels Klev's [3] distinct argument that definitional identities should not figure in the identity-introduction rules.

[25] We are not denying that identity is the finest equivalence relation – it is. But for any *terms* a and b, $a = b$ is derivable whenever $a \equiv b$ is, but the converse is not always true.

[26] Our argument, that harmony requires the derivability of $a \equiv b$ from $a = b$, assumed that $a \equiv b$ is already derivable from the canonical grounds for $a = b$ as specified in (=-CanDer). But is that true? Klev writes at one point ([4]: 878) that derivations of definitional identities occurring in a derivation D, for instance $\mathcal{D}_1$ and $\mathcal{D}_2$ in (=-CanDer), should not be considered sub-derivations of D. A defender of Klev might then insist that $a \equiv a'$ and $a' \equiv b$ are strictly *not* amongst the canonical grounds on which $a = b$ is derived in (=-CanDer); so it was illegitimate for us to assume that $a \equiv b$ was derivable from those grounds.

That would be trying to have your cake and eat it. The selling point of Klev's approach was that it can justify (LL) i.e. it can show that $A[b]$ is derivable from $A[a]$ together with any grounds on which $a = b$ was canonically derivable. When trying to establish *this* part of the harmony requirement we must grant that $a \equiv a'$ and $a' \equiv b$ *are* among the grounds on which $a = b$ was canonically derivable. It would be inconsistent to retract that concession when we are trying to establish the other part.

the converse: the elimination rule permits us to infer *less* from $a = b$ than would be justified given that the introduction rule is (Refl). Klev's solution goes further than Read's in addressing the alleged problem of E-*strong* disharmony, because Klev's rules make an inferential difference. But that was the wrong problem to be addressing in the first place.

What are the prospects for a harmonious account of identity i.e. on which the elimination rules exploit only *and all* the powers that the introduction rules confer? We suspect that no such rules are available. There are many grounds on which we assert identities: consider the grounds on which we assert e.g. the identity of sets, of numbers, of rivers and of persons. Why think these can be captured in simple rules like those governing conjunction? And why think that grasp of such rules is necessary for *understanding* identity? If anything is necessary for that, we suspect that it is Leibniz's Law; and that the only thing uniting the open-ended set of *grounds* for identity statements is not that *they* all instantiate some schema but rather that they justify that elimination-rule. For instance, our grounds for '*A* is the same person as *B*' ought to justify the inference from '*A* wore a hat at *t*' to '*B* wore a hat at *t*'. But then there may be no tidy introduction rules for identity, and no prospect of establishing harmony between them and the elimination rule; so by inferentialist criteria identity is not a logical constant. Of course this is little more than a gesture in what we think is a good direction for further inquiry. Since we can't yet do better than that, here is a sensible place to stop.[27]

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