A Note on the Size of the Largest Ball Inside a Convex Polytope

Imre Bárány

Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences P.O. Box 127, Hungary H-1364 E-mail: barany@renyi.hu, and Department of Mathematics University College London Gower Street, London, WC1E 6BT, UK

Nándor Simányi

University of Alabama at Birmingham
Department of Mathematics
Campbell Hall, Birmingham, AL 35294 U.S.A.
E-mail: simanyi@math.uab.edu

Abstract. Let m > 1 be an integer, B_m the set of all unit vectors of \mathbb{R}^m pointing in the direction of a nonzero integer vector of the cube $[-1, 1]^m$. Denote by s_m the radius of the largest ball contained in the convex hull of B_m . We determine the exact value of s_m and obtain the asymptotic equality $s_m \sim \frac{2}{\sqrt{\log m}}$.

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§1. Introduction

Let $m \geq 2$ be an integer, and consider the sets

$$A_m = \{-1, 0, 1\}^m \setminus \{\vec{0}\}, \text{ and } B_m = \left\{\frac{v}{||v||} \middle| v \in A_m\right\}.$$

Let C_m be the convex hull of B_m , and s_m the radius of the largest ball contained in C_m . (Due to the apparent symmetries of C_m , such a largest ball is necessarily centered at the origin.) In the paper [B-M-S(2005)] (dealing with rotation numbers/vectors of billiards) we needed sharp lower and upper estimates for the extremal radius s_m . Here we determine the exact value of s_m which, of course, implies such estimates.

Theorem.

$$s_m = \left(\sum_{k=1}^m \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2}\right)^{-1/2}$$

The Theorem implies that

$$\frac{1}{4}\log m < s_m^{-2} < \frac{1}{4}\log m + \frac{5}{4}.$$

As an immediate corollary, the quantity s_m is asymptotically equal to $\frac{2}{\sqrt{\log m}}$.

§2. Proof of the Theorem

The proof will be split into of a few lemmas. The first one of them is a trivial observation.

Lemma 1. The set of vertices B_m of the convex polytope C_m , and hence C_m itself, is invariant under the action of the full isometry group G of the cube $[-1,1]^m$. (The group G is generated by all permutations of the coordinates in \mathbb{R}^m , and by all reflections across the coordinate hyperplanes.)

We will use the notation $v_k = \frac{1}{\sqrt{k}} \sum_{i=1}^k e_i \ (k=1,\ldots,m)$ for some specific vertices of C_m . (Here e_i stands for the *i*-th standard unit vector of \mathbb{R}^m .)

Lemma 2. The simplex S, spanned by the linearly independent vectors v_k (k = 1, ..., m) as vertices, is a face of the polytope C_m whose outer normal vector is $u = (u_1, ..., u_m)$ with the coordinates $u_i = \sqrt{i} - \sqrt{i-1}$.

Proof. Consider the scalar product function $\langle v, u \rangle$ $(v \in B_m)$ restricted to the set B_m of vertices of the polytope C_m . Elementary inspection shows that this scalar product function can only attain its maximum value at the vertices v_k , and actually,

$$\langle v_k, u \rangle = 1$$

for each k = 1, ..., m. This proves all claims of the lemma. \square

Lemma 3. For any face F of the polytope C_m there exists a congruence $g \in G$ such that g(F) = S.

Proof. Fix a non-zero vector $w = (w_1, \ldots, w_m)$ whose ray $R(w) = \{\lambda w | \lambda \geq 0\}$ intersects the interior of the face F. By selecting w in a generic manner, we can assume that the absolute values $|w_i|$ of its coordinates are distinct and all different from zero. Therefore, by applying a suitable element $g \in G$, we can even assume that

$$(2) w_1 > w_2 > \dots > w_m > 0.$$

We claim that g(F) = S. Indeed, by (2) we have the linear expansion

$$w = \sum_{k=1}^{m} \sqrt{k}(w_k - w_{k+1})v_k.$$

of w in the basis $\{v_1, \ldots, v_m\}$ with positive coefficients. (With the natural convention $w_{m+1} = 0$.) This proves that some positive multiple of w is a convex linear combination of the vertices of S with non-zero coefficients, so the face g(F) shares an interior point with S. \square

It follows from the previous lemma that the radius s_m of the inscribed sphere is actually the distance between S and the origin. However, this distance is equal to $s_m = \langle u, e_1 \rangle / ||u|| = 1/||u||$ by (1). It is clear that

$$||u||^2 = \sum_{k=1}^m \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2}.$$

finishing the proof of our theorem. \Box

Define $R_m = \sum_{k=1}^m \frac{1}{k}$. For the asymptotic value of s_m we use the elementary fact that $\log m < R_m < \log m + 1$.

$$\frac{1}{4}\log m < \sum_{k=1}^{m} \frac{1}{4k} < \sum_{k=1}^{m} \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2} = ||u||^2$$

$$<1+\sum_{k=2}^{m}\frac{1}{4(k-1)}<1+\frac{1}{4}(\log m+1)=\frac{1}{4}\log m+\frac{5}{4}.$$

Remark 1. Let K be the convex cone generated by the vectors v_k , k = 1, ..., m. The meaning of Lemma 3 is that the cones g(K) $(g \in G)$ form a triangulation of the space \mathbb{R}^m . As a matter of fact, the intersections of the cones g(K) with the standard (m-1)-simplex

$$S_{m-1} = \left\{ x \in \mathbb{R}^m \middle| \sum_{i=1}^m x_i = 1, \ x_i \ge 0 \text{ for all } i \right\}$$

form the baricentric subdivision of S_{m-1} .

Remark 2. The following natural question has been considered in several papers, for instance in [B-F(1988)] and [B-W(2003)]. What is the maximal radius r(m, N) of the inscribed ball of the convex hull of N points chosen from the unit ball of \mathbb{R}^m ? In our case $N = 3^m - 1$ and one may wonder how close s_m and B_m are to the maximal radius and best arrangement. It turns out that they are very far: it follows from the results of [B-F(1988)] and [B-W(2003)] that, in the given range $N = 3^m - 1$,

$$r(m, N) = \left(\frac{8}{9}\right)^{1/2} (1 + o(1))$$

as $m \to \infty$. So the optimal radius is much larger than s_m . This also shows that, as expected, B_m is far from being distributed uniformly on the unit sphere.

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