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#### Abstract

Let $m>1$ be an integer, $B_{m}$ the set of all unit vectors of $\mathbb{R}^{m}$ pointing in the direction of a nonzero integer vector of the cube $[-1,1]^{m}$. Denote by $s_{m}$ the radius of the largest ball contained in the convex hull of $B_{m}$. We determine the exact value of $s_{m}$ and obtain the asymptotic equality $s_{m} \sim \frac{2}{\sqrt{\log m}}$.


Primary subject classification: 52B11
Secondary subject classification: 52B12

## §1. Introduction

Let $m \geq 2$ be an integer, and consider the sets

$$
A_{m}=\{-1,0,1\}^{m} \backslash\{\overrightarrow{0}\}, \text { and } B_{m}=\left\{\left.\frac{v}{\|v\|} \right\rvert\, v \in A_{m}\right\} .
$$

Let $C_{m}$ be the convex hull of $B_{m}$, and $s_{m}$ the radius of the largest ball contained in $C_{m}$. (Due to the apparent symmetries of $C_{m}$, such a largest ball is necessarily centered at the origin.) In the paper [B-M-S(2005)] (dealing with rotation numbers/vectors of billiards) we needed sharp lower and upper estimates for the extremal radius $s_{m}$. Here we determine the exact value of $s_{m}$ which, of course, implies such estimates.

## Theorem.

$$
s_{m}=\left(\sum_{k=1}^{m} \frac{1}{(\sqrt{k}+\sqrt{k-1})^{2}}\right)^{-1 / 2}
$$

The Theorem implies that

$$
\frac{1}{4} \log m<s_{m}^{-2}<\frac{1}{4} \log m+\frac{5}{4}
$$

As an immediate corollary, the quantity $s_{m}$ is asymptotically equal to $\frac{2}{\sqrt{\log m}}$.

## §2. Proof of the Theorem

The proof will be split into of a few lemmas. The first one of them is a trivial observation.

Lemma 1. The set of vertices $B_{m}$ of the convex polytope $C_{m}$, and hence $C_{m}$ itself, is invariant under the action of the full isometry group $G$ of the cube $[-1,1]^{m}$. (The group $G$ is generated by all permutations of the coordinates in $\mathbb{R}^{m}$, and by all reflections across the coordinate hyperplanes.)

We will use the notation $v_{k}=\frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_{i}(k=1, \ldots, m)$ for some specific vertices of $C_{m}$. (Here $e_{i}$ stands for the $i$-th standard unit vector of $\mathbb{R}^{m}$.)

Lemma 2. The simplex $S$, spanned by the linearly independent vectors $v_{k}$ ( $k=$ $1, \ldots, m)$ as vertices, is a face of the polytope $C_{m}$ whose outer normal vector is $u=\left(u_{1}, \ldots, u_{m}\right)$ with the coordinates $u_{i}=\sqrt{i}-\sqrt{i-1}$.

Proof. Consider the scalar product function $\langle v, u\rangle\left(v \in B_{m}\right)$ restricted to the set $B_{m}$ of vertices of the polytope $C_{m}$. Elementary inspection shows that this scalar product function can only attain its maximum value at the vertices $v_{k}$, and actually,

$$
\begin{equation*}
\left\langle v_{k}, u\right\rangle=1 \tag{1}
\end{equation*}
$$

for each $k=1, \ldots, m$. This proves all claims of the lemma.
Lemma 3. For any face $F$ of the polytope $C_{m}$ there exists a congruence $g \in G$ such that $g(F)=S$.

Proof. Fix a non-zero vector $w=\left(w_{1}, \ldots, w_{m}\right)$ whose ray $R(w)=\{\lambda w \mid \lambda \geq 0\}$ intersects the interior of the face $F$. By selecting $w$ in a generic manner, we can assume that the absolute values $\left|w_{i}\right|$ of its coordinates are distinct and all different from zero. Therefore, by applying a suitable element $g \in G$, we can even assume that

$$
\begin{equation*}
w_{1}>w_{2}>\cdots>w_{m}>0 \tag{2}
\end{equation*}
$$

We claim that $g(F)=S$. Indeed, by (2) we have the linear expansion

$$
w=\sum_{k=1}^{m} \sqrt{k}\left(w_{k}-w_{k+1}\right) v_{k}
$$

of $w$ in the basis $\left\{v_{1}, \ldots, v_{m}\right\}$ with positive coefficients. (With the natural convention $w_{m+1}=0$.) This proves that some positive multiple of $w$ is a convex linear combination of the vertices of $S$ with non-zero coefficients, so the face $g(F)$ shares an interior point with $S$.

It follows from the previous lemma that the radius $s_{m}$ of the inscribed sphere is actually the distance between $S$ and the origin. However, this distance is equal to $s_{m}=\left\langle u, e_{1}\right\rangle /\|u\|=1 /\|u\|$ by (1). It is clear that

$$
\|u\|^{2}=\sum_{k=1}^{m} \frac{1}{(\sqrt{k}+\sqrt{k-1})^{2}} .
$$

finishing the proof of our theorem.
Define $R_{m}=\sum_{k=1}^{m} \frac{1}{k}$. For the asymptotic value of $s_{m}$ we use the elementary fact that $\log m<R_{m}<\log m+1$.

$$
\begin{aligned}
& \frac{1}{4} \log m<\sum_{k=1}^{m} \frac{1}{4 k}<\sum_{k=1}^{m} \frac{1}{(\sqrt{k}+\sqrt{k-1})^{2}}=\|u\|^{2} \\
< & 1+\sum_{k=2}^{m} \frac{1}{4(k-1)}<1+\frac{1}{4}(\log m+1)=\frac{1}{4} \log m+\frac{5}{4} .
\end{aligned}
$$

Remark 1. Let $K$ be the convex cone generated by the vectors $v_{k}, k=1, \ldots, m$. The meaning of Lemma 3 is that the cones $g(K)(g \in G)$ form a triangulation of the space $\mathbb{R}^{m}$. As a matter of fact, the intersections of the cones $g(K)$ with the standard ( $m-1$ )-simplex

$$
S_{m-1}=\left\{x \in \mathbb{R}^{m} \mid \sum_{i=1}^{m} x_{i}=1, x_{i} \geq 0 \text { for all } i\right\}
$$

form the baricentric subdivision of $S_{m-1}$.

Remark 2. The following natural question has been considered in several papers, for instance in [B-F(1988)] and [B-W(2003)]. What is the maximal radius $r(m, N)$ of the inscribed ball of the convex hull of $N$ points chosen from the unit ball of $\mathbb{R}^{m}$ ? In our case $N=3^{m}-1$ and one may wonder how close $s_{m}$ and $B_{m}$ are to the maximal radius and best arrangement. It turns out that they are very far: it follows from the results of $[\mathrm{B}-\mathrm{F}(1988)]$ and $[\mathrm{B}-\mathrm{W}(2003)]$ that, in the given range $N=3^{m}-1$,

$$
r(m, N)=\left(\frac{8}{9}\right)^{1 / 2}(1+o(1))
$$

as $m \rightarrow \infty$. So the optimal radius is much larger than $s_{m}$. This also shows that, as expected, $B_{m}$ is far from being distributed uniformly on the unit sphere.

Acknowledgement. The authors express their sincere gratitude to Michał Misiurewicz (Indiana University Purdue University at Indianapolis) for posing the above problem during his joint research with the second author and Alexander Blokh (University of Alabama at Birmingham.) The first named author was partially supported by Hungarian National Foundation Grants T 037846 and T 046246.

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