# LARGE WEIGHT DOES NOT YIELD AN IRREDUCIBLE BASE 

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#### Abstract

Answering a question of Juhász, Soukup and Szentmiklssy, we show that it is consistent that some first countable space of uncountable weight does not contain an uncountable subspace which has an irreducible base.


## 1. Introduction

For a topological space $X, w(X)$ is the minimal cardinality of a base for $X, \chi(p, X)=\min \{|u|: u$ is a neighbourhood base of $p\}$, and $\chi(X)=\sup \{\chi(p, X): p \in X\}$.

In [1] the following problem was investigated: What makes a space have weight larger than its character? The notion of irreducible base was introduced, and it was proved [1, Lemma 2.6] that if a topological space $X$ has an irreducible base then $w(X)=|X| \cdot \chi(X)$. The following question was formulated:
Problem 1. Does every first countable space of uncountable weight contain an uncountable subspace which has an irreducible base?

We show that the answer is consistently NO. We thank Lajos Soukup for actually writing the paper.

Definition 1.1. Let $X$ be a topological space. A base $\mathcal{U}$ of $X$ is called irreducible if it has an irreducible decomposition $\mathcal{U}=\bigcup\left\{\mathcal{U}_{x}: x \in X\right\}$, i.e, (i) and (ii) below hold:
(i) $\mathcal{U}_{x}$ is a neighbourhood base of $x$ in $X$ for each $x \in X$.
(ii) for each $x \in X$ the family $\mathcal{U}_{x}^{-}=\bigcup_{y \neq x} \mathcal{U}_{y}$ is not a base of $X$.

Theorem 1.2. There is a c.c.c poset $P=\langle P, \leq\rangle$ of size $\omega_{1}$ such that in $V^{P}$ there is a first countable space $X=\left\langle\omega_{1}, \tau\right\rangle$ of uncountable weight which does not contain an uncountable subspace which has an irreducible base.

Proof. The elements of the poset $P$ will be finite "approximations" of a base $\left\{U(\alpha, n): \alpha<\omega_{1}, n<\omega\right\}$ of $X$.

[^0]We define the poset $P=\langle P, \leq\rangle$ as follows. The underlying set of $P$ consists of the triples $\langle A, n, U\rangle$ satisfying (P1)-(P3) below:
(P1) $A \in\left[\omega_{1}\right]^{<\omega}, n \in \omega$ and $U$ is a function, $U: A \times n \rightarrow \mathcal{P}(A)$,
(P2) $\alpha \in U(\alpha, i) \subset U(\alpha, i-1)$ for each $\alpha \in A$ and $i<n$,
(P3) If $\beta \in U(\alpha, i) \subset U(\beta, 0)$ for some $i<n$, then $\beta \leq \alpha$.
For $p \in P$ write $p=\left\langle A_{p}, n_{p}, U_{p}\right\rangle$. Let us remark that property (P3) will guarantee that $\mathrm{w}(X)=\omega_{1}$.

Define the order $\leq$ on $P$ as follows. For $p, q \in P$ we put $q \leq p$ if
(a) $A_{p} \subset A_{q}$,
(b) $n_{p} \leq n_{q}$,
(c) $U_{p}(\alpha, i)=U_{q}(\alpha, i) \cap A_{p}$ for each $\langle\alpha, i\rangle \in A_{p} \times n_{p}$,
(d) for each $\langle\alpha, i\rangle,\langle\beta, j\rangle \in A_{p} \times n_{p}$,

$$
\begin{aligned}
& \text { if } U_{p}(\alpha, i) \cap U_{p}(\beta, j)=\emptyset \text { then } U_{q}(\alpha, i) \cap U_{q}(\beta, j)=\emptyset \text {, } \\
& \quad \text { if } U_{p}(\alpha, i) \subset U_{p}(\beta, j) \text { then } U_{q}(\alpha, i) \subset U_{q}(\beta, j) \text {. }
\end{aligned}
$$

We say that the conditions $p_{0}=\left\langle A_{0}, n_{0}, U_{0}\right\rangle$ and $p_{1}=\left\langle A_{1}, n_{1}, U_{1}\right\rangle$ are twins iff $n_{0}=n_{1},\left|A_{0}\right|=\left|A_{1}\right|$ and denoting by $\sigma$ the unique $<_{\mathrm{On}^{-}}{ }^{-}$ preserving bijection between $A_{0}$ and $A_{1}$ we have
(I1) $\sigma \upharpoonright A_{0} \cap A_{1}=\operatorname{id}_{A_{0} \cap A_{1}}$,
(I2) $\sigma$ is an isomorphism between $p_{0}$ and $p_{1}$, i.e. for each $\alpha \in A_{0}$ and $i<n_{0}$ we have $U_{1}(\sigma(\alpha), i)=\sigma^{\prime \prime} U_{0}(\alpha, i)$.
We say that $\sigma$ is the twin function between $p_{0}$ and $p_{1}$. Define the smashing function $\underline{\sigma}$ of $p_{0}$ and $p_{1}$ as follows: $\underline{\sigma}=\sigma^{-1} \cup \operatorname{id}_{A_{0}}$. The function $\sigma^{*}$ defined by the formula $\sigma^{*}=\sigma \cup \sigma^{-1}$ is called the exchange function of $p_{0}$ and $p_{1}$.

The burden of the proof is to verify the next lemma.
Amalgamation Lemma 1.3. Assume that $p_{0}=\left\langle A_{0}, n_{0}, U_{0}\right\rangle$ and $p_{1}=$ $\left\langle A_{1}, n_{1}, U_{1}\right\rangle$ are twins, $A_{0} \cap A_{1}<A_{0} \backslash A_{1}<A_{1} \backslash A_{0}, \xi_{0} \in A_{0} \backslash A_{1}$, $\xi_{1}=\sigma\left(\xi_{0}\right)$, where $\sigma$ is the twin function between $p_{0}$ and $p_{1}$, and let $k<m<n_{0}$. Then $p_{0}$ and $p_{1}$ have a common extension $p=\langle A, n, U\rangle$ in $P$ such that

$$
\begin{equation*}
\xi_{0} \in U\left(\xi_{1}, m\right) \subset U\left(\xi_{1}, k\right) \subset U\left(\xi_{0}, k\right) \tag{*}
\end{equation*}
$$

Proof. Write $n=n_{0}=n_{1}, D=A_{0} \cap A_{1}$ and $A^{*}=A_{0} \cup A_{1}$. Unfortunately we can not assume that $A=A^{*}$ because in this case we can not guarantee (P3) for $p$. So we need to add further elements to $A^{*}$ to get a large enough $A$ as follows. Choose a set $B \subset \omega_{1} \backslash A^{*}$ of cardinality $\left|A^{*} \times n\right|$ and fix a bijection $\rho$ between $A^{*} \times n$ and $B$. We will take $A=A^{*} \cup B$. To simplify the notation we will write $\langle\alpha, i\rangle$ for $\rho(\alpha, i)$, for all $\alpha \in A^{*}$ and $i<n$, i.e. we identify the elements of $B$ and of $A^{*} \times n$.

The idea of the proof is the following: for each $\langle\alpha, i\rangle \in A^{*} \times n$ we put the element $\langle\alpha, i\rangle$ into $U(\alpha, i)$. On the other hand, we try to keep $U(\alpha, i)$ small, so we put $\langle\beta, j\rangle$ into $U(\alpha, i)$ if and only if we can "derive" from the property (d2) that $U(\beta, j) \subset U(\alpha, i)$ should hold in
any condition $p=\langle A, n, U\rangle$ which is a common extension of $p_{0}$ and $p_{1}$ and which satisfies $(*)$.

The condition $p$ will be constructed in two steps. First we construct a condition $p^{\prime}=\left\langle A, n, U^{\prime}\right\rangle$ extending both $p_{0}$ and $p_{1}$. This $p^{\prime}$ can be considered as the minimal amalgamation of $p_{0}$ and $p_{1}$. Then, in the second step, we carry out small modifications on the function $U^{\prime}$, namely we increase its value on certain places to guarantee $(*)$.

Now we carry out our construction. For $\varepsilon<2$ and $\langle\beta, j\rangle \in A_{\varepsilon} \times n$ let

$$
\begin{equation*}
V_{\varepsilon}(\beta, j)=\left\{\langle\alpha, i\rangle \in A_{\varepsilon} \times n: U_{\varepsilon}(\alpha, i) \subset U_{\varepsilon}(\beta, j)\right\} \tag{1}
\end{equation*}
$$

and

$$
\begin{align*}
& W_{\varepsilon}(\beta, j)=\left\{\langle\alpha, i\rangle \in A_{1-\varepsilon} \times n: \exists\langle\gamma, l\rangle \in D \times n\right. \\
&\left.U_{1-\varepsilon}(\alpha, i) \subset U_{1-\varepsilon}(\gamma, l) \wedge U_{\varepsilon}(\gamma, l) \subset U_{\varepsilon}(\beta, j)\right\} \tag{2}
\end{align*}
$$

If we want to define $p^{\prime}$ in such a way that $p^{\prime} \leq p_{0}, p_{1}$, then ( d 2 ) implies that $U^{\prime}(\alpha, i) \subset U^{\prime}(\beta, j)$ should hold whenever $\langle\alpha, i\rangle \in V(\beta, j) \cup W(\beta, j)$.

Now we are ready to define the function $U^{\prime}$. For $\varepsilon<2, \beta \in A_{\varepsilon}$ and $j<n$ let

$$
\begin{equation*}
U^{\prime}(\beta, j)=U_{\varepsilon}(\beta, j) \cup U_{1-\varepsilon}\left(\sigma^{*}(\beta), j\right) \cup V_{\varepsilon}(\beta, j) \cup W_{\varepsilon}(\beta, j) . \tag{3}
\end{equation*}
$$

For $\langle\alpha, i\rangle \in A^{*} \times n$ and $j<n$ let

$$
\begin{equation*}
U^{\prime}(\langle\alpha, i\rangle, j)=\{\langle\alpha, i\rangle\} . \tag{4}
\end{equation*}
$$

Let us remark that $U^{\prime}(\delta, j)$ is well-defined even for $\delta \in A_{0} \cap A_{1}$. Indeed, in this case $\sigma^{*}(\delta)=\delta$ and $V_{\varepsilon}(\delta, j)=W_{1-\varepsilon}(\delta, j)$, and so

$$
U^{\prime}(\delta, j)=U_{0}(\delta, j) \cup U_{1}(\delta, j) \cup V_{0}(\delta, j) \cup V_{1}(\delta, j)
$$

Now put

$$
p^{\prime}=\left\langle A, n, U^{\prime}\right\rangle
$$

Claim 1.4. If $\alpha \in U^{\prime}(\beta, j)$ then $\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j)$.
Indeed, if $\beta \in A_{\varepsilon}$ then $U^{\prime}(\beta, j) \cap A^{*}=U_{\varepsilon}(\beta, j) \cup U_{1-\varepsilon}\left(\sigma^{*}(\beta), j\right)$.
Claim 1.5. If $\langle\alpha, i\rangle \in U^{\prime}(\beta, j)$ then $\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j)$.
Proof of the Claim. Assume that $\beta \in A_{\varepsilon}$. If $\langle\alpha, i\rangle \in V_{\varepsilon}(\beta, j)$ then $\alpha \in$ $U_{\varepsilon}(\alpha, i) \subset U_{\varepsilon}(\beta, j)$ and $U_{\varepsilon}(\beta, j) \subset U^{\prime}(\beta, j)$. So we have $\alpha \in U^{\prime}(\beta, j)$ which implies $\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j)$ by Claim 1.4.
If $\langle\alpha, i\rangle \in W_{1-\varepsilon}(\beta, j)$ then for some $\langle\gamma, l\rangle \in D \times n$ we have $U_{1-\varepsilon}(\alpha, i) \subset$ $U_{1-\varepsilon}(\gamma, l) \wedge U_{\varepsilon}(\gamma, l) \subset U_{\varepsilon}(\beta, j)$. Thus $\alpha \in U_{\varepsilon}(\beta, j) \subset U^{\prime}(\beta, j)$, which implies $\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j)$ by Claim 1.4.
Claim 1.6. $p^{\prime} \in P$.
Proof of the claim 1.6.
(P11) and (P2) clearly hold, so we need to check only (P3).

Assume on the contrary that (P3) fails for $p^{\prime}$. Since $U^{\prime}(\langle\nu, s\rangle, j)=$ $\{\langle\nu, s\rangle\}$ by (4) for each $\langle\nu, s\rangle \in B$ and $j<n$, we can assume that some
$\alpha<\beta \in A^{*}$ and $i<n$ witness that (P3) fails, i.e. $\beta \in U^{\prime}(\alpha, i) \subset$ $U^{\prime}(\beta, 0)$. Then $\underline{\sigma}(\beta) \in U_{0}(\underline{\sigma}(\alpha), i) \subset U^{\prime}(\underline{\sigma}(\beta), 0)$ by Claim 1.4. Since $p_{0}$ satisfies (P3) it follows that $\underline{\sigma}(\beta) \leq \underline{\sigma}(\alpha)$, and so $\alpha \in A_{0} \backslash A_{1}$ and $\beta \in A_{1} \backslash A_{0}$. Consider the element $u=\langle\alpha, i\rangle \in A \backslash A^{*}$. Then $u \in U^{\prime}(\alpha, i)$ and so $u \in U^{\prime}(\beta, 0)$ as well. By the definition of $U^{\prime}(\beta, 0)$ this means that $\langle\alpha, i\rangle \in W_{1}(\beta, 0)$, that is, there is $\langle\gamma, l\rangle \in D \times n$ such that $U_{0}(\alpha, i) \subset U_{0}(\gamma, l)$ and $U_{1}(\gamma, l) \subset U_{1}(\beta, j)$. Thus

$$
\begin{equation*}
\underline{\sigma}(\beta) \in U_{0}(\alpha, i) \subset U_{0}(\gamma, l) \subset U_{0}(\underline{\sigma}(\beta), 0) \tag{5}
\end{equation*}
$$

by Claim 1.4. Thus $\underline{\sigma}(\beta) \in U_{0}(\gamma, l) \subset U_{0}(\underline{\sigma}(\beta), 0)$ and so $\underline{\sigma}(\beta) \leq \gamma$ because $p_{0}$ satisfies (P3). But this is a contradiction because $\gamma \in D=$ $A_{0} \cap A_{1}, \underline{\sigma}(\beta) \in A_{0} \backslash A_{1}$ and we assumed that $\left(A_{0} \cap A_{1}\right)<\left(A_{0} \backslash A_{1}\right)$.

Claim 1.7. $p^{\prime} \leq p_{0}, p_{1}$.
Proof of claim 1.7. Conditions (a) and (b) are clear.
To check (c) assume that $\alpha \in A_{\varepsilon}$ and $i \in n$. By (3),

$$
\begin{aligned}
& U^{\prime}(\alpha, i) \cap A_{\varepsilon}=\left(U_{\varepsilon}(\alpha, i) \cup U_{1-\varepsilon}(\alpha, i)\right) \cap A_{\varepsilon}= \\
& U_{\varepsilon}(\alpha, i) \cup\left(U_{1-\varepsilon}(\alpha, i) \cap A_{\varepsilon}\right)=U_{\varepsilon}(\alpha, i)
\end{aligned}
$$

because $U_{1-\varepsilon}(\alpha, i)=\sigma^{*}\left[U_{\varepsilon}(\alpha, i)\right]$.
To check (d1) assume that $\beta, \gamma \in A_{\varepsilon}$ and $j, k<n$ such that $U^{\prime}(\beta, j) \cap$ $U^{\prime}(\gamma, k) \neq \emptyset$. Fix $x \in U^{\prime}(\beta, j) \cap U^{\prime}(\gamma, k)$. Then

$$
\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j) \cap U_{0}(\underline{\sigma}(\gamma), k)
$$

by Claim 1.4 if $x=\alpha \in A^{*}$, and by Claim 1.5 if $x=\langle\alpha, i\rangle \in A \backslash A^{*}$.
If $\varepsilon=0$ then $\underline{\sigma}(\beta)=\beta$ and $\underline{\sigma}(\gamma)=\gamma$, so $\underline{\sigma}(\alpha) \in U_{\varepsilon}(\beta, j) \cap U_{\varepsilon}(\gamma, k)$.
If $\varepsilon=1$ then $\underline{\sigma}(\beta)=\sigma^{*}(\beta)$ and $\underline{\sigma}(\gamma)=\sigma^{*}(\gamma)$, and so $\sigma^{*}(\underline{\sigma}(\alpha)) \in$ $U_{\varepsilon}(\beta, j) \cap U_{\varepsilon}(\gamma, k)$.

Finally to check (d2) assume that $\beta, \gamma \in A_{\varepsilon}$ and $j, k<n$ such that $U_{\varepsilon}(\beta, j) \subset U_{\varepsilon}(\gamma, k)$. Then clearly

$$
U_{1-\varepsilon}(\beta, j)=\sigma^{*}\left[U_{\varepsilon}(\beta, j)\right] \subset \sigma^{*}\left[U_{\varepsilon}(\gamma, k)\right]=U_{1-\varepsilon}(\gamma, k),
$$

moreover $V_{\varepsilon}(\beta, j) \subset V_{\varepsilon}(\gamma, k)$ by (1), and $W_{\varepsilon}(\beta, j) \subset W_{\varepsilon}(\gamma, k)$ by (2), and so $U^{\prime}(\beta, j) \subset U^{\prime} \varepsilon(\gamma, k)$ by (4).

Now carry out the promised modification of $U^{\prime}$ to obtain $U$ as follows. If $z \in A$ and $j<n$ let

$$
U(z, j)= \begin{cases}U^{\prime}(z, j) \cup U^{\prime}\left(\xi_{1}, k\right) & \text { if } U_{0}\left(\xi_{0}, k\right) \subset U_{0}(z, j), \\ U^{\prime}(z, j) & \text { otherwise. }\end{cases}
$$

Put

$$
p=\langle A, n, U\rangle
$$

If $U_{0}\left(\xi_{0}, k\right) \subset U_{0}(z, j)$ then $U_{1}\left(\xi_{1}, k\right) \subset U_{1}\left(\sigma^{*}(z), j\right) \subset U^{\prime}(z, j)$ and $W_{1}\left(\xi_{1}, k\right) \subset V_{0}\left(\xi_{0}, k\right) \subset U^{\prime}(z, j)$. So

$$
\begin{equation*}
U(z, j) \backslash U^{\prime}(z, j) \subset V_{1}(\xi, k) \tag{6}
\end{equation*}
$$

Moreover

$$
U(z, j)= \begin{cases}U^{\prime}(z, j) \cup V_{1}\left(\xi_{1}, k\right) & \text { if } U_{0}\left(\xi_{0}, k\right) \subset U_{0}(z, j)  \tag{7}\\ U^{\prime}(z, j) & \text { otherwise }\end{cases}
$$

Claim 1.8. If $\langle\alpha, i\rangle \in U(\beta, j)$ then $\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j)$.
Indeed, if $\langle\alpha, i\rangle \in U(\beta, j)$ then $\langle\alpha, i\rangle \in U^{\prime}(\beta, j)$ or $\langle\alpha, i\rangle \in U^{\prime}\left(\sigma^{*}(\beta), j\right)$, and now apply Claim 1.5.

Claim 1.9. $p \in P$.
Proof of claim 1.9. (P1) and (P2) clearly hold, so we need to check (P3) only.

Assume on the contrary that (P3) fails for $p$. Since $U(\langle\nu, s\rangle, j)=$ $\{\langle\nu, s\rangle\}$ for each $\langle\nu, s\rangle \in A \backslash A^{*}$ and $j<n$ we can assume that there are $\alpha<\beta \in A^{*}$ and $i<n$ witness that (P[3) fails, i.e.

$$
\begin{equation*}
\beta \in U(\alpha, i) \subset U(\beta, 0) \tag{8}
\end{equation*}
$$

Then $\underline{\sigma}(\beta) \in U_{0}(\underline{\sigma}(\alpha), i) \subset U(\underline{\sigma}(\beta), 0)$. But $p_{0}$ satisfies $(\mathrm{P}(3)$ so $\underline{\sigma}(\beta) \leq$ $\underline{\sigma}(\alpha)$, and so $\alpha \in A_{0} \backslash A_{1}$ and $\beta \in A_{1} \backslash A_{0}$. Thus $U_{0}(\beta, j)$ is undefined, and so

$$
\begin{equation*}
U^{\prime}(\beta, 0)=U(\beta, 0) \text { and } U(\alpha, i) \backslash U^{\prime}(\alpha, i) \subset A \backslash A^{*} \tag{9}
\end{equation*}
$$

by (7). So (8) yields

$$
\beta \in U^{\prime}(\alpha, i) \subset U^{\prime}(\beta, 0)
$$

However this is a contradiction because $p^{\prime}$ satisfies (P3).
Claim 1.10. $p \leq p_{0}, p_{1}$.
Proof. (a) and (b) are trivial. (c) also holds because $p^{\prime} \leq p_{\varepsilon}$ and $\left(U(\alpha, i) \backslash U^{\prime}(\alpha, i)\right) \cap A_{\varepsilon}=\emptyset$ by (6)

To check (d1) assume that $\beta, \gamma \in A_{\varepsilon}$ and $j, k<n$ such that $U(\beta, j) \cap$ $U(\gamma, k) \neq \emptyset$. Pick $x \in U(\beta, j) \cap U(\gamma, k)$. Then

$$
\underline{\sigma}(\alpha) \in U_{0}(\underline{\sigma}(\beta), j) \cap U_{0}(\underline{\sigma}(\gamma), k)
$$

by Claim 1.4 if $x=\alpha \in A^{*}$, and by Claim 1.8 if $x=\langle\alpha, i\rangle \in A \backslash A^{*}$.
If $\varepsilon=0$ then $\underline{\sigma}(\beta)=\beta$ and $\underline{\sigma}(\gamma)=\gamma$, so $\underline{\sigma}(\alpha) \in U_{\varepsilon}(\beta, j) \cap U_{\varepsilon}(\gamma, k)$.
If $\varepsilon=1$ then $\underline{\sigma}(\beta)=\sigma^{*}(\beta)$ and $\underline{\sigma}(\gamma)=\sigma^{*}(\gamma)$, and so $\sigma^{*}(\underline{\sigma}(\alpha)) \in$ $U_{\varepsilon}(\beta, j) \cap U_{\varepsilon}(\gamma, k)$.

Finally to check (d2) assume that $\beta, \gamma \in A_{\varepsilon}$ and $i, j<n$ such that $U_{\varepsilon}(\beta, i) \subset U_{\varepsilon}(\gamma, j)$. Since $p^{\prime} \leq p_{\varepsilon}$ we have $U^{\prime}(\beta, i) \subset U^{\prime}(\gamma, j)$. If $U(\beta, i)=U^{\prime}(\beta, i)$, we are done. So we can assume that $U(\beta, i)=$ $U^{\prime}(\beta, i) \cup V\left(\xi_{1}, k\right)$. Then $\varepsilon=0$ and $U_{0}\left(\xi_{0}, k\right) \subset U_{0}(\beta, i)$. But then $U_{0}\left(\xi_{0} . k\right) \subset U_{0}(\gamma, j)$ and so $U(\gamma, j)=U^{\prime}(\gamma, j) \cup V\left(\xi_{1}, k\right)$, and so $U(\beta, i) \subset$ $U(\gamma, j)$.

Since $p$ satisfies $(*)$, the amalgamation lemma is proved. the theorem.

By standard $\Delta$-system argument, any uncountable set of conditions contains two elements, $p_{0}$ and $p_{1}$, which are twins. So, by Lemma 1.3, they have a common extension $p$. So $P$ satisfies c.c.c.

If $\mathcal{G}$ is a generic filter, for $\alpha<\omega_{1}$ and $i<\omega$ put

$$
\begin{equation*}
U(\alpha, i)=\cup\left\{U_{p}(\alpha, i): p \in \mathcal{G}, \alpha \in A_{p}, i<n_{p}\right\}, \tag{10}
\end{equation*}
$$

and let $\mathcal{U}_{\alpha}=\{U(\alpha, i): i<\omega\}$ be the base of the point $\alpha$ in $X=\left\langle\omega_{1}, \tau\right\rangle$.
By (P3), a countable subfamily of $\left\{U(\alpha, i): \alpha<\omega_{1}, i<\omega\right\}$ is not a base of $X$. So $\mathrm{w}(X)=\omega_{1}$.

Finally we show that $X$ does not contain an uncountable subspace which has an irreducible base.

Assume on the contrary that
$r \Vdash$ the subspace $\dot{Y}=\left\{\dot{y}_{\xi}: \xi<\omega_{1}\right\}$ has an irreducible base $\mathcal{B}$, and $\left\{\dot{\mathcal{B}}_{y_{\xi}}: \xi<\omega_{1}\right\}$ is an irreducible decomposition of $\dot{\mathcal{B}}$.

We can assume that $r \Vdash \dot{y}_{\xi} \geq \check{\xi}$.
For each $\xi<\omega_{1}$ pick a condition $r_{\xi}$ and $k_{\xi} \in \omega$ such that

$$
\begin{equation*}
r_{\xi} \Vdash \text { "if } V \in \mathcal{B} \text { with } \dot{y}_{\xi} \in V \subset U\left(\dot{y}_{\xi}, \check{k}_{\xi}\right) \text { then } V \in \mathcal{B}_{y_{\xi}} \text { ". } \tag{11}
\end{equation*}
$$

For each $\xi<\omega_{1}$ pick a condition $p_{\xi} \leq r_{\xi}$, an ordinal $\alpha_{\xi} \geq \xi$, a name $\dot{V}_{\xi}$ and a natural number $m_{\xi}<\omega$ such that $\alpha_{\xi} \in A_{p_{\xi}}$ and

$$
\begin{equation*}
p_{\xi} \Vdash \dot{y}_{\xi}=\check{\alpha}_{\xi}, \dot{V}_{\xi} \in \dot{\mathcal{B}}_{\alpha_{\xi}} \text { and } U\left(\check{\alpha}_{\xi}, \check{m}_{\xi}\right) \subset \dot{V}_{\xi} \subset U\left(\check{\alpha}_{\xi}, \check{k}_{\xi}\right) \text {. } \tag{12}
\end{equation*}
$$

By standard argument find $I \in\left[\omega_{1}\right]^{\omega_{1}}$ such that
(i) $m_{\xi}=m$ and $k_{\xi}=k$ for each $\xi \in I$,
(ii) the sequence $\left\{\alpha_{\xi}: \xi \in I\right\}$ is strictly increasing,
(iii) the conditions $\left\{p_{\xi}: \xi \in I\right\}$ are pairwise twins,
(iv) $\sigma_{\xi, \eta}\left(\alpha_{\xi}\right)=\alpha_{\eta}$ for $\{\xi, \eta\} \in[I]^{2}$, where $\sigma_{\xi, \eta}$ is the twin function.

Pick $\xi<\eta$ from $I$. By the Amalgamation Lemma there is a common extension $p$ of $p_{\xi}$ and $p_{\eta}$ such that

$$
\begin{equation*}
\alpha_{\xi} \in U_{p}\left(\alpha_{\eta}, m\right) \wedge U_{p}\left(\alpha_{\eta}, k\right) \subset U_{p}\left(\alpha_{\xi}, k\right) \tag{13}
\end{equation*}
$$

Then, by (d2),

$$
\begin{equation*}
p \Vdash \check{\alpha}_{\xi} \in U\left(\check{\alpha}_{\eta}, \check{m}\right) \wedge U\left(\check{\alpha}_{\eta}, \check{k}\right) \subset U\left(\check{\alpha}_{\xi}, \check{k}\right) . \tag{14}
\end{equation*}
$$

Then, by (12),

$$
\begin{equation*}
p \Vdash \dot{V}_{\eta} \in \mathcal{B}_{\alpha_{\eta}} \text { and } \check{\alpha}_{\xi} \in U\left(\check{\alpha}_{\eta}, \check{m}\right) \subset \dot{V}_{\eta} \subset U\left(\check{\alpha}_{\eta}, \check{k}\right) \subset U\left(\check{\alpha}_{\xi}, \check{k}\right) \tag{15}
\end{equation*}
$$

which contradicts (11).
This completes the proof of the Theorem.

## References

[1] Juhsz, I.; Soukup, L.; Szentmiklssy, Z. What makes a space have large weight? Topology Appl. 57 (1994), no. 2-3, 271-285.

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