## An analogue of a van der Waerden's theorem and its application to two-distance preserving mappings

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## Abstract

The van der Waerden's theorem reads that an equilateral pentagon in Euclidean 3-space  $\mathbb{E}^3$  with all diagonals of the same length is necessarily planar and its vertex set coincides with the vertex set of some convex regular pentagon. We prove the following many-dimensional analogue of this theorem: for  $n \ge 2$ , every *n*-dimensional cross-polytope in  $\mathbb{E}^{2n-2}$  with all diagonals of the same length and all edges of the same length necessarily lies in  $\mathbb{E}^n$  and hence is a convex regular cross-polytope. We also apply our theorem to the study of two-distance preserving mappings of Euclidean spaces.

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1. van der Waerden's theorem and its many-dimensional analogue. In 1970 B.L. van der Waerden has shown that an equilateral and isogonal pentagon in Euclidean 3-space  $\mathbb{E}^3$  is necessarily planar and its vertex set coincides with the vertex set of some convex regular pentagon. For more details about this theorem we refer to [12] and [5] and references given there. More recent results related to this theorem may be found in [3], [8], and [9].

The van der Waerden's theorem may be reformulated as follows: an equilateral pentagon in Euclidean 3-space  $\mathbb{E}^3$  with all diagonals of the same length is necessarily planar and its vertex set coincides with the vertex set of some convex regular pentagon. The aim of this paper is to find a many-dimensional analogue of this statement by replacing a pentagon by a polyhedron which, under some conditions on the lengths of its edges and diagonals, is 'surprisingly flat.' The latter means that we start with a polyhedron located in  $\mathbb{E}^N$  and prove that it necessarily lies in  $\mathbb{E}^n$  for some n < N.

Let  $n \ge 1$  be an integer and  $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$  be an orthonormal basis in Euclidean *n*-space  $\mathbb{E}^n$ . Denote by  $V_n$  the set of the end-points of the vectors  $\pm \mathbf{e}_1, \pm \mathbf{e}_2, \dots, \pm \mathbf{e}_n$ . The convex hull of  $V_n$  is called the *standard n-dimensional cross-polytope* in  $\mathbb{E}^n$  and is denoted by  $S_n$ . Obviously,  $V_n$  is the set of vertices of  $S_n$ .

Let N and n be positive integers. In this paper, every injective mapping  $f: V_n \to \mathbb{E}^N$  is called an *n*-dimensional cross-polytope in  $\mathbb{E}^N$ . We prefer this definiton for the following two reasons:

(i) using f, the reader may easily reconstruct an abstract simplicial complex C which is combinatorially equivalent to  $S_n$  and whose vertex set is  $f(V_n)$  (it suffice to put by definiton that a simplex  $\Delta$  (of arbitrary dimension) with vertices  $w_1, w_2, \ldots, w_k \in f(V_n)$  belongs to C if and only if the simplex with vertices  $f^{-1}(w_1), f^{-1}(w_2), \ldots, f^{-1}(w_k)$  is a face of  $S_n$ ) and

(ii) from our definition the reader may easily see that the realization of the abstract simplicial complex C described in (i) may be degenerate (e.g., the vertices  $w_1, w_2, \ldots, w_k \in f(V_n)$  may lay on a single line in  $\mathbb{E}^N$ ).

For every two points  $u, v \in V_n \subset S_n$  there are only two possibilities: either u and v are joint together by an edge of  $S_n$  or u + v = 0. In the first case the straight-line segment with the points

f(u) and f(v) in  $\mathbb{E}^N$  is called an *edge* of the *n*-dimensional cross-polytope  $f: V_n \to \mathbb{E}^N$ , while in the second case this segment is called a *diagonal* of f.

The main result of this paper is the following theorem.

**Theorem 1.** Let  $n \ge 2$  be an integer, let a, b be positive numbers, and let  $f: V_n \to \mathbb{E}^{2n-2}$  be an n-dimensional cross-polytope such that the length of every edge of f is equal to a and the length of every diagonal of f is equal to b. Then f is isometric to a homothetic copy of  $S_n$ , the standard n-dimensional cross-polytope in  $\mathbb{E}^n$ . In particular,  $b = \sqrt{2a}$ .

**Proof:** As soon as we know distances between every two vertices of f, we may treat f a simplex with 2n vertices

 $f(\mathbf{e}_1), f(-\mathbf{e}_1), f(\mathbf{e}_2), f(-\mathbf{e}_2), \dots, f(\mathbf{e}_n), f(-\mathbf{e}_n).$  (1)

In general, a simplex with 2n vertices and prescribed edge lengths is located in  $\mathbb{E}^{2n-1}$ . According to assumptions of Theorem 1, f is located in  $\mathbb{E}^{2n-2}$  and, thus, its (2n-1)-dimensional volume is equal to 0.

Let's make use of the Cayley–Menger formula for the k-dimensional volume,  $vol_k$ , of a simplex with k + 1 vertices in  $\mathbb{E}^k$  (see, e.g., [4, p. 98]):

$$(-1)^{k+1}2^{k}(k!)^{2}(\operatorname{vol}_{k})^{2} = \begin{vmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & d_{12}^{2} & d_{13}^{2} & \dots & d_{1,k+1}^{2} \\ 1 & d_{21}^{2} & 0 & d_{23}^{2} & \dots & d_{2,k+1}^{2} \\ 1 & d_{31}^{2} & d_{32}^{2} & 0 & \dots & d_{3,k+1}^{2} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & d_{k+1,1}^{2} & d_{k+1,2}^{2} & d_{k+1,3}^{2} & \dots & 0 \end{vmatrix} .$$
(2)

Here we assume that the vertices are labeled with numbers from 1 to k+1 and  $d_{ij}$  is the Euclidean distance between *i*-th and *j*-th vertices.

Enumerating the vertices of f according to (1) and taking into account that  $\operatorname{vol}_{2n-1} = 0$ , we obtain from (2)

Note that the matrix in (3) contains n blocks of the form

$$\left[\begin{array}{cc} 0 & b^2 \\ b^2 & 0 \end{array}\right].$$

In (3), these blocks are boxed for better visibility.

Denote the determinant in (3) by  $D_n$ . We are going to compute  $D_n$  and, thus, replace (3) by an explicit relation involving a and b.

Multiply the first row of  $D_n$  by  $(-a^2)$  and add the result to every other row. This yields

$$D_{n} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -a^{2} & b^{2} - a^{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & b^{2} - a^{2} & -a^{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & -a^{2} & b^{2} - a^{2} & \dots & 0 & 0 \\ 1 & 0 & 0 & b^{2} - a^{2} & -a^{2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & -a^{2} & b^{2} - a^{2} \\ 1 & 0 & 0 & 0 & 0 & \dots & -a^{2} & b^{2} - a^{2} \\ 1 & 0 & 0 & 0 & 0 & \dots & b^{2} - a^{2} & -a^{2} \end{vmatrix} .$$
(4)

In (4), subtract the second row from the third, the forth from the fifth, ..., and the (2n)th from the (2n + 1)th. We obtain

$$D_{n} = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & -a^{2} & b^{2} - a^{2} \\ 1 & b^{2} & -b^{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & -a^{2} & b^{2} - a^{2} \\ 1 & 0 & 0 & b^{2} & -b^{2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & -a^{2} & b^{2} - a^{2} \\ 1 & 0 & 0 & 0 & 0 & \dots & -a^{2} & b^{2} - a^{2} \\ 1 & 0 & 0 & 0 & 0 & \dots & b^{2} & -b^{2} \end{vmatrix} .$$
(5)

In (5), add the second column to the third, the forth to the fifth, ..., and the (2n)th to the (2n+1)th. We get

$$D_{n} = \begin{vmatrix} 0 & 1 & 2 & 1 & 2 & \dots & 1 & 2 \\ 1 & -a^{2} & b^{2} - 2a^{2} & 0 & 0 & \dots & 0 & 0 \\ 0 & b^{2} & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & -a^{2} & b^{2} - 2a^{2} & \dots & 0 & 0 \\ 0 & 0 & 0 & b^{2} & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \ddots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & -a^{2} & b^{2} - 2a^{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & b^{2} & 0 \end{vmatrix} .$$
(6)

Expand the determinant in (6) along the third row, the fifth row, ..., and the (2n+1)th row. The result is

$$D_n = (-1)^n b^{2n} \begin{vmatrix} 0 & 2 & 2 & \dots & 2 \\ 1 & b^2 - 2a^2 & 0 & \dots & 0 \\ 1 & 0 & b^2 - 2a^2 & \dots & 0 \\ \dots & \dots & \dots & \ddots & \dots \\ 1 & 0 & 0 & \dots & b^2 - 2a^2 \end{vmatrix} .$$
(7)

Note that the size of the matrix in (7) is reduced to n + 1.

If  $b^2 - 2a^2 = 0$ , the determinant in (7) is equal to zero (e.g., because the second row is equal to the third). If  $b^2 - 2a^2 \neq 0$ , rewrite (7) in the form

$$D_{n} = 2 \frac{(-1)^{n} b^{2n}}{b^{2} - 2a^{2}} \begin{vmatrix} 0 & 1 & 1 & \dots & 1 \\ b^{2} - 2a^{2} & b^{2} - 2a^{2} \\ b^{2} - 2a^{2} & 0 & b^{2} - 2a^{2} \\ \dots & \dots & \dots & \ddots \\ b^{2} - 2a^{2} & 0 & 0 & \dots \\ b^{2} - 2a^{2} & 0 & 0 & \dots \\ b^{2} - 2a^{2} & 0 & 0 & \dots \\ b^{2} - 2a^{2} \end{vmatrix} .$$
(8)

In (8), subtract the second, the third, ..., and the (n + 1)st column from the first column. This yields

$$D_n = 2 \frac{(-1)^n b^{2n}}{b^2 - 2a^2} \begin{vmatrix} -n & 1 & 1 & \dots & 1 \\ 0 & \underline{b^2 - 2a^2} & 0 & \dots & 0 \\ 0 & 0 & \underline{b^2 - 2a^2} & \dots & 0 \\ \dots & \dots & \ddots & \ddots & \dots \\ 0 & 0 & 0 & \dots & \underline{b^2 - 2a^2} \end{vmatrix} = 2nb^{2n}(2a^2 - b^2)^{n-1}.$$

Hence, (3) is equivalent to  $b^2 - 2a^2 = 0$ . This means that the cross-polytope f is congruent to a homothetic copy of the standard *n*-dimensional cross-polytope  $S_n$  in  $\mathbb{E}^n$ . Q.E.D.

**2. Two-distance preserving mappings.** A mapping  $g : \mathbb{E}^n \to \mathbb{E}^m$  is said to be *unit distance preserving* if, for all  $x, y \in \mathbb{E}^n$ , the equality |x - y| = 1 implies the equality |g(x) - g(y)| = 1. Here by |x| we denote the Euclidean norm of a vector  $x \in \mathbb{E}^n$ .

In 1953, F.S. Beckman and D.A. Quarles [1] proved that, for  $n \ge 2$ , every unit distance preserving mapping  $g: \mathbb{E}^n \to \mathbb{E}^n$  is an isometry of  $\mathbb{E}^n$  (i.e., g preserves all distances). Since that time, the problem 'does a unit distance preserving mapping is an isometry' was studied for spaces of various types (hyperbolic [10], Banach [11],  $\mathbb{Q}^n$  [13], just to name a few).

In 1985, B.V. Dekster [7] found a mapping  $g : \mathbb{E}^2 \to \mathbb{E}^6$ , which is unit distance preserving but is not an isometry. That example motivated geometers to look for other conditions that make the statement 'every unit distance preserving mapping  $g : \mathbb{E}^n \to \mathbb{E}^m$  is an isometry' correct even if  $m \neq n$ . One of the possible sets of such conditions uses the notions of cable and strut defined as follows.

Given a mapping  $g : \mathbb{E}^n \to \mathbb{E}^m$ , a positive real number c is called a *cable* of g if, for all  $x, y \in \mathbb{E}^n$ , the equality |x - y| = c implies  $|g(x) - g(y)| \leq c$  and a positive real number s is called a *strut* of g if, for all  $x, y \in \mathbb{E}^n$ , the equality |x - y| = s implies  $|g(x) - g(y)| \geq s$ .

In 1999, K. Bezdek and R. Connelly [2] proved that if  $n \ge 2$ , c is a cable of a mapping  $g: \mathbb{E}^n \to \mathbb{E}^m$ , s is a strut of g and  $c/s < (\sqrt{5}-1)/2$ , then g is an isometry.

As a corollary of Theorem 1, we prove the following theorem.

**Theorem 2.** Let  $n \ge 6$  and  $0 \le m \le 2n-2$  be integers, let A and B be positive real numbers, and let  $g : \mathbb{E}^n \to \mathbb{E}^m$  be a mapping such that, for all  $x, y \in \mathbb{E}^n$ , the equality |x - y| = A implies |g(x) - g(y)| = A and the equality  $|x - y| = \sqrt{2}A$  implies |g(x) - g(y)| = B. Then g is an isometry. **Proof:** First, observe that B is necessarily equal to  $\sqrt{2}A$ .

In fact, given  $x, y \in \mathbb{E}^n$  such that  $|x - y| = \sqrt{2}A$ , find an *n*-dimensional cross-polytope P in  $\mathbb{E}^n$  with the following properties:

(i) P is congruent to a homothetic copy of the standard n-dimensional cross-polytope  $S_n$  in  $\mathbb{E}^n$  with the scale factor  $A/\sqrt{2}$ ;

(ii) x and y belong to the vertex set of P;

(iii) the straight line segment with the endpoints x and y is a diagonal of P.

Since  $g: \mathbb{E}^n \to \mathbb{E}^m$  and  $m \leq 2n-2$ , Theorem 1 yields that the image of P under the mapping g is congruent to P. In particular, this means that  $|g(x) - g(y)| = \sqrt{2}A$  and, thus,  $B = \sqrt{2}A$ .

Now, let's prove that the real number  $c = 2A/\sqrt{n}$  is a cable of the mapping g, i.e., let's prove that if  $v_1, v_2 \in \mathbb{E}^n$  are such that  $|v_1 - v_2| = c = 2A/\sqrt{n}$  then  $|g(v_1) - g(v_2)| \leq c$ .

Let L be the (n-1)-dimensional plane in  $\mathbb{E}^n$  that passes thought the point  $(v_1 + v_2)/2$  and is orthogonal to the vector  $v_1 - v_2$ . Let  $v_3, v_4, \ldots, v_{n+2}$  be the vertices of an (n-1)-dimensional regular simplex in L with edge lengths  $\sqrt{2}A$  and circumcenter at the point  $(v_1 + v_2)/2$ . The latter means that the point  $(v_1 + v_2)/2$  is the center of an (n-2)-dimensional sphere which passes through all the vertices  $v_3, v_4, \ldots, v_{n+2}$ . Since the radius of this (n-2)-sphere is equal to  $R = A\sqrt{1-1/n}$  (see, e.g., [6, pp. 294–295]), Pythagora's Theorem gives  $|v_i - v_j| = A$  for all i = 1, 2 and  $j = 3, 4, \ldots, n+2$ .

It follows from conditions of Theorem 2 and the relation  $B = \sqrt{2}A$  that, for every i = 1, 2, the *n*-simplex  $\Delta_i$  with vertices  $v_i, v_3, v_4, \ldots, v_{n+2}$  is congruent to the *n*-simplex with vertices  $g(v_i), g(v_3), g(v_4), \ldots, g(v_{n+2})$ . For short, denote the latter simplex by  $g(\Delta_i)$ . Denote by  $\lambda$  the (n-1)-dimensional plane containing the points  $g(v_3), g(v_4), \ldots, g(v_{n+2})$ .

Since  $g(\Delta_1)$  is congruent to  $\Delta_1$ , it follows that  $m \ge n$ .

If m = n, the non-degenerate *n*-simplices  $g(\Delta_1)$  and  $g(\Delta_2)$  lie either in the same half-space of  $\mathbb{E}^m$  determined by  $\lambda$  (in this case  $|g(v_1) - g(v_2)| = 0 < c$ ) or in the different half-spaces of  $\mathbb{E}^m$  determined by  $\lambda$  (in this case  $|g(v_1) - g(v_2)| = |v_1 - v_2| = c$ ). In both cases,  $|g(v_1) - g(v_2)| \leq c$ .

If m > n, the *n*-simplices  $g(\Delta_1)$  and  $g(\Delta_2)$  may be obtained from  $\Delta_1$  and  $\Delta_2$  in two steps: first, we apply to  $\Delta_1$  and  $\Delta_2$  such an isometry  $h : \mathbb{E}^n \to \mathbb{E}^m$  that  $h(v_j) = g(v_j)$  for all  $j = 1, 3, 4, \ldots, n+2$ and then rotate the simplex  $h(\Delta_2)$  around  $\lambda$  in such a way that  $h(v_2)$  coincides with  $g(v_2)$ . From this description we conclude that  $|g(v_1) - g(v_2)| \leq |h(v_1) - h(v_2)| = |v_1 - v_2| = c$ . Hence,  $c = 2A/\sqrt{n}$ is a cable.

Obviously, we may consider  $s = \sqrt{2}A$  as a strut of g.

Since  $n \ge 6$ ,  $c/s = \sqrt{2/n} < (\sqrt{5} - 1)/2$ . Thus, according to the above-cited theorem by K. Bezdek and R. Connelly, g is an isometry. Q.E.D.

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