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CHARACTERIZING LINEAR MAPPINGS THROUGH ZERO PRODUCTS OR ZERO JORDAN PRODUCTS

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Abstract. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule, we study the local properties of $*$-derivations and $*$-Jordan derivations from $\mathcal{A}$ into $\mathcal{M}$ under the following orthogonality conditions on elements in $\mathcal{A}$: $ab^*=0$, $ab^*+b^*a=0$ and $ab^*=b^*a=0$. We characterize the mappings on zero product determined algebras and zero Jordan product determined algebras. Moreover, we give some applications on C^*-algebras, group algebra, matrix algebras, algebras of locally measurable operators and von Neumann algebras.

1. Introduction

Throughout this paper, let $\mathcal{A}$ be an associative algebra over the complex field $\mathbb{C}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. For each a, b in $\mathcal{A}$, we define the *Jordan product* by $a\circ b=ab+ba$. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *derivation* if $\delta(ab)=a\delta(b)+\delta(a)b$ for each a,b in $\mathcal{A}$; and δ is called a *Jordan derivation* if $\delta(a\circ b)=a\circ\delta(b)+\delta(a)\circ b$ for each a,b in $\mathcal{A}$. It follows from the results in [9, 20, 21] that every Jordan derivation from a C^*-algebra into its Banach bimodule is a derivation.

By an *involution* on an algebra $\mathcal{A}$, we mean a mapping $*$ from $\mathcal{A}$ into itself, such that

$$(\lambda a+\mu b)^*=\bar{\lambda}a^*+\bar{\mu}b^*,\ (ab)^*=b^*a^* \text{ and } (a^*)^*=a,$$

whenever a,b in $\mathcal{A}$, λ,μ in $\mathbb{C}$ and $\bar{\lambda},\bar{\mu}$ denote the conjugate complex numbers. An algebra $\mathcal{A}$ equipped with an involution is called a $*$-algebra. Moreover, let $\mathcal{A}$ be a $*$-algebra, an $\mathcal{A}$-bimodule $\mathcal{M}$ is called a $*$-$\mathcal{A}$-bimodule if $\mathcal{M}$ equipped with a $*$-mapping from $\mathcal{M}$ into itself, such that

$$(\lambda m+\mu n)^*=\bar{\lambda}m^*+\bar{\mu}n^*,\ (am)^*=m^*a^*,\ (ma)^*=a^*m^* \text{ and } (m^*)^*=m,$$

whenever a in $\mathcal{A}$, m,n in $\mathcal{M}$ and λ,μ in $\mathbb{C}$. An element a in $\mathcal{A}$ is called *self-adjoint* if $a^*=a$; an element p in $\mathcal{A}$ is called an *idempotent* if $p^2=p$; and p is called a *projection* if p is both a self-adjoint element and an idempotent.

In [24], A. Kishimoto studies the $*$-derivations on a C^*-algebra. Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. A derivation δ from $\mathcal{A}$ into $\mathcal{M}$ is called a $*$-*derivation* if $\delta(a^*)=\delta(a)^*$ for every a in $\mathcal{A}$. Obviously, every derivation δ is a linear combination of two $*$-derivations. In fact, we can define a linear mapping $\hat{\delta}$ from $\mathcal{A}$ into $\mathcal{M}$ by $\hat{\delta}(a)=\delta(a^*)^*$ for every a in $\mathcal{A}$, therefore $\delta=\delta_1+i\delta_2$, where

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$\delta_1 = \frac{1}{2}(\delta + \hat{\delta})$ and $\delta_2 = \frac{1}{2i}(\delta - \hat{\delta})$. It is easy to show that δ_1 and δ_2 are both $*$-derivations. Similarly, we can define the $*$-Jordan derivations.

For $*$-derivations and $*$-Jordan derivations, in [3, 13, 17, 18], the authors characterize the following two conditions on a linear mapping δ from a $*$-algebra $\mathcal{A}$ into its $*$-bimodule $\mathcal{M}$:

$$(\mathbb{D}_1)\ a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0;$$

$$(\mathbb{D}_2)\ a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = \delta(b)^*a + b^*\delta(a) = 0;$$

where $\mathcal{A}$ is a C^*-algebra, a zero product determined algebra or a group algebra $L^1(G)$.

Let $\mathcal{J}$ be an ideal of $\mathcal{A}$, we say that $\mathcal{J}$ is a *right separating set* or *left separating set* of $\mathcal{M}$ if for every m in $\mathcal{M}$, $\mathcal{J}m = \{0\}$ implies $m = 0$ or $m\mathcal{J} = \{0\}$ implies $m = 0$, respectively. We denote by $\mathfrak{J}(\mathcal{A})$ the subalgebra of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$.

In Section 2, we suppose that $\mathcal{A}$ is a $*$-algebra and $\mathcal{M}$ is a $*$-$\mathcal{A}$-bimodule that satisfy one of the following conditions:
(1) $\mathcal{A}$ is a zero product determined Banach $*$-algebra with a bounded approximate identity and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule;
(2) $\mathcal{A}$ is a von Neumann algebra and $\mathcal{M} = \mathcal{A}$;
(3) $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule with a left or right separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$;
and we investigate whether the linear mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying the condition $\mathbb{D}_1$ characterize $*$-derivations. In particular, we generalize some results in [13, 17, 18].

An $\mathcal{A}$-bimodule $\mathcal{M}$ is said to have the *property* $\mathbb{M}$, if there is an ideal $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ of $\mathcal{A}$ such that

$$\{m \in \mathcal{M} : xmx = 0 \text{ for every } x \in \mathcal{J}\} = \{0\}.$$

It is clear that if $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{M}$ has property $\mathbb{M}$.

For $*$-Jordan derivations, we can study the following conditions on a linear mapping δ from a $*$-algebra $\mathcal{A}$ into its $*$-$\mathcal{A}$-bimodule $\mathcal{M}$:

$$(\mathbb{D}_3)\ a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0.$$

$$(\mathbb{D}_4)\ a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0.$$

It is obvious that the condition $\mathbb{D}_2$ or $\mathbb{D}_3$ implies the condition $\mathbb{D}_4$.

In Section 3, we suppose that $\mathcal{A}$ is a $*$-algebra and $\mathcal{M}$ is a $*$-$\mathcal{A}$-bimodule that satisfy one of the following conditions:
(1) $\mathcal{A}$ is a unital zero Jordan product determined $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule;
(2) $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule such that the property $\mathbb{M}$;
(3) $\mathcal{A}$ is a C^*-algebra (not necessary unital) and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule;
and we investigate whether the linear mappings from $\mathcal{A}$ into $\mathcal{M}$ satisfying the condition $\mathbb{D}_3$ or $\mathbb{D}_4$ characterize $*$-Jordan derivations. In particular, we improve some results in [13, 17, 18].

2. $*$-DERIVATIONS ON SOME ALGEBRAS

A (Banach) algebra $\mathcal{A}$ is said to be *zero product determined* if every (continuous) bilinear mapping ϕ from $\mathcal{A}\times\mathcal{A}$ into any (Banach) linear space $\mathcal{X}$ satisfying

$$\phi(a,b)=0,\ \text{whenever } ab=0$$

can be written as $\phi(a,b)=T(ab)$, for some (continuous) linear mapping T from $\mathcal{A}$ into $\mathcal{X}$. In [7], M. Brešar shows that if $\mathcal{A}=\mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is a zero product determined, and in [1], the authors prove that every C^*-algebra $\mathcal{A}$ is zero product determined.

Let $\mathcal{A}$ be a Banach $*$-algebra and $\mathcal{M}$ be a Banach $*$-$\mathcal{A}$-bimodule. Denote by $\mathcal{M}^{\sharp\sharp}$ the second dual space of $\mathcal{M}$. In the following, we show that $\mathcal{M}^{\sharp\sharp}$ is also a Banach $*$-$\mathcal{A}$-bimodule.

Since $\mathcal{M}$ is a Banach $*$-$\mathcal{A}$-bimodule, $\mathcal{M}^{\sharp\sharp}$ turns into a dual Banach $\mathcal{A}$-bimodule with the operation defined by

$$a\cdot m^{\sharp\sharp}=\lim_{\mu} am_{\mu}\ \text{and}\ m^{\sharp\sharp}\cdot a=\lim_{\mu} m_{\mu}a$$

for every a in $\mathcal{A}$ and every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, where (m_μ) is a net in $\mathcal{M}$ with $\|m_\mu\|\leqslant\|m^{\sharp\sharp}\|$ and $(m_\mu)\to m^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{M}^{\sharp\sharp},\mathcal{M}^{\sharp})$.

We define an involution $*$ in $\mathcal{M}^{\sharp\sharp}$ by

$$(m^{\sharp\sharp})^*(\rho)=\overline{m^{\sharp\sharp}(\rho^*)},\ \ \rho^*(m)=\overline{\rho(m^*)},$$

where $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, ρ in $\mathcal{M}^{\sharp}$ and m in $\mathcal{M}$. Moreover, if (m_μ) is a net in $\mathcal{M}$ and $m^{\sharp\sharp}$ is an element in $\mathcal{M}^{\sharp\sharp}$ such that $m_\mu\to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp},\mathcal{M}^{\sharp})$, then for every ρ in $\mathcal{M}^{\sharp}$, we have that

$$\rho(m_\mu)=m_\mu(\rho)\to m^{\sharp\sharp}(\rho).$$

It follows that

$$(m_\mu^*)(\rho)=\rho(m_\mu^*)=\overline{\rho^*(m_\mu)}\to\overline{m^{\sharp\sharp}(\rho^*)}=(m^{\sharp\sharp})^*(\rho)$$

for every ρ in $\mathcal{M}^{\sharp}$. It means that the involution $*$ in $\mathcal{M}^{\sharp\sharp}$ is continuous in $\sigma(\mathcal{M}^{\sharp\sharp},\mathcal{M}^{\sharp})$. Thus we can obtain that

$$(a\cdot m^{\sharp\sharp})^*=(\lim_{\mu} am_\mu)^*=\lim_{\mu} m_\mu^* a^*=(m^{\sharp\sharp})^*\cdot a^*,$$

Similarly, we can show that $(m^{\sharp\sharp}\cdot a)^*=a^*\cdot(m^{\sharp\sharp})^*$. It implies that $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}$-bimodule.

Let $\mathcal{A}$ be a Banach $*$-algebra, a *bounded approximate identity* for $\mathcal{A}$ is a net $(e_i)_{i\in\Gamma}$ of self-adjoint elements in $\mathcal{A}$ such that $\lim\limits_i\|ae_i-a\|=\lim\limits_i\|e_ia-a\|=0$ for every a in $\mathcal{A}$ and $\sup_{i\in\Gamma}\|e_i\|\leq k$ for some $k>0$.

In [18], H. Ghahramani and Z. Pan prove that if $\mathcal{A}$ is a unital zero product determined $*$-algebra and a linear mapping δ from $\mathcal{A}$ into itself satisfies the condition

$$(\mathbb{D}_1)\ a,b\in\mathcal{A},\ ab^*=0\Rightarrow a\delta(b)^*+\delta(a)b^*=0$$

then $\delta(a)=\Delta(a)+\delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation.

For general zero product determined Banach $*$-algebra with a bounded approximate identity, we have the following result.

Theorem 2.1. *Suppose that $\mathcal{A}$ is a zero product determined Banach $*$-algebra with a bounded approximate identity, and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0$$

then there exist a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and an element ξ in $\mathcal{M}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi \cdot a$ for every a in $\mathcal{A}$. Furthermore, ξ can be chosen in $\mathcal{M}$ in each of the following cases:
(1) *$\mathcal{A}$ is a unital $*$-algebra.*
(2) *$\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.*

Proof. Let $(e_i)_{i\in\Gamma}$ be a bounded approximate identity of $\mathcal{A}$. Since δ is continuous, the net $(\delta(e_i))_{i\in\Gamma}$ is bounded and we can assume that it converges to ξ in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

Since $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule, we know that the nets $(e_i m)_{i\in\Gamma}$ and $(me_i)_{i\in\Gamma}$ converge to m with the norm topology for every m in $\mathcal{M}$. Thus we have that

$$\mathrm{Ann}_{\mathcal{M}}(\mathcal{A}) = \{m \in \mathcal{M} : amb = 0 \text{ for each } a, b \in \mathcal{A}\} = \{0\}.$$

By the hypothesis, we can obtain that

$$a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \Rightarrow a\delta(b)^*c = 0.$$

It follows that

$$a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow c^*b^* = b^*a^* = 0 \Rightarrow c^*\delta(b)^*a^* = 0 \Rightarrow a\delta(b)c = 0. \quad (2.1)$$

By (2.1) and [1, Theorem 4.5], we know that

$$\delta(ab) = \delta(a)b + a\delta(b) - a \cdot \xi \cdot b$$

for each a, b in $\mathcal{A}$, and ξ can be chosen in $\mathcal{M}$ if $\mathcal{A}$ is a unital $*$-algebra or $\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \xi \cdot a$$

for every a in $\mathcal{A}$. It is easy to show that Δ is a norm-continuous derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and we only need to show that $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$.

First we claim that $\Delta(e_i) = \delta(e_i) - \xi \cdot e_i$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. In fact, since $(e_i)_{i\in\Gamma}$ is bounded in $\mathcal{A}$, we assume $(e_i)_{i\in\Gamma}$ converges to ζ in $\mathcal{A}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. For every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, define

$$m^{\sharp\sharp} \cdot \zeta = \lim_i m^{\sharp\sharp} \cdot e_i.$$

Thus $m \cdot \zeta = m$ for every m in $\mathcal{M}$. By [10, Proposition A.3.52], we know that the mapping $m^{\sharp\sharp} \mapsto m^{\sharp\sharp} \cdot \zeta$ from $\mathcal{M}^{\sharp\sharp}$ into itself is $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-continuous, and by the $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-denseness of $\mathcal{M}$ in $\mathcal{M}^{\sharp\sharp}$, we have that

$$m^{\sharp\sharp} \cdot \zeta = m^{\sharp\sharp} \quad (2.2)$$

for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Hence $\Delta(e_i) = \delta(e_i) - \xi \cdot e_i$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

Next we prove $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$. By the definition of Δ, we know that $a\Delta(b)^* + \Delta(a)b^* = 0$ for each a, b in $\mathcal{A}$ with $ab^* = 0$. Define a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ by

$$\phi(a, b) = a\Delta(b^*)^* + \Delta(a)b.$$

Thus $ab = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is a zero product determined algebra, there exists a norm-continuous linear mapping T from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ such that

$$T(ab) = \phi(a, b) = a\Delta(b^*)^* + \Delta(a)b \tag{2.3}$$

for each a, b in $\mathcal{A}$. Let $b = e_i$ be in (2.3), we can obtain that

$$T(ae_i) = a\Delta(e_i)^* + \Delta(a)e_i.$$

By the continuity of T and (2.2), it follows that $T(a) = \Delta(a)$ for every a in $\mathcal{A}$. Thus

$$T(ab) = \Delta(ab) = a\Delta(b^*)^* + \Delta(a)b.$$

Since Δ is a derivation, we have that $a\Delta(b^*)^* = a\Delta(b)$ and $\Delta(b^*)a^* = \Delta(b)^*a^*$. Let $a = e_i$ and taking $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-limits, by (2.2), it follows that $\Delta(b^*) = \Delta(b)^*$ for every b in $\mathcal{A}$. $\square$

Let G be a locally compact group. The group algebra and the measure convolution algebra of G, are denoted by $L^1(G)$ and $M(G)$, respectively. The convolution product is denote by $\cdot$ and the involution is denoted by $*$. It is well known that $M(G)$ is a unital Banach $*$-algebra, and $L^1(G)$ is a closed ideal in $M(G)$ with a bounded approximate identity. By [3, Lemma 1.1], we know that $L^1(G)$ is zero product determined. By [10, Theorem 3.3.15(ii)], it follows that $M(G)$ with respect to convolution product is the dual of $C_0(G)$ as a Banach $M(G)$-bimodule.

By [26, Corollary 1.2], we know that every continuous derivation Δ from $L^1(G)$ into $M(G)$ is an inner derivation, that is, there exists μ in $M(G)$ such that $\Delta(f) = f \cdot \mu - \mu \cdot f$ for every f in $L^1(G)$. Thus by Theorem 2.1, we can prove [17, Theorem 3.1(ii)] as follows.

Corollary 2.2. *Let G be a locally compact group. If δ is a continuous linear mapping from $L^1(G)$ into $M(G)$ such that*

$$f, g \in L^1(G),\ \ f \cdot g^* = 0 \Rightarrow f \cdot \delta(g)^* + \delta(f) \cdot g^* = 0$$

then there are μ, ν in $M(G)$ such that

$$\delta(f) = f \cdot \mu - \nu \cdot f$$

for every f in $L^1(G)$ and $\mathrm{Re}\mu \in \mathcal{Z}(M(G))$.

Proof. By Theorem 2.1, we know that there exist a $*$-derivation Δ from $L^1(G)$ into $M(G)$ and an element ξ in $M(G)$ such that $\delta(f) = \Delta(f) + \xi \cdot f$ for every f in $L^1(G)$. By [26, Corollary 1.2], it follows that there exists μ in $M(G)$ such that $\Delta(f) = f \cdot \mu - \mu \cdot f$. Since $\Delta(f^*) = \Delta(f)^*$, we have that

$$f^* \cdot \mu - \mu \cdot f^* = \mu^* \cdot f^* - f^* \cdot \mu^*$$

for every f in $L^1(G)$. By [3, Lemma 1.3(ii)], we know $\mathrm{Re}\mu = \frac{1}{2}(\mu + \mu^*) \in \mathcal{Z}(M(G))$. Let $\nu = \mu - \xi$, from the definition of Δ, we have that $\delta(f) = f \cdot \mu - \nu \cdot f$ for every f in $L^1(G)$. $\square$

For a general C^*-algebra $\mathcal{A}$, in [13], B. Fadaee and H. Ghahramani prove that if δ is a continuous linear mapping from $\mathcal{A}$ into its second dual space $\mathcal{A}^{\sharp\sharp}$ such that the condition $\mathbb{D}_1$, then there exist a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{A}^{\sharp\sharp}$ and an element ξ in $\mathcal{A}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi a$ for every a in $\mathcal{A}$.

In [1], the authors prove that every C^*-algebra $\mathcal{A}$ is zero product determined, and it is well known that $\mathcal{A}$ has a bounded approximate identity. Thus by Theorem 2.1, we can improve the result in [13] for any essential Banach $*$-bimodule.

Corollary 2.3. *Suppose that $\mathcal{A}$ is a C^*-algebra and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0$$

then there exist a $*$-derivation Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ and an element ξ in $\mathcal{M}^{\sharp\sharp}$ such that $\delta(a) = \Delta(a) + \xi \cdot a$ for every a in $\mathcal{A}$. Furthermore, ξ can be chosen in $\mathcal{M}$ in each of the following cases:
(1) *$\mathcal{A}$ has an identity.*
(2) *$\mathcal{M}$ is a dual $*$-$\mathcal{A}$-bimodule.*

For von Neumann algebras, we have the following result.

Theorem 2.4. *Suppose that $\mathcal{A}$ is a von Neumann algebra. If δ is a linear mapping from $\mathcal{A}$ into itself such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0,$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Proof. Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \delta(1)a$$

for every a in $\mathcal{A}$. In the following we show that Δ is a $*$-derivation. It is clear that $\Delta(1) = 0$ and $ab^* = 0$ can implies that $a\Delta(b)^* + \Delta(a)b^* = 0$.

Case 1: Suppose that $\mathcal{A}$ is an abelian von Neumann algebra. First we show that Δ satisfies that

$$a, b \in \mathcal{A},\ ab = 0 \Rightarrow a\Delta(b) = 0.$$

It is well known that $\mathcal{A} \cong C(X)$, where X is a compact Hausdorff space and $C(X)$ denotes the C^*-algebra of all continuous complex-valued functions on X. Thus we have that $ab = 0$ if and only if $ab^* = 0$ for each a, b in $\mathcal{A}$. Indeed, let f and g be two functions in $C(X)$ corresponding to a and b, respectively, we can obtain that

$$ab^* = 0 \Leftrightarrow f \cdot \bar{g} = 0 \Leftrightarrow f \cdot g = 0 \Leftrightarrow ab = 0.$$

Let a and b be in $\mathcal{A}$ with $ab^* = ab = 0$, we have that $a\Delta(b)^* + \Delta(a)b^* = 0$. Multiply a from the left side of above equation, we can obtain that $a^2\Delta(b)^* = 0$. Let f and h be two functions in $C(X)$ corresponding to a and $\Delta(b)$, then we have that

$$0 = f^2\bar{g} = f^2 g = fg.$$

It implies that $a\Delta(b) = 0$. By [23, Theorem 3], we know that Δ is continuous. By [19, Lemma 2.5] and $\Delta(1) = 0$, we know that $\Delta(a) = \Delta(1)a = 0$ for every a in $\mathcal{A}$.

Case 2: Suppose that $\mathcal{A} \cong M_n(\mathcal{B})$, where $\mathcal{B}$ is also a von Neumann algebra and $n \geqslant 2$. By [6, 7] we know that $\mathcal{A}$ is a zero product determined algebra. Thus by [18, Theorem 3.1] it follows that Δ is a $*$-derivation.

Case 3: Suppose that $\mathcal{A}$ is a general von Neumann algebra. It is well known that $\mathcal{A} \cong \sum_{i=1}^{n} \bigoplus \mathcal{A}_i$ (n is a finite integer or infinite), where each $\mathcal{A}_i$ coincides with either Case 1 or Case 2. Denote the unit element of $\mathcal{A}_i$ by 1_i and the restriction of Δ in $\mathcal{A}_i$ by Δ_i. Since $1_i(1-1_i) = 0$ and $\Delta(1) = 0$, we have that

$$1_i\Delta(1-1_i)^* + \Delta(1_i)(1-1_i) = 0.$$

It follows that

$$-1_i\Delta(1_i)^* + \Delta(1_i) - \Delta(1_i)1_i = 0. \tag{2.4}$$

Multiplying 1_i from the left side of (2.4) and by $1_i\Delta(1_i) = \Delta(1_i)1_i$, we have that $1_i\Delta(1_i)^* = 0$. It implies that $\Delta(1_i) = 0$. For every a in $\mathcal{A}$, we write $a = \sum_{i=1}^{n} a_i$ with a_i in $\mathcal{A}_i$. Since $a_i(1-1_i) = 0$, we have that $\Delta(a_i)(1-1_i) = 0$, which means that $\Delta(a_i) \in \mathcal{A}_i$. Let a_i, b_i be in $\mathcal{A}_i$ with $a_ib_i^* = 0$, we have that

$$\Delta(a_i)b_i^* + a_i\Delta(b_i)^* = \Delta_i(a_i)b_i^* + a_i\Delta_i(b_i)^* = 0.$$

By Cases 1 and 2, we know that every Δ_i is a $*$-derivation. Thus Δ is a $*$-derivation. □

In the following, we characterize a linear mapping δ satisfies the condition $\mathbb{D}_1$ from a unital $*$-algebra into a unital $*$-$\mathcal{A}$-bimodule with a right or left separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$.

Lemma 2.5. [7, Theorem 4.1] *Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{X}$ is a linear space. If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ ab = 0 \Rightarrow \phi(a, b) = 0,$$

then we have that

$$\phi(a, x) = \phi(ax, 1) \text{ and } \phi(x, a) = \phi(1, xa)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Theorem 2.6. *Suppose that $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule with a right or left separating set $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Proof. Since $\mathcal{A}$ is a unital $*$-algebra and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule, we know that $\mathcal{J} \subseteq \mathfrak{J}(\mathcal{A})$ is a right separating set of $\mathcal{M}$ if and only if $\mathcal{J}^* = \{x^* : x \in \mathcal{J}\} \subseteq \mathfrak{J}(\mathcal{A})$ is a left separating set of $\mathcal{M}$. Thus without loss of generality, we can assume that $\mathcal{J}$ is a left separating set of $\mathcal{A}$, otherwise, we replace $\mathcal{J}$ by $\mathcal{J}^*$.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a) = \delta(a) - \delta(1)a$$

for every a in $\mathcal{A}$. In the following we show that Δ is a $*$-derivation.

It is clear that $\Delta(1) = 0$ and $ab^* = 0$ can implies that $a\Delta(b)^* + \Delta(a)b^* = 0$. Define a bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a, b) = a\Delta(b^*)^* + \Delta(a)b$$

for each a and b in $\mathcal{A}$. By the assumption we know that $ab = 0$ implies $\phi(a, b) = 0$.

Let a, b be in $\mathcal{A}$ and x be in $\mathcal{J}$. By Lemma 2.5, we can obtain that

$$\phi(x, 1) = \phi(1, x) \text{ and } \phi(a, x) = \phi(ax, 1).$$

Hence we have the following two identities:

$$x\Delta(1)^* + \Delta(x) = \Delta(x^*)^* + \Delta(1)x \tag{2.5}$$

and

$$a\Delta(x^*)^* + \Delta(a)x = ax\Delta(1)^* + \Delta(ax). \tag{2.6}$$

By (2.5) and $\Delta(1) = 0$, we know that $\Delta(x)^* = \Delta(x^*)$. Thus by (2.6), it implies that

$$\Delta(ax) = a\Delta(x) + \Delta(a)x.$$

Similar to the proof of [4, Theorem 2.3], we can obtain that $\Delta(ab) = a\Delta(b)+\Delta(a)b$ for each a and b in $\mathcal{A}$.

It remains to show that $\Delta(a)^* = \Delta(a^*)$ for every a in $\mathcal{A}$. Indeed, for every a in $\mathcal{A}$ and every x in $\mathcal{J}$, we have that $\Delta(ax)^* = \Delta((ax)^*)$. It implies that

$$(\Delta(a)x + a\Delta(x))^* = \Delta(x^*)a^* + x^*\Delta(a^*).$$

Thus we can obtain that $x^*(\Delta(a)^* - \Delta(a^*)) = 0$, hence $(\Delta(a) - \Delta(a^*)^*)x = 0$. It follows that $\Delta(a)^* = \Delta(a^*)$ for every a in $\mathcal{A}$. □

Remark 1. Let $\mathcal{A}$ be a $*$-algebra, $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule, and δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$. Similar to the condition $\mathbb{D}_1$ which we have characterized in Section 2:

$$(\mathbb{D}_1)\ a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + \delta(a)b^* = 0,$$

we can consider the condition $\mathbb{D}_1'$

$$(\mathbb{D}_1')\ a, b \in \mathcal{A},\ a^*b = 0 \Rightarrow a^*\delta(b) + \delta(a)^*b = 0.$$

Through the minor modifications, we can obtain the corresponding results.

Remark 2. A linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *local derivation* if for every a in $\mathcal{A}$, there exists a derivation δ_a (depending on a) from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(a) = \delta_a(a)$. It is clear that every local derivation satisfies the following condition:

$$(\mathbb{H})\ a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow a\delta(b)c = 0.$$

In [1], the authors prove that every continuous linear mapping from a unital C^*-algebra into its unital Banach bimodule such that the condition $\mathbb{H}$ and $\delta(1) = 0$ is a derivation.

Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. The natural way to translate the condition $\mathbb{H}$ to the context of $*$-derivations is to consider the following condition

$$(\mathbb{H}')\ a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \Rightarrow a\delta(b)^*c = 0.$$

However, the conditions $\mathbb{H}'$ and $\mathbb{H}$ are equivalent. Indeed, if condition $\mathbb{H}'$ holds, we have that

$$a, b, c \in \mathcal{A},\ ab = bc = 0 \Rightarrow c^*b^* = b^*a^* = 0 \Rightarrow c^*\delta(b)^*a^* = 0 \Rightarrow a\delta(b)c = 0,$$

and if the condition $\mathbb{H}$ holds, we have that

$$a, b, c \in \mathcal{A},\ ab^* = b^*c = 0 \Rightarrow c^*b = ba^* = 0 \Rightarrow c^*\delta(b)a^* = 0 \Rightarrow a\delta(b)^*c = 0.$$

It means that the condition $\mathbb{H}'$ and $\delta(1) = 0$ can not implies that δ is a $*$-derivation.

3. $*$-Jordan derivations on some algebras

A (Banach) algebra $\mathcal{A}$ is said to be *zero Jordan product determined* if every (continuous) bilinear mapping ϕ from $\mathcal{A} \times \mathcal{A}$ into any (Banach) linear space $\mathcal{X}$ satisfying

$$\phi(a, b) = 0,\ \text{whenever } a \circ b = 0$$

can be written as $\phi(a, b) = T(a \circ b)$, for some (continuous) linear mapping T from $\mathcal{A}$ into $\mathcal{X}$. In [5], we show that if $\mathcal{A}$ is a unital algebra with $\mathcal{A} = \mathfrak{J}(\mathcal{A})$, then $\mathcal{A}$ is a zero Jordan product determined algebra.

Theorem 3.1. *Suppose that $\mathcal{A}$ is a unital zero Jordan product determined $*$-algebra, and $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule. If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that*

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1) = 0$.

Proof. Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - \delta(1)a$ for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-Jordan derivation.

It is clear that $\Delta(1) = 0$, and by $\delta(1)a = a\delta(1)$ we have that

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0.$$

Define a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a, b) = a \circ \Delta(b^*)^* + \Delta(a) \circ b.$$

Thus $a \circ b = 0$ implies $\phi(a, b) = 0$. Since $\mathcal{A}$ is a zero Jordan product determined algebra, we know that there exists a linear mapping T from $\mathcal{A}$ into $\mathcal{M}$ such that

$$T(a \circ b) = \phi(a, b) = a \circ \Delta(b^*)^* + \Delta(a) \circ b \tag{3.1}$$

for each a, b in $\mathcal{A}$. Let $a = 1$ and $b = 1$ be in (3.1), respectively. By $\Delta(1) = 0$, we can obtain that

$$T(a) = \Delta(a) \text{ and } T(b) = \Delta(b^*)^*.$$

It follows that $\Delta(a^*) = \Delta(a)^*$ for every a in $\mathcal{A}$. By (3.1), we have that

$$T(a \circ b) = \Delta(a \circ b) = \phi(a, b) = a \circ \Delta(b) + \Delta(a) \circ b.$$

It means that Δ is a $*$-Jordan derivation. □

In [5], we prove that the matrix algebra $M_n(\mathcal{B})(n \geq 2)$ is zero Jordan product determined, where $\mathcal{B}$ is a unital algebra. In [16], H. Ghahramani show that every Jordan derivation from $M_n(\mathcal{B})(n \geq 2)$ into its unital bimodule $\mathcal{M}$ is a derivation. Hence we have the following result.

Corollary 3.2. *Suppose that $\mathcal{B}$ is a unital $*$-algebra, $M_n(\mathcal{B})$ is a matrix algebra with $n \geq 2$, and $\mathcal{M}$ is a unital $*$-$M_n(\mathcal{B})$-bimodule. If δ is a linear mapping from $M_n(\mathcal{B})$ into $\mathcal{M}$ such that*

$$a, b \in M_n(\mathcal{B}),\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $M_n(\mathcal{B})$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Let $\mathcal{H}$ be a complex Hilbert space and $B(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Suppose that $\mathcal{A}$ is a von Neumann algebra on $\mathcal{H}$ and $LS(\mathcal{A})$ the set of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{A}$.

In [27], M. Muratov and V. Chilin prove that $LS(\mathcal{A})$ is a unital $*$-algebra and $\mathcal{A} \subset LS(\mathcal{A})$. By [25, Proposition 21.20, Exercise 21.18], we know that if $\mathcal{A}$ is a von Neumann algebra without direct summand of type I_1, and $\mathcal{B}$ is a $*$-algebra with $\mathcal{A} \subseteq \mathcal{B} \subseteq LS(\mathcal{A})$, then $\mathcal{B} \cong \sum_{i=1}^{k} \bigoplus M_{n_i}(\mathcal{B}_i)$ (k is a finite integer or infinite), where $\mathcal{B}_i$ is a unital algebra. By Theorem 3.1, we have the following result.

Corollary 3.3. *Suppose that $\mathcal{A}$ is a von Neumann algebra without direct summand of type I_1, and $\mathcal{B}$ is a $*$-algebra with $\mathcal{A} \subseteq \mathcal{B} \subseteq LS(\mathcal{A})$. If δ is a linear mapping from $\mathcal{B}$ into $LS(\mathcal{A})$ such that*

$$a, b \in \mathcal{B},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{B}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1) = 0$.

For von Neumann algebras, by Corollary 3.2 and similar to the proof of Theorem 2.4, we can easily obtain the following result and we omit the proof.

Corollary 3.4. *Suppose that $\mathcal{A}$ is a von Neumann algebra. If δ is a linear mapping from $\mathcal{A}$ into itself with such that*

$$a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0 \text{ and } \delta(1)a = a\delta(1),$$

then $\delta(a) = \Delta(a) + \delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-derivation. In particular, δ is a $*$-derivation when $\delta(1) = 0$.

Lemma 3.5. [5, Theorem 2.1] *Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{X}$ is a linear space. If ϕ is a bilinear mapping from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{X}$ such that*

$$a, b \in \mathcal{A},\ a \circ b = 0 \Rightarrow \phi(a, b) = 0,$$

then we have that

$$\phi(a,x)=\frac{1}{2}\phi(ax,1)+\frac{1}{2}\phi(xa,1)$$

for every a in $\mathcal{A}$ and every x in $\mathfrak{J}(\mathcal{A})$.

Suppose that $\mathcal{A}$ is a unital algebra and $\mathcal{M}$ is a unital $\mathcal{A}$-bimodule satisfying that

$$\{m\in\mathcal{M}: xmx=0 \text{ for every } x\in\mathcal{J}\}=\{0\},$$

where $\mathcal{J}$ is an ideal of $\mathcal{A}$ linear generated by idempotents in $\mathcal{A}$. In [15, Theorem 4.3], H. Ghahramani studies the linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$ satisfies

$$a,b\in\mathcal{A},\ a\circ b=0\Rightarrow a\circ\delta(b)+\delta(a)\circ b=0,$$

and show that δ is a generalized Jordan derivation. In the following, we suppose that $\mathcal{J}$ is an ideal of $\mathcal{A}$ generated algebraically by all idempotents in $\mathcal{A}$, and have the following result.

Theorem 3.6. *Suppose that $\mathcal{A}$ is a unital $*$-algebra, $\mathcal{M}$ is a unital $*$-$\mathcal{A}$-bimodule, and $\mathcal{J}\subseteq\mathfrak{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$ such that*

$$\{m\in\mathcal{M}: xmx=0 \text{ for every } x\in\mathcal{J}\}=\{0\}.$$

If δ is a linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that

$$a,b\in\mathcal{A},\ a\circ b^*=0\Rightarrow a\circ\delta(b)^*+\delta(a)\circ b^*=0 \text{ and } \delta(1)a=a\delta(1),$$

then $\delta(a)=\Delta(a)+\delta(1)a$ for every a in $\mathcal{A}$, where Δ is a $*$-Jordan derivation. In particular, δ is a $*$-Jordan derivation when $\delta(1)=0$.

Proof. Let $\widehat{\mathcal{J}}$ be an algebra generated algebraically by $\mathcal{J}$ and $\mathcal{J}^*$. Since $\mathcal{J}\subseteq\mathfrak{J}(\mathcal{A})$ is an ideal of $\mathcal{A}$, it is easy to show that $\widehat{\mathcal{J}}\subseteq\mathfrak{J}(\mathcal{A})$ is also an ideal of $\mathcal{A}$, and such that

$$\{m\in\mathcal{M}: xmx=0 \text{ for every } x\in\widehat{\mathcal{J}}\}=\{0\}.$$

Thus without loss of generality, we can assume that $\mathcal{J}$ is a self-adjoint ideal of $\mathcal{A}$, otherwise, we may replace $\mathcal{J}$ by $\widehat{\mathcal{J}}$.

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by

$$\Delta(a)=\delta(a)-\delta(1)a$$

for every a in $\mathcal{A}$. In the following we show that Δ is a $*$-derivation.

It is clear that $\Delta(1)=0$, and by $\delta(1)a=a\delta(1)$ we have that $a\circ b^*=0$ implies that $a\circ\Delta(b)^*+\Delta(a)\circ b^*=0$.

Define a bilinear mapping ϕ from $\mathcal{A}\times\mathcal{A}$ into $\mathcal{M}$ by

$$\phi(a,b)=a\circ\Delta(b^*)^*+\Delta(a)\circ b$$

for each a and b in $\mathcal{A}$. By the assumption we know that $a\circ b=0$ implies $\phi(a,b)=0$.

Let a, b be in $\mathcal{A}$ and x be in $\mathcal{J}$. By Lemma 3.5, we can obtain that

$$\phi(x,1)=\phi(1,x).$$

It follows that

$$x\circ\Delta(1)^*+\Delta(x)\circ 1=1\circ\Delta(x^*)^*+\Delta(1)\circ x. \tag{3.2}$$

By (3.2) and $\Delta(1)=0$, we know that $\Delta(x)^*=\Delta(x^*)$. Again by Lemma 3.5, it follows that

$$a\circ\Delta(x^*)^*+\Delta(a)\circ x=\frac{1}{2}[\Delta(ax)\circ 1+\Delta(xa)\circ 1]. \tag{3.3}$$

By (3.3) and $\Delta(x)^*=\Delta(x^*)$, it is easy to show that

$$\Delta(a\circ x)=a\circ\Delta(x)+\Delta(a)\circ x. \tag{3.4}$$

Next, we prove that Δ is a Jordan derivation.

Define $\{a,m,b\}=amb+bma$ and $\{a,b,m\}=\{m,b,a\}=abm+mba$ for each a, b in $\mathcal{A}$ and every m in $\mathcal{M}$. Let a be in $\mathcal{A}$ and x,y be in $\mathcal{M}$.

By the technique of the proof of [15, Theorem 4.3] and (3.4), we have the following two identities:

$$\Delta\{x,a,y\}=\{\Delta(x),a,y\}+\{x,\Delta(a),y\}+\{x,a,\Delta(y)\}, \tag{3.5}$$

and

$$\Delta\{x,a^2,y\}=\{\Delta(x),a^2,y\}+\{x,a\circ\Delta(a),y\}+\{x,a^2,\Delta(y)\}. \tag{3.6}$$

On the other hand, by (3.5) we have that

$$\Delta\{x,a^2,x\}=\{\Delta(x),a^2,x\}+\{x,\Delta(a^2),x\}+\{x,a^2,\Delta(x)\}. \tag{3.7}$$

By comparing (3.6) and (3.7), it follows that $\{x,\Delta(a^2),x\}=\{x,a\circ\Delta(a),x\}$. That is $x(\Delta(a^2)-a\circ\Delta(a))x=0$. By the assumption, it implies that $\Delta(a^2)-a\circ\Delta(a)=0$ for every a in $\mathcal{A}$.

It remains to show that $\Delta(a)^*=\Delta(a^*)$ for every a in $\mathcal{A}$. Indeed, for every a in $\mathcal{A}$ and every x in $\mathcal{J}$, we have that $\Delta(xax)^*=\Delta((xax)^*)$. Since Δ is a Jordan derivation, it implies that

$$(\Delta(x)ax+x\Delta(a)x+xa\Delta(x))^*=\Delta(x^*)a^*x^*+x^*\Delta(a^*)x^*+x^*a^*\Delta(x^*).$$

Thus we can obtain that $x^*(\Delta(a)^*-\Delta(a^*))x^*=0$. Since $\mathcal{J}$ is a self-adjoint ideal of $\mathcal{A}$, it follows that $\Delta(a)^*=\Delta(a^*)$. $\square$

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{M}$ be a Banach $*$-$\mathcal{A}$-bimodule. Denote by $\mathcal{A}^{\sharp\sharp}$ and $\mathcal{M}^{\sharp\sharp}$ the second dual space of $\mathcal{A}$ and $\mathcal{M}$, respectively. By [11, p.26], we can define a product $\diamond$ in $\mathcal{A}^{\sharp\sharp}$ by

$$a^{\sharp\sharp}\diamond b^{\sharp\sharp}=\lim_{\lambda}\lim_{\mu}\alpha_\lambda\beta_\mu$$

for each $a^{\sharp\sharp}$, $b^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$, where (α_λ) and (β_μ) are two nets in $\mathcal{A}$ with $\|\alpha_\lambda\|\leqslant\|a^{\sharp\sharp}\|$ and $\|\beta_\mu\|\leqslant\|b^{\sharp\sharp}\|$, such that $\alpha_\lambda\to a^{\sharp\sharp}$ and $\beta_\mu\to b^{\sharp\sharp}$ in the weak*-topology $\sigma(\mathcal{A}^{\sharp\sharp},\mathcal{A}^{\sharp})$. Moreover, we can define an involution $*$ in $\mathcal{A}^{\sharp\sharp}$ by

$$(a^{\sharp\sharp})^*(\rho)=\overline{a^{\sharp\sharp}(\rho^*)},\ \rho^*(a)=\overline{\rho(a^*)},$$

where $a^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$, ρ in $A^{\sharp}$ and a in $\mathcal{A}$. By [22, p.726], we know that $\mathcal{A}^{\sharp\sharp}$ is a von Neumann algebra under the product $\diamond$ and the involution $*$.

Since $\mathcal{M}$ is a Banach $\mathcal{A}$-bimodule, $\mathcal{M}^{\sharp\sharp}$ turns into a dual Banach $(\mathcal{A}^{\sharp\sharp},\diamond)$-bimodule with the operation defined by

$$a^{\sharp\sharp}\cdot m^{\sharp\sharp}=\lim_{\lambda}\lim_{\mu}a_\lambda m_\mu \text{ and } m^{\sharp\sharp}\cdot a^{\sharp\sharp}=\lim_{\mu}\lim_{\lambda}m_\mu a_\lambda$$

for every $a^{\sharp\sharp}$ in $\mathcal{A}^{\sharp\sharp}$ and every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$, where (a_λ) is a net in $\mathcal{A}$ with $\|a_\lambda\| \leqslant \|a^{\sharp\sharp}\|$ and $(a_\lambda) \to a^{\sharp\sharp}$ in $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$, (m_μ) is a net in $\mathcal{M}$ with $\|m_\mu\| \leqslant \|m^{\sharp\sharp}\|$ and $(m_\mu) \to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$.

We remarked, in the discussion preceding Theorem 2.1, that $\mathcal{M}^{\sharp\sharp}$ has an involution $*$ and it is continuous in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. By [1, p.553], we know that every continuous bilinear map φ from $\mathcal{A} \times \mathcal{M}$ into $\mathcal{M}$ is Arens regular, which means that

$$\lim_\lambda \lim_\mu \varphi(a_\lambda, m_\mu) = \lim_\mu \lim_\lambda \varphi(a_\lambda, m_\mu)$$

for every $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$-convergent net (a_λ) in $\mathcal{A}$ and every $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$-convergent net (m_μ) in $\mathcal{M}$. Thus we can obtain that

$$(a^{\sharp\sharp} \cdot m^{\sharp\sharp})^* = (\lim_\lambda \lim_\mu a_\lambda m_\mu)^* = \lim_\lambda \lim_\mu m_\mu^* a_\lambda^* = \lim_\mu \lim_\lambda m_\mu^* a_\lambda^* = (m^{\sharp\sharp})^* \cdot (a^{\sharp\sharp})^*,$$

where (a_λ) is a net in $\mathcal{A}$ with $(a_\lambda) \to a^{\sharp\sharp}$ in $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$ and (m_μ) is a net in $\mathcal{M}$ with $(m_\mu) \to m^{\sharp\sharp}$ in $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. Similarly, we can show that $(m^{\sharp\sharp} \cdot a^{\sharp\sharp})^* = (a^{\sharp\sharp})^* \cdot (m^{\sharp\sharp})^*$. It implies that $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule.

A projection p in $\mathcal{A}^{\sharp\sharp}$ is called *open* if there exists an increasing net (a_α) of positive elements in $\mathcal{A}$ such that $p = \lim_\alpha a_\alpha$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$. If p is open, we say the projection $1 - p$ is *closed*.

For a unital C^*-algebra, we have the following result.

Theorem 3.7. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{M}$ is a unital Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that $\delta(1)a = a\delta(1)$ for every a in $\mathcal{A}$, then the following three statements are equivalent:*
(1) $a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(2) $a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(3) $\delta(a) = \Delta(a) + \delta(1)a$ *for every a in $\mathcal{A}$, where Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}$.*

Proof. It is clear that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). It is sufficient the prove that (2) $\Rightarrow$ (3).

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}$ by $\Delta(a) = \delta(a) - \delta(1)a$ for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-derivation. First we prove that $\Delta(a^*) = \Delta(a)^*$ for every a in $\mathcal{A}$.

By assumption, we can easily to show that

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0 \text{ and } \Delta(1) = 0,$$

In the following, we verify $\Delta(b) = \Delta(b)^*$ for every self-adjoint element b in $\mathcal{A}$.

Since Δ is a norm continuous linear mapping form $\mathcal{A}$ into $\mathcal{M}$, we know that $\Delta^{\sharp\sharp} : (\mathcal{A}^{\sharp\sharp}, \diamond) \to \mathcal{M}^{\sharp\sharp}$ is the weak*-continuous extension of Δ to the double duals of $\mathcal{A}$ and $\mathcal{M}$.

Let b be a non-zero self-adjoint element in $\mathcal{A}$, $\sigma(b) \subseteq [-\|b\|, \|b\|]$ be the spectrum of b and $r(b) \in \mathcal{A}^{\sharp\sharp}$ be the range projection of b.

Denote by $\mathcal{A}_b$ the C^*-subalgebra of $\mathcal{A}$ generated by b, and by $C(\sigma(b))$ the C^*-algebra of all continuous complex-valued functions on $\sigma(b)$. By Gelfand theory we know that there is an isometric $*$ isomorphism between $\mathcal{A}_b$ and $C(\sigma(b))$.

For every n in $\mathbb{N}$, let p_n be the projection in $\mathcal{A}_b^{\sharp\sharp} \subseteq \mathcal{A}^{\sharp\sharp}$ corresponding to the characteristic function $\chi_{([-\|b\|,-\frac{1}{n}]\cup[\frac{1}{n},\|b\|])\cap\sigma(b)}$ in $C(\sigma(b))$, and let b_n be in $\mathcal{A}_b$ such that

$$b_n p_n = p_n b_n = b_n = b_n^* \text{ and } \|b_n - b\| < \frac{1}{n}.$$

By [28, Section 1.8], we know that (p_n) converges to $r(b)$ in the strong*-topology of $\mathcal{A}^{\sharp\sharp}$, and hence in the weak*-topology.

It is well known that p_n is a closed projection in $\mathcal{A}_b^{\sharp\sharp} \subseteq \mathcal{A}^{\sharp\sharp}$ and $1-p_n$ is an open projection in $\mathcal{A}_b^{\sharp\sharp}$. Thus there exists an increasing net (z_λ) of positive elements in $((1-p_n)\mathcal{A}^{\sharp\sharp}(1-p_n)) \cap \mathcal{A}$ such that

$$0 \leq z_\lambda \leq 1 - p_n$$

and (z_λ) converges to $1-p_n$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$. Since

$$0 \leq ((1-p_n) - z_\lambda)^2 \leq (1-p_n) - z_\lambda \leq (1-p_n),$$

we have that (z_λ) also converges to $1-p_n$ in the strong*-topology of $\mathcal{A}^{\sharp\sharp}$.

By $b_n = b_n^*$ and $z_\lambda b_n = b_n z_\lambda = 0$, it follows that

$$z_\lambda \circ \Delta^{\sharp\sharp}(b_n)^* + \Delta^{\sharp\sharp}(z_\lambda) \circ b_n = 0. \tag{3.8}$$

Taking weak*-limits in (3.8) and since $\Delta^{\sharp\sharp}$ is weak*-continuous, we have that

$$(1-p_n) \circ \Delta^{\sharp\sharp}(b_n)^* + \Delta^{\sharp\sharp}((1-p_n)) \circ b_n = 0. \tag{3.9}$$

Since (p_n) converges to $r(b)$ in the weak*-topology of $\mathcal{A}^{\sharp\sharp}$ and (b_n) converges to b in the norm-topology of $\mathcal{A}$, by (3.9), we have that

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(b)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ b = 0. \tag{3.10}$$

Since the range projection of every power b^m with $m \in \mathbb{N}$ coincides with the $r(b)$, and by (3.10), it follows that

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(b^m)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ b^m = 0$$

for every $m \in \mathbb{N}$, and by the linearity and norm continuity of the product we have that

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(z)^* + \Delta^{\sharp\sharp}(1-r(b)) \circ z = 0$$

for every $z = z^*$ in $\mathcal{A}_b$. A standard argument involving weak*-continuity of $\Delta^{\sharp\sharp}$ gives

$$(1-r(b)) \circ \Delta^{\sharp\sharp}(r(b))^* + \Delta^{\sharp\sharp}(1-r(b)) \circ r(b) = 0. \tag{3.11}$$

By (3.11), we can obtain that

$$(\Delta^{\sharp\sharp}(r(b))^* + \Delta^{\sharp\sharp}(r(b)) - \Delta^{\sharp\sharp}(1)) \circ r(b) = 2\Delta^{\sharp\sharp}(r(b))^*.$$

By $\Delta(1) = 0$, we have that $\Delta^{\sharp\sharp}(1) = 0$. It implies that

$$\Delta^{\sharp\sharp}(r(b))^* = \Delta^{\sharp\sharp}(r(b)). \tag{3.12}$$

It is clear that every characteristic function

$$p = \chi_{([-\|b\|,-\alpha]\cup[\alpha,\|b\|])\cap\sigma(b)} \tag{3.13}$$

in $C_0(\sigma(b))^{\sharp\sharp}$ with $0 < \alpha < \|b\|$, is the range projection of a function in $C(\sigma(b))$. Moreover, every projection of the form

$$q = \chi_{([-\beta,-\alpha]\cup[\alpha,\beta])\cap\sigma(b)} \tag{3.14}$$

in $C_0(\sigma(b))^{\sharp\sharp}$ with $0 < \alpha < \beta < \|b\|$ can be written as the difference of two projections of the type in (3.13).

Since $\mathcal{A}_b$ and $C(\sigma(b))$ are isometric $*$ isomorphism, and by $\Delta^{\sharp\sharp}(r(b))^* = \Delta^{\sharp\sharp}(r(b))$ for range projection of b in $\mathcal{A}^{\sharp\sharp}$, we have that $\Delta^{\sharp\sharp}(p)^* = \Delta^{\sharp\sharp}(p)$ for every projection p of the type in (3.13). It follows that $\Delta^{\sharp\sharp}(q)^* = \Delta^{\sharp\sharp}(q)$ for every projection q of the type in (3.14).

It is well known that b can be approximated in norm by finite linear combinations of mutually orthogonal projections q_j of the type in (3.14), and Δ is continuous, we have that $\Delta(b)^* = \Delta(b)$. Thus for every a in $\mathcal{A}$, we can obtain that $\Delta(a)^* = \Delta(a)$.

By the assumption, it follows that

$$a, b \in \mathcal{A},\ ab = ba = 0 \Rightarrow a \circ \Delta(b) + \Delta(a) \circ b = 0.$$

By [2, Theorem 4.1], we know that Δ is a $*$-derivation. $\square$

In the following we consider general C^*-algebras $\mathcal{A}$. Let $(e_i)_{i\in\Gamma}$ be a bounded approximate identity of $\mathcal{A}$, $\mathcal{M}$ be an essential Banach $*$-$\mathcal{A}$-bimodule, and δ be a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$, then $(\delta(e_i))_{i\in\Gamma}$ is bounded and we can assume that it converges to ξ in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$. It follows the next result.

Theorem 3.8. *Suppose that $\mathcal{A}$ is a C^*-algebra (not necessary unital) and $\mathcal{M}$ is an essential Banach $*$-$\mathcal{A}$-bimodule. If δ is a continuous linear mapping from $\mathcal{A}$ into $\mathcal{M}$ such that $\xi \cdot a = a \cdot \xi$ for every a in $\mathcal{A}$, then the following three statements are equivalent:*
(1) $a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(2) $a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0$;
(3) $\delta(a) = \Delta(a) + \xi \cdot a$ *for every a in $\mathcal{A}$, where Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$.*

Proof. It is clear that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1). It is only need to prove that (2) $\Rightarrow$ (3).

Define a linear mapping Δ from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$ by

$$\Delta(a) = \delta(a) - \xi \cdot a$$

for every a in $\mathcal{A}$. It is sufficient to show that Δ is a $*$-derivation.

By the definition of Δ and $\xi \cdot a = a \cdot \xi$ for every a in $\mathcal{A}$, we can easily to show that

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \Delta(b)^* + \Delta(a) \circ b^* = 0.$$

By [10, Proposition 2.9.16], we know that $(e_i)_{i\in\Gamma}$ converges to the identity 1 in $\mathcal{A}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{A}^{\sharp\sharp}, \mathcal{A}^{\sharp})$. By the proof of Theorem 2.1, we know that

$\Delta(e_i) = \delta(e_i) - e_i \cdot \xi$ converges to zero in $\mathcal{M}^{\sharp\sharp}$ with the topology $\sigma(\mathcal{M}^{\sharp\sharp}, \mathcal{M}^{\sharp})$, and we can obtain that

$$m^{\sharp\sharp} \cdot 1 = m^{\sharp\sharp}$$

for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Since $\mathcal{M}^{\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule, we have that

$$1 \cdot m^{\sharp\sharp} = m^{\sharp\sharp}$$

for every $m^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp}$. Since Δ is a norm-continuous linear mapping form $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$, $\Delta^{\sharp\sharp} : (\mathcal{A}^{\sharp\sharp}, \diamond) \to \mathcal{M}^{\sharp\sharp\sharp\sharp}$ is the weak*-continuous extension of Δ to the double duals of $\mathcal{A}$ and $\mathcal{M}^{\sharp\sharp}$ such that $\Delta^{\sharp\sharp}(1) = 0$.

By [10, Proposition A.3.52], we know that the mapping $m^{\sharp\sharp\sharp\sharp} \mapsto m^{\sharp\sharp\sharp\sharp} \cdot 1$ from $\mathcal{M}^{\sharp\sharp\sharp\sharp}$ into itself is $\sigma(\mathcal{M}^{\sharp\sharp\sharp\sharp}, \mathcal{M}^{\sharp\sharp\sharp})$-continuous, and by the $\sigma(\mathcal{M}^{\sharp\sharp\sharp\sharp}, \mathcal{M}^{\sharp\sharp\sharp})$-denseness of $\mathcal{M}^{\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$, we have that

$$m^{\sharp\sharp\sharp\sharp} \cdot 1 = m^{\sharp\sharp\sharp\sharp}$$

for every $m^{\sharp\sharp\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$. Since $\mathcal{M}^{\sharp\sharp\sharp\sharp}$ is a Banach $*$-$\mathcal{A}^{\sharp\sharp}$-bimodule, we have that

$$1 \cdot m^{\sharp\sharp\sharp\sharp} = m^{\sharp\sharp\sharp\sharp}$$

for every $m^{\sharp\sharp\sharp\sharp}$ in $\mathcal{M}^{\sharp\sharp\sharp\sharp}$.

Finally, we use the same proof of Theorem 3.7 and show that Δ is a $*$-derivation from $\mathcal{A}$ into $\mathcal{M}^{\sharp\sharp}$. ☐

Remark 3. In [12], A. Essaleh and A. Peralta introduce the concept of a triple derivation on C^*-algebras. Suppose that $\mathcal{A}$ is a C^*-algebra. Let a, b and c be in $\mathcal{A}$, define the *ternary product* by $\{a, b, c\} = \frac{1}{2}(ab^*c + cb^*a)$. A linear mapping δ from $\mathcal{A}$ into itself is called a *triple derivation* if

$$\delta\{a, b, c\} = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}$$

for each a, b and c in $\mathcal{A}$. Let z be an element in $\mathcal{A}$. δ is called *triple derivation at z* if

$$a, b, c \in \mathcal{A},\ \{a, b, c\} = z \Rightarrow \delta(z) = \{\delta(a), b, c\} + \{a, \delta(b), c\} + \{a, b, \delta(c)\}.$$

In [12], A. Essaleh and A. Peralta prove that every continuous linear mapping δ which is triple derivations at zero from a unital C^*-algebra into itself with $\delta(1) = 0$ is a $*$-derivation.

On the other hand, it is apparent to show that if δ is triple derivation at zero, then δ satisfies that

$$a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a \circ \delta(b)^* + \delta(a) \circ b^* = 0.$$

Thus Theorem 3.7 generalizes [12, Corollary 2.10].

Remark 4. In [8], M. Brešar and J. Vukman introduce the left derivations and Jordan left derivations. A linear mapping δ from an algebra $\mathcal{A}$ into its bimodule $\mathcal{M}$ is called a *left derivation* if $\delta(ab) = a\delta(b) + b\delta(a)$ for each a, b in $\mathcal{A}$; and δ is called a *Jordan left derivation* if $\delta(a \circ b) = 2a\delta(b) + 2b\delta(a)$ for each a, b in $\mathcal{A}$.

Let $\mathcal{A}$ be a $*$-algebra and $\mathcal{M}$ be a $*$-$\mathcal{A}$-bimodule. A left derivation (Jordan left derivation) δ from $\mathcal{A}$ into $\mathcal{M}$ is called a *$*$-left derivation* (*$*$-Jordan left derivation*) if $\delta(a^*) = \delta(a)^*$ for every a in $\mathcal{A}$.

We also can investigate the following conditions on a linear mapping δ from $\mathcal{A}$ into $\mathcal{M}$:

$$(\mathbb{J}_1)\ a, b \in \mathcal{A},\ ab^* = 0 \Rightarrow a\delta(b)^* + b^*\delta(a) = 0;$$
$$(\mathbb{J}_2)\ a, b \in \mathcal{A},\ a \circ b^* = 0 \Rightarrow a\delta(b)^* + b^*\delta(a) = 0;$$
$$(\mathbb{J}_3)\ a, b \in \mathcal{A},\ ab^* = b^*a = 0 \Rightarrow a\delta(b)^* + b^*\delta(a) = 0.$$

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