Perfect Broad-Band Invisibility in Isotropic Media with Gain and Loss

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We offer a simple route to perfect omnidirectional invisibility in a spectral band of desired width. Our approach is based on the observation that in two dimensions a complex potential $v(x, y)$ is invisible for incident plane waves with a wavenumber not exceeding a preassigned value α provided that its Fourier transform with respect to y, which we denote by $\tilde{v}(x, \mathfrak{K}_y)$, vanishes for $\mathfrak{K}_y \leq 2\alpha$. We can fulfil this condition for potentials modelling the permittivity profile of an optical slab. Such a slab is perfectly invisible for any transverse electric wave whose wavenumber is in the range $[0, \alpha]$. Our results also apply to transverse magnetic waves propagating in a medium with a relative permittivity $\hat{\varepsilon}(x, y)$ that is a smooth bounded function with a positive real part.

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The search for scattering potentials that are invisible in a spectral band of arbitrarily large width is of great theoretical and practical importance. The use of conformal mappings [1], metamaterials [2, 3], and anisotropic material [4] has led to some remarkable progress in this subject. But achieving perfect (non-approximate) broad-band invisibility for ordinary nonmagnetic isotropic material has remained out of reach. This letter proposes a simple method for accomplishing this goal.

In one dimension, if a real or complex potential $v(x)$ decays exponentially (or more rapidly) as $x \to \pm\infty$, the reflection amplitudes are complex-analytic functions of the wavenumber k, [5]. Because a nonzero complex-analytic function can vanish only at a set of isolated points in the complex plane, $v(x)$ can be reflectionless either in the entire frequency spectrum (full-band) or at a discrete set of isolated values of the frequency. This means that reflectionlessness and invisibility in a spectral band of finite width (finite-band) are forbidden for such short-range potentials.

The problem of finding full-band reflectionless real potentials in one dimension has been addressed in the 1950's [6]. The outcome is a class of potentials with an asymptotic exponential decay, which have recently found applications in designing antireflection coatings [7, 8].

For a complex scattering potential, the reflection coefficients for the left and right incident waves need not be equal. In particular a complex potential can be invisible from one direction and visible from the other [9]. This observation has recently attracted a lot of attention and led to a detailed study of the phenomenon of unidirectional invisibility [10–21].

Consider the Schrödinger equation $-\psi'' + v(x)\psi = k^2\psi$ for a complex potential of the form:

$$v(x) = \chi_a(x + \tfrac{a}{2}) f(x), \qquad \chi_a(x) := \begin{cases} 1 & \text{for } x \in [0, a], \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

where $f(x)$ is a periodic potential with period $K := 2\pi/a$. We can express $f(x)$ in terms of its Fourier series, $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inKx}$. It turns out that if $v(x)$ is sufficiently weak, so that the first Born approximation is reliable, and $c_0 = c_{-m} = 0 \neq c_m$ for some integer $m > 0$, then $v(x)$ is unidirectionally invisible from the left for the wavenumber $k = \pi m/a$, [14]. The simplest example is $f(x) = c_1 e^{iKx}$ whose investigation led to the discovery of unidirectional invisibility [9, 22–24].

As noted in Ref. [14], if $c_n = 0$ for all $n \leq 0$, then $v(x)$ is invisible from the left for all wavenumbers that are integer multiples of π/a. The $a \to \infty$ limit of this result suggests the full-band left-invisibility of any potential whose Fourier transform $\tilde{v}(\mathfrak{K})$ vanishes for $\mathfrak{K} \leq 0$. Surprisingly this result holds true even for the potentials that are not weak [17], i.e., they enjoy perfect left-invisibility.

The vanishing of $\tilde{v}(\mathfrak{K})$ for $\mathfrak{K} \leq 0$ is equivalent to requiring the real and imaginary parts of $v(x)$ to be related by the Kramers-Kronig relations [17]. The potentials of this type are generally long-range and their Schrödinger equation might not admit asymptotically plane-wave (Jost) solutions. This makes their physical realization more difficult and leads to problems with the application of the standard scattering theory [18]. These difficulties do not however overshadow the significance of their discovery. For example, this discovery has paved the way for the construction of finite-band unidirectionally [19] and bidirectionally invisible potentials [20] in one dimension, and led to the design and experimental realization of ceratin broad-band metamaterial absorbers [25].

The condition, $c_n = 0$ for $n \leq 0$, for the unidirectional invisibility of weak locally periodic potentials of the form (1) follows as a simple byproduct of a dynamical formulation of scattering theory where the transfer matrix of the potential is given by the solution of a dynamical equation [26]. We have recently developed a multi-dimensional extension of this formulation [27] and employed it in the study of unidirectional invisibility in two and three dimensions [28]. Here we use it as a basic framework for exploring finite-band invisibility in two dimensions.

Let $v(x, y)$ be a scattering potential in two dimensions, and suppose that the solutions of the Schrödinger equation

$$-\nabla^2\psi(x,y) + v(x,y)\psi(x,y) = k^2\psi(x,y), \qquad \textbf{(2)}$$

have the asymptotic form:

$$\frac{1}{2\pi}\int_{-k}^{k} dp\, e^{ipy}\left[A_\pm(p)e^{i\varpi(p)x} + B_\pm(p)e^{-i\varpi(p)x}\right], \qquad \textbf{(3)}$$

for $x \to \pm\infty$, where $A_\pm(p)$ and $B_\pm(p)$ are functions vanishing for $|p| > k$, $\varpi(p) := \sqrt{k^2 - p^2}$, and the x-axis is the scattering axis. We can write the wavevector for a left-incident wave that makes an angle θ_0 with the x-axis in the form $\vec{k}_0 = \varpi(p_0)\hat{e}_x + p_0\hat{e}_y$, where $\hat{e}_x$ and $\hat{e}_y$ are respectively the unit vectors pointing along the x- and y-axes, and $p_0 := k\sin\theta_0$ (See Fig. 1.) For such an incident wave, $A_-(p) = 2\pi\delta(p - p_0)$, $B_+(p) = 0$, and the scattering solution of (2) satisfies:

$$\psi(\vec{r}) \to e^{i\vec{k}_0\cdot\vec{r}} + \sqrt{i/kr}\, e^{ikr} f(\theta) \;\text{ as }\; r \to \infty,$$

where $\vec{r} := x\hat{e}_x + y\hat{e}_x$, (r, θ) are the polar coordinates of $\vec{r}$, and $f(\theta)$ is the scattering amplitude.

The transfer matrix of the potential $v(x, y)$ is the 2×2 matrix (operator) $\mathbf{M}(p)$ fulfilling

$$\mathbf{M}(p)\begin{bmatrix} A_-(p) \\ B_-(p) \end{bmatrix} = \begin{bmatrix} A_+(p) \\ B_+(p) \end{bmatrix}.$$

Its entries $M_{ij}(p)$ are linear operators acting on the functions $A_-(p)$ and $B_-(p)$. In Ref. [27] we show that $\mathbf{M}(p)$ stores all the information about the scattering features of $v(x, y)$. In particular, if we set $T_-(p) := B_-(p)$ and $T_+(p) := A_+(p) - A_-(p)$, we can show that

$$T_-(p) = -2\pi M_{22}(p)^{-1}M_{21}(p)\delta(p - p_0), \qquad \textbf{(4)}$$

$$T_+(p) = M_{12}(p)T_-(p) + 2\pi[M_{11}(p) - 1]\delta(p - p_0), \qquad \textbf{(5)}$$

$$f(\theta) = -\frac{ik|\cos\theta|}{\sqrt{2\pi}} \times \begin{cases} T_-(k\sin\theta) & \text{for } \cos\theta < 0 \\ T_+(k\sin\theta) & \text{for } \cos\theta \geq 0 \end{cases}. \qquad \textbf{(6)}$$

A practically important property of the transfer matrix $\mathbf{M}(p)$ is that it has the same composition property as its one-dimensional analog [27]. This follows from the remarkable fact that

$$\mathbf{M}(p) = \mathbf{U}(\infty, p), \qquad \textbf{(7)}$$

where $\mathbf{U}(x, p)$ is the evolution operator for an effective non-Hermitian Hamiltonian operator $\mathbf{H}(x, p)$ with x playing the role of an evolution parameter. To make this statement more precise, we first introduce $v(x, i\partial_p)$ as the operator defined by

$$v(x, i\partial_p)\phi(p) := \frac{1}{2\pi}\int_{-k}^{k} dq\, \tilde{v}(x, p - q)\phi(q), \qquad \textbf{(8)}$$

where $\phi(p)$ is a test function vanishing for $|p| > k$, and $\tilde{v}(x, \mathfrak{K}_y)$ is the Fourier transform of $v(x, y)$ with respect to y, i.e.,

$$\tilde{v}(x, \mathfrak{K}_y) := \int_{-\infty}^{\infty} dy\, e^{-i\mathfrak{K}_y y}v(x, y). \qquad \textbf{(9)}$$

Equation (7) holds provided that we identify $\mathbf{U}(x, p)$ with the solution of

$$i\partial_x\mathbf{U}(x, p) = \mathbf{H}(x, p)\mathbf{U}(x, p), \quad \mathbf{U}(-\infty, p) = \mathbf{I}, \qquad \textbf{(10)}$$

where

$$\mathbf{H}(x, p) := \frac{1}{2\varpi(p)}\, e^{-i\varpi(p)x\sigma_3}v(x, i\partial_p)\,\boldsymbol{\mathcal{K}}\, e^{i\varpi(p)x\sigma_3}, \qquad \textbf{(11)}$$

$\mathbf{I}$ is the 2×2 identity matrix, $\boldsymbol{\sigma}_i$ are the Pauli matrices, and $\boldsymbol{\mathcal{K}} := \sigma_3 + i\sigma_2$, [27].

It is important to realize that all the quantities we have introduced, in particular $\mathbf{M}(p)$ and $\mathbf{H}(x, p)$, depend on the wavenumber k. If $\mathbf{H}(x, p)$ equals the zero operator $\mathbf{0}$ for a value of k, then (7) and (10) imply $\mathbf{M}(p) = \mathbf{I}$ for this value of k. In light of (4), (5), and (6), this gives $f(\theta) = 0$ for all θ_0, i.e., the potential is invisible for any incident plane wave with this wavenumber. Because this argument does not rely on any approximation, this invisibility is perfect. Furthermore if this property holds for a range of values of k, then the potential will be perfectly invisible for any wave packet that is constructed by superposing the plane waves with wavenumber belonging to this range.

Now, suppose that there is some $\alpha > 0$ such that $\tilde{v}(x, \mathfrak{K}_y) = 0$ for all $\mathfrak{K}_y \leq 2\alpha$. Then in view of (8) and (11), $\mathbf{H}(x, p) = \mathbf{0}$ for all $k \leq \alpha$, and the argument of the preceding paragraph proves the following result.

Theorem 1: Let $\alpha > 0$ and $v(x, y)$ be a scattering potential such that

$$\tilde{v}(x, \mathfrak{K}_y) = 0 \;\text{ for all }\; \mathfrak{K}_y \leq 2\alpha. \qquad \textbf{(12)}$$

Then $v(x, y)$ is perfectly invisible for any incident plane wave with wavenumber $k \leq \alpha$.

According to this theorem, we can achieve perfect invisibility in the spectral band $[0, \alpha]$, if we can construct a potential satisfying (12). This is actually quite easy. With the help of (9), we can express every such potential in the form

$$v(x, y) = e^{i2\alpha y}u(x, y), \qquad \textbf{(13)}$$

where $u(x, y)$ satisfies $\tilde{u}(x, \mathfrak{K}_y) = 0$ for all $\mathfrak{K}_y \leq 0$, i.e., for each fixed value of x, $u_x(y) := u(x, y)$ is one of the potentials considered in [17–20]. Clearly,

$$u(x, y) = \frac{1}{2\pi}\int_0^{\infty} dq e^{iqy}\tilde{u}(x, q), \qquad \textbf{(14)}$$

where for $x \to \pm\infty$, $|\tilde{u}(x, q)| \to 0$ sufficiently fast so that the solutions of (2) have the asymptotic expression (3). This condition is clearly satisfied for

$$\tilde{u}(x, q) = \chi_a(x)\tilde{f}(x, q), \quad q \geq 0, \qquad \textbf{(15)}$$

where χ_a is the function defined in (1) and $\tilde{f}$ is an arbitrary function fulfilling $\int_0^{\infty} dq|\tilde{f}(x, q)| < \infty$. As an example, let

$\tilde{f}(x,q) = \tilde{\mathfrak{z}}\, e^{-Lq} q^n$, where $\tilde{\mathfrak{z}}$ and L are real parameters, $L > 0$, and n is a nonnegative integer. Then (14) and (15) give

$$u(x,y) = \mathfrak{z}\, \chi_a(x) \left(\frac{y}{L} + i\right)^{-n-1}, \tag{16}$$

where $\mathfrak{z} := n!\, \tilde{\mathfrak{z}}/2\pi(-iL)^{n+1}$. Note that for $|y| \to \infty$, $|v(x,y)| \propto |L/y|^{n+1}$.

Next, we explore optical realizations of the perfect invisibility discussed in Theorem 1. Consider a nonmagnetic optical medium with translational symmetry along the z-axis, so that its properties are described by a relative permittivity $\hat{\varepsilon}$ that depends only on x and y. A z-polarized transverse electric (TE) wave propagating in this medium has an electric field of the form $\vec{E}(x,y,z) = E_0\, e^{-ikct} \psi(x,y) \hat{e}_z$, where E_0 is a constant, c is the speed of light in vacuum, $\hat{e}_z$ is the unit vector pointing along the z-axis, and ψ solves the Helmholtz equation $[\nabla^2 + k^2 \hat{\varepsilon}(x,y)]\psi = 0$. The equivalence of this equation and the Schrödinger equation (2) for the optical potential:

$$v(x,y) = k^2[1 - \hat{\varepsilon}(x,y)], \tag{17}$$

together with Theorem 1 prove the following result.

Theorem 2: Let $u(x,y)$ be a function such that $\tilde{u}(x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 0$. Then a nonmagnetic optical medium described by the permittivity profile

$$\hat{\varepsilon}(x,y) = 1 + e^{2i\alpha y} u(x,y), \tag{18}$$

is perfectly invisible for any incident TE wave with wavenumber $k \le \alpha$, [29].

In particular, if (14) and (15) hold, (18) describes an optical slab of thickness a that is invisible for these waves.

Next, consider choosing $\tilde{f}$ in (15) in such a way that $u(x,y)$ decays rapidly for $y \to \pm\infty$. Then (18) describes a slab of finite extension along both x- and y-axes. For example, the permittivity profile (18) with $u(x,y)$ given by (16) models a slab with a rectangular cross section,

$$D = \{(x,y) | x \in [0,a], y \in [-b,b]\}, \tag{19}$$

provided that $(L/b)^{n+1} \ll 1$. Figure 1 shows the plot of the real and imaginary parts of $\hat{\varepsilon}(x,y)$ for $u(x,y)$ given by (16), $\alpha = 2\pi/500$ nm, $\mathfrak{z} = 10^{-3}$, $L = 1\ \mu$m, and $n = 4$. These values yield $|\hat{\varepsilon}(x,y) - 1| < 7 \times 10^{-6}$ for $|y| > 2.5\ \mu$m. Therefore, we can use $\hat{\varepsilon}(x,y)$ to model a slab of cross section D with $b \ge 2.5\ \mu$m, which is invisible for TE waves of wavelength $\lambda := 2\pi/k \ge 500$ nm.

In order to provide a graphical demonstration of the invisibility of the above system for TE waves with wavelength $\lambda \ge 500$ nm, we compute its scattering amplitude using the first Born approximation. This is a reliable approximation scheme, because the corresponding optical potential is sufficiently weak.

Performing the first Born approximation corresponds to solving the dynamical equation (10) for the transfer matrix using the first-order perturbation theory, i.e., $\mathbf{M}(p) \approx \mathbf{I} - i\int_{-\infty}^{\infty} dx \mathbf{H}(x,p)$, [14, 28]. Substituting (11) in this equation, we obtain explicit formulas for the action of $M_{ij}(p)$ on test functions $\phi(p)$. These together with (4) – (6) imply

$$f(\theta) \approx \frac{-1}{2\sqrt{2\pi}}\, \tilde{\tilde{v}}\big(k(\cos\theta - \cos\theta_0), k(\sin\theta - \sin\theta_0)\big), \tag{20}$$

where $\tilde{\tilde{v}}(\mathfrak{K}_x,\mathfrak{K}_y) := \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{-i(\mathfrak{K}_x x + \mathfrak{K}_y y)} v(x,y)$ is the two-dimensional Fourier transform of $v(x,y)$.

For a scattering potential $v(x,y)$ satisfying $\tilde{v}(x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 2\alpha$, we have $\tilde{\tilde{v}}(\mathfrak{K}_x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 2\alpha$. In view of this observation and the fact that $|\sin\theta - \sin\theta_0| \le 2$, the right-hand side of (20) vanishes. This provides a first-order perturbative verification of Theorem 1 which holds to all orders of perturbation theory.

We can use (20) to determine the wavelengths λ at which a weak optical potential is invisible for TE waves. Figure 2 shows regions in the θ-λ plane where $f(\theta) \ne 0$ for some TE waves that propagate in a medium with permittivity profile given by (16), (18), $\mathfrak{z} = 10^{-3}$, $a = 100\ \mu$m, $L = 1\ \mu$m, $\alpha = 2\pi/500$ nm, and $n = 4$. This profile, which can be realized using a slab of thickness $a = 100\ \mu$m and width $2b \ge 5\ \mu$m placed in vacuum, is invisible for the TE waves with an arbitrary incidence angle θ_0 and wavelength $\lambda \ge 500$ nm.

Next, we study the propagation of a transverse magnetic (TM) wave in a nonmagnetic isotropic medium described by a relative permittivity profile $\hat{\varepsilon}(x,y)$. The magnetic field for this wave has the form $\vec{H}(x,y,z) = H_0\, e^{-ikct} \phi(x,y) \hat{e}_z$, where H_0 is a constant and ϕ is a function. Imposing Maxwell's equations, we find that ϕ satisfies

$$\hat{\varepsilon}^{-1}\nabla^2\phi + \vec{\nabla}(\hat{\varepsilon}^{-1}) \cdot \vec{\nabla}\phi + k^2\phi = 0. \tag{21}$$

This becomes equivalent to the Schrödinger equation (2) provided that we set $\psi := \phi/\sqrt{\hat{\varepsilon}}$ and

$$v := -k^2\eta + \frac{3|\vec{\nabla}\eta|^2}{4(1+\eta)^2} - \frac{\nabla^2\eta}{2(1+\eta)}, \tag{22}$$

where $\eta := \hat{\varepsilon} - 1$. For a permittivity profile of the form (18), $\eta(x,y) = e^{2i\alpha y} u(x,y)$. Therefore, $\tilde{\eta}(x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 2\alpha$ provided that $\tilde{u}(x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 0$.

A careful mathematical analysis of (22) shows that if $\hat{\varepsilon}$ is bounded and its real part exceeds a positive value, i.e., there are positive real numbers m and M such that $m \le \mathrm{Re}(\hat{\varepsilon}) \le |\hat{\varepsilon}| \le M$, then the vanishing of $\tilde{\eta}(x,\mathfrak{K}_y)$ for $\mathfrak{K}_y \le 2\alpha$ implies that the same holds for the potential (22), [30]. Therefore it satisfies the invisibility condition (12), and we are led to the following result.

Theorem 3: A nonmagnetic optical medium described by a smooth relative permittivity profile of the form (18) is perfectly invisible for incident TM waves of wavenumber $k \le \alpha$ provided that $\tilde{u}(x,\mathfrak{K}_y) = 0$ for $\mathfrak{K}_y \le 0$ and there are positive numbers m and M such that $m \le \mathrm{Re}[\hat{\varepsilon}(x,y)] \le |\hat{\varepsilon}(x,y)| \le M$, [29].

If the hypothesis of this theorem holds except that we allow u to have discontinuities along boundaries of certain connected regions D_α, then we need to solve (21) in D_α and patch the solutions for adjacent D_α by imposing the standard electromagnetic interface conditions along their common boundaries. Because the interface conditions involve $\hat{\varepsilon}$, the presence of the discontinuities can make the system visible even for $k \le \alpha$. This is true unless the resulting jumps in the value of $|\hat{\varepsilon}|$ are negligibly small. For example consider the optical slab we examined in our discussion of the TE waves, and suppose that $u(x,y)$ is given by the right-hand side of (16) multiplied by $e^{-(2x-a)^2/\sigma^2}$. Then, for $\sigma \ll a$, we can safely ignore the contribution of the discontinuity of $\hat{\varepsilon}$ along the boundaries of the slab and conclude that it is practically invisible for both TE and TM waves with $k \le \alpha$.

In summary, we have introduced a simple criterion for perfect finite-band invisibility in two dimensions and explored some of its optical realizations. In contrast to the criteria for broadband invisibility in one dimension [17–20], ours does

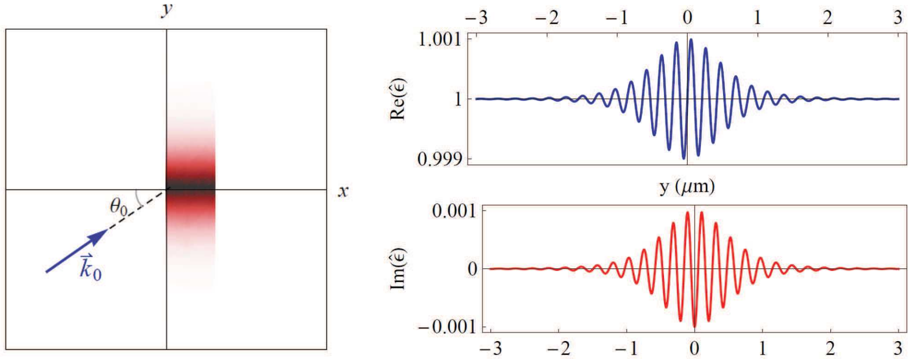

Fig. 1. A schematic view of an oblique wave incident upon an inhomogeneous medium confined between the planes $x = 0$ and $x = a$ (on the left), and plots of the real and imaginary parts of its relative permittivity $\hat{\varepsilon}$ that is given by (16) and (18) with $\alpha = 2\pi/500$ nm, $L = 1$ μm, $n = 4$, and $x \in [0, a]$ (on the right).

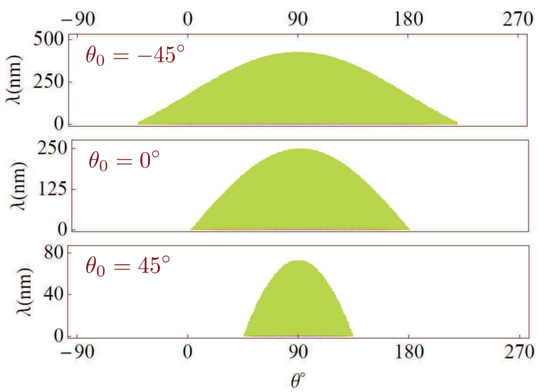

Fig. 2. Visibility domains of the permittivity profile $\hat{\varepsilon} = 1 + \mathfrak{z}\,\chi_a(x)e^{2i\alpha y}\,(y/L + i)^{-5}$ for TE waves: The colored regions correspond to values of λ and θ for which $f(\theta) \neq 0$. The top, middle, and bottom graphs correspond to TE waves with incidence angle $\theta_0 = -45°, 0°$, and $45°$, respectively. Here $\mathfrak{z} = 10^{-3}$, $a = 100$ μm, $L = 1$ μm, and $\alpha = 2\pi/500$ nm. For all values of θ_0 the system is invisible for $\lambda \geq 500$ nm. As one increases θ_0, the system becomes invisible above a critical wavelength that is smaller than 500 nm.

not put an upper bound on the asymptotic decay rate of the potential along the scattering axis [31]. This is a key feature of our route to broadband invisibility that allows for its realization using optical slabs with a finite thickness. Our results are not confined to optical waves and apply to any wave whose behavior is described using a scattering potential. Furthermore, they admit a straightforward extension to three dimensions. This is simply because the dynamical formulation of scattering in three dimensions [27] involves an effective Hamiltonian operator that has the same structure as its two-dimensional analog. We expect the resulting broadband invisibility in three dimensions to find interesting applications in acoustics.

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29. This statement holds also for $\hat{\varepsilon}(x, y) = \hat{\varepsilon}_b[1 + e^{2i\alpha y}u(x, u)]$ with $\hat{\varepsilon}_b > 0$ provided that we replace k by $\sqrt{\hat{\varepsilon}_b}\,k$.
30. This follows from the fact that in this case we can expand the right-hand side of (22) in a convergent power series involving products of powers and derivative of $\eta(x, y)$ and the fact that their Fourier transform with respect to y has the same property as $\tilde{\eta}(x, \mathfrak{K}_y)$.
31. The practical implementation of broadband invisibility in one dimension [17–20] requires truncating the potential along the scattering axis. Numerical evidence shows that this does not drastically affect their invisibility. The implementation of our results do not require a truncation along the scattering axis.