


On 2D integro-differential systems. Stability and sensitivity analysis

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Abstract In the paper a two-dimensional integro-differential system is considered. Using some variational methods we give sufficient conditions for the existence and uniqueness of a solution to the considered system. Moreover, we show that the system is stable and robust.

Keywords 2D integro-differential system · Sensitivity · Robustness · Stability

1 Introduction

We will denote by Q the unit interval in \mathbb{R}^2 , i.e.

$$Q = \{(x, y) \in \mathbb{R}^2 : x \in [0, 1] \text{ and } y \in [0, 1]\}. \quad (1)$$

General continuous 2D differential system has the following form

$$z_{xy}(x, y) = f(x, y, z(x, y), z_x(x, y), z_y(x, y)), \quad (2)$$

$$z(x, 0) = a(x), z(0, y) = b(y), a(0) = b(0), \quad (3)$$

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where $f : Q \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, $n \geq 1$, $a, b : [0, 1] \rightarrow \mathbb{R}^n$ are given functions. The linear case of system (2) can be written as

$$z_{xy}(x, y) = A_0(x, y)z(x, y) + A_1(x, y)z_x(x, y) + A_2(x, y)z_y(x, y) + w(x, y), \quad (4)$$

where $A_0, A_1, A_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^{n \times n}$ are given functions (which values are matrices of the dimension $n \times n$), w is a given n -dimensional vector function and z satisfies boundary conditions (3). Continuous 2D systems correspond to the discrete model of Fornasini–Marchesini type, which has the following form (see Fornasini and Marchesini 1976)

$$z(i+1, j+1) = A_0(i, j)z(i, j) + A_1(i, j)z(i+1, j) + A_2(i, j)z(i, j+1) + w(i, j), \quad (5)$$

$$z(i, 0) = a(i), z(0, j) = b(j), a(0) = b(0), \quad (6)$$

$i, j = 0, 1, 2, \dots$

Two-dimensional discrete systems (5)–(6) and continuous systems (2)–(3) play an essential role in mathematical modeling of many technical, physical, biological and other phenomena. For example, in the paper by Fornasini (see Fornasini 1991) 2D space models of the form (5)–(6) were applied to the investigation of the process of pollution and self purification of a river. Application of the 2D discrete models to image processing and transmission were studied in the book of Bracewell (see Bracewell 1995). The 2D continuous systems of the form (2)–(3) were adopted to investigation of the gas filtration model (see Bors and Walczak 2012). Other applications of discrete and continuous 2D systems in the theory of automatic control, stability, robotics and optimization can be found in papers of Galkowski et al. (2003), Paszke et al. (2004), Lomadze et al. (2008), Kaczorek (2001), Dey and Kar (2011), Singh (2008), Idczak and Walczak (2000) and in the monograph of Kaczorek (1985).

In the paper, we investigate 2D integro-differential system of the form

$$z_{xy}(x, y) + f^1(x, y, z(x, y)) + \int_0^x \int_0^y (f^2(s, t, z(s, t)) + A^1(s, t)z_x(s, t) + A^2(s, t)z_y(s, t))dsdt = v(x, y) \quad (7)$$

with the following boundary conditions

$$z(x, 0) = 0 \text{ for } x \in [0, 1] \text{ and } z(0, y) = 0 \text{ for } y \in [0, 1], \quad (8)$$

where $f^1, f^2 : Q \times \mathbb{R}^n \rightarrow \mathbb{R}$, $A^1, A^2 : Q \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ are given functions (for more details see Sect. 3). We shall consider the above system in the space of absolutely continuous functions of two variables. The definition and basic properties of absolutely continuous functions defined on the interval Q are presented in Sect. 2.

In the paper we prove, under assumptions (C1)–(C3) (see Sect. 3), that for any square integrable function v system (7)–(8) possesses a unique solution z_v which continuously depends on v and the operator $v \mapsto z_v$ is differentiable in the Fréchet sense, i.e. the considered system is well-posed and robust. It should be noticed that robustness can be understood in many aspects and meanings. However, usually it is *the ability of a system to resist change without adapting its initial stable configuration* Fukuda (2015). If the operator $u \mapsto z_v$ is differentiable then the change of the solution caused by the change of control v is not rapid. The speed of this change can be expressed by the derivative of the operator $u \mapsto z_v$, which measures the resistance of the system to the change of v .

The proof of the main result, given in Sect. 4, is based on the global diffeomorphism theorem (Theorem 1). In the final part of the paper we give an example and compare the method used in the paper with the methods based on the contraction principle and the Schauder fixed point theorem.

2 Preliminaries

In this section we introduce some mathematical backgrounds necessary to understand further considerations. In particular, we define the space of absolutely continuous functions which is the space of solutions and present its basic properties.

We begin with the following theorem on a diffeomorphism between Banach and Hilbert spaces which is the main tool in the proof of the main result.

Theorem 1 (see Idczak and Skowron 2012) *Let Z be a real Banach space, V be a real Hilbert space, $F : Z \rightarrow V$ be an operator of C^1 class. If*

- (a) *for any $v \in V$ the functional $\varphi(z) = \frac{1}{2} \|F(z) - v\|_V^2$ satisfies Palais–Smale condition ((PS)–condition),*
- (b) *for any $v \in V$ the equation $F'(z)h = v$ possesses a unique solution,*

then

1. *for any $v \in V$ there exists exactly one solution $z_v \in Z$ to the system $F(z) = v$,*
2. *the operator $V \ni v \rightarrow z_v \in Z$ is differentiable in the Fréchet sense.*

In other words, the operator F is a diffeomorphism between Banach space Z and Hilbert space V .

We recall that functional ϕ satisfies (PS)–condition if whenever there is a sequence $\{z^n\} \subset Z$ with $|\phi(z^n)| \leq \text{const}$ and $\phi'(z^n) \rightarrow 0$ in the space Z^* of linear and continuous functionals defined on Z , then in the closure of the set $\{z^n : n \in N\}$, there is some point \bar{z} where $\phi'(\bar{z}) = 0$ (see Aubin and Ekeland 2006).

From the bounded inverse theorem (see Limaye 1996, Theorem 11.1) it follows that for any $z \in Z$ there exists a constant $\alpha_z > 0$ such that $\|F'(z)h\|_V \geq \alpha_z \|h\|_Z$. Therefore it follows easily that the above theorem is equivalent to Idczak and Skowron (2012, Theorem 3.1), with $f = F$.

Let us denote by $AC(Q, \mathbb{R}^n)$ the space of absolutely continuous vector functions $z = (z_1, z_2, \dots, z_n)$ defined on the interval Q . The geometrical definition of the space $AC(Q, \mathbb{R})$ can be found in papers Berkson and Gillespie (1984) and Walczak (1987). In this paper we need necessary and sufficient conditions for $z : Q \rightarrow \mathbb{R}^n$ to be absolutely continuous on Q i.e. $z \in AC(Q, \mathbb{R}^n)$. We have the following theorem (see Berkson and Gillespie 1984; Walczak 1987).

Theorem 2 *A function z belongs to the space $AC(Q, \mathbb{R}^n)$ if and only if there exist functions $l \in L^1(Q, \mathbb{R}^n)$, $l^1, l^2 \in L^1([0, 1], \mathbb{R}^n)$ ¹ and a constant $c \in \mathbb{R}^n$ such that*

$$z(x, y) = \int_0^x \int_0^y l(s, t) ds dt + \int_0^x l^1(s) ds + \int_0^y l^2(t) dt + c.$$

¹ Throughout the paper $L^p(Q, \mathbb{R}^n)$, $p \geq 1$ denotes the space of all functions $z : Q \rightarrow \mathbb{R}^n$ which are measurable and $\int_Q |z(z)|^p dx < \infty$

Moreover the function z possesses partial derivatives z_x, z_y, z_{xy} , for a.e. $(x, y) \in Q$ and

$$\begin{aligned} z_x(x, y) &= \int_0^y l(x, t) dt + l^1(x), \\ z_y(x, y) &= \int_0^x l(s, y) ds + l^2(y), \\ z_{xy}(x, y) &= l(x, y). \end{aligned}$$

Theorem 2 follows directly from (Berkson and Gillespie 1984, Theorem 4) and (Šremr 2010, Proposition 3.5) (see also (Walczak 1987, Theorem 2), (Walczak 1998, Theorem 1)).

It is easy to check that if the function z satisfies homogeneous boundary conditions, i.e. $z(x, 0) = 0$ for $x \in [0, 1]$ and $z(0, y) = 0$ for $y \in [0, 1]$ then $l^1 = 0, l^2 = 0, c = 0$ and consequently we can write

$$z(x, y) = \int_0^x \int_0^y l(s, t) ds dt = \int_0^x \int_0^y z_{xy}(s, t) ds dt. \quad (9)$$

By $AC_0^2(Q, \mathbb{R}^n)$ we shall denote the space of absolutely continuous functions on the interval Q which satisfy the homogeneous boundary conditions $z(x, 0) = z(0, y) = 0$ for $x, y \in [0, 1]$ and such that $z_{xy} \in L^2(Q, \mathbb{R}^n)$. The space AC_0^2 is a Hilbert space with the inner product given by formula

$$\langle z, \bar{z} \rangle = \int_0^1 \int_0^1 \langle z_{xy}(x, y), \bar{z}_{xy}(x, y) \rangle dx dy. \quad (10)$$

In the space $AC_0^2(Q, \mathbb{R}^n)$ we introduce two norms. The first one is a classical norm given by the formula

$$\|z\| = \left(\int_0^1 \int_0^1 |z_{xy}(x, y)|^2 dx dy \right)^{\frac{1}{2}} = \|z_{xy}\|_{L^2} \quad (11)$$

and the second one is defined by the integral with exponential weight

$$\|z\|_{AC_{0,m}^2} = \left(\int_0^1 \int_0^1 e^{-m(x+y)} |z_{xy}(x, y)|^2 dx dy \right)^{\frac{1}{2}}, \quad m > 0. \quad (12)$$

Exponential norm (12) was introduced by Bielecki (1956). The space $AC_0^2(Q, \mathbb{R}^n)$ with norm (12) will be denoted by $AC_{0,m}^2(Q, \mathbb{R}^n)$.

It is easy to notice that

$$e^{-2m}\|z\| \leq \|z\|_{AC_{0,m}^2} \leq \|z\|.$$

Thus the norms given by formulas (11) and (12) are equivalent.

Similarly, in the space $L^2(Q, \mathbb{R}^n)$ of square integrable functions on Q we introduce two equivalent norms:

$$\|v\|_{L^2} = \left(\int_0^1 \int_0^1 |v(x, y)|^2 dx dy \right)^{\frac{1}{2}}$$

and

$$\|v\|_{L_m^2} = \left(\int_0^1 \int_0^1 e^{-m(x+y)} |v(x, y)|^2 dx dy \right)^{\frac{1}{2}}. \quad (13)$$

The space of square integrable functions with norm (13) will be denoted by $L_m^2(Q, \mathbb{R}^n)$.

3 Basic assumptions and lemmas

In this section we formulate basic assumptions and prove lemmas. Lemma 1 has a technical character, whereas Lemmas 2 and 3 are fundamental for understanding the proof of the main theorem—Theorem 3.

On the functions defining system (7) we assume that

- (C1) the functions $f^1(\cdot, \cdot, z)$, $f^2(\cdot, \cdot, z)$ are measurable on Q for every $z \in \mathbb{R}^n$ and $f^1(x, y, \cdot)$, $f^2(x, y, \cdot)$ are continuously differentiable on \mathbb{R}^n for a.e. $(x, y) \in Q$, the function $A^1(\cdot, y)$ is differentiable for a.e. $y \in [0, 1]$, the function $A^2(x, \cdot)$ is differentiable for a.e. $x \in [0, 1]$, the functions A^1 , A^2 , A_x^1 , A_y^2 are measurable on Q and essentially bounded on Q , the function $v \in L^2(Q, \mathbb{R}^n)$;
- (C2) there exist a constant $B > 0$ and a function $b \in L^2(Q, \mathbb{R}^+)$ such that

$$|f^1(x, y, z)|, |f^2(x, y, z)| \leq B|z| + b(x, y)$$

and

$$|A^1(x, y)|, |A^2(x, y)|, |A_x^1(x, y)|, |A_y^2(x, y)| \leq B$$

for $z \in \mathbb{R}^n$ and a.e. $(x, y) \in Q$;

- (C3) the functions f_z^1 , f_z^2 are bounded on bounded sets, i.e. for any $\varrho > 0$ there exists a constant M_ϱ , such that

$$|f_z^1(x, y, z)|, |f_z^2(x, y, z)| \leq M_\varrho$$

for $(x, y) \in Q$ and $|z| \leq \varrho$.

In the following lemma we prove some estimates for functions from the space $AC_0^2(Q, \mathbb{R}^n)$.

Lemma 1 *If the function $z \in AC_0^2(Q, \mathbb{R}^n)$ then*

$$\|z\|_{L_m^2} \leq \frac{2}{m} \|z\|_{AC_{0,m}^2}, \quad (14)$$

$$\|w_0\|_{L_m^2} \leq \frac{2}{m} \|z\|_{AC_{0,m}^2}, \quad (15)$$

$$\|w_1\|_{L_m^2} \leq \frac{2}{m} \|z\|_{AC_{0,m}^2}, \quad (16)$$

$$\|w_2\|_{L_m^2} \leq \frac{2}{m} \|z\|_{AC_{0,m}^2}, \quad (17)$$

where $w_0(x, y) = \int_0^x \int_0^y |z(s, t)| ds dt$, $w_1(x, y) = \int_0^x \int_0^y |z_x(s, t)| ds dt$, $w_2(x, y) = \int_0^x \int_0^y |z_y(s, t)| ds dt$.

Remark 1 The norms $\|\cdot\|_{L_m^2}$ and $\|\cdot\|_{AC_{0,m}^2}$ are defined by (12) and (13) respectively.

Proof Let z be an arbitrary function from the space $AC_0^2(Q, \mathbb{R}^n)$. By (9) and applying Cauchy–Schwarz inequality we get

$$\begin{aligned}\|z\|_{L_m^2}^2 &= \int_0^1 \int_0^1 e^{-m(x+y)} |z(x, y)|^2 dx dy \\ &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_{xy}(s, t)| ds dt \right)^2 dx dy \\ &\leq \int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s, t)|^2 ds dt \cdot \int_0^x \int_0^y ds dt \right) dx dy \\ &\leq \int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s, t)|^2 ds dt \right) dx dy.\end{aligned}$$

Applying integrating by parts formula for the inner integral (with respect to x) we obtain

$$\begin{aligned}&\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s, t)|^2 ds dt \right) dx dy \\ &= \int_0^1 \left(\int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s, t)|^2 ds dt \right) dx \right) dy \\ &= \int_0^1 \left[\left(\frac{-1}{m} e^{-m(x+y)} \int_0^x \int_0^y |z_{xy}(s, t)|^2 ds dt \right) \right]_{x=0}^{x=1} \\ &\quad - \int_0^1 \left(\frac{-1}{m} e^{-m(x+y)} \int_0^y |z_{xy}(x, t)|^2 dt \right) dx dy \\ &= \int_0^1 \left(\frac{-1}{m} e^{-m(1+y)} \int_0^1 \int_0^y |z_{xy}(s, t)|^2 ds dt \right) dy \\ &\quad + \int_0^1 \left(\int_0^1 \left(\frac{1}{m} e^{-m(x+y)} \int_0^y |z_{xy}(x, t)|^2 dt \right) dx \right) dy.\end{aligned}\tag{18}$$

Changing the order of integrals (thanks to Fubini's theorem) and again applying integrating by parts formula (for the integrals with respect to y), we get that the last term in (18) equals to

$$\begin{aligned}&\int_0^1 \left(\frac{-1}{m} e^{-m(1+y)} \int_0^y \left(\int_0^1 |z_{xy}(s, t)|^2 ds \right) dt \right) dy \\ &+ \int_0^1 \left(\int_0^1 \left(\frac{1}{m} e^{-m(x+y)} \int_0^y |z_{xy}(x, t)|^2 dt \right) dy \right) dx \\ &= \left[\frac{1}{m^2} e^{-m(1+y)} \int_0^y \int_0^1 |z_{xy}(s, t)|^2 ds dt \right]_{y=0}^{y=1} \\ &+ \int_0^1 \left(\frac{1}{m^2} e^{-m(1+y)} \int_0^1 |z_{xy}(s, y)|^2 ds \right) dy \\ &+ \int_0^1 \left(\left[\frac{1}{m^2} e^{-m(x+y)} \int_0^y |z_{xy}(x, t)|^2 dt \right]_{y=0}^{y=1} \right. \\ &\quad \left. + \int_0^1 \frac{1}{m^2} e^{-m(x+y)} |z_{xy}(x, y)|^2 dy \right) dx\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{m^2} e^{-2m} \int_0^1 \int_0^1 |z_{xy}(s, t)|^2 ds dt + \frac{1}{m^2} \int_0^1 \int_0^1 e^{-m(1+y)} |z_{xy}(s, y)|^2 ds dy \\
 &+ \frac{1}{m^2} \int_0^1 \int_0^1 e^{-m(x+1)} |z_{xy}(x, t)|^2 dt dx + \frac{1}{m^2} \int_0^1 e^{-m(x+y)} |z_{xy}(x, y)|^2 dy dx \\
 &\leq \frac{4}{m^2} \int_0^1 \int_0^1 e^{-m(x+y)} |z_{xy}(x, t)|^2 dx dy = \frac{4}{m^2} \|z\|_{AC_{0,m}^2}^2.
 \end{aligned}$$

We have thus proved inequality (14). Making use of (9) and from what has already been proved, it follows that

$$\begin{aligned}
 \|w_0\|_{L_m^2}^2 &= \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z(s, t)| ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y \left(\int_0^s \int_0^t |z_{xy}(\sigma, \tau)| d\sigma d\tau \right) ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y \left(\int_0^x \int_0^y |z_{xy}(\sigma, \tau)| d\sigma d\tau \right) ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_{xy}(\sigma, \tau)| d\sigma d\tau \cdot \int_0^x \int_0^y ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_{xy}(\sigma, \tau)| d\sigma d\tau \right)^2 dx dy \leq \frac{4}{m^2} \|z\|_{AC_{0,m}^2}^2.
 \end{aligned}$$

Let us prove the next inequality. By (9) and by already proved inequality we have

$$\begin{aligned}
 \|w_1\|_{L_m^2}^2 &= \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_x(s, t)| ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y \left(\int_0^t |z_{xy}(s, \tau)| d\tau \right) ds dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^y \left(\int_0^x \int_0^t |z_{xy}(s, \tau)| d\tau ds \right) dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^y \left(\int_0^x \int_0^y |z_{xy}(s, \tau)| d\tau ds \right) dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_{xy}(s, \tau)| d\tau ds \cdot \int_0^y dt \right)^2 dx dy \\
 &\leq \int_0^1 \int_0^1 e^{-m(x+y)} \left(\int_0^x \int_0^y |z_{xy}(s, t)| ds dt \right)^2 dx dy = \|w_0\|_{L_m^2}^2 \leq \frac{4}{m^2} \|z\|_{AC_{0,m}^2}^2.
 \end{aligned}$$

We conclude similarly that

$$\|w_2\|_{L_m^2}^2 \leq \frac{4}{m^2} \|z\|_{AC_{0,m}^2}^2.$$

□

Denote by $F : AC_0^2(Q, \mathbb{R}^n) \rightarrow L^2(Q, \mathbb{R}^n)$ the operator:

$$\begin{aligned}
 F(z)(x, y) &= z_{xy}(x, y) + f^1(x, y, z(x, y)) \\
 &+ \int_0^x \int_0^y (f^2(s, t, z(s, t)) + A^1(s, t)z_x(s, t) + A^2(s, t)z_y(s, t)) ds dt. \quad (19)
 \end{aligned}$$

We will prove that the norm of F is coercive.

Lemma 2 *If the functions f^1, f^2, A^1, A^2 satisfy assumptions (C1) and (C2) then the functional $z \mapsto \|F(z)\|_{L^2}$ is coercive, i.e.*

$$\|F(z)\|_{L^2} \rightarrow \infty \text{ whenever } \|z\|_{AC_0^2} \rightarrow \infty. \quad (20)$$

Proof Let us take $m > 8B$ (cf. (C2)). By (19) and assumptions (C1)–(C2) we have

$$\|F(z)\|_{L_m^2} \geq \|z_{xy}\|_{L_m^2} - \left(B\|z\|_{L_m^2} + B\|w_0\|_{L_m^2} + B\|w_1\|_{L_m^2} + B\|w_2\|_{L_m^2} \right) - D,$$

where $D = 2\|b\|_{L_m^2}$. By Lemma 1 and thanks to (12) it follows that

$$\|F(z)\|_{L_m^2} \geq \|z\|_{AC_{0,m}^2} - \frac{8B}{m} \|z\|_{AC_{0,m}^2} - D = \|z\|_{AC_{0,m}^2} \left(1 - \frac{8B}{m} \right) - D.$$

Inequality $m > 8B$ implies that $\|F(z)\|_{L_m^2} \rightarrow \infty$ if $\|z\|_{AC_{0,m}^2} \rightarrow \infty$. Since the pairs of the norms $\|\cdot\|_{L^2}, \|\cdot\|_{L_m^2}$ and $\|\cdot\|_{AC_0^2}, \|\cdot\|_{AC_{0,m}^2}$ are equivalent, we conclude that (20) holds. \square

Let $\{z^k\}_{k=0}^\infty \subset AC_0^2$ be an arbitrary sequence. Denote by $\{g^k\} \subset L^2(Q, \mathbb{R}^n)$ a sequence of functions defined by

$$\begin{aligned} g^k(x, y) = & f^1(x, y, z^k(x, y)) \\ & + \int_0^x \int_0^y \left(f^2(s, t, z^k(s, t)) + A^1(s, t) z_x^k(s, t) + A^2(s, t) z_y^k(s, t) \right) ds dt \\ & - v(x, y), \end{aligned} \quad (21)$$

for $k = 0, 1, 2, \dots$

Lemma 3 *If*

1. *the functions f^1, f^2, A^1, A^2 satisfy assumptions (C1) and (C2);*
2. *the sequence $\{z^k\}_{k=0}^\infty \subset AC_0^2(Q, \mathbb{R}^n)$ tends to $z^0 \in AC_0^2(Q, \mathbb{R}^n)$ weakly in $AC_0^2(Q, \mathbb{R}^n)$*

then

- (a) *the sequence of functions $\{z^k\}$ tends uniformly to z^0 on the interval Q ;*
- (b) *the sequence $\{g^k\}$ tends to g^0 for $(x, y) \in Q$ a.e.*

Moreover, there exists a function $b^0 \in L^2(Q, \mathbb{R}^+)$ such that

$$|g^k(x, y)| \leq b^0(x, y)$$

for a.e. $(x, y) \in Q$ and $k = 1, 2, \dots$

Proof We first prove that the weak convergence of the sequence $\{z^k\}$ to z^0 in the space $AC_0^2(Q, \mathbb{R}^n)$ implies the uniform convergence of the sequence $\{z^k\}$ to z^0 on the interval Q . By the definition of the inner product [see (10)] the weak convergence of the sequence $\{z^k\}$ to z^0 in the space $AC_0^2(Q, \mathbb{R}^n)$ is equivalent to the weak convergence of mixed second order derivatives $\{z_{xy}^k\}$ to z_{xy}^0 in the space $L^2(Q, \mathbb{R}^n)$. Without loss of generality we can assume that $z^0 = 0$. Suppose that z^k does not converge uniformly to $z^0 = 0$ while it converges to 0 weakly in $AC_0^2(Q, \mathbb{R}^n)$. Therefore, there exists $\varepsilon_0 > 0$ such that for any $n \in \mathbb{N}$ there is a point $(x^n, y^n) \in Q$ such that

$$|z^n(x^n, y^n)| > \varepsilon_0. \quad (22)$$

The sequence $\{(x^n, y^n)\} \subset Q$ is compact. Passing if necessary to a subsequence we can assume, that (x^n, y^n) tends to some $(\tilde{x}, \tilde{y}) \in Q$. Denote by χ^n the characteristic function of the interval²

$$\{(x, y) \in Q : 0 \leq x < x^n, 0 \leq y < y^n\}$$

and by $\tilde{\chi}$ the characteristic function of the interval

$$\{(x, y) \in Q : 0 \leq x < \tilde{x}, 0 \leq y < \tilde{y}\}.$$

It is easy to notice that χ^n tends to $\tilde{\chi}$ on Q a.e. This implies the following inequalities

$$\begin{aligned} \lim_{n \rightarrow \infty} |z^n(x^n, y^n)| &\leq \lim_{n \rightarrow \infty} |z^n(x^n, y^n) - z^n(\tilde{x}, \tilde{y})| + \lim_{n \rightarrow \infty} |z^n(\tilde{x}, \tilde{y})| \\ &= \lim_{n \rightarrow \infty} \left| \int_0^1 \int_0^1 \chi^n(s, t) z_{xy}^n(s, t) ds dt - \int_0^1 \int_0^1 \tilde{\chi}(s, t) z_{xy}^n(s, t) ds dt \right| \\ &\quad + \lim_{n \rightarrow \infty} \left| \int_0^1 \int_0^1 \tilde{\chi}(s, t) z_{xy}^n(s, t) ds dt \right|. \end{aligned}$$

Since z_{xy}^n tends to zero weakly in $L^2(Q, \mathbb{R}^n)$ the last limit is equal zero. Therefore

$$\begin{aligned} \lim_{n \rightarrow \infty} |z^n(x^n, y^n)| &\leq \lim_{n \rightarrow \infty} \int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)| |z_{xy}^n(s, t)| ds dt \\ &\leq \lim_{n \rightarrow \infty} \left(\int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)|^2 ds dt \right)^{\frac{1}{2}} \cdot \left(\int_0^1 \int_0^1 |z_{xy}^n(s, t)|^2 ds dt \right)^{\frac{1}{2}} \\ &\leq C \cdot \lim_{n \rightarrow \infty} \left(\int_0^1 \int_0^1 |\chi^n(s, t) - \tilde{\chi}(s, t)|^2 ds dt \right)^{\frac{1}{2}} = 0, \end{aligned}$$

where $C > 0$ is some constant such that $\|z_{xy}^n\| \leq C$. Consequently, $\lim_{n \rightarrow \infty} |z^n(x^n, y^n)| = 0$. This contradicts our assumption (22). Thus z^k tends to z^0 uniformly on Q .

Next we prove the assertion (b) of Lemma 3. By assumptions (C1) the functions f^1 and f^2 are continuous with respect to the third variable; assumption (C2) allows to find an appropriate dominating integrable function and to apply Lebesgue's dominated convergence theorem, consequently we get that

$$\lim_{k \rightarrow \infty} f^1(x, y, z^k(x, y)) = f^1(x, y, z^0(x, y)) \quad (23)$$

for a.e. $(x, y) \in Q$ and

$$\lim_{k \rightarrow \infty} \int_0^x \int_0^y f^2(s, t, z^k(s, t)) ds dt = \int_0^x \int_0^y f^2(s, t, z^0(s, t)) ds dt$$

for a.e. $(x, y) \in Q$. Thanks to the fact that z^k tends uniformly to z^0 the sequence $\{z^k(x, y)\}$ is uniformly bounded, therefore there exists $\bar{C} > 0$ such that

$$|z^k(x, y)| \leq \bar{C} \quad (24)$$

² the characteristic function χ of a set $X \subset \mathbb{R}^n$ is the function such that $\chi(x) = 1 \Leftrightarrow x \in X$ and $\chi(x) = 0$ otherwise.

for $(x, y) \in Q$ and $k = 1, 2, \dots$. Taking into account the above and again **(C2)**, we get that there exists a function $b^1 \in L^2(Q, \mathbb{R}^+)$ such that

$$|f^1(x, y, z^k(x, y))|, \left| \int_0^x \int_0^y f^2(s, t, z^k(s, t)) ds dt \right| \leq b^1(x, y)$$

for a.e. $(x, y) \in Q$ and $k = 1, 2, \dots$. Integrating by parts, thanks **(C1)** and Fubini's theorem, we obtain

$$\begin{aligned} \int_0^x \int_0^y A^1(s, t) z_x^k(s, t) ds dt &= \int_0^y \left(\int_0^x A^1(s, t) z_x^k(s, t) ds \right) dt \\ &= \int_0^y A^1(x, t) z^k(x, t) dt - \int_0^x \int_0^y A_x^1(s, t) z^k(s, t) ds dt \end{aligned} \quad (25)$$

for $k = 0, 1, \dots$. Since z^k tends to z^0 uniformly on Q provided that z^k converges to z^0 weakly in $AC_0^2(Q, \mathbb{R}^n)$, we get by (25) that

$$\begin{aligned} &\lim_{k \rightarrow \infty} \int_0^x \int_0^y A^1(s, t) z_x^k(s, t) ds dt \\ &= \int_0^y \lim_{k \rightarrow \infty} A^1(x, t) z^k(x, t) dt - \int_0^x \int_0^y \lim_{k \rightarrow \infty} A_x^1(s, t) z^k(s, t) ds dt \\ &= \int_0^y A^1(x, t) z^0(x, t) dt - \int_0^x \int_0^y A_x^1(s, t) z^0(s, t) ds dt \\ &= \int_0^x \int_0^y A^1(s, t) z_x^0(s, t) ds dt \end{aligned} \quad (26)$$

for $(x, y) \in Q$. Similarly, we can show that

$$\lim_{k \rightarrow \infty} \int_0^x \int_0^y A^2(s, t) z_y^k(s, t) ds dt = \int_0^x \int_0^y A^2(s, t) z_y^0(s, t) ds dt \quad (27)$$

for $(x, y) \in Q$. By (25), (24) and thanks to **(C1)** we have

$$\begin{aligned} &\left| \int_0^x \int_0^y A^1(s, t) z_x^k(s, t) ds dt \right| \\ &\leq \int_0^y |A^1(x, t)| \cdot |z^k(x, t)| dt + \int_0^x \int_0^y |A_x^1(s, t)| \cdot |z^k(s, t)| ds dt \\ &\leq \int_0^y B \bar{C} dt + \int_0^x \int_0^y B \bar{C} ds dt \leq B \bar{C} \int_0^1 dt + B \bar{C} \int_0^1 \int_0^1 ds dt = C_1 \end{aligned} \quad (28)$$

for some constant $C^1 > 0$, all $(x, y) \in Q$ and $k = 1, 2, \dots$. Similar bounds can be found for the integral $\int_0^x \int_0^y A^2(s, t) z_y^k(s, t) ds dt$. From (23), (26), (27) and (28) it follows that

$$\lim_{k \rightarrow \infty} g^k(x, y) = g^0(x, y)$$

for a.e. $(x, y) \in Q$. Moreover, there exists a function $b^0 \in L^2(Q, \mathbb{R}^+)$ such that $|g^k(x, y)| \leq b^0(x, y)$ for $k = 1, 2, \dots$ and a.e. $(x, y) \in Q$. This completes the proof. \square

4 Main result and example

Let us consider a functional $\varphi : AC_0^2 \rightarrow \mathbb{R}$ given by the formula

$$\varphi(z) = \frac{1}{2} \|F(z) - v\|_{L^2}^2, \quad (29)$$

where F is the operator defined by (19) and v is a fixed function from the space $L^2(Q, \mathbb{R}^n)$. We begin by proving some lemmas.

Lemma 4 *If the functions f^1, f^2, A^1, A^2 satisfy assumptions (C1)–(C2), then the functional φ given by (29) satisfies (PS)—condition.*

Proof Let $\{z^k\} \subset AC_0^2$ be an arbitrary (PS)—sequence for the functional φ . By Lemma 2 φ is coercive. It implies that the sequence $\{z^k\}$ is weakly compact in AC_0^2 . Passing if necessary to a subsequence we can assume, that z^k tends to some z^0 weakly in AC_0^2 . We claim that $\{z^k\}$ is compact with respect to the norm topology of the space AC_0^2 . Thanks to assumptions (C1)–(C2) it is easy to check that the functional φ is Fréchet differentiable and

$$\begin{aligned} \langle \varphi'(z^k), h \rangle &= \int_0^1 \int_0^1 \left(h_{xy}(x, y) + f_z^1(x, y, z^k(x, y)) \right) h(x, y) \\ &\quad + \int_0^x \int_0^y \left(f_z^2(s, t, z^k(s, t)) h(s, t) + A^1(s, t) h_x(s, t) \right. \\ &\quad \left. + A^2(s, t) h_y(s, t) \right) ds dt, z_{xy}^k(x, y) + g^k(x, y) \Big) dx dy, \end{aligned} \quad (30)$$

where the sequence $\{g^k\} \subset L^2(Q, \mathbb{R}^n)$ is given by formula (21). Let us put $h^k = z^k - z^0$, $k = 1, 2, \dots$. From (30) it follows that

$$\begin{aligned} \langle \varphi'(z^k) - \varphi'(z^0), z^k - z^0 \rangle &= \langle z_{xy}^k - z_{xy}^0, h_{xy}^k \rangle + \sum_{i=1}^5 V^i(z^k) \\ &= \|z^k - z^0\|_{AC_0^2}^2 + \sum_{i=1}^5 V^i(z^k), \end{aligned} \quad (31)$$

where

$$\begin{aligned} V^1(z^k) &= \langle z_{xy}^k - z_{xy}^0, g^k - g^0 \rangle \\ &= \int_0^1 \int_0^1 \left(z_{xy}^k(x, y) - z_{xy}^0(x, y), g^k(x, y) - g^0(x, y) \right) dx dy, \\ V^2(z^k) &= \int_0^1 \int_0^1 \left(f_z^1(x, y, z^k(x, y)) (z^k(x, y) - z^0(x, y)) \right. \\ &\quad \left. + \int_0^x \int_0^y f_z^2(s, t, z^k(s, t)) (z^k(s, t) - z^0(s, t)) ds dt, z_{xy}^k(x, y) \right. \\ &\quad \left. + g^k(x, y) \right) dx dy, \\ V^3(z^k) &= - \int_0^1 \int_0^1 \left(f_z^1(x, y, z^0(x, y)) (z^k(x, y) - z^0(x, y)) \right. \\ &\quad \left. + \int_0^x \int_0^y f_z^2(s, t, z^0(s, t)) (z^k(s, t) - z^0(s, t)) ds dt, z_{xy}^0(x, y) \right) \end{aligned}$$

$$\begin{aligned}
 & -g^0(x, y) \Big\rangle dx dy, \\
 V^4(z^k) &= \int_0^1 \int_0^1 \left\langle \int_0^x \int_0^y A^1(s, t) (z_x^k(s, t) - z_x^0(s, t)) ds dt, \right. \\
 & \quad \left. z_{xy}^k(x, y) + g^k(x, y) \right\rangle dx dy \\
 & + \int_0^1 \int_0^1 \left\langle \int_0^x \int_0^y A^2(s, t) (z_y^k(s, t) - z_y^0(s, t)) ds dt, \right. \\
 & \quad \left. z_{xy}^k(x, y) + g^k(x, y) \right\rangle dx dy, \\
 V^5(z^k) &= - \int_0^1 \int_0^1 \left\langle \int_0^x \int_0^y A^1(s, t) (z_x^k(s, t) - z_x^0(s, t)) ds dt, \right. \\
 & \quad \left. z_{xy}^0(x, y) + g^0(x, y) \right\rangle dx dy \\
 & - \int_0^1 \int_0^1 \left\langle \int_0^x \int_0^y A^2(s, t) (z_y^k(s, t) - z_y^0(s, t)) ds dt, \right. \\
 & \quad \left. z_{xy}^0(x, y) + g^0(x, y) \right\rangle dx dy.
 \end{aligned}$$

By the Cauchy–Schwarz inequality we have the following estimation

$$\begin{aligned}
 |V^1(z^k)|^2 &\leq \int_0^1 \int_0^1 |z_{xy}^k(x, y) - z_{xy}^0(x, y)|^2 dx dy \cdot \\
 &\quad \int_0^1 \int_0^1 |g^k(x, y) - g^0(x, y)|^2 dx dy.
 \end{aligned}$$

Since $z_{xy}^k - z_{xy}^0$ converges weakly to zero in $L^2(Q, \mathbb{R}^n)$, therefore there exists a constant $C > 0$ such that

$$|V^1(z^k)|^2 \leq C \int_0^1 \int_0^1 |g^k(x, y) - g^0(x, y)|^2 dx dy.$$

By Lemma 2 and Lebesgue dominated convergence theorem it follows that $V^1(z^k) \rightarrow 0$ as $k \rightarrow \infty$. We have proved that $z^k(x, y)$ tends to $z^0(x, y)$ uniformly on Q (see Lemma 2). Therefore, it is easy to notice that $V^2(z^k)$ and $V^3(z^k)$ converge to zero as $k \rightarrow \infty$.

Let us consider the functional V^4 . By (25) we have

$$\begin{aligned}
 V^4(z^k) &= \int_0^1 \int_0^1 \left\langle \int_0^y A^1(x, t) (z^k(x, t) - z^0(x, t)) dt \right. \\
 & \quad \left. - \int_0^x \int_0^y A_x^1(s, t) (z^k(s, t) - z^0(s, t)) ds dt, \right. \\
 & \quad \left. z_{xy}^k(x, y) + g^k(x, y) \right\rangle dx dy \\
 & + \int_0^1 \int_0^1 \left\langle \int_0^x A^2(s, y) (z^k(s, y) - z^0(s, y)) ds \right. \\
 & \quad \left. - \int_0^x \int_0^y A_y^2(s, t) (z^k(s, t) - z^0(s, t)) ds dt, \right.
 \end{aligned}$$

$$z_{xy}^k(x, y) + g^k(x, y) \Big) dx dy.$$

Using the Cauchy–Schwarz inequality and Lemma 2 it is easy to show that $V^4(z^k) \rightarrow 0$ as $k \rightarrow \infty$.

Similar considerations can be applied to $V^5(z^k)$. Thus $\lim_{k \rightarrow \infty} \sum_{i=1}^5 V^i(z^k) = 0$.

Now, let us observe that

$$\lim_{k \rightarrow \infty} \varphi'(z^k) (z^k - z^0) = 0$$

because $\{z^k\}$ is the (PS)—sequence for the functional φ and the sequence $\{z^k - z^0\}$ is bounded. Moreover,

$$\lim_{k \rightarrow \infty} \varphi'(z^0) (z^k - z^0) = 0$$

since z^k tends weakly to z^0 in AC_0^2 . Combining these equalities and (31) we conclude that

$$\lim_{k \rightarrow \infty} \|z^k - z^0\|_{AC_0^2}^2 = 0.$$

This gives us the desired conclusion that the functional φ given by (29) satisfies (PS)—condition. \square

Next, we prove the following

Lemma 5 *If the functions f^1, f^2, A^1, A^2 satisfy assumptions (C1)–(C3) then for any $v \in L^2(Q, \mathbb{R}^n)$ there exists a unique solution $h_v \in AC_0^2$ to the system*

$$F'(z^0)h = v, \quad (32)$$

where the operator $F : AC_0^2 \rightarrow L^2(Q, \mathbb{R}^n)$ is given by (19) and $z^0 \in AC_0^2$ is an arbitrary function.

Proof Let us put

$$h(x, y) = \int_0^x \int_0^y g(s, t) ds dt,$$

where $g \in L^2(Q, \mathbb{R}^n)$. Substituting the above into (32) we obtain

$$Hg = v,$$

where

$$\begin{aligned} Hg(x, y) = & g(x, y) + f_z^1(x, y, z^0(x, y)) \cdot \int_0^x \int_0^y g(s, t) ds dt \\ & + \int_0^x \int_0^y \left(f_z^2(s, t, z^0(s, t)) \int_0^s \int_0^t g(\sigma, \tau) d\sigma d\tau \right. \\ & \left. + A^1(s, t) \int_0^t g(s, \tau) d\tau + A^2(s, t) \int_0^s g(\sigma, t) d\sigma \right) ds dt. \end{aligned}$$

Let us denote by \tilde{H} the operator defined by

$$\tilde{H}g = Hg - g - v. \quad (33)$$

We will restrict our investigation of the operator \tilde{H} to the space $L_m^2(Q, \mathbb{R}^n)$. We prove that for sufficiently large $m > 0$ the mapping \tilde{H} is contracting with respect to the norm $\|\cdot\|_{L_m^2}$ defined by (13). Under assumptions (C2) and (C3), there exists a constant $d > 0$ such that

$$\begin{aligned} & \|\tilde{H}(g^1 - g^2)\|_{L_m^2} \\ & \leq d \left(\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |g^1(s, t) - g^2(s, t)|^2 ds dt \right) dx dy \right)^{\frac{1}{2}} \\ & + d \left(\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y \left(\int_0^s \int_0^t |g^1(\sigma, \tau) - g^2(\sigma, \tau)|^2 d\sigma d\tau \right) ds dt \right) dx dy \right)^{\frac{1}{2}} \\ & + d \left(\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y \left(\int_0^t |g^1(s, \tau) - g^2(s, \tau)|^2 d\tau \right) ds dt \right) dx dy \right)^{\frac{1}{2}} \\ & + d \left(\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y \left(\int_0^s |g^1(\sigma, t) - g^2(\sigma, t)|^2 d\sigma \right) ds dt \right) dx dy \right)^{\frac{1}{2}} \\ & \leq 4d \left(\int_0^1 \int_0^1 \left(e^{-m(x+y)} \int_0^x \int_0^y |g^1(s, t) - g^2(s, t)|^2 ds dt \right) dx dy \right)^{\frac{1}{2}}. \end{aligned}$$

Integrating by parts twice, in much the same way as in the proof of inequality (Kaczorek 2001), we obtain

$$\|\tilde{H}(g^1 - g^2)\|_{L_m^2} \leq \frac{4d}{m^2} \|g^1 - g^2\|_{L_m^2}.$$

Hence for sufficiently large m , i.e. $m > 2\sqrt{d}$, the operator \tilde{H} is contracting and, consequently, has a unique fixed point. It means that, there exists exactly one point $g^0 \in L^2(Q, \mathbb{R}^n)$ such that $g^0 = \tilde{H}g^0$. By (33) we get $Hg^0 = v$ and it follows easily that a function h_v given by

$$h_v(x, y) = \int_0^x \int_0^y g^0(s, t) ds dt$$

is a solution of (32) for fixed $v \in L^2(Q, \mathbb{R}^n)$. \square

We are now in a position to show the main result of the work.

Theorem 3 *If the functions f^1, f^2, A^1, A^2 satisfy assumptions (C1)–(C3) then for any $v \in L^2(Q, \mathbb{R}^n)$ the integro-differential system (7)–(8) has a unique solution $z_v \in AC_0^2$. The solution z_v continuously depends on v with respect to the norm topology in the spaces $L^2(Q, \mathbb{R}^n)$ and AC_0^2 . Moreover, the operator*

$$L^2(Q, \mathbb{R}^n) \ni v \mapsto z_v \in AC_0^2$$

is differentiable (in Fréchet sense).

Proof It follows from Lemmas 4 and 5 that the operator F given by (19) meets assumptions of Theorem 1. Thus system (7)–(8) has a solution z_v which satisfies the requirements of our theorem. \square

We now give an example of integro-differential system of the form (7)–(8) which satisfies assumptions of Theorem 3. For simplicity we put $n = 1$.

Example 1 Consider 2D integro-differential system

$$\begin{aligned} z_{xy}(x, y) + w^1(x, y) \left(\frac{z^3(x, y)}{1 + z^2(x, y)} + \psi^1(z(x, y)) \right) \\ + \int_0^x \int_0^y \left(w^2(s, t) \frac{z(s, t) - 1}{1 + z^2(x, y)} + \psi^2(z(x, y)) \right) \\ + A^1(s, t) z_x(s, t) + A^2(s, t) z_y(s, t) ds dt = v(x, y), \end{aligned} \quad (34)$$

where w^1, w^2, A^1, A^2 are some polynomials, $v \in L^2(Q, \mathbb{R})$ and ψ^1, ψ^2 are some C^1 -class functions with unbounded derivatives. For example one can take $\psi^1(z) = \cos z^k$ and $\psi^2(z) = \sin z^l$, where $k, l > 1$. This simple and theoretical example allows us to emphasize the difference between our work and some other methods of nonlinear analysis.

It is easy to see that system (34) satisfies assumptions (C1)–(C3). Hence by Theorem 3 for any $v \in L^2(Q, \mathbb{R})$ there exists a solution $z_v \in AC_0^2$ to the system (34) with the following properties:

1. the solution z_v is unique,
2. z_v continuously depends on v with respect to the norm topology of the spaces $L^2(Q, \mathbb{R}^n)$ and AC_0^2 , i.e. system (34) is stable,
3. the operator $L^2(Q, \mathbb{R}) \ni v \mapsto z_v \in AC_0^2$ is differentiable in Fréchet sense, i.e. system (34) is robust.

Let us notice that the functions $f^1(x, y, z) = w^1(x, y) \left(\frac{z^3}{1+z^2} + \psi^1(z) \right)$ and $f^2(x, y, z) = w^2(x, y) \frac{z-1}{1+z^2} + \psi^2(z)$ are not Lipschitz functions ($\sin z^l$ and $\cos z^k$ with $k, l \geq 1$ have "fast variation" when $|z| \rightarrow \infty$) and consequently we cannot apply the Banach contraction principle. In this case the Schauder fixed point theory may be applicable. But even using sophisticated fixed point theorems we get only the existence of a solution to system (34) and can hardly say anything related to properties (1)–(3).

5 Concluding remarks

In the paper two-dimensional integro-differential system was investigated. The main result of this work is theorem 3 on the stability and robustness of a solution to considered system (7)–(8). As far as we know 2D integro-differential systems have not been studied before. One-dimensional integro-differential systems described by ordinary differential operators were examined in many works (see monograph Lakshmikantham 1995 and references therein). It is important to notice that integro-differential operators can be used in mathematical modeling of systems with "memory", i.e. systems where the state at each moment t depends on its behavior on some interval $[t_0, t)$. In our opinion 2D integro-differential systems have the potential to play a similar role.

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