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DOI:
10.1007/s11045-017-0537-1

Document Version
Accepted author manuscript

Citation for published version (APA):
Nadarajah, S., & Kwong, H. S. (2017). A note on “On the ratio of independent complex Gaussian random variables”. *Multidimensional Systems and Signal Processing*, 1-5. https://doi.org/10.1007/s11045-017-0537-1

Published in:
Multidimensional Systems and Signal Processing

A note on "On the ratio of independent complex Gaussian random variables"

by

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Abstract: Nadimi, Ramezani and Blanes-Vidal [Multidimensional Systems and Signal Processing, 2017, doi: 10.1007/s11045-017-0519-3] studied the distribution of the ratio of two independent complex Gaussian random variables. The expressions provided for the distribution involved a hypergeometric function and an infinite sum. Here, we derive simpler and more manageable expressions. The practical usefulness of the expressions in terms of computational time is illustrated.

Keywords: Bessel function; Elementary function; Horn confluent hypergeometric function

1 Introduction

A random variable X is said to have the complex Gaussian distribution with parameters $\left(\nu \exp\left(\mathrm{j}\phi\right), \sigma^2\right)$ if its amplitude R and phase Θ have the joint probability density function (pdf)

$$f_{R,\Theta}(r,\theta) = \frac{r}{\pi\sigma^2} \exp\left[-\frac{\left|r\exp(\mathrm{j}\theta) - \nu\exp\left(\mathrm{j}\phi\right)\right|^2}{\sigma^2}\right]$$

for $r > 0$ and $-\pi < \theta < \pi$, where $\mathrm{j} = \sqrt{-1}$.

Suppose X and Y are independent complex Gaussian random variables with parameters $\left(\nu_X \exp\left(\mathrm{j}\phi_X\right), \sigma_X^2\right)$ and $\left(\nu_Y \exp\left(\mathrm{j}\phi_Y\right), \sigma_Y^2\right)$, respectively. Nadimi et al. (2017) derived expressions for the distribution of the ratio $Z = X/Y$. Theorem 1 in Nadimi et al. (2017) gave an expression for the joint distribution of the amplitude and phase of Z. The expression involved a hypergeometric function. Theorem 2 in Nadimi et al. (2017) gave an expression for the distribution of the amplitude of Z. The expression was an infinite sum with each term involving a finite sum.

The aim of this note is to derive simpler and more manageable expressions for the distribution of Z. We derive an elementary expression for the joint distribution of the amplitude and phase of Z. We also derive a closed form expression (involving a known special function) for the distribution of the amplitude of Z. These expressions are given in Section 2. Section 3 performs a computational study to show practical values of the expressions. We show in particular that the expressions are computationally less time consuming than the expressions obtained in Nadimi et al. (2017, Theorems 1 and 2).

The calculations in this note involve several special functions, including the gamma function defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}\exp(-t)dt;$$

the modified Bessel function of the first kind of order zero defined by

$$I_0(x) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left(\frac{x}{2}\right)^{2k};$$

the generalized Laguerre polynomial defined by

$$L_n^{\lambda}(x) = \frac{x^{-\lambda}\exp(x)}{n!} \frac{d^n}{dx^n} \left[x^{n+\lambda}\exp(-x)\right];$$

the Laguerre polynomial defined by

$$L_n(x) = \frac{\exp(x)}{n!} \frac{d^n}{dx^n} \left[x^n \exp(-x)\right];$$

and the Horn confluent hypergeometric function defined by

$$\Psi_2\left(a, b, c, x, y\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(a)_{n+k}}{(b)_n (c)_k n! k!} x^n y^k,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Erdelyi et al. (1981), Srivastava and Karlsson (1985), Prudnikov et al. (1986) and Gradshteyn and Ryzhik (2000). In-built routines for computing them are available in packages like Maple, Matlab and Mathematica.

2 Main results

Throughout this section, we assume X and Y are independent complex Gaussian random variables with parameters $\left(\nu_X \exp\left(\mathrm{j}\phi_X\right), \sigma_X^2\right)$ and $\left(\nu_Y \exp\left(\mathrm{j}\phi_Y\right), \sigma_Y^2\right)$, respectively. We let R_X (respectively, R_Y) and Θ_X (respectively, Θ_Y) denote the amplitude and phase of X (respectively, Y). Similarly, we let R_Z and Θ_Z denote the amplitude and phase of $Z = X/Y$. We let $k_X = \nu_X/\sigma_X$ and $k_Y = \nu_Y/\sigma_Y$.

Theorem 1 derives an elementary expression for the joint pdf of R_Z and Θ_Z. The corresponding expression given in Nadimi et al. (2017, Theorem 1) is

$$f_{R_Z,\Theta_Z}\left(r_Z, \theta_Z\right) = \frac{r_Z \sigma_X^2 \sigma_Y^2 \exp\left(-k_X^2 - k_Y^2\right)}{\pi\left(r_Z^2 \sigma_Y^2 + \sigma_X^2\right)^2} {}_1F_1\left(2; 1; \frac{r_Z^2 k_X^2 \sigma_Y^2 + k_Y^2 \sigma_X^2 + 2 r_Z \nu_X \nu_Y \cos\left(\theta_Z - \phi_X + \phi_Y\right)}{r_Z^2 \sigma_Y^2 + \sigma_X^2}\right), \quad (1)$$

where ${}_1F_1(a; b; x)$ denotes the confluent hypergeometric function. Theorem 2 expresses the pdf of R_Z in terms of the Horn confluent hypergeometric function. The corresponding expression given in Nadimi et al. (2017, Theorem 2) is

$$f_{R_Z}\left(r_Z\right) = \frac{2 r_Z \sigma_X^2 \sigma_Y^2 \exp\left(-k_X^2 - k_Y^2\right)}{\pi\left(r_Z^2 \sigma_Y^2 + \sigma_X^2\right)^2} \sum_{m=0}^{\infty} c_m (m+1)! \left(\frac{\sigma_X^2 \sigma_Y^2}{r_Z^2 \sigma_Y^2 + \sigma_X^2}\right)^m, \quad (2)$$

where

$$c_m = \sum_{\ell=0}^{m} \frac{1}{\ell!^2 (m-\ell)!^2} \left(\frac{r_Z \nu_X}{\sigma_X^2}\right)^{2\ell} \left(\frac{\nu_Y}{\sigma_Y^2}\right)^{2(m-\ell)}.$$

Theorem 1 *The joint pdf of R_Z and Θ_Z can be expressed as*

$$f_{R_Z,\Theta_Z}\left(r_Z,\theta_Z\right)=\frac{r_Z\exp\left(-k_X^2-k_Y^2\right)}{\pi\sigma_X^2\sigma_Y^2c^2}\left(1+\frac{\gamma_1^2}{c}\right)\exp\left(\frac{\gamma_1^2}{c}\right) \tag{3}$$

for $r_Z>0$ and $-\pi<\theta_Z<\pi$, where

$$c=\frac{r_Z^2}{\sigma_X^2}+\frac{1}{\sigma_Y^2}$$

and

$$\gamma_1=\sqrt{\left(\frac{r_Z\nu_X}{\sigma_X^2}\right)^2+\left(\frac{\nu_Y}{\sigma_Y^2}\right)^2+\frac{2r_Z\nu_X\nu_Y\cos\left(\theta_Z-\phi_X+\phi_Y\right)}{\sigma_X^2\sigma_Y^2}}.$$

Proof: By equation (11) in Nadimi et al. (2017), the joint pdf of R_Z and Θ_Z can be written as

$$f_{R_Z,\Theta_Z}\left(r_Z,\theta_Z\right)=\frac{2r_Z\exp\left(-k_X^2-k_Y^2\right)}{\pi\sigma_X^2\sigma_Y^2}\int_0^\infty t^3I_0\left(2\gamma_1t\right)\exp\left(-ct^2\right)dt. \tag{4}$$

Nadimi et al. (2017) say that the integral in (4) is not tractable. In fact, by equation (2.15.5.4) in Prudnikov et al. (1986, volume 2),

$$f_{R_Z,\Theta_Z}\left(r_Z,\theta_Z\right)=\frac{r_Z\exp\left(-k_X^2-k_Y^2\right)}{\pi\sigma_X^2\sigma_Y^2c^2}\exp\left(\frac{\gamma_1^2}{c}\right)L_1^0\left(-\frac{\gamma_1^2}{c}\right). \tag{5}$$

By the definition of Lagurre polynomial,

$$L_1^0\left(-\frac{\gamma_1^2}{c}\right)=L_1\left(-\frac{\gamma_1^2}{c}\right).$$

Furthermore, using the property $L_1(x)=1-x$,

$$L_1^0\left(-\frac{\gamma_1^2}{c}\right)=1+\frac{\gamma_1^2}{c}. \tag{6}$$

The result follows by substituting (6) into (5). The proof is complete. $\square$

Theorem 2 *The pdf of R_Z can be expressed as*

$$f_{R_Z}\left(r_Z\right)=\frac{4r_Z\exp\left(-k_X^2-k_Y^2\right)}{\sigma_X^2\sigma_Y^2c^2}\Psi_2\left(2,1,1,\frac{r_Z^2\nu_X^2}{\sigma_X^4c},\frac{\nu_Y^2}{\sigma_Y^4c}\right) \tag{7}$$

for $r_Z>0$.

Proof: By equation (15) in Nadimi et al. (2017), the joint pdf of R_Z and R_Y can be written as

$$f_{R_Z,R_Y}\left(r_Z,t\right)=\frac{4r_Z\exp\left(-k_X^2-k_Y^2\right)}{\sigma_X^2\sigma_Y^2}\exp\left(-ct^2\right)\sum_{n=0}^\infty\sum_{k=0}^\infty\frac{1}{(n!)^2(k!)^2}\left(\frac{r_Z\nu_X}{\sigma_X^2}\right)^{2n}\left(\frac{\nu_Y}{\sigma_Y^2}\right)^{2k}t^{2n+2k+3}. \tag{8}$$

Integrating out t in (8) and using

$$\int_0^\infty t^{2n+2k+3}\exp\left(-ct^2\right)dt=c^{-n-k-2}\Gamma(n+k+2),$$

we obtain

$$f_{R_Z}(r_Z) = \frac{4r_Z \exp\left(-k_X^2 - k_Y^2\right)}{\sigma_X^2 \sigma_Y^2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{(n!)^2 (k!)^2} \left(\frac{r_Z \nu_X}{\sigma_X^2}\right)^{2n} \left(\frac{\nu_Y}{\sigma_Y^2}\right)^{2k} c^{-n-k-2} \Gamma(n+k+2). \quad (9)$$

Using the property that $(a)_k = \Gamma(a+k)/\Gamma(a)$, we can rewrite (9) as

$$f_{R_Z}(r_Z) = \frac{4r_Z \exp\left(-k_X^2 - k_Y^2\right)}{\sigma_X^2 \sigma_Y^2 c^2} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{n!k!} \left(\frac{r_Z^2 \nu_X^2}{\sigma_X^4 c}\right)^{n} \left(\frac{\nu_Y^2}{\sigma_Y^4 c}\right)^{k} \frac{(2)_{n+k}}{(1)_n (1)_k}.$$

Now the result follows from the definition of the Horn confluent hypergeometric function. The proof is complete. $\square$

3 Computational issues

Here, we illustrate computational efficiency of the expressions derived in Section 2. Computational efficiency is assessed in terms of time.

We computed $f_{R_Z,\Theta_Z}(r_Z, \theta_Z)$ one hundred times for each $r_Z = 0.1, 0.2, \ldots, 100$ using (1) and (3). The corresponding central processing unit times are plotted in Figure 1. We also computed $f_{R_Z}(r_Z)$ one hundred times for each $r_Z = 0.1, 0.2, \ldots, 100$ using (2) and (7). The corresponding central processing unit times are plotted in Figure 2. We have taken $\nu_X = 1$, $\nu_Y = 1$, $\phi_X = 1$, $\phi_Y = 1$, $\sigma_X = 1$, $\sigma_Y = 1$ and $\theta_Z = \pi/2$.

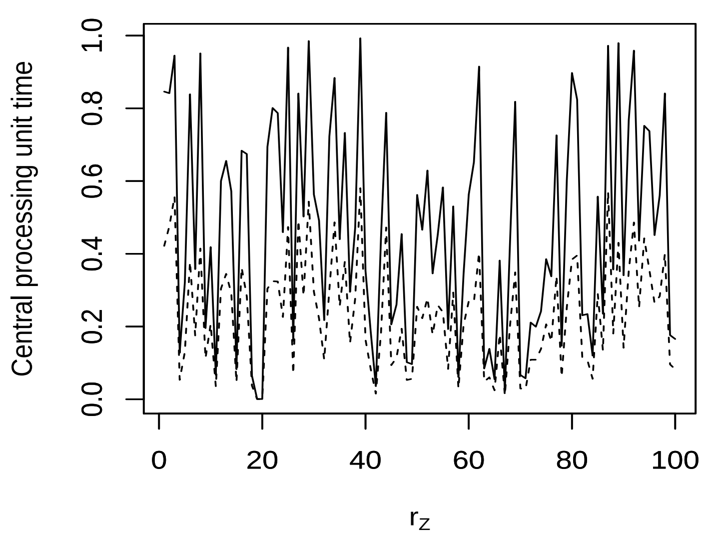

Figure 1: Central processing unit times in seconds taken to compute (1) (solid curve) and (3) (broken curve) versus r_Z.

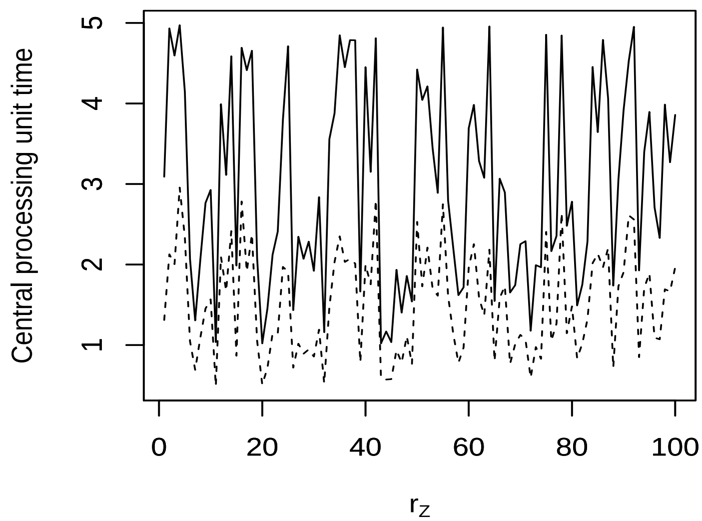

Figure 2: Central processing unit times in seconds taken to compute (2) (solid curve) and (7) (broken curve) versus r_Z.

We see that our expressions in (3) and (7) are computationally more efficient for all values of r_Z. The central processing unit times for our expressions appear about two times smaller. There is no evidence that the computational times change significantly with respect to r_Z.

In Figures 1 and 2, we took $\nu_X = 1$, $\nu_Y = 1$, $\phi_X = 1$, $\phi_Y = 1$, $\sigma_X = 1$, $\sigma_Y = 1$ and $\theta_Z = \pi/2$. But the same results held for a wide range of values of ν_X, ν_Y, ϕ_X, ϕ_Y, σ_X, σ_Y and θ_Z. In particular, the central processing unit times for our expressions always appeared about two times smaller.

The Mathematica software was used for computations. Mathematica like other algebraic manipulation packages allows for arbitrary precision, so the accuracy of computed values was not an issue. That is, the values computed can be considered as exact.

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