On the computational complexity of spiking neural P systems

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Abstract. It is shown that there is no standard spiking neural P system that simulates Turing machines with less than exponential time and space overheads. The spiking neural P systems considered here have a constant number of neurons that is independent of the input length. Following this we construct a universal spiking neural P system with exhaustive use of rules that simulates Turing machines in linear time and has only 10 neurons.

1 Introduction

Since their inception inside of the last decade P systems [16] have spawned a variety of hybrid systems. One such hybrid, that of spiking neural P systems [3], results from a fusion with spiking neural networks. It has been shown that these systems are computationally universal. Here the time/space computational complexity of spiking neural P systems is examined. We begin by showing that counter machines simulate standard spiking neural P systems with linear time and space overheads. Fischer et al. [2] have previously shown that counter machines require exponential time and space to simulate Turing machines. Thus it immediately follows that there is no spiking neural P system that simulates Turing machines with less than exponential time and space overheads. These results are for spiking neural P systems that have a constant number of neurons independent of the input length.

Extended spiking neural P systems with exhaustive use of rules were proved computationally universal in [4]. Zhang et al. [18] gave a small universal spiking neural P system with exhaustive use of rules (without delay) that has 125 neurons. The technique used to prove universality in [4] and [18] involved simulation of counter machines and thus suffers from an exponential time overhead when simulating Turing machines. In an earlier version [10] of the work we present here, we gave an extended spiking neural P system with exhaustive use of rules that simulates Turing machines in polynomial time and has 18 neurons. Here we improve on this result to give an extended spiking neural P system with exhaustive use of rules that simulates Turing machines in linear time and has only 10 neurons.

The brief history of small universal spiking neural P systems is given in Table 1. Note that, to simulate an arbitrary Turing machine that computes in time t, all of the small universal spiking neural P systems prior to our results require time that is exponential in t. An arbitrary Turing machine that uses space of s is simulated by the universal systems given in [4,11,18] in space that is doubly exponential in s, and by the universal systems given in [3,10,15,19] in space that is exponential in s.

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number of	simulation	type	exhaustive	author	
neurons	time/space	of rules	use of rules	es	
84	exponential	standard	no	Păun and Păun [15]	
67	exponential	standard	no	Zhang et al. [19]	
49	exponential	extended†	no	Păun and Păun [15]	
41	exponential	extended†	no	Zhang et al. [19]	
12	double-exponential	extended†	no	Neary [11]	
18	exponential	extended	no	Neary [11,12]*	
17	exponential	standard†	no	[9]	
14	double-exponential	standard†	no	[9]	
5	exponential	extended†	no	[9]	
4	double-exponential	extended†	no	[9]	
3	double-exponential	extended‡	no	[9]	
125	exponential/	extended†	yes	Zhang et al. [18]	
	double-exponential				
18	polynomial/exponential	extended	yes	Neary [10]	
10	linear/exponential	extended	yes	Section 5	

Table 1. Small universal SN P systems. The "simulation time" column gives the overheads used by each system we simulating a standard single tape Turing machine. † indicates that there is a restriction of the rules as delay is not used and ‡ indicates that a more generalised output technique is used. *The 18 neuron system is not explicitly given in [11]; it is however mentioned at the end of the paper and is easily derived from the other system presented in [11]. Also, its operation and its graph were presented in [12].

Chen et al. [1] have shown that with exponential pre-computed resources SAT is solvable in constant time with spiking neural P systems. Leporati et al. [7] gave a semi-uniform family of extended spiking neural P systems that solve the Subset Sum problem in constant time. In later work, Leporati et al. [8] gave a uniform family of maximally parallel spiking neural P systems with more general rules that solve the Subset Sum problem in polynomial time. All the above solutions to NP-hard problems rely on families of spiking neural P systems. Specifically, the size of the problem instance determines the number of neurons in the spiking neural P system that solves that particular instance. This is similar to solving problems with uniform circuits families where each input size has a specific circuit that solves it. Ionescu and Dragos [5] have shown that spiking neural P systems simulate circuits in linear time.

In the next two sections we give definitions for spiking neural P systems and counter machines and explain the operation of both. Following this, in Section 4, we prove that counter machines simulate spiking neural P systems in linear time. Thus proving that there exists no universal spiking neural P system that simulates Turing machines in less than exponential time. In Section 5 we present our universal spiking neural P system, with exhaustive use of rules, that simulates Turing machine in linear time and has only 10 neurons. Finally, we end the paper with some discussion and conclusions.

2 Spiking neural P systems

Definition 1 (Spiking neural P systems). A spiking neural P system is a tuple $\Pi = (O, \sigma_1, \sigma_2, \dots, \sigma_m, syn, in, out)$, where:

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    O = {s} is the unary alphabet (s is known as a spike),
    σ<sub>1</sub>, σ<sub>2</sub>, ··· , σ<sub>m</sub> are neurons, of the form σ<sub>i</sub> = (n<sub>i</sub>, R<sub>i</sub>), 1 ≤ i ≤ m, where:

            (a) n<sub>i</sub> ≥ 0 is the initial number of spikes contained in σ<sub>i</sub>,
            (b) R<sub>i</sub> is a finite set of rules of the following two forms:

                  i. E/s<sup>b</sup> → s; d, where E is a regular expression over s, b ≥ 1 and d ≥ 1,
                  ii. s<sup>e</sup> → λ; 0 where λ is the empty word, e ≥ 1, and for all E/s<sup>b</sup> → s; d from R<sub>i</sub> s<sup>e</sup> ∉ L(E) where L(E) is the language defined by E,
                 syn ⊆ {1, 2, ··· , m} × {1, 2, ··· , m} are the set of synapses between neurons, where i ≠ j for all (i, j) ∈ syn,
                  in, out ∈ {σ<sub>1</sub>, σ<sub>2</sub>, ··· , σ<sub>m</sub>} are the input and output neurons respectively.
                  is the input and output neurons respectively.
                  in out ∈ {σ<sub>1</sub>, σ<sub>2</sub>, ··· , σ<sub>m</sub>}
                  in put and output neurons respectively.
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In the same manner as in [15], spikes are introduced into the system from the environment by reading in a binary sequence (or word) $w \in \{0,1\}^*$ via the input neuron σ_1 . The sequence w is read from left to right one symbol at each timestep. If the read symbol is 1 then a spike enters the input neuron on that timestep.

A firing rule $r=E/s^b\to s;d$ is applicable in a neuron σ_i if there are $j\geqslant b$ spikes in σ_i and $s^j\in L(E)$ where L(E) is the set of words defined by the regular expression E. If, at time t, rule r is executed then b spikes are removed from the neuron, and at time t+d-1 the neuron fires. When a neuron σ_i fires a spike is sent to each neuron σ_j for every synapse (i,j) in Π . Also, the neuron σ_i remains closed and does not receive spikes until time t+d-1 and no other rule may execute in σ_i until time t+d. We note here that in 2b(i) it is standard to have a $d\geqslant 0$. However, we have $d\geqslant 1$ as it simplifies explanations throughout the paper. This does not effect the operation as the neuron fires at time t+d-1 instead of t+d. A forgeting rule $r'=s^e\to \lambda;0$ is applicable in a neuron σ_i if there are exactly e spikes in σ_i . If r' is executed then e spikes are removed from the neuron. At each timestep t a rule must be applied in each neuron if there is one or more applicable rules at time t. Thus while the application of rules in each individual neuron is sequential the neurons operate in parallel with each other.

Note from 2b(i) of Definition 1 that there may be two rules of the form $E/s^b \to s$; d, that are applicable in a single neuron at a given time. If this is the case then the next rule to execute is chosen non-deterministically. The output is the time between the first and second spike in the output neuron σ_m .

An extended spiking neural P system [15] has more general rules of the form $E/s^b \to s^p$; d, where $b \ge p \ge 0$. Note if p=0 then $E/s^b \to s^p$; d is a forgetting rule. An extended spiking neural P system with exhaustive use of rules [4] applies its rules as follows. If a neuron σ_i contains k spikes and the rule $E/s^b \to s^p$; d is applicable, then the neuron σ_i sends out gp spikes after d timesteps leaving u spikes in σ_i , where k=bg+u, u < b and $k,g,u \in \mathbb{N}$. Thus, a synapse in a spiking neural P system with exhaustive use of rules may transmit an arbitrary number of spikes in a single timestep. In the sequel we allow the input neuron of a system with exhaustive use of rules to receive an arbitrary number of spikes in a single timestep. This is a generalisation on the input allowed by Ionescu et al. [4]. We discuss why we think this generalisation is natural for this model at the end of the paper.

In earlier work [15], Korec's notion of strong universality was adopted for small SN P systems. Analogously, some small SN P systems could be described as what Korec refers to

as weak universality. However, as we noted in other work [9], it could be considered that Korec's notion of strong universality is somewhat arbitrary and we also pointed out some inconsistency in his notion of weak universality. Hence, in this work we rely on time/space complexity analysis to compare the encodings used by the small SN P system in Table 1.

In the sequel each spike in a spiking neural P system represents a single unit of space. The maximum number of spikes in a spiking neural P system at any given timestep during a computation is the space used by the system.

3 Counter machines

The definition we give for counter machine is similar to that of Fischer et al. [2].

Definition 2 (Counter machine).

A counter machine is a tuple $C = (z, c_m, Q, q_0, q_h, \Sigma, f)$, where z gives the number of counters, c_m is the output counter, $Q = \{q_0, q_1, \dots, q_h\}$ is the set of states, $q_0, q_h \in Q$ are the initial and halt states respectively, Σ is the input alphabet and f is the transition function

$$f: (\Sigma \times Q \times g(i)) \to (\{Y, N\} \times Q \times \{INC, DEC, NULL\})$$

where g(i) is a binary valued function and $0 \le i \le z$, Y and N control the movement of the input read head, and INC, DEC, and NULL indicate the operation to carry out on counter c_i .

Each counter c_i stores a natural number value x. If x>0 then g(i) is true and if x=0 then g(i) is false. The input to the counter machine is read in from an input tape with alphabet Σ . The movement of the scanning head on the input tape is one-way so each input symbol is read only once. When a computation begins the scanning head is over the leftmost symbol α of the input word $\alpha w \in \Sigma^*$ and the counter machine is in state q_0 . We give three examples below to explain the operation of the transition function f.

- $-f(\alpha, q_j, g(i)) = (Y, q_k, INC(h))$ move the read head right on the input tape to read the next input symbol, change to state q_k and increment the value x stored in counter c_i by
- $-f(\alpha, q_j, g(i)) = (N, q_k, DEC(h))$ do not move the read head, change to state q_k and decrement the value x stored in counter c_i by 1. Note that g(i) must evaluate to true for this rule to execute.
- $-f(\alpha,q_i,g(i))=(N,q_k,NULL)$ do not move the read head and change to state q_k .

A single application of f is a timestep. Thus in a single timestep only one counter may be incremented or decremented by 1.

Our definition for counter machine, given above, is more restricted than the definition given by Fischer [2]. In Fischer's definition INC and DEC may be applied to every counter in the machine in a single timestep. Clearly the more general counter machines of Fischer simulate our machines with no extra space or time overheads. Fischer has shown that counter machines are exponentially slow in terms of computation time as the following theorem illustrates.

Theorem 1 (Fischer [2]). There is a language L, real-time recognizable by a one-tape TM, which is not recognizable by any k-CM in time less than $T(n) = 2^{\frac{n}{2k}}$.

In Theorem 1 a one-tape TM is an offline Turing machine with a single read only input tape and a single work tape, a k-CM is a counter machine with k counters, n is the input length and real-time recognizable means recognizable in n timesteps. For his proof Fischer noted that the language $L = \{waw^r \mid w \in \{0,1\}^*\}$, where w^r is w reversed, is recognisable in n timesteps on a one-tape offline Turing machine. He then noted, that time of $2^{\frac{n}{2k}}$ is required to process input words of length n due to the unary data storage used by the counters of the k-CM. Note that Theorem 1 also holds for non-deterministic counter machines as they use the same unary storage method.

4 Non-deterministic counter machines simulate spiking neural P systems in linear time

Theorem 2. Let Π be a spiking neural P system with m neurons that completes its computation in time T and space S. Then there is a non-deterministic counter machine C_{Π} that simulates the operation of Π in time $O(T(x_r)^2m + Tm^2)$ and space O(S) where x_r is a constant dependant on the rules of Π .

Proof idea Before we give the proof of Theorem 2 we give the main idea behind the proof. Each neuron σ_i from the spiking neural P system Π is simulated by a counter c_i from the counter machine C_{Π} . If a neuron σ_i contains y spikes, then the counter will have value y. A single synchronous update of all the neurons at a given timestep t is simulated as follows. If the number of spikes in a neuron σ_i is deceasing by b spikes in-order to execute a rule, then the value y stored in the simulated neuron c_i is decremented b times using DEC(i) to give y-b. This process is repeated for each neuron that executes a rule at time t. If neuron σ_i fires at time t and has synapses to neurons $\{\sigma_{i_1}, \ldots, \sigma_{i_v}\}$ then for each open neuron σ_{i_j} in $\{\sigma_{i_1}, \ldots, \sigma_{i_v}\}$ at time t we increment the simulated neuron c_{i_j} using $INC(i_j)$. This process is repeated until all firing neurons have been simulated. This simulation of the synchronous update of Π at time t is completed by C_{Π} in constant time. Thus we get the linear time bound given in Theorem 2.

Proof. Let $\Pi = (O, \sigma_1, \sigma_2, \dots, \sigma_m, syn, in, out)$ be a spiking neural P system where $in = \sigma_1$ and $out = \sigma_2$. We explain the operation of a non-deterministic counter machine C_{Π} that simulates the operation of Π in time $O(T(x_r)^2m + Tm^2)$ and space O(S).

There are m+1 counters $c_1, c_2, c_3, \dots, c_m, c_{m+1}$ in C_{II} . Each counter c_i emulates the activity of a neuron σ_i . If σ_i contains y spikes then counter c_i will store the value y. The states of the counter machine are used to control which neural rules are simulated in each counter and also to synchronise the operations of the simulated neurons (counters).

Input encoding It is sufficient for C_{Π} to have a binary input tape. The value of the binary word $w \in \{1,0\}^*$ that is placed on the terminal to be read into C_{Π} is identical to the binary sequence read in from the environment by the input neuron σ_i . A single symbol is read from the terminal at each simulated timestep. The counter c_1 (the simulated input neuron) is incremented only on timesteps when a 1 (a simulated spike) is read. As such at each simulated timestep t, a simulated spike is received by c_1 if and only if a spike is received by the input neuron σ_1 . At the start of the computation, before the input is read in, each counter simulating σ_i is incremented n_i times to simulated the n_i spikes in each neuron given by 2(a) of Definition 1. This takes a constant amount of time.

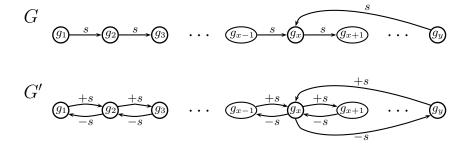


Fig. 1. Finite state machine G decides if a particular rule is applicable in a neuron given the number of spikes in the neuron at a given time in the computation. Each s represents a spike in the neuron. Machine G' keeps track of the movement of spikes into and out of the neuron and decides whither or not a particular rule is applicable at each timestep in the computation. +s represents a single spike entering the neuron and -s represents a single spike exiting the neuron.

Storing neural rules in the counter machine states Recall from Definition 1 that the applicability of a rule in a neuron is dependant on a regular expression over a unary alphabet. Let $r = E/s^b \to s$; d be a rule in neuron σ_i . Then there is a finite state machine G that accepts language L(E) and thus decides if the number of spikes in σ_i permits the application of r in σ_i at a given time in the computation. G is given in Figure 1. If g_i is an accept state in G then j > b. This ensures that there is enough spikes to execute r. We also place the restriction on G that x > b. During a computation we may use G to decide if r is applicable in σ_i by passing an s to G each time a spike enters σ_i . However, G may not give the correct result if spikes leave the neuron as it does not record spikes leaving σ_i . Thus using G we may construct a second machine G' such that G' records the movement of spikes going into and out of the neuron. G' is construct as follows; G' has all the same states (including accept states) and transitions as G along with an extra set of transitions that record spikes leaving the neuron. This extra set of transitions are given as follows for each transition on s from a state g_i to a state g_i in G there is a new transition on -s going from state g_i to g_i in G' that records the removal of a spike from G'. By recording the dynamic movement of spikes, G' is able to decide if the number of spikes in σ_i permits the application of r in σ_i at each timestep during the computation. G' is also given in Figure 1. Note that forgetting rules $s^e \to \lambda$; 0 are dependant on simpler regular expressions thus we will not give a machine G' for forgetting rules here.

Let neuron σ_i have the greatest number l of rules of any neuron in Π . Thus the applicability of rules r_1, r_2, \dots, r_l in σ_i is decided by the automata G'_1, G'_2, \dots, G'_l . We record if a rule may be simulated in a neuron at any given timestep during the computation by recording the current state of its G' automaton (Figure 1) in the states of the counter machine. There are m neuron in Π . Thus each state in our counter machine remembers the current states of at most ml different G' automata in order to determine which rules are applicable in each neuron at a given time.

Recall that in each rule of the form $r = E/s^b \to s$; d that d specifies the number of timestep between the removal of b spikes from the neuron and the spiking of the neuron. The number of timesteps < d remaining until a neuron will spike is recorded in the states of the C_{II} . Each state in our counter machine remembers at most m different values < d.

Algorithm overview Next we explain the operation of C_{II} by explaining how it simulates the synchronous update of all neurons in II at an arbitrary timestep t. The algorithm has 3 stages. A single iteration of Stage 1 identifies which applicable rule to simulate in a simulated open neuron. Then the correct number y of simulated spikes are removed by decrementing the counter y times (y = b or y = e in 2b of Definition 1). Stage 1 is iterated until all simulated open neurons have had the correct number of simulated spikes removed. A single iteration of Stage 2 identifies all the synapses leaving a firing neuron and increments every counter that simulates an open neuron at the end of one of these synapses. Stage 2 is iterated until all firing neurons have been simulated by incrementing the appropriate counters. Stage 3 synchronises each neuron with the global clock and increments the output counter if necessary. If the entire word w has not been read from the input tape the next symbol is read.

Stage 1. Identify rules to be simulated and remove spikes from neurons Recall that d=0 indicates a neuron is open and the value of d in each neuron is recorded in the states of the counter machine. Thus our algorithm begins by determining which rule to simulate in counter c_{i_1} where $i_1 = min\{i \mid d=0 \text{ for } \sigma_i\}$ and the current state of the counter machine encodes an accept state for one or more of the G' automata for the rules in σ_{i_1} at time t. If there is more than one rule applicable the counter machine non-deterministically chooses which rule to simulate. Let $r = E/s^b \to s$; d be the rule that is to be simulated. Using the $DEC(i_1)$ instruction, counter c_{i_1} is decremented b times. With each decrement of c_{i_1} the new current state of each automaton G'_1, G'_2, \cdots, G'_l is recorded in the counter machine's current state. After b decrements of c_i the simulation of the removal of b spikes from neuron σ_{i_1} is complete. Note that the value of d from rule r is recorded in the counter machine state.

There is a case not covered by the above paragraph. To see this note that in G' in Figure 1 there is a single non-deterministic choice to be made. This choice is at state g_x if a spike is being removed (-s). Thus, if one of the automata is in such a state g_x our counter machine resolves this be decrementing the counter x times using the DEC instruction. If $c_{i_1} = 0$ after the counter has been decremented x times then the counter machine simulates state g_{x-1} otherwise state g_y is simulated. Immediately after this the counter is incremented x-1 times to restore it to the correct value.

When the simulation of the removal of b spikes from neuron σ_{i_1} is complete, the above process is repeated with counter c_{i_2} where $i_2 = \min\{i \mid i_2 > i_1, d = 0 \text{ for } \sigma_i\}$ and the current state of the counter machine encodes an accept state for one or more of the G' automata for the rules in σ_{i_2} at time t. This process is iterated until every simulated open neuron with an applicable rule at time t has had the correct number of simulated spikes removed.

Stage 2. Simulate spikes This stage of the algorithm begins by simulating spikes traveling along synapses of the form (i_1, j) where $i_1 = min\{i \mid d = 1 \text{ for } \sigma_i\}$ (if d = 1 the neuron is firing). Let $\{(i_1, j_1), (i_1, j_2), \cdots, (i_1, j_k)\}$ be the set of synapses leaving σ_i where $j_u < j_{u+1}$ and $d \leq 1$ in σ_{j_u} at time t (if $d \leq 1$ the neuron is open and may receive spikes). Then the following sequence of instructions are executed INC (j_1) , INC (j_2) , \cdots , INC (j_k) , thus incrementing any counter (simulated neuron) that receives a simulated spike.

The above process is repeated for synapses of the form (i_2, j) where $i_2 = min\{i \mid i_2 > i_1, d = 1 \text{ for } \sigma_i\}$. This process is iterated until every simulated neuron c_i that is open has been incremented once for each spike σ_i receives at time t.

Stage 3. Reading input, decrementing d, updating output counter and halting If the entire word w has not been read from the input tape then the next symbol is read. If this

is the case and the symbol read is a 1 then counter c_1 is incremented thus simulating a spike being read in by the input neuron. In this stage the state of the counter machine changes to record the fact that each $k \leq d$ that records the number of timesteps until a currently closed neuron will fire is decremented to k-1. If the counter c_m , which simulates the output neuron, has spiked only once prior to the simulation of timestep t+1 then this stage will also increment output counter c_{m+1} . If during the simulation of timestep t counter t_m has simulated a spike for the second time in the computation, then the counter machine enters the halt state. When the halt state is entered the number stored in counter t_m is equal to the unary output that is given by time between the first two spikes in t_m .

Space analysis The input word on the binary tape of C_{II} is identical to the length of the binary sequence read in by the input neuron of II. Counters c_1 to c_m uses the same space as neurons σ_1 to σ_m . Counter c_{m+1} uses the same amount of space as the unary output of the computation of II. Thus C_{II} simulates II in space of O(S).

Time analysis The simulation involves 3 stages. Recall that x > b. Let x_r be the maximum value for x of any G' automaton thus x_r is greater than the maximum number of spikes deleted in a neuron.

Stage 1. In order to simulate the deletion of a single spike in the worst case the counter will have to be decremented x_r times and incremented $x_r - 1$ times as in the special case. This is repeated a maximum of $b < x_r$ times (where b is the number of spikes removed). Thus a single iteration of Stage 1 take $O(x_r^2)$ time. Stage 1 is iterated a maximum of m times per simulated timestep giving $O(x_r^2 m)$ time.

Stage 2. The maximum number of synapses leaving a neuron i is m. A single spike traveling along a neuron is simulated in one step. Stage 2 is iterated a maximum of m times per simulated timestep giving $O(m^2)$ time.

Stage 3. Takes a small constant number of steps.

Thus a single timestep of Π is simulated by C_{Π} in $O((x_r)^2 m + m^2)$ time and T timesteps of Π are simulated in linear time $O(T(x_r)^2 m + Tm^2)$ by C_{Π} .

The following is an immediate corollary of Theorems 1 and 2.

Corollary 1. There exist no universal spiking neural P system that simulates Turing machines with less than exponential time and space overheads.

5 A universal spiking neural P system that is both small and time efficient

In this section we construct a universal spiking neural P system that applies exhaustive use of rules, has only 10 neurons, and simulates any Turing machine in linear time.

Theorem 3. Let M be a single tape Turing machine with |A| symbols and |Q| states that runs in time T. Then there is a universal spiking neural P system Π_M with exhaustive use of rules that simulates the computation of M in time O(|A||Q|T) and space $O([2^{\log_2\lceil 2|Q||A|+2|A|\rceil}]^T)$ and has only 10 neurons.

If the reader would like to get a quick idea of how our spiking neural P system with 10 neurons operates they should skip to the algorithm overview in Subsection 5.3 of the proof.

Proof. We give a spiking neural P system Π_M that simulates an arbitrary Turing machine M in linear time and exponential space. Π_M is given by Figure 3 and Tables 2 and 3. The algorithm for Π_M is deterministic and is mainly concerned with the simulation of an arbitrary transition rule. Without loss of generality we insist that M always finishes its computation with the tape head at the leftmost end of the tape contents. Let M be any single tape Turing machine with symbols $\alpha_1, \alpha_2, \ldots, \alpha_{|A|}$ and states $q_1, q_2, \ldots q_{|Q|}$, blank symbol α_1 , and halt state $q_{|Q|}$.

5.1 Encoding a configuration of Turing machine M

Each configuration of M is encoded as three natural numbers using a well known technique. A configuration of M is given by the following equation

$$C_k = \mathbf{q_r}, \cdots \alpha_1 \alpha_1 \alpha_1 a_{-x} \cdots a_{-3} a_{-2} a_{-1} a_0 a_1 a_2 a_3 \cdots a_y \alpha_1 \alpha_1 \alpha_1 \cdots$$
 (1)

where q_r is the current state, each a_i is a tape cell of M and the tape head of M, given by an underline, is over a_0 . Also, tape cells a_{-x} and a_y both contain α_1 , and the cells between a_{-x} and a_y include all of the cells on M's tape that have either been visited by the tape head prior to configuration C_k or contain part of the input to M.

In the sequel the encoding of object p is given by $\langle p \rangle$. The tape symbols $\alpha_1, \alpha_2, \ldots, \alpha_{|A|}$ of M are encoded as $\langle \alpha_1 \rangle = 1, \langle \alpha_2 \rangle = 3, \ldots, \langle \alpha_{|A|} \rangle = 2|A|-1$, respectively, and the states $q_1, q_2, \ldots, q_{|Q|}$ are encoded as $\langle q_1 \rangle = 2A, \langle q_2 \rangle = 4A, \ldots, \langle q_{|Q|} \rangle = 2|Q|A$, respectively. The contents of each tape cell a_i in configuration C_k is encoded as $\langle a_i \rangle = \langle \alpha \rangle$ where α is a tape symbol of M. The tape contents in Equation (1) to the left and right of the tape head are respectively encoded as the numbers $X = \sum_{i=1}^x z^i \langle a_{-i} \rangle$ and $Y = \sum_{j=1}^y z^j \langle a_j \rangle$ where $z = 2^v$ and $v = \lceil \log_2(2|Q||A|+2|A|) \rceil$. Thus the entire configuration C_k is encoded as three natural numbers via the equation

$$\langle C_k \rangle = (X, Y, \langle q_r \rangle + \langle \alpha_i \rangle)$$
 (2)

where $\langle C_k \rangle$ is the encoding of C_k from Equation (1) and α_i is the symbol being read by the tape head in cell a_0 .

A transition rule $q_r, \alpha_i, \alpha_j, D, q_u$ of M is executed on C_k as follows. If the current state is q_r and the tape head is reading the symbol α_i in cell a_0, α_j the write symbol is printed to cell a_0 , the tape head moves one cell to the left to a_{-1} if D = L or one cell to the right to a_1 if D = R, and q_u becomes the new current state. A simulation of transition rule $q_r, \alpha_i, \alpha_j, D, q_u$ on the encoded configuration $\langle C_k \rangle$ from Equation (2) is given by the equation

$$\langle C_{k+1} \rangle = \begin{cases} \left(\frac{X}{z} - (\frac{X}{z} \mod z), \ zY + z \langle \alpha_j \rangle, \ \langle q_u \rangle + (\frac{X}{z} \mod z) \right) \\ \left(zX + z \langle \alpha_j \rangle, \ \frac{Y}{z} - (\frac{Y}{z} \mod z), \ \langle q_u \rangle + (\frac{Y}{z} \mod z) \right) \end{cases}$$
(3)

where configuration C_{k+1} results from executing a single transition rule on configuration C_k , and $(b \mod c) = d$ where d < c, b = ec + d and $b, c, d, e \in \mathbb{N}$. In Equation (3) the top case is simulating a left move transition rule and the bottom case is simulating a right move transition rule. In the top case, following the left move, the sequence to the right of the tape head is longer by 1 tape cell, as cell a_0 is added to the right sequence. Cell a_0 is overwritten with the write symbol α_j and thus we compute $zY + z\langle\alpha_j\rangle$ to simulate cell a_0 becoming part of the right sequence. Also, in the top case the sequence to the left of the tape head is getting shorter by 1 tape cell thus we compute $\frac{X}{z} - (\frac{X}{z} \mod z)$. The rightmost cell of the

left sequence a_{-1} is the new tape head location and the tape symbol it contains is encoded as $(\frac{X}{z} \mod z)$. Thus the value $(\frac{X}{z} \mod z)$ is added to the new encoded current state $\langle q_u \rangle$. For the bottom case, a right move, the sequence to the right gets shorter which is simulated by $\frac{Y}{z} - (\frac{Y}{z} \mod z)$ and the sequence to the left gets longer which is simulated by $zX + z\langle \alpha_j \rangle$. The leftmost cell of the right sequence a_1 is the new tape head location and the tape symbol it contains is encoded as $(\frac{Y}{z} \mod z)$.

5.2 Input to Π_M

Here we give an explanation of how the input is read into Π_M . We also give a rough outline of how the input to Π_M is encoded in linear time.

A configuration C_k given by Equation (2) is read into Π_M as follows. All the neurons of the system initially have no spikes with the exception of σ_{10} which has 31 spikes. The input neuron σ_5 receives X+2 spikes at the first timestep t_1 , Y spikes at time t_2 , and $\langle q_r \rangle + \langle \alpha_i \rangle$ spikes at time t_4 . We explain how the system is initialised to encode an initial configuration of M by giving the number of spikes in each neuron and the rule that is to be applied in each neuron at time t_1 we have

$$t_1: \sigma_5 = X + 2,$$
 $s^2(s^z)^*/s \to s; 1,$ $\sigma_{10} = 31,$ $s^{31}/s^{16} \to \lambda; 0.$

where on the left $\sigma_j = k$ gives the number k of spikes in neuron σ_j at time t_i and on the right is the next rule that is to be applied at time t_i if there is an applicable rule at that time. Thus from Figure 3 when we apply the rule $s^2(s^z)^*/s \to s$; 1 in neuron σ_5 and the rule $s^{31}/s^{16} \to \lambda$; 0 in neuron σ_{10} at time t_1 we get

$$t_{2}: \sigma_{4} = X + 2, \qquad s^{2}(s^{z})^{*}/s^{z} \to s^{z}; 2,$$

$$\sigma_{5} = Y, \qquad s^{2z}(s^{z})^{*}/s \to s; 1,$$

$$\sigma_{6}, \sigma_{7}, \sigma_{8}, \sigma_{9} = X + 2, \qquad s^{15}/s^{8} \to \lambda; 0.$$

$$t_{3}: \sigma_{4} = X + 2, \qquad s^{2}(s^{z})^{*}/s^{z} \to s^{z}; 1,$$

$$\sigma_{6} = Y, \qquad (s^{z})^{*}/s \to s; 1,$$

$$\sigma_{7}, \sigma_{8}, \sigma_{9} = Y, \qquad (s^{z})^{*}/s \to \lambda; 0,$$

$$\sigma_{10} = 7, \qquad s^{7}/s^{4} \to \lambda; 0.$$

$$t_{4}: \sigma_{1} = X, \qquad s^{2}/s^{2} \to \lambda; 0,$$

$$\sigma_{5} = \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad (s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \to s; 1,$$

$$\sigma_{10} = 3, \qquad s^{3}/s^{2} \to \lambda; 0.$$

$$t_5: \sigma_1 = X,$$

$$\sigma_2 = Y,$$

$$\sigma_4, \sigma_6 = \langle q_r \rangle + \langle \alpha_i \rangle,$$

$$\sigma_7, \sigma_8, \sigma_9 = \langle q_r \rangle + \langle \alpha_i \rangle,$$

$$\sigma_{10} = 1,$$

$$s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to \lambda; 0,$$

$$s/s \to s; \log_2(z) + 3.$$

Forgetting rules are applied to get rid of superfluous spikes (for example see neurons σ_7, σ_8 , and σ_9 at time t_2). Note that σ_4 is closed at time t_2 as there is a delay of 2 on the rule $(s^2(s^z)^*/s^z \to s^z; 2)$ to be executed in σ_4 . This prevents the Y spikes from entering neuron σ_4 when σ_5 fires at time t_2 . At time t_5 the spiking neural P system has X spikes in σ_1 , Y spikes in σ_2 , and $\langle q_r \rangle + \langle \alpha_i \rangle$ spikes in σ_4 and σ_6 . Thus at time t_5 the spiking neural P system encodes an initial configuration of M.

In this paragraph we will show that given an initial configuration of M it is encoded as input to our spiking neural P system in Figure 3 in linear time. In order to do this we must compute the three numbers that give $\langle C_k \rangle$ from Equation 2 in linear time. The number X is computed as follows: given a sequence $a_{-x}a_{-x+1}\dots a_{-2}a_{-1}$ the sequence w= $\langle a_{-x} \rangle 0^{\log_2(z)-1} \langle a_{-x+1} \rangle 0^{\log_2(z)-1} \dots \langle a_{-2} \rangle 0^{\log_2(z)-1} \langle a_{-1} \rangle 0^{\log_2(z)-1} 2$ is easily computed in time that is linear in x. The spiking neural P system Π_{input} in Figure 2 takes the sequence w and converts it into the X spikes that form part of the input to our system in Figure 3. We give a rough idea of how Π_{input} operates (if the reader wishes to pursue a more detailed view the rules for Π_{input} are to be found in Table 4). The input neuron of Π_{input} receives the sequence w as a sequence of spikes and no-spikes. On each timestep where $\langle a \rangle$ is read $\langle a \rangle$ spikes are passed to the input neuron σ_1 , and on each timestep where 0 is read no spikes are passed to the input neuron. Thus at timestep t_1 neuron σ_1 receives $\langle a_{-x} \rangle$ spikes, and at timestep t_2 neurons σ_2 , σ_3 , and σ_4 receive $\langle a_{-x} \rangle$ spikes from σ_1 . Following timestep t_2 , the number of spikes in neurons σ_2 , σ_3 , and σ_4 double with each timestep. So at timestep $t_{\log_2(z)+1}$ the number of spikes in each of the neurons σ_2 , σ_3 , and σ_4 is $\frac{z}{2}\langle a_{-x}\rangle$. At timestep $t_{\log_2(z)+1}$ neurons σ_2 , σ_3 and σ_4 also receive $\langle a_{-x+1} \rangle$ spikes from σ_1 giving a total of $z \langle a_{-x} \rangle + \langle a_{-x+1} \rangle$ spikes in each of these neurons at time $t_{\log_2(z)+2}$. Proceeding to time $t_{2\log_2(z)+2}$ neurons σ_2 , σ_3 and σ_4 have $z^2\langle a_{-x}\rangle+z\langle a_{-x+1}\rangle+\langle a_{-x+2}\rangle$ spikes. This process continues until $X=\sum_{i=1}^x z^i\langle a_{-i}\rangle$ is computed. The end of the process is signaled when the rightmost number in the sequence is read. When this number (2) is read it allows the result to be passed to σ_6 via σ_5 . Following this σ_6 sends X spikes out of the system. Note that prior to this 2 being read only forgetting rules are executed in σ_6 thus preventing any spikes from being sent out of the system. Π_{input} computes X in time $x \log_2(z) + 3$. Recall from Section 5.1 that the value of z is dependent on the number of states and symbols in M thus X is computed in time that is linear in x. In a similar manner, the value Y is computed by Π_{input} in time linear in y. The number $\langle q_r \rangle + \langle \alpha_i \rangle$ is computed in constant time. Thus the input $\langle C_k \rangle$ for Π_M is computed in linear time.

5.3 Algorithm overview

To help simplify the explanation, some of the rules given here differ slightly from those in the more detailed simulation that follows this overview. The numbers from Equation (2), encoding a Turing machine configuration, are stored in the neurons of our system as X, Y

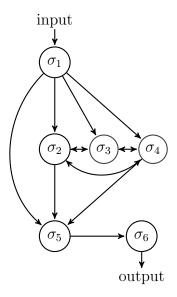


Fig. 2. Spiking neural P system Π_{input} . Each circle is a neuron and each arrow represents the direction spikes move along a synapse between a pair of neurons. The rules for Π_{input} are to be found in Table 4.

and $\langle q_r \rangle + \langle \alpha_i \rangle$ spikes. Equation (3) is implemented in Figure 3 to give a spiking neural P system Π_M that simulates the transition rules of M. The two values X and Y are stored in neurons σ_1 and σ_2 , respectively. If X or Y is to be multiplied the spikes that encode X or Y are sent down through the network of neurons from either σ_1 or σ_2 respectively, until they reach σ_{10} . Note in Figure 3 that each neuron from σ_7, σ_8 and σ_9 has incoming synapses coming from the other two neurons in σ_7, σ_8 and σ_9 . Thus if σ_7, σ_8 and σ_9 each contain N spikes at time t_k , and they each fire sending N spikes, then each of the neurons σ_7, σ_8 and σ_9 will contain 2N spikes at time t_{k+1} . Given Y the value $zY = 2^vY$ is computed as follows: First we calculate 2Y by firing σ_7, σ_8 and σ_9 , then 4Y by firing σ_7, σ_8 , and σ_9 again. After v timesteps the value zY is computed. zX is computed using the same technique.

Now, we give the general idea of how the neurons compute $\frac{X}{z} - (\frac{X}{z} \mod z)$ and $(\frac{X}{z} \mod z)$ from Equation (3) (a slightly different strategy is used in the simulation). We begin with X spikes in σ_1 . The rule $(s^z)^*/s^z \to s; 1$ is applied in σ_1 sending $\frac{X}{z}$ spikes to σ_4 . Following this $(s^z)^*s^{(\frac{X}{z} \mod z)}/s^z \to s^z; 1$ is applied in σ_4 which sends $\frac{X}{z} - (\frac{X}{z} \mod z)$ to σ_1 leaving $(\frac{X}{z} \mod z)$ spikes in σ_4 . The values $\frac{Y}{z} - (\frac{Y}{z} \mod z)$ and $(\frac{Y}{z} \mod z)$ are computed in a similar manner.

Finally, using the encoded current state $\langle q_r \rangle$ and the encoded read symbol $\langle \alpha_i \rangle$ the values $z \langle \alpha_j \rangle$ and $\langle q_u \rangle$ from Equation (3) are computed. Using the technique outlined in the first paragraph of the algorithm overview the value $z(\langle q_r \rangle + \langle \alpha_i \rangle)$ is computed by sending $\langle q_r \rangle + \langle \alpha_i \rangle$ spikes from σ_5 to σ_{10} in Figure 3. Then the rule $s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)}/s^{z(\langle q_r \rangle + \langle \alpha_i \rangle) - \langle q_u \rangle} \rightarrow s^{z\langle \alpha_j \rangle}$; 1 is applied in σ_{10} which sends $z \langle \alpha_j \rangle$ spikes out to neurons σ_4 and σ_6 . This rule uses $z(\langle q_r \rangle + \langle \alpha_i \rangle) - \langle q_u \rangle$ spikes thus leaving $\langle q_u \rangle$ spikes remaining in σ_{10} . This completes our sketch of how Π_M in Figure 3 computes the values in Equation (3) to simulate a transition rule. A more detailed simulation of a transition rule follows.

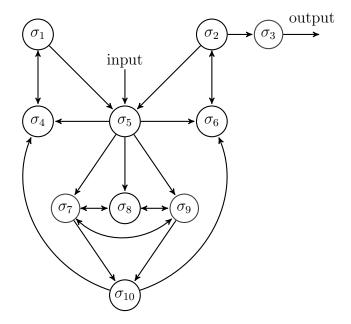


Fig. 3. Universal spiking neural P system Π_M . Each circle is a neuron and each arrow represents the direction spikes move along a synapse between a pair of neurons. The rules for Π_M are to be found in Tables 2 and 3.

5.4 Simulation of $q_r, \alpha_i, \alpha_j, L, q_u$ (top case of Equation (3))

The simulation of the transition rule begins at time t_k with X spikes in σ_1 , Y spikes in σ_2 , $\langle q_r \rangle + \langle \alpha_i \rangle$ spikes in σ_4 and σ_6 , and 1 spike in σ_{10} . As before we explain the simulation by giving the number of spikes in each neuron and the rule that is to be applied in each neuron at time t. So at time t_k we have

$$\begin{split} t_k: \sigma_1 &= X, \\ \sigma_2 &= Y, \\ \sigma_4, \sigma_6 &= \langle q_r \rangle + \langle \alpha_i \rangle, \\ \sigma_{10} &= 1, \end{split} \qquad \begin{aligned} s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to s; 1, \\ s / s \to s; \log_2(z) + 3. \end{aligned}$$

Thus from Figure 3 when we apply the rule $s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to s; 1$ in neurons σ_4 and σ_6 at time t_k we get

$$t_{k+1}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; \log_2(z) + 6,$$

$$\sigma_2 = Y + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad (s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \to s; \log_2(z) + 2.$$

$$t_{k+2}: \sigma_{1} = X + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{2z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s^{z} \rightarrow s; \log_{2}(z) + 5,$$

$$\sigma_{3} = Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad (s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s^{z} \rightarrow s^{z}; 1,$$

$$\text{if } \langle q_{r} \rangle = \langle q_{|Q|} \rangle \qquad (s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \rightarrow \lambda; 0,$$

$$\sigma_{5} = Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad (s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \rightarrow \lambda; 0,$$

$$\sigma_{6} = Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \rightarrow \lambda; 0,$$

$$\sigma_{10} = 1, \qquad s^{2z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s^{z} \rightarrow s; \log_{2}(z) + 4,$$

$$\sigma_{4}, \sigma_{6} = Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \rightarrow \lambda; 0,$$

$$\sigma_{7}, \sigma_{8}, \sigma_{9} = Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s \rightarrow s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \rightarrow s; \log_{2}(z).$$

In timestep t_{k+2} above σ_3 the output neuron fires if and only if the encoded current state encodes the halt state $q_{|Q|}$. Recall that when M halts the entire tape contents are to the right of the tape head, thus only Y the encoding of the right sequence is sent out of the system. Thus the unary output is a number of spikes that encodes the tape contents of M.

Note that at timestep t_{k+3} the neuron σ_7 receives $Y + \langle q_r \rangle + \langle \alpha_i \rangle$ spikes from each of the two neurons σ_8 and σ_9 . Thus at time t_{k+4} neuron σ_7 contains $2(Y + \langle q_r \rangle + \langle \alpha_i \rangle)$ spikes. In a similar manner σ_8 and σ_9 also receive $2(Y + \langle q_r \rangle + \langle \alpha_i \rangle)$ spikes at timestep t_{k+3} . The number of spikes in each of the neurons σ_7 , σ_8 and σ_9 doubles at each timestep between t_{k+3} and $t_{k+\log_2(z)+2}$.

$$t_{k+4}: \sigma_{1} = X + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{2z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s^{z} \rightarrow s; \log_{2}(z) + 3,$$

$$\sigma_{7}, \sigma_{8}, \sigma_{9} = 2(Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle), \qquad s^{z}(s^{z})^{*}s^{2(\langle q_{r} \rangle + \langle \alpha_{i} \rangle)}/s \rightarrow s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \rightarrow s; \log_{2}(z) - 1.$$

$$t_{k+5}: \sigma_{1} = X + \langle q_{r} \rangle + \langle \alpha_{i} \rangle, \qquad s^{2z}(s^{z})^{*}s^{\langle q_{r} \rangle + \langle \alpha_{i} \rangle}/s^{z} \rightarrow s; \log_{2}(z) + 2,$$

$$\sigma_{7}, \sigma_{8}, \sigma_{9} = 4(Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle), \qquad s^{z}(s^{z})^{*}s^{4(\langle q_{r} \rangle + \langle \alpha_{i} \rangle)}/s \rightarrow s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \rightarrow s; \log_{2}(z) + 1,$$

$$\sigma_{7}, \sigma_{8}, \sigma_{9} = 8(Y + \langle q_{r} \rangle + \langle \alpha_{i} \rangle), \qquad s^{z}(s^{z})^{*}s^{8(\langle q_{r} \rangle + \langle \alpha_{i} \rangle)}/s \rightarrow s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \rightarrow s; \log_{2}(z) - 3.$$

The number of spikes in neurons σ_7 , σ_8 , and σ_9 continues to double until timestep $t_{k+\log_2(z)+2}$. When neurons σ_7 and σ_9 fire at timestep $t_{k+\log_2(z)+2}$ they send $\frac{z}{2}(Y+\langle q_r\rangle+\langle \alpha_i\rangle)$ spikes each to neuron σ_{10} which has opened at time $t_{k+\log_2(z)+2}$ (for the first time in the transition rule

simulation). Thus at time $t_{k+\log_2(z)+3}$ neuron σ_{10} contains $z(Y+\langle q_r\rangle+\langle \alpha_i\rangle)$ spikes.

$$t_{k+\log_2(z)+2}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; 5,$$

$$\sigma_7, \sigma_8, \sigma_9 = \frac{z}{2} (Y + \langle q_r \rangle + \langle \alpha_i \rangle), \qquad s^z(s^z)^* s^{\frac{z}{2} (\langle q_r \rangle + \langle \alpha_i \rangle)} / s \to s; 1,$$

$$\sigma_{10} = 1, \qquad s/s \to s; 1.$$

$$t_{k+\log_2(z)+3}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; 4,$$

$$\sigma_4, \sigma_6 = 1, \qquad s/s \to \lambda; 0,$$

$$\sigma_7, \sigma_8, \sigma_9 = z(Y + \langle q_r \rangle + \langle \alpha_i \rangle), \qquad (s^z)^* / s \to \lambda; 0,$$

$$\sigma_{10} = z(Y + \langle q_r \rangle + \langle \alpha_i \rangle), \qquad (s^z)^* s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)} / s^{z^z} \to s^z^z; 1.$$

Note that $(zY \mod z^2) = 0$ and also that $z(\langle q_r \rangle + \langle \alpha_i \rangle) < z^2$. Thus in neuron σ_{10} at time $t_{k+\log_2(z)+3}$ the rule $(s^{z^2})^* s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)} / s^{z^2} \to s^{z^2}$; 1 separates the encoding of the right side of the tape s^{zY} and the encoding of the current state and read symbol $s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)}$. To see this note the number of spikes in neurons σ_6 and σ_{10} at time $t_{k+\log_2(z)+4}$.

The rule $s^{z(\langle q_r\rangle+\langle \alpha_i\rangle)}/s^{z(\langle q_r\rangle+\langle \alpha_i\rangle)-\langle q_u\rangle-1}\to s^{z\langle \alpha_j\rangle};1$, applied in σ_{10} at timestep $t_{k+\log_2(z)+4},$ computes the new encoded current state $\langle q_u\rangle$ and the encoded write symbol $z\langle \alpha_j\rangle$. To see this note the number of spikes in neurons σ_6 and σ_{10} at time $t_{k+\log_2(z)+5}$. Note that neuron σ_1 is preparing to execute the rule $s^{2z}(s^z)^*s^{\langle q_r\rangle+\langle \alpha_i\rangle}/s^z\to s;1$ at timestep $t_{k+\log_2(z)+6},$ and so at timesteps $t_{k+\log_2(z)+4}$ and $t_{k+\log_2(z)+5}$ neuron σ_1 remains closed. Thus the spikes sent out from σ_4 at these times do not enter σ_1 .

$$t_{k+\log_2(z)+4}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; 3,$$

$$\sigma_4, \sigma_6 = zY, \qquad (s^z)^* / s \to s; 1,$$

$$\sigma_{10} = z(\langle q_r \rangle + \langle \alpha_i \rangle), \qquad s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)} / s^{z(\langle q_r \rangle + \langle \alpha_i \rangle) - \langle q_u \rangle - 1} \to s^{z\langle \alpha_j \rangle}; 1.$$

$$t_{k+\log_2(z)+5}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; 2,$$

$$\sigma_2 = zY, \qquad (s^z)^* / s \to s; 1,$$

$$\sigma_1 = \langle q_u \rangle + 1, \qquad s^{\langle q_u \rangle + 1} / s^{\langle q_u \rangle} \to s^{\langle q_u \rangle}; 4.$$

$$t_{k+\log_2(z)+6}: \sigma_1 = X + \langle q_r \rangle + \langle \alpha_i \rangle, \qquad s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; 1,$$

$$\sigma_2 = zY + z\langle \alpha_j \rangle, \qquad s^{\langle q_u \rangle + 1} / s^{\langle q_u \rangle} \to s^{\langle q_u \rangle}; 3.$$

At time $t_{k+\log_2(z)+7}$ in neuron σ_4 the rule $s^z(s^z)^*s^{(\frac{X}{z}\mod z)}/s^z \to s^z; 1$ is applied sending $\frac{X}{z} - (\frac{X}{z}\mod z)$ spikes to σ_1 and leaving $(\frac{X}{z}\mod z)$ spikes in σ_4 . At the same time in neuron

 σ_5 the rule $s^z(s^z)^*s^{(\frac{X}{z} \mod z)}/s^z \to \lambda; 0$ is applied leaving only $(\frac{X}{z} \mod z)$ spikes in σ_5 .

$$\begin{split} t_{k+\log_2(z)+7} &: \sigma_1 = \langle q_r \rangle + \langle \alpha_i \rangle, & s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to \lambda; 0, \\ \sigma_2 &= zY + z \langle \alpha_j \rangle, \\ \sigma_4 &= \frac{X}{z}, & s^z (s^z)^* s^{(\frac{X}{z} \mod z)} / s^z \to s^z; 1, \\ \sigma_5 &= \frac{X}{z}, & s^z (s^z)^* s^{(\frac{X}{z} \mod z)} / s^z \to \lambda; 0, \\ \sigma_{10} &= \langle q_u \rangle + 1, & s^{\langle q_u \rangle + 1} / s^{\langle q_u \rangle} \to s^{\langle q_u \rangle}; 2. \end{split}$$

$$t_{k+\log_2(z)+8} : \sigma_1 = \frac{X}{z} - (\frac{X}{z} \mod z),$$

$$\sigma_2 = zY + z\langle \alpha_j \rangle,$$

$$\sigma_4 = \frac{X}{z} \mod z,$$

$$\sigma_5 = \frac{X}{z} \mod z,$$

$$\sigma_{10} = \langle q_u \rangle + 1,$$

$$s^{(\frac{X}{z} \mod z)} / s \to s; 1,$$

$$\begin{split} t_{k+\log_2(z)+9} : \sigma_1 &= \frac{X}{z} - (\frac{X}{z} \mod z), \\ \sigma_2 &= zY + z \langle \alpha_j \rangle, \\ \sigma_4 &= \langle q_u \rangle + (\frac{X}{z} \mod z), \qquad \qquad s^{\langle q_u \rangle + (\frac{X}{z} \mod z) / s \to s; 1,} \\ \sigma_6 &= \langle q_u \rangle + (\frac{X}{z} \mod z), \qquad \qquad s^{\langle q_u \rangle + (\frac{X}{z} \mod z) / s \to s; 1,} \\ \sigma_7, \sigma_8, \sigma_9 &= \frac{X}{z} \mod z, \qquad \qquad s^{(\frac{X}{z} \mod z) / s (\frac{X}{z} \mod z) \to \lambda; 0,} \\ \sigma_{10} &= 1, \qquad \qquad s/s \to s; \log_2(z) + 3. \end{split}$$

The simulation of the left moving transition rule is now complete. Note that the number of spikes in σ_1 , σ_2 , σ_4 , and σ_6 at timestep $t_{k+\log_2(z)+9}$ are the values given by the top case of Equation (3) and encode the configuration after the left move transition rule.

The case of when the tape head moves onto a part of the tape that is to the left of a_{-x+1} in Equation (1) is not covered by the simulation. For example when the tape head is over cell a_{-x+1} , then X=z (recall a_{-x} contains α_1). If the tape head moves to the left then from the top case of Equation (3) the new value for the left sequence is X=0. Therefore we increase the length of X to simulate the infinite blank symbols (α_1 symbols) to the left as follows. The rule $s^{z+\langle q_r\rangle+\langle \alpha_i\rangle}/s^z\to s^z;1$ is applied in σ_1 at time $t_{k+\log_2(z)+6}$. Then at time $t_{k+\log_2(z)+7}$ the rule $(s^z)^*/s\to s;1$ is applied in σ_4 and the rule $s^z/s^{z-1}\to \lambda;0$ is applied in σ_5 . Thus at time $t_{k+\log_2(z)+8}$ there are z spikes in σ_1 which simulates another α_1 symbol to the left, and there is 1 spike in σ_5 to simulate the current read symbol α_1 .

We have shown how to simulate an arbitrary left moving transition rule of M. Right moving transition rules are also simulated in $\log_2(z)+9$ timesteps in a manner similar to that of left moving transition rules. Thus a single transition rule of M is simulated by Π_M in $\log_2(z)+9$ timesteps. Recall from Section 5.1 $z=2^{\log_2\lceil 2|Q||A|+2|A|\rceil}$ thus the entire computation of M

is simulated in O(|A||Q|T) time. From Section 5.1 M is simulated in $O([2^{\log_2\lceil 2|Q||A|+2|A|\rceil}]^T)$ space.

While the small universal spiking neural P system in Figure 3 simulates Turing machines with a linear time overhead it *requires* an exponential space overhead. This *requirement* may be shown by proving it is simulated by a counter machine using the same space. However, it is not unreasonable to expect efficiency from simple universal systems as many of the simplest computationally universal models have polynomial time and space overheads [13,14,17].

It was mentioned in Section 2 that we generalised the previous definition of spiking neural P systems with exhaustive use of rules to allow the input neuron to receive an arbitrary number of spikes in a single timestep. If the synapses of the system can transmit an arbitrary number of spikes in a single timestep, then it does not seem unreasonable to allow an arbitrary number of spikes to enter the input neuron in a single timestep. If the input is restricted to a constant number of spikes, as is the case with earlier spiking neural P systems, then the system will remain exponentially slow due to the time required to read the unary input into the system.

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neuron	rules		
σ_1	$(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to s; 1 \text{ if D=R}$		
	$s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; \log_2(z) + 6 \text{ if D=L}$		
	$s^{z+\langle q_r\rangle+\langle \alpha_i\rangle}/s^z \to s^z; \log_2(z)+6 \text{ if D=L}$		
	$s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to \lambda; 0 \text{ if D=L}$ $(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to s; 1 \text{ if D=L or } \langle q_r \rangle = \langle q_{ Q } \rangle$		
σ_2	$(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to s; 1 \text{ if D=L or } \langle q_r \rangle = \langle q_{ Q } \rangle$		
	$s^{2z}(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s; \log_2(z) + 6 \text{ if D=R}$		
	$s^{z+\langle q_r \rangle + \langle \alpha_i \rangle}/s^z \to s^z; \log_2(z) + 6 \text{ if D=R}$		
	$s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to \lambda; 0 \text{ if D=R}$ $(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s^z \to s^z; 1, \text{ if } \langle q_r \rangle = \langle q_{ Q } \rangle$		
σ_3	$(s)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s^z \to s^z; 1, \text{ if } \langle q_r \rangle = \langle q_{ Q } \rangle$		
	$(s^z)^* s^{\langle q_r \rangle + \langle \alpha_i \rangle} / s \to \lambda; 0, \text{ if } \langle q_r \rangle \neq \langle q_{ Q } \rangle$		
	$(s^z)^* s^{(\frac{X}{z} \mod z)} / s \to \lambda; 0$		
	$s^z/s \to \lambda; 0$		
σ_4	$s^2(s^z)^*/s^z \to s^z; 2$		
	$(s^z)^*/s \to s; 1$		
	$s^2/s^2 \to \lambda; 0$		
	$s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to s; 1$		
	$s^{z}(s^{z})^{*}s^{\langle q_{r}\rangle+\langle \alpha_{i}\rangle}/s \to \lambda;0$		
	$s/s \to \lambda; 0$		
	$s^{z}(s^{z})^{*}s^{(\frac{X}{z} \mod z)}/s^{z} \to s^{z}; 1$		
	$s^{\left(\frac{X}{z} \mod z\right)}/s \to \lambda; 0$		
σ_5	$s^2(s^z)^*/s \rightarrow s; 1$		
	$s^{2z}(s^z)^*/s \to s; 1$ $(s^z)^*s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to s; 1$		
	$(s) s^{(x)} + (s^{(z)}) + (s$		
	$s(s) s^{2} \Rightarrow \lambda; 0$ $s^{z}/s^{z-1} \Rightarrow \lambda; 0$		
	$s^{\left(\frac{X}{z} \mod z\right)}/s \to s; 1$		
σ_6	$s^{2}(s^{z})^{*}/s \rightarrow \lambda; 0$		
06	$(s^z)^*/s \to s; 1$		
	$s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to s; 1$		
	$s^{z}(s^{z})^{*}s^{\langle q_{r}\rangle+\langle \alpha_{i}\rangle}/s \to \lambda;0$		
	$s/s \rightarrow \lambda; 0$		
	$s^{z}(s^{z})^{*}s^{(\frac{Y}{z} \mod z)}/s^{z} \to s^{z}; 1$		
	$s^{(\frac{Y}{z} \mod z)}/s \to \lambda; 0$		

Table 2. This table gives the rules in each of the neurons σ_1 to σ_6 of Π_M . In the rules above q_r is the current state, α_i is the read symbol, α_j is the write symbol, D is the move direction, and q_u is the next state of some transition rule $q_r, \alpha_i, \alpha_j, D, q_u$ of M. Note that $(\frac{X}{z} \mod z)), (\frac{Y}{z} \mod z)) \in \langle A \rangle$ the set of encodings for the symbols of M (see Section 5.1).

neuron	rules	
$\sigma_7, \sigma_8, \sigma_9$	$s^{2}(s^{z})^{*}/s \to \lambda; 0$ $(s^{z})^{*}/s \to \lambda; 0$ $s^{\langle q_{r}\rangle + \langle \alpha_{i}\rangle}/s \to \lambda; 0$	
	$(s^z)^*/s \to \lambda; 0$	
	$s^{\langle q_r \rangle + \langle \alpha_i \rangle}/s \to \lambda; 0$	
	$s^z(s^z)^*s^{\frac{z}{m}(\langle q_r \rangle + \langle \alpha_i \rangle)}/s \to s; 1$ for all $m = 2^k, \ 2 \leqslant m \leqslant z$ and $k \in \mathbb{N}$	
	$s^{(\frac{X}{z} \mod z)}/s^{(\frac{X}{z} \mod z)} \to \lambda; 0$	
σ_{10}	$s^{31}/s^{16} \to \lambda; 0$	
	$s^{15}/s^8 o \lambda; 0$	
	$s^7/s^4 \to \lambda; 0$	
	$s^3/s^2 \rightarrow \lambda$: 0	
	$s/s \rightarrow s; \log_2(z) + 3$	
	$(s^{z^2})^* s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)} / s^{z^2} \to s^{z^2}; 1$	
	$s^{z(\langle q_r \rangle + \langle \alpha_i \rangle)} / s^{z(\langle q_r \rangle + \langle \alpha_i \rangle) - \langle q_u \rangle - 1} \to s^{z(\langle \alpha_j \rangle)}; 1$	
	$s^{\langle q_u \rangle + 1}/s^{\langle q_u \rangle} o s^{\langle q_u \rangle}; 4$	

Table 3. This table gives the rules in each of the neurons σ_7 to σ_{10} of Π_M . See Table 2 for some further explanation.

neuron	rules
σ_1	$s^*/s \to s; 1$
$\sigma_2, \sigma_3, \sigma_4$	$s^*/s \to s; 1$
σ_5	$(s^z)^* s^{\langle \alpha \rangle} / s \to s; \log_2(z)$
	$(s^z)^*s^2/s \to s; 1$
σ_6	$(s^z)^* s^{\langle a \rangle} / s \to \lambda; 0$
	$(s^z)^* s^2 / s^z \to s^z; 1$

Table 4. This table gives the rules in each of the neurons of Π_{input} .