Discretization of continuous-time quantum walks via the staggered model with Hamiltonians

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May 24, 2018

Abstract

We characterize a close connection between the continuous-time quantum-walk model and a discrete-time quantum-walk version, based on the staggered model with Hamiltonians in a class of Cayley graphs, which can be considered as a discretization of continuous-time quantum walks. This connection provides examples of perfect state transfer and instantaneous uniform mixing in the staggered model. On the other hand, we provide some more examples of perfect state transfer and instantaneous uniform mixing in the staggered model that cannot be reproduced by the continuous-time model.

^{*}Acknowledges the support of grants FAPESP 15/16339-2 and FAPESP 03447-6.

[†]Acknowledges the support of CNPq grant 474143/2013-9.

1 Introduction

Quantization versions of classical random walks have been actively studied for the past two decades after the seminal papers [2, 10, 24]. In this work we focus on extending some properties of the recent-proposed model of staggered quantum walks [22, 20, 21]. This model is endowed with many interesting features, remarkably, it provides a generalization of the Szegedy's quantum walk model [24] and exactly simulates the instances of flip-flop coined quantum walks that employ the Hadamard or Grover coins [19]. An implementation of this model in the class of triangle-free graphs has also been recently proposed [17].

Connections between continuous-time quantum walks and discrete-time coined quantum walks have been extensively analyzed [23, 8, 16, 18] without providing a discretization procedure. In this paper, we show that in a special class of graphs and under certain conditions, the staggered quantum walk model is able to provide a natural discretization of a continuous-time quantum walk when the adjacency matrix can be written as a sum of commuting matrices related to a graph factorization.

Perfect state transfer has been demonstrated in spin chains [5, 14] and can be mathematically modeled using the continuous-time quantum walk on graphs [15, 9]. In this paper, we show that there is a large class of graphs on which the staggered quantum walk allows perfect state transfer. Part of this class is obtained by discretizing the examples from the continuous-time quantum walk. The remaining part shows that the staggered model allows for certain interesting features in terms of perfect state transfer that the continuous case does not possess. We then proceed to show that related phenomena, such as uniform mixing and periodicity, are also present.

The remainder of the paper is organized as follows. In section 2, we review the staggered quantum walk model. In section 3, we present a discretization of continuous-time quantum walks. In section 4, we describe a especial kind of tessellation on Cayley graphs. In section 5, we provide a large class of examples of staggered quantum walks on Cayley graphs of abelian groups that admits perfect state transfer. We also discuss staggered quantum walks on Cayley graphs of non-abelian groups. In section 6, we provide a large class of examples of staggered quantum walks on Cayley graphs that admits instantaneous uniform mixing. In section 7, we draw our conclusions.

2 Staggered quantum walks with Hamiltonians

We introduce some of the nomenclature needed along the text. We model a quantum walk on a finite graph by associating particle positions with vertices and hopping directions with edges. In this fashion, the quantum system is a graph, typically denoted as X = (V, E), where V is the set of vertices and $E \subseteq \binom{V}{2}$ is the set of edges.

A tessellation is a partition of V into tiles $C_1, ..., C_k$ such that if any two vertices a and b belong to a tile, then $\{a, b\} \in E$. Given a tessellation, a unitary operator H is defined by attaching to each tile C_i a unitary vector u_i such that the support of u_i is precisely the set of vertices in C_i , and then making

$$H = 2\sum_{i=1}^{k} |u_i\rangle\langle u_i| - I.$$

This way, H is forcibly unitary and Hermitian, that is, a reflection operator.

The staggered quantum walk model proposed in [22] consists in defining (at least) two tessellations, say T_1 and T_2 , such that each edge of the graph belongs to at least one of them, along with the unitary vectors associated to each. The evolution operator is therefore given by

$$U=H_1H_2.$$

A variation of this model more suitable for practical implementation was introduced in [21]. Again, provided tessellations with their corresponding unitary matrices, and two parameters θ_1 and θ_2 , the evolution operator is given by

$$U = \exp(i\theta_1 H_1) \exp(i\theta_2 H_2). \tag{1}$$

The staggered model with Hamiltonians takes advantage of the dual nature of reflection operators; they can either be used as propagators or Hamiltonians. Henceforth in this paper, we consider the model described in equation (1) and its generalization with more than two tessellations.

The greatest difficulty in dealing with this model comes from the fact that the spectrum of U has little to do with the structure of the underlying graph in the general case. In fact, other than unitary, little can be said about U at all. For this reason, we enforce extra properties. These might come

in two flavours: to restrict the choices for the unitary vectors used at each tessellation, and to see that the structure of the tessellations are somehow dependent on each other. In this direction, we focus on the following case

- The unit vectors attached to each tile in a tessellation are real and uniform, that is, if the tile contains γ vertices, the vector has entries equal $1/\sqrt{\gamma}$ at each vertex in the tile, and 0 otherwise.
- The matrices H_i , corresponding to each tessellation T_i , commute.

Perhaps quite surprisingly, there is a large class of graphs satisfying both properties above.

3 A discretization of continuous-time quantum walks on special graphs

Given a graph X, a tessellation covering of X consists in a set of tessellations such that each edge of X belongs to at least one tessellation. A tessellation covering is a factorization if each edge belongs to precisely one tessellation. A tessellation covering is uniform if each tessellation induces a partition into cliques of the same size. Note that although all graphs admit a tessellation factorization, in most cases these tessellations will not be uniform. A typical example of a uniform tessellation factorization is a factorization into perfect matchings, also known as a 1-factorization.

Assume X has a uniform tessellation covering into k tessellations, let A_i denote the adjacency matrix of the subgraph induced by the ith tessellation and let H_i denote the unitary matrix obtained from each tessellation using the real and positive uniform superposition. Let γ_i denote the size of the cliques in the ith tessellation, and define $\gamma = \sum_{i=1}^k \gamma_i$.

Note that

$$\gamma_i H_i = 2A_i + (2 - \gamma_i)I. \tag{2}$$

Now assume $\{A_1, ..., A_k\}$ are commuting matrices. This means that $\{H_1, ..., H_k\}$ are also commuting matrices, hence

$$U^{T} = \left(\prod_{i=1}^{k} \exp(\mathrm{i}\theta_{i} H_{i})\right)^{T} = \prod_{i=1}^{k} \exp(\mathrm{i}\theta_{i} H_{i})^{T},$$

and the quantum dynamics can be analysed locally at each H_i . Moreover, if the covering is a factorization, then $A(X) = \sum_{i=1}^{k} A_i$. Thus it follows that

$$U = \prod_{i=1}^{k} \exp(i\theta_i H_i)$$

$$= \exp\left(i \sum_{i=1}^{k} \theta_i \frac{2A_i + (2 - \gamma_i)I}{\gamma_i}\right)$$

$$= \exp\left(i \sum_{i=1}^{k} \frac{\theta_i}{\gamma_i} (2 - \gamma_i)\right) \exp\left(2i \sum_{i=1}^{k} \frac{\theta_i}{\gamma_i} A_i\right),$$

and upon choosing θ_i to be a constant multiple of γ_i , say $\theta_i = \theta \gamma_i$, it follows that, for any $T \in \mathbb{Z}_+$,

$$U^T = \exp(i\theta(2k - \gamma)T)\exp(2i\theta TA).$$

This shows that, when a uniform tessellation factorization exists, it is possible to use the staggered quantum walk model from [21] to emulate a continuous quantum walk model, that is, $\{U^0, U^1, U^2, ...\}$ provides a discretization of $\exp(itA)$ in units of 2θ , up to a global phase.

This immediately raises the question of how common such structures are. We devote the next section to study a broad class of examples.

4 Tessellation-factorizations in Cayley graphs

Given a group \mathcal{G} and a subset \mathcal{C} of group elements which does not contain the identity id and is closed under taking the inverse, the Cayley graph $\operatorname{Cay}(\mathcal{G},\mathcal{C})$ is the graph whose vertex set is \mathcal{G} , and two vertices g and h are adjacent if $gh^{-1} \in \mathcal{C}$. The set \mathcal{C} is typically called the connection set.

Consider a staggered quantum walk on the Cayley graph $X = \text{Cay}(\mathcal{G}, \mathcal{C})$, where \mathcal{G} is a finite abelian group and \mathcal{C} is a connection set with the following properties

(1)
$$\mathcal{G} = \langle \mathcal{C} \rangle$$
, and

(2)
$$g \in \mathcal{C} \implies g^k \in \mathcal{C} \text{ for } 0 < k < \operatorname{ord}(g),$$

where $\operatorname{ord}(g)$ is the order of g. Property (1) ensures that the graph is connected. Property (2) implies that if $g \in \mathcal{C}$ then set $\{g^k : 0 \le k < \operatorname{ord}(g)\}$ is a clique in X and can be considered as a tile of some uniform tessellation of X.

If \mathcal{C} is partitioned as $\mathcal{C}_1 \cup ... \cup \mathcal{C}_k$, such that each $\mathcal{C}_i \cup id$ is a subgroup of \mathcal{G} , then this partition induces a uniform factorization of X with k tessellations in the following way. The cosets of $\mathcal{C}_i \cup id$ in \mathcal{G} are cliques (tessellation tiles) and the union of those cliques is the ith tessellation. The tessellation-factorization has no edges in the tessellation intersection and defines a staggered quantum walk on X with k tessellations.

As before, if A_i is the adjacency matrix of the subgraph induced by the ith tessellation, that is, if

$$A_i = A(Cay(\mathcal{G}, \mathcal{C}_i)),$$

then, as the group is abelian, these matrices will commute. Defining H_i , γ_i and $\theta_i = \theta/\gamma_i$ and U as in the previous section, then powers of U provide a discretization of the continuous-time quantum walk $\exp(itA)$. As a result, phenomena such as perfect state transfer or uniform mixing in continuous time quantum walks ([6, 7, 4, 13]) can also be observed in (coinless) discrete time quantum walks.

However, we will see in the examples below that we are able to impose more control over the quantum dynamics if we vary the θ_i s and if we do not require the tessellations to provide a factorization. As a result, we can construct new examples of perfect state transfer, uniform mixing, and related phenomena.

5 Perfect state transfer

For a detailed introduction in the topic of perfect state transfer in graphs according to the continuous-time quantum walk model, we refer the reader to [9, Chapter 2], or [12].

5.1 Theorem. Let $X = Cay(\mathcal{G}, \mathcal{C})$. Suppose \mathcal{C} is partitioned as $\mathcal{C}_1, ..., \mathcal{C}_k$, each $\mathcal{C}_i \cup id$ a subgroup, and at least one \mathcal{C}_i has exactly one element. Define Hermitian commuting matrices H_i to each \mathcal{C}_i as in Eq. (2). Define

$$U = \prod_{i=1}^{k} \exp(\mathrm{i}\theta_i H_i).$$

By conveniently selecting the θ_i , U^T admits perfect state transfer at any chosen time T.

Proof. Suppose C_1 has only one element, which must be of order 2. Denote by $\{A_1, ..., A_k\}$ the adjacency matrices of each tessellation induced by the cosets of $C_i \cup id$ in G. Since A_1 corresponds to C_1 , which comprises one order-2 element, it follows that A_1 is a perfect matching. Let γ_i be the size of the cliques in A_i , that is, γ_i is the cardinality of $C_i \cup id$. It follows that the distinct eigenvalues of each A_i are $\gamma_i - 1$ and -1, hence these are integral graphs, and thus periodic. This means that if $2T\theta_i/\gamma_i$ is an even multiple of π , then $\exp(2i(\theta_i/\gamma_i)TA_i) = I$. Upon choosing θ_1 such that $T\theta_1 = \pi/2$, it follows that

$$U^T = \alpha A_1$$
,

where α is complex number of absolute value 1, depending on the θ_i s and γ_i s. Hence perfect state transfer occurs in this case. This construction can be carried out in any abelian group of even order, as an element of order 2 necessarily exists.

If more elements of order 2 are available in the connection set, say $g_1, ..., g_\ell$, perfect state transfer between the vertices corresponding to id and any product of the g_i s can be manufactured by conveniently selecting the θ_i s for a given T.

In Figure 1, we depict a Cayley graph for $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$, with connection set $\mathcal{C} = \{(1,0,0), (0,1,0), (0,0,1), (0,0,2)\}$. Let H_1 , H_2 and H_3 be the unitary Hermitian matrices obtained from the uniform superposition on the tessellations given by $\mathcal{C}_1 = \{(1,0,0)\}$, $\mathcal{C}_2 = \{(0,1,0)\}$ and $\mathcal{C}_3 = \{(0,0,1), (0,0,2)\}$ respectively. The corresponding tessellations consist of horizontal edges (including curved ones), vertical edges and edges in the triangles, respectively. Define U as

$$U = \exp(i\theta_1 H_1) \exp(i\theta_2 H_2) \exp(i\theta_3 H_3).$$

If $\theta_1 = \theta_2 = \theta$, and $\theta_3 = 12\theta$, then perfect state transfer occurs between a_i and b_i at time $T = \pi/2\theta$. By making $\theta_2 = 2\theta$, then perfect state transfer occurs between a_i and c_i .

In any example of perfect state transfer at time T based on Theorem 5.1, the quantum walk will be periodic with period 2T, because $U^{2T} = I$ modulo a global phase. It is possible to have periodicity without perfect state transfer if θ is a rational multiple of π and if the perfect state transfer based on Theorem 5.1 would be produced at a rational (non-integer) number of steps,

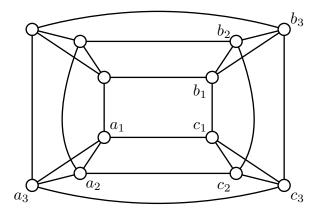


Figure 1: Example of Cayley graph admitting perfect state transfer in the staggered quantum walk model.

say T = m/n, where m, n are coprime integers. In this case, the staggered quantum walk does not have perfect state transfer, but it is periodic with period m.

5.1 Example with an edge in the tessellation intersection

It is not necessary for the connection set to be partitioned into disjoint subsets. As long as each element in the connection set belongs to a subgroup entirely contained in the connection set, a set of tessellations containing all edges can be defined. However in this case, we no longer have a tessellation factorization, but simply a (uniform) tessellation covering.

Here is an example. Let $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_4$ and

$$\mathcal{C} = \{(1,0), (0,1), (0,2), (0,3), (1,1), (1,3)\}.$$

Suppose that $C_1 = \{(1,0)\}$, $C_2 = \{(0,1), (0,2), (0,3)\}$, and $C_3 = \{(1,1), (0,2), (1,3)\}$. Notice that $C_2 \cap C_3$ is nonempty, which means that their corresponding tessellations will have an edge in the intersection. Figure 2 depicts the three tessellations separately.

If $\theta_1 = \theta$ and $\theta_2 = \theta_3 = 4\theta$ then perfect state transfer occurs between the vertices connected by the edges that belong to C_1 at time $T = \pi/2\theta$, for

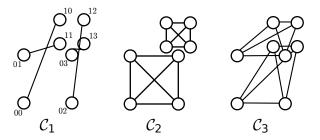


Figure 2: Example of a tessellation covering with edges in the intersection in which perfect state transfer still occurs.

instance, from vertex (1,2) to (0,2). Notice again that we have to choose θ so that T is a positive integer. Otherwise, if $\pi/2\theta$ is a rational number m/n with n > 1, there is no perfect state transfer, but the walk is periodic modulo a global phase with period m.

In this example, edges associated with generator (0, 2) in the Cayley graph \mathcal{G} belong to tessellations \mathcal{C}_2 and \mathcal{C}_3 . Examples with edges in the tessellation intersection are interesting because staggered quantum walks based on those tessellations do not reduce to the coined or Szegedy's models [20].

5.2 Non-abelian case

So far we have shown examples using Cayley graphs of abelian groups. Examples with non-abelian groups can be provided as well after taking an extra care of checking that the matrices H_i used in Theorem 5.1 commute. As a guide to choose connection sets, we have the following result.

5.2 Theorem. Let A_1 and A_2 be the adjacency matrices of $Cay(\mathcal{G}, \mathcal{C}_1)$ and $Cay(\mathcal{G}, \mathcal{C}_2)$, respectively, where $C_1 = \mathcal{C}_1 \cup \mathbf{id}$ and $C_2 = \mathcal{C}_2 \cup \mathbf{id}$ are subgroups of \mathcal{G} . A_1 and A_2 commute if and only if $C_1 \cdot C_2 = C_2 \cdot C_1$.

Proof. First note that A_1 and A_2 commute if and only if (A_1+I) and (A_2+I) commute. Now let g and h be elements of \mathcal{G} , and hence also vertices of $Cay(\mathcal{G}, \mathcal{C}_1)$ and $Cay(\mathcal{G}, \mathcal{C}_2)$.

By the definition of adjacency matrix of a Cayley graph, $(A_1 + I)_{ga} = 1 \Leftrightarrow a \in C_1 \cdot g$ and $(A_2 + I)_{ga} = 1 \Leftrightarrow a \in C_2 \cdot g$. Then,

$$[(A_1+I)(A_2+I)]_{gh} = \sum_{a \in \mathcal{G}} (A_1+I)_{ga}(A_2+I)_{ah} = |(C_1 \cdot g) \cap (C_2 \cdot h)|.$$

Likewise, $\sum_{a\in\mathcal{G}}(A_2+I)_{ga}(A_1+I)_{ah}=|(C_2\cdot g)\cap (C_1\cdot h)|$. Moreover, note that if $|(C_1\cdot g)\cap (C_2\cdot h)|\neq 0$, then there is $a_1\in C_1$ and $a_2\in C_2$ such that $a_1g=a_2h$. Then $C_1\cdot g=C_1\cdot (a_1^{-1}a_2h)=C_1\cdot a_2h$. Thus,

$$|(C_1 \cdot g) \cap (C_2 \cdot h)| = |(C_1 \cdot a_2 h) \cap (C_2 \cdot h)| = |(C_1 \cdot a_2) \cap C_2|$$
$$= |C_1 \cap C_2 \cdot a_2^{-1}| = |C_1 \cap C_2|. \tag{3}$$

Likewise, if $|(C_1 \cdot h) \cap (C_2 \cdot g)| \neq 0$ then $|(C_1 \cdot h) \cap (C_2 \cdot g)| = |C_1 \cap C_2|$. We now proceed to show that $C_1 \cdot C_2 = C_2 \cdot C_1$ if and only if, for all $g, h \in \mathcal{G}$, $|(C_1 \cdot g) \cap (C_2 \cdot h)| = |(C_2 \cdot g) \cap (C_1 \cdot h)|$, which because of (3) is equivalent to showing that these intersections are either both empty or both non-empty.

Suppose that $|(C_1 \cdot g) \cap (C_2 \cdot h)| \neq 0$ and $|(C_1 \cdot h) \cap (C_2 \cdot g)| = 0$ (the opposite case is analogous). There is $a_1 \in C_1$ and $a_2 \in C_2$ such that $a_1g = a_2h$. Then clearly $hg^{-1} \in C_2 \cdot C_1$. However, if $hg^{-1} \in C_1 \cdot C_2$ then $(C_1 \cdot h) \cap (C_2 \cdot g)$ is also non-empty. Hence, $C_2 \cdot C_1 \neq C_1 \cdot C_2$. This shows that $C_2 \cdot C_1 = C_1 \cdot C_2$ implies $A_1A_2 = A_2A_1$.

Now assume that $|C_1 \cdot C_2| > |C_2 \cdot C_1|$ (the opposite case is analogous). Let $a_1 \in C_1$ and $a_2 \in C_2$ be such that $a_1 a_2 \notin C_2 \cdot C_1$. It follows that $|(C_1 \cdot a_2^{-1}) \cap (C_2 \cdot a_1)| = 0$, then $(A_2 + I)(A_1 + I)_{a_1 a_2^{-1}} = 0$, however

$$(C_1 \cdot a_1) \cap (C_2 \cdot a_2^{-1}) = C_1 \cap C_2,$$

which contains at least the identity element. Thus $(A_1 + I)(A_2 + I)_{a_1a_2^{-1}} \neq 0$. This shows that $A_1A_2 = A_2A_1$ implies $C_1 \cdot C_2 = C_2 \cdot C_1$.

If \mathcal{G} is non-abelian, a staggered quantum walk on $\operatorname{Cay}(\mathcal{G}, \mathcal{C})$ admits PST if the connection set \mathcal{C} can be written as $\mathcal{C} = \mathcal{C}_1 \cup ... \cup \mathcal{C}_k$ so that $C_i = \mathcal{C}_i \cup \operatorname{id}$ and $C_j = \mathcal{C}_j \cup \operatorname{id}$ are pairwise commuting subgroups of \mathcal{G} for all i and j and at least one \mathcal{C}_i has exactly one element.

Theorem 5.2 poses a strong restriction on the possible choices of connection sets. It seems that after taking into account this restriction any instance of perfect state transfer with a non-abelian group \mathcal{G} can be reproduced by using some abelian groups of the same order in place of \mathcal{G} . Notice that in our construction it does not matter whether C_i is abelian or not.

6 Instantaneous uniform mixing

We say that the discrete time quantum walk described by the $n \times n$ evolution operator U admits instantaneous uniform mixing at time T if U^T is a flat

complex matrix, that is, all its entries have absolute value equal to $1/\sqrt{n}$. In continuous time quantum walks, instantaneous uniform mixing has been studied in, for instance, [6], [13], [1], [3], [11]. In [3], it was shown the instantaneous uniform mixing occurs in a complete graph K_n if and only if n = 2, 3, 4.

6.1 Theorem. Let $X = Cay(\mathcal{G}, \mathcal{C})$ be a Cayley graph for an abelian group. Suppose \mathcal{C} is partitioned as $\mathcal{C}_1, ..., \mathcal{C}_k$, each $\mathcal{C}_i \cup id$ a group of order at most 4, and $\prod_{i=1}^k (|\mathcal{C}_i| + 1) = |\mathcal{G}|$. Let A_i be the adjacency matrix of each $Cay(\mathcal{G}, \mathcal{C}_i)$, and define Hermitian matrices H_i to each \mathcal{C}_i as in Eq. (2). Thus, for a suitable choice of θ_i s,

$$U = \prod_{i=1}^{k} \exp(\mathrm{i}\theta_i H_i)$$

admits instantaneous uniform mixing at any chosen time T.

Proof. First note that because the groups $C_i \cup id$ have order at most 4, the matrices A_i will be a collection of disjoint edges, disjoint triangles, or disjoint complete graphs of size 4, denoted by K_4 . Each of these components admits instantaneous uniform mixing. For each of these graphs define H_i as in equation (2), thus the matrices H_i admit uniform mixing at times $\pi/4$, $\pi/3$, or $\pi/2$, respectively. This is because the matrix H_i is the direct sum of n/γ_i Grover operators of dimension γ_i and the absolute value of any diagonal entry of $\exp(i\theta_i H_i)$ is equal to a nonzero nondiagonal entry if and only if $|\sin \theta_i| = \sqrt{n}/2$, which indeed has real solutions only for $n \leq 4$. Therefore, selecting

$$\theta_{i} = \begin{cases} \frac{\pi}{4T}, & \text{if } A_{i} \text{ is a collection of disjoint edges,} \\ \frac{\pi}{3T}, & \text{if } A_{i} \text{ is a collection of disjoint triangles,} \\ \frac{\pi}{2T}, & \text{if } A_{i} \text{ is a collection of disjoint } K_{4}s, \end{cases}$$
(4)

it follows that, for all i, $\exp(i\theta_i H_i T)$ contains entries that are either 0 or complex numbers of absolute value equal to $1/\sqrt{\gamma_i}$, where γ_i is the order of $\mathcal{C}_i \cup id$. Moreover, an entry (a, b) is non-zero if and only if $a^{-1} \cdot b \in \mathcal{C}_i \cup id$. Now we proceed to show that U as defined in the statement admits instantaneous uniform mixing.

Given a subgroup S of G, let M_S be a matrix of dimension |G| such that entry (a,b) is a complex number of absolute value $1/\sqrt{|S|}$ if $a^{-1} \cdot b \in S$, 0 otherwise. Let T be another subgroup and define M_T analogously. Let $\mathcal{R} = S \cdot \mathcal{T}$.

Claim 1: Each element of \mathcal{R} can be uniquely written as a_1a_2 with $a_1 \in \mathcal{S}$ and $a_2 \in \mathcal{T}$ if and only if $|\mathcal{R}| = |\mathcal{S}||\mathcal{T}|$. In fact, $|\mathcal{S}||\mathcal{T}| \geq |\mathcal{R}|$, and equality holds if and only if there are no duplicates in the $\mathcal{S} \cdot \mathcal{T}$. Note that both conditions are equivalent to the intersection of \mathcal{S} and \mathcal{T} being trivial.

Claim 2: If the conditions in Claim 1 hold, then the product $M_{\mathcal{R}} = M_{\mathcal{S}} M_{\mathcal{T}}$ is a matrix of dimension $|\mathcal{G}|$ such that entry (a, b) is a complex number of absolute value $1/\sqrt{|\mathcal{R}|}$ if $a^{-1} \cdot b \in \mathcal{R}$, 0 otherwise. In fact, note that

$$(M_{\mathcal{S}} M_{\mathcal{T}})_{\mathrm{id},a} = \sum_{b \in \mathcal{G}} (M_{\mathcal{S}})_{\mathrm{id},b} (M_{\mathcal{T}})_{b,a}.$$

If $a \in \mathcal{R}$ then, from Claim 1 and the definitions of $M_{\mathcal{S}}$ and $M_{\mathcal{T}}$, there is a unique $b \in \mathcal{S}$ such that $a = b \cdot (b^{-1} \cdot a)$, and $(b^{-1} \cdot a) \in \mathcal{T}$. That is, a unique $b \in \mathcal{G}$ such that $(M_{\mathcal{S}})_{id,b} \neq 0 \neq (M_{\mathcal{T}})_{b,a}$. Moreover, the product of these entries will be a complex number of absolute value $1/\sqrt{|\mathcal{S}||\mathcal{T}|}$. From Claim 1, this is $1/\sqrt{|\mathcal{R}|}$. On the other hand, if $a \notin \mathcal{R}$, there is no b such that $(M_{\mathcal{S}})_{id,b} \neq 0 \neq (M_{\mathcal{T}})_{b,a}$, otherwise $b \cdot (b^{-1} \cdot a) = a \in \mathcal{S} \cdot \mathcal{T} = \mathcal{R}$.

Recall the hypothesis that the product of the cardinalities of the subgroups $C_i \cup id$ is equal to $|\mathcal{G}|$. Because these subgroups generate \mathcal{G} , we have

$$|\mathcal{G}| = \left|\prod_{i=1}^k (\mathcal{C}_i \cup \mathtt{id})
ight| \leq \prod_{i=1}^k |\mathcal{C}_i \cup \mathtt{id}| = |\mathcal{G}|.$$

Thus each element of \mathcal{G} is written uniquely as a product of elements in each $\mathcal{C}_i \cup \mathtt{id}$. Therefore we can recursively split the product into parts, and using Claims 1 and 2, it follows that U is a matrix in which all entries are complex numbers of absolute value $1/\sqrt{|\mathcal{G}|}$.

Note that the hypothesis that $|\mathcal{G}| = \prod (|\mathcal{C}_i| + 1)$ is equivalent to saying that $\mathcal{G} \cong \bigotimes \mathcal{C}_i \cup id$. The fact that these factors are also determining the connection set implies that the theorem generates, up to isomorphism, a unique Cayley graph admitting uniform mixing for each abelian group of order $2^k 3^m$ that has no element of orders 8 or 9.

We note however that this hypothesis is not strictly necessary to achieve uniform mixing as described, given that we do know examples in which the connection set is denser and yet uniform mixing occurs. For example, any hypercube in which antipodal points have been made adjacent [6]. However we also know examples where the hypothesis fails and uniform mixing cannot occur.

Finally, recall that if n > 4 then K_n does not admit instantaneous uniform mixing in the continuous-time model. The arguments in the proof above also show that if \mathcal{C} is partitioned into subgroups that generate each element of \mathcal{G} uniquely and at least one of these subgroups has an element of order larger than 4, then there is no choice of θ_i s that would allow for instantaneous uniform mixing to occur in $\operatorname{Cay}(\mathcal{G}, \mathcal{C})$ using the staggered quantum walk model.

7 Conclusions

The main results of this work are Theorems 5.1 and 6.1, which describe a large class of examples of staggered quantum walks on Cayley graphs of abelian groups that admit either perfect state transfer or instantaneous uniform mixing. We have also discussed related topics such as periodicity and extensions using non-abelian groups.

Throughout the text, we presented a close connection between the continuoustime and the staggered quantum walk model. Under some assumptions, when we can cover the graph with a uniform tessellation, the staggered model provides a natural discretization of continuous-time quantum walks.

References

- [1] William Adamczak et al. "Non-Uniform Mixing Of Quantum Walk On Cycles". In: *International Journal Of Quantum Information* 05.06 (2007), pp. 781–793 (cit. on p. 11).
- [2] Y. Aharonov, L. Davidovich, and N. Zagury. "Quantum random walks". In: *Physical Review A* 48.2 (1993), pp. 1687–1690 (cit. on p. 2).
- [3] Amir Ahmadi et al. "On mixing in continuous-time quantum walks on some circulant graphs". In: Quantum Information & Computation 3.6 (2003), pp. 611–618 (cit. on p. 11).

- [4] Milan Bašić. "Characterization of quantum circulant networks having perfect state transfer". In: *Quantum Information Processing* 12.1 (2013), pp. 345–364 (cit. on p. 6).
- [5] Sougato Bose. "Quantum communication through spin chain dynamics: an introductory overview". In: *Contemporary Physics* 48.1 (2007), pp. 13–30 (cit. on p. 2).
- [6] Ada Chan. "Complex Hadamard Matrices, Instantaneous Uniform Mixing and Cubes". In: ArXiv e-prints (2013). arXiv: 1305.5811 (cit. on pp. 6, 11, 13).
- [7] Wang-Chi Cheung and Chris D Godsil. "Perfect state transfer in cubelike graphs". In: *Linear Algebra and its Applications* 435.10 (2011), pp. 2468–2474 (cit. on p. 6).
- [8] Andrew M Childs. "On the Relationship Between Continuous- and Discrete-Time Quantum Walk". In: *Communications in Mathematical Physics* 294 (2010), pp. 581–603. arXiv: 0810.0312 [quant-ph] (cit. on p. 2).
- [9] Gabriel Coutinho. "Quantum State Transfer in Graphs". PhD dissertation. University of Waterloo, 2014 (cit. on pp. 2, 6).
- [10] Edward Farhi and Sam Gutmann. "Quantum computation and decision trees". In: *Physical Review A* 58.2 (1998), pp. 915–928 (cit. on p. 2).
- [11] Chris Godsil and Hanmeng Zhan. "Uniform Mixing on Cayley Graphs". In: (2016). arXiv: arXiv:1504.00721v2 (cit. on p. 11).
- [12] Chris D Godsil. "State transfer on graphs". In: *Discrete Mathematics* 312.1 (2012), pp. 129–147 (cit. on p. 6).
- [13] Chris D Godsil, Natalie Mullin, and Aidan Roy. "Uniform Mixing and Association Schemes". In: *ArXiv e-prints* (2013). arXiv: 1301.5889 (cit. on pp. 6, 11).
- [14] Alastair Kay. "Perfect, efficient, state transfer and its application as a constructive tool". In: *International Journal of Quantum Information* 08.04 (2010), pp. 641–676 (cit. on p. 2).
- [15] Vivien M Kendon and Christino Tamon. "Perfect State Transfer in Quantum Walks on Graphs". In: *Journal of Computational and Theoretical Nanoscience* 8.3 (2011), pp. 422–433 (cit. on p. 2).

- [16] Dheeraj M N and Todd A. Brun. "Continuous limit of discrete quantum walks". In: *Phys. Rev. A* 91 (6 2015), p. 062304 (cit. on p. 2).
- [17] Jalil Khatibi Moqadam, Marcos Cesar de Oliveira, and Renato Portugal. "Staggered quantum walks with superconducting microwave resonators". In: (2016), p. 11. arXiv: 1609.09844 (cit. on p. 2).
- [18] Pascal Philipp and Renato Portugal. "Exact simulation of coined quantum walks with the continuous-time model". In: Quantum Information Processing 16.1 (2016), p. 14 (cit. on p. 2).
- [19] Renato Portugal. "Establishing the equivalence between Szegedy's and coined quantum walks using the staggered model". In: Quantum Information Processing 15.4 (2016), pp. 1387–1409 (cit. on p. 2).
- [20] Renato Portugal. "Staggered quantum walks on graphs". In: *Phys. Rev.* A 93 (6 2016), p. 062335 (cit. on pp. 2, 9).
- [21] Renato Portugal, Marcos Cesar de Oliveira, and Jalil Khatibi Moqadam. "Staggered Quantum Walks with Hamiltonians". In: (2016). arXiv: 1605.02774 (cit. on pp. 2, 3, 5).
- [22] Renato Portugal et al. "The staggered quantum walk model". In: Quantum Information Processing 15.1 (2016), pp. 85–101 (cit. on pp. 2, 3).
- [23] Frederick W. Strauch. "Connecting the discrete- and continuous-time quantum walks". In: *Phys. Rev. A* 74 (3 2006), p. 030301 (cit. on p. 2).
- [24] Mario Szegedy. "Quantum Speed-Up of Markov Chain Based Algorithms". In: 45th Annual IEEE Symposium on Foundations of Computer Science. IEEE, pp. 32–41 (cit. on p. 2).