

$G^{k,l}$ -constrained multi-degree reduction of Bézier curves

Przemysław Gospodarczyk*, Stanisław Lewanowicz, Paweł Woźny

Institute of Computer Science, University of Wrocław, ul. Joliot-Curie 15, 50-383 Wrocław, Poland

Abstract

We present a new approach to the problem of $G^{k,l}$ -constrained ($k, l \leq 3$) multi-degree reduction of Bézier curves with respect to the least squares norm. First, to minimize the least squares error, we consider two methods of determining the values of geometric continuity parameters. One of them is based on quadratic and nonlinear programming, while the other uses some simplifying assumptions and solves a system of linear equations. Next, for prescribed values of these parameters, we obtain control points of the multi-degree reduced curve, using the properties of constrained dual Bernstein basis polynomials. Assuming that the input and output curves are of degree n and m , respectively, we determine these points with the complexity $O(mn)$, which is significantly less than the cost of other known methods. Finally, we give several examples to demonstrate the effectiveness of our algorithms.

Keywords: Constrained dual Bernstein basis, Bézier curves, Multi-degree reduction, Geometric continuity, Quadratic programming, Nonlinear programming.

1. Introduction

Let Π_n^d denote the space of all parametric polynomials in $\mathbb{R}^d$ of degree at most n ; $\Pi_n^1 := \Pi_n$.

A Bézier curve $P_n \in \Pi_n^d$ of degree $n \in \mathbb{N}$ is the following parametric curve:

$$P_n(t) := \sum_{i=0}^n p_i B_i^n(t) \quad (0 \leq t \leq 1), \quad (1.1)$$

where $p_0, p_1, \dots, p_n \in \mathbb{R}^d$ are so-called *control points*, and $B_0^n, B_1^n, \dots, B_n^n$ are the *Bernstein polynomials* of degree n given by

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i} \quad (0 \leq i \leq n). \quad (1.2)$$

In this paper, we consider the following problem.

Problem 1.1. [$G^{k,l}$ -constrained multi-degree reduction]

For a given Bézier curve P_n of degree n , find a Bézier curve R_m of lower degree m ,

$$R_m(t) := \sum_{i=0}^m r_i B_i^m(t) \quad (0 \leq t \leq 1), \quad (1.3)$$

so that the following conditions are satisfied:

(i) P_n and R_m are $G^{k,l}$ -continuous ($-1 \leq k, l \leq 3$ and $k + l < m - 1$) at the endpoints, i.e.,

$$\left. \begin{aligned} \frac{d^i}{dt^i} R_m(t) &= \frac{d^i}{dt^i} P_n(\varphi(t)) & (t=0; i=0, 1, \dots, k), \\ \frac{d^j}{dt^j} R_m(t) &= \frac{d^j}{dt^j} P_n(\varphi(t)) & (t=1; j=0, 1, \dots, l), \end{aligned} \right\} \quad (1.4)$$

where $\varphi : [0, 1] \rightarrow [0, 1]$ is a strictly increasing function with $\varphi(0) = 0$ and $\varphi(1) = 1$;

*Corresponding author. Fax +48 71 3757801

Email addresses: pgo@ii.uni.wroc.pl (Przemysław Gospodarczyk), Stanislaw.Lewanowicz@ii.uni.wroc.pl (Stanisław Lewanowicz), Pawel.Wozny@ii.uni.wroc.pl (Paweł Woźny)

(ii) value of the squared L_2 -error

$$\|P_n - R_m\|_{L_2}^2 := \int_0^1 (1-t)^{\alpha} t^{\beta} \|P_n(t) - R_m(t)\|^2 dt \quad (\alpha, \beta > -1)$$

is minimized in the space Π_m^d , where $\|\cdot\|$ is the Euclidean vector norm.

Problems of the above type have been recently discussed in several papers [6, 9, 10, 11, 12, 16, 17], usually under simplifying assumptions $\varphi'(0) = \varphi'(1) = 1$, which implied, for example, the hybrid $C^{1,1}/G^{2,2}$ -constrained degree reduction, meaning that we impose constraints of $C^{1,1}$ -continuity, followed by $G^{2,2}$ -continuity, at the endpoints. Most of the known algorithms solve a system of normal equations to get control points of the multi-degree reduced curve (1.3). Consequently, solution depends on the inverse of a certain matrix, so the obtained formulas are not truly explicit and the cost of the method is high (see, e.g., [6, 10, 12]). For extensive lists of references, see the recent papers of Lu [6], or Rababah and Mann [12]. The conventional problem of degree reduction differs from Problem 1.1 in considering, instead of condition (i), the $C^{k,l}$ -continuity at the endpoints of curves, i.e.,

$$\left. \begin{aligned} R_m^{(i)}(0) &= P_n^{(i)}(0) & (i = 0, 1, \dots, k), \\ R_m^{(j)}(1) &= P_n^{(j)}(1) & (j = 0, 1, \dots, l). \end{aligned} \right\} \quad (1.5)$$

In the past 30 years, many papers dealing with this problem have been published (see, e.g., [2, 3, 13, 14, 15]). In particular, in [15], two of us have proposed a method based on the use of the so-called *dual Bernstein polynomials*, which has complexity $O(mn)$, the least among the existing algorithms. In the present paper, we apply an extended version of this method as an essential part of the algorithms of solving Problem 1.1. Such an approach allows us to avoid matrix inversion. Assuming that $-1 \leq k, l \leq 3$ and including the hybrid cases, there are 37 continuity cases which require computation of the continuity parameters. Those variants of the problem differ, and we have not proven that in each case a unique solution exists.

The outline of the paper is as follows. Section 2 contains a preliminary material. In Section 3, we relate the $G^{k,l}$ -continuity conditions with the control points of the curves P_n and R_m . Section 4 brings complete solutions of Problem 1.1, with and without the simplifying assumptions. Section 5 deals with algorithmic implementation of the proposed methods. In Section 6, we give some examples showing efficiency of our methods. Conclusions are given in Section 7.

2. Preliminaries

In this section, we introduce necessary definitions and notation.

We define the inner product $\langle \cdot, \cdot \rangle_{\alpha, \beta}$ by

$$\langle f, g \rangle_{\alpha, \beta} := \int_0^1 (1-t)^{\alpha} t^{\beta} f(t)g(t)dt \quad (\alpha, \beta > -1). \quad (2.1)$$

There is a unique *dual Bernstein polynomial basis of degree n*

$$D_0^n, D_1^n, \dots, D_n^n \in \Pi_n,$$

associated with the basis (1.2), so that

$$\langle D_i^n, B_j^n \rangle_{\alpha, \beta} = \delta_{ij} \quad (i, j = 0, 1, \dots, n),$$

where δ_{ij} equals 1 if $i = j$, and 0 otherwise.

Given the integers k, l such that $k, l \geq -1$ and $k + l < n - 1$, let $\Pi_n^{(k,l)}$ be the space of all polynomials of degree at most n , whose derivatives of orders $0, 1, \dots, k$ at $t = 0$, as well as derivatives of orders $0, 1, \dots, l$ at $t = 1$, vanish. We use the convention that derivative of order 0 of a function is the function itself. Clearly, $\dim \Pi_n^{(k,l)} = n - k - l - 1$, and the Bernstein polynomials $\{B_{k+1}^n, B_{k+2}^n, \dots, B_{n-l-1}^n\}$ form a basis of this space. There is a unique *dual constrained Bernstein polynomial basis of degree n*

$$\{D_{k+1}^{(n,k,l)}, D_{k+2}^{(n,k,l)}, \dots, D_{n-l-1}^{(n,k,l)}\} \subset \Pi_n^{(k,l)}$$

satisfying the relation $\langle D_i^{(n,k,l)}, B_j^n \rangle_{\alpha, \beta} = \delta_{ij} \quad (i, j = k+1, k+2, \dots, n-l-1)$. Obviously, we have

$D_i^{(n,-1,-1)} = D_i^n$, which corresponds to the case without any constraints. For properties of the polynomials B_i^n and D_i^n , see [15, Appendix A].

Forward difference operator is given by

$$\Delta^0 q_i := q_i, \quad \Delta^k q_i := \Delta^{k-1} q_{i+1} - \Delta^{k-1} q_i \quad (k = 1, 2, \dots).$$

We use $C^{p,q}/G^{k,l}$ notation to describe the hybrid constraints, where $p, q \in \{-1, 1\}$ and $(k \geq 2 \text{ or } l \geq 2)$. In the case of $k \geq 2$ and $p = 1$, we set $\varphi'(0) := 1$. Similarly, for $l \geq 2$ and $q = 1$, we set $\varphi'(1) := 1$. Setting $p := -, q := -$ means that we do not fix $\varphi'(0)$, $\varphi'(1)$, respectively. Clearly, $C^{-,-}/G^{k,l}$ denotes $G^{k,l}$.

3. Geometric continuity

In this section, we relate the $G^{k,l}$ -continuity conditions (1.4) with the control points. We limit ourselves to $k, l \leq 3$ cases, which are the most important from a practical point of view.

Remark that the control points $r_1, r_2, \dots, r_k$ depend on the parameters

$$\lambda_j := \varphi^{(j)}(0) \quad (j = 1, 2, \dots, k),$$

while the points $r_{m-1}, r_{m-2}, \dots, r_{m-l}$ depend on

$$\mu_j := \varphi^{(j)}(1) \quad (j = 1, 2, \dots, l).$$

Now, let us recall the well known formulas (see [7], also [6, 11]). When $k = 3$, we have:

$$r_0 = p_0, \quad r_1 = p_0 + \frac{n}{m} \lambda_1 \Delta p_0, \quad (3.1)$$

$$r_2 = p_0 + \frac{n}{m} \left[2\lambda_1 + \frac{1}{m-1} \lambda_2 \right] \Delta p_0 + \frac{(n-1)_2}{(m-1)_2} \lambda_1^2 \Delta^2 p_0, \quad (3.2)$$

$$\begin{aligned} r_3 = p_0 + \frac{n}{m} \left[3\lambda_1 + \frac{3}{m-1} \lambda_2 + \frac{1}{(m-2)_2} \lambda_3 \right] \Delta p_0 \\ + 3 \frac{(n-1)_2}{(m-1)_2} \left[\lambda_1^2 + \frac{1}{m-2} \lambda_1 \lambda_2 \right] \Delta^2 p_0 + \frac{(n-2)_3}{(m-2)_3} \lambda_1^3 \Delta^3 p_0. \end{aligned} \quad (3.3)$$

In the case of $k = 2$, we use (3.1) and (3.2). For $k = 1$, formulas (3.1) hold. Analogously, when $l = 3$, we have:

$$r_m = p_n, \quad r_{m-1} = p_n - \frac{n}{m} \mu_1 \Delta p_{n-1}, \quad (3.4)$$

$$r_{m-2} = p_n - \frac{n}{m} \left[2\mu_1 - \frac{1}{m-1} \mu_2 \right] \Delta p_{n-1} + \frac{(n-1)_2}{(m-1)_2} \mu_1^2 \Delta^2 p_{n-2}, \quad (3.5)$$

$$\begin{aligned} r_{m-3} = p_n - \frac{n}{m} \left[3\mu_1 - \frac{3}{m-1} \mu_2 + \frac{1}{(m-2)_2} \mu_3 \right] \Delta p_{n-1} \\ + 3 \frac{(n-1)_2}{(m-1)_2} \left[\mu_1^2 - \frac{1}{m-2} \mu_1 \mu_2 \right] \Delta^2 p_{n-2} - \frac{(n-2)_3}{(m-2)_3} \mu_1^3 \Delta^3 p_{n-3}. \end{aligned} \quad (3.6)$$

In the case of $l = 2$, we use (3.4) and (3.5). For $l = 1$, formulas (3.4) hold.

4. $G^{k,l}$ -constrained multi-degree reduction problem

4.1. Multi-degree reduction of Bézier curves with prescribed boundary control points

First, we discuss the following *model problem* of constrained multi-degree reduction:

Problem 4.1. [Multi-degree reduction with prescribed boundary control points]

Given a Bézier curve $P_n \in \Pi_n^d$,

$$P_n(t) := \sum_{i=0}^n p_i B_i^n(t),$$

we look for a Bézier curve $R_m \in \Pi_m^d$ ($m < n$),

$$R_m(t) := \sum_{i=0}^m r_i B_i^m(t), \quad (4.1)$$

having the prescribed control points $r_0, r_1, \dots, r_k$ and $r_{m-l}, r_{m-l+1}, \dots, r_m$, that gives minimum value of the error

$$E^{(\alpha, \beta)} := \|P_n - R_m\|_{L_2}^2 = \int_0^1 (1-t)^\alpha t^\beta \|P_n(t) - R_m(t)\|^2 dt \quad (\alpha, \beta > -1). \quad (4.2)$$

Given the points $p_i := (p_{i1}, p_{i2}, \dots, p_{id}) \in \mathbb{R}^d$ ($i = 0, 1, \dots, n$) and $r_i := (r_{i1}, r_{i2}, \dots, r_{id}) \in \mathbb{R}^d$ ($i = 0, 1, \dots, m$), we use notation $\mathbf{p}^h, \mathbf{r}^h$ for the vectors of h th coordinates of the points $p_0, p_1, \dots, p_n$ and $r_0, r_1, \dots, r_m$, respectively:

$$\mathbf{p}^h := [p_{0h}, p_{1h}, \dots, p_{nh}], \quad \mathbf{r}^h := [r_{0h}, r_{1h}, \dots, r_{mh}] \quad (h = 1, 2, \dots, d).$$

As an extension of the result given in [15] (see also [5]), we obtain the following theorem.

Theorem 4.2. *The inner control points $r_i = (r_{i1}, r_{i2}, \dots, r_{id})$ ($k+1 \leq i \leq m-l-1$) of the curve (4.1), being the solution of the Problem 4.1, are given by*

$$r_i = \sum_{j=0}^n v_j \phi_{ij} \quad (i = k+1, k+2, \dots, m-l-1), \quad (4.3)$$

where

$$\phi_{ij} := \left\langle B_j^n, D_i^{(m, k, l)} \right\rangle_{\alpha, \beta}, \quad (4.4)$$

and

$$v_j := p_j - \binom{n}{j}^{-1} \left(\sum_{h=0}^k + \sum_{h=m-l}^m \right) \binom{n-m}{j-h} \binom{m}{h} r_h \quad (j = 0, 1, \dots, n). \quad (4.5)$$

The squared L_2 -error (4.2) is given by

$$E^{(\alpha, \beta)} = \sum_{h=1}^d \{ I_{nn}(\mathbf{p}^h, \mathbf{p}^h) + I_{mm}(\mathbf{r}^h, \mathbf{r}^h) - 2I_{nm}(\mathbf{p}^h, \mathbf{r}^h) \}, \quad (4.6)$$

where for $\mathbf{a} := [a_0, a_1, \dots, a_N]$ and $\mathbf{b} := [b_0, b_1, \dots, b_M]$, we define

$$I_{NM}(\mathbf{a}, \mathbf{b}) := \frac{B(\alpha+1, \beta+1)}{(\alpha+\beta+2)_{N+M}} \sum_{i=0}^N \sum_{j=0}^M \binom{N}{i} \binom{M}{j} (\alpha+1)_{N+M-i-j} (\beta+1)_{i+j} a_i b_j,$$

where $B(\alpha, \beta) := \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function.

PROOF. Let us write

$$R_m(t) = S_m(t) + T_m(t),$$

where

$$S_m(t) := \sum_{i=k+1}^{m-l-1} r_i B_i^m(t), \quad T_m(t) := \left(\sum_{i=0}^k + \sum_{i=m-l}^m \right) r_i B_i^m(t).$$

Using the degree elevation formula (see, e.g., [4, §6.10]; we adopt the usual convention that $\binom{u}{v} = 0$ if $v < 0$ or $v > u$)

$$B_i^m(t) = \binom{m}{i} \sum_{h=0}^n \binom{n-m}{h-i} \binom{n}{h}^{-1} B_h^n(t),$$

we write

$$T_m(t) = \sum_{j=0}^n d_j B_j^n(t),$$

where

$$d_j := \binom{n}{j}^{-1} \left(\sum_{h=0}^k + \sum_{h=m-l}^m \right) \binom{n-m}{j-h} \binom{m}{h} r_h.$$

Now, we observe that

$$\|P_n - R_m\|_{L_2}^2 = \|W_n - S_m\|_{L_2}^2 = \sum_{h=1}^d \int_0^1 (1-t)^{\alpha t^\beta} [W_n^h(t) - S_m^h(t)]^2 dt,$$

where

$$\begin{aligned} W_n(t) &:= [W_n^1(t), W_n^2(t), \dots, W_n^d(t)] = P_n(t) - T_m(t) = \sum_{i=0}^n v_i B_i^n(t), \\ S_m(t) &:= [S_m^1(t), S_m^2(t), \dots, S_m^d(t)], \end{aligned}$$

with

$$v_i := p_i - d_i.$$

Thus, we are looking for the best least squares approximation for W_n^h ($h = 1, 2, \dots, d$) in the space $\Pi_m^{(k,l)}$. Remembering that B_i^m and $D_i^{(m,k,l)}$ ($k+1 \leq i \leq m-l-1$) are the dual bases in the space $\Pi_m^{(k,l)}$, we obtain

$$r_i = \sum_{j=0}^n v_j \left\langle B_j^n, D_i^{(m,k,l)} \right\rangle_{\alpha, \beta} = \sum_{j=0}^n v_j \phi_{ij} \quad (i = k+1, k+2, \dots, m-l-1),$$

which is the formula (4.3).

Proof of (4.6) uses an argument similar to the one given in [15]. $\square$

Remark 4.3. Let us define the quantities ψ_{ij} ($i = k+1, k+2, \dots, m-l-1$; $j = 0, 1, \dots, n$), related to the coefficients ϕ_{ij} (cf. (4.4)) by the following formula:

$$\phi_{ij} := \binom{m-k-l-2}{i-k-1} \binom{m}{i}^{-1} \binom{n}{j} \frac{(\alpha+l+2)_{n-j}(\beta+k+2)_j}{(\alpha+l+2)_{l+1}(\beta+k+2)_{k+1}} \psi_{ij}. \quad (4.7)$$

Observe that the quantities ψ_{ij} can be put in a rectangular table and the entries of this ψ -table can be computed using [5, Algorithm 4.2], assuming that $c_1 := k+1$, $c_2 := l+1$, $\alpha_1 := \alpha$ and $\alpha_2 := \beta$. Note that the complexity of this algorithm is $O(mn)$.

4.2. $G^{k,l}$ -constrained multi-degree reduction

Coming back to the problem of $G^{k,l}$ -constrained multi-degree reduction (see Problem 1.1), let us notice that the formulas (3.1)–(3.6) with *fixed parameters* λ_i and μ_j (cf. §3) constitute constraints of the form demanded in Problem 4.1. As a result, the control points (4.3) depend on these parameters.

Now, the *optimum values* of the parameters can be obtained by minimizing the error function (4.6),

$$E^{(\alpha, \beta)} \equiv E^{(\alpha, \beta)}(\lambda_1, \lambda_2, \dots, \lambda_k, \mu_1, \mu_2, \dots, \mu_l), \quad (4.8)$$

depending on $\{\lambda_i\}$ and $\{\mu_j\}$ via formulas (3.1)–(3.6) and (4.3).

For a minimum of function (4.8), it is necessary that the derivatives of $E^{(\alpha, \beta)}$ with respect to the parameters are zero, which yields the system

$$\begin{aligned} \sum_{h=1}^d \sum_{j=u}^{m-l-1} [F_{mj}(\mathbf{r}^h) - F_{nj}(\mathbf{p}^h)] \frac{\partial r_{jh}}{\partial \lambda_u} &= 0 \quad (u = 1, 2, \dots, k), \\ \sum_{h=1}^d \sum_{j=k+1}^{m-v} [F_{mj}(\mathbf{r}^h) - F_{nj}(\mathbf{p}^h)] \frac{\partial r_{jh}}{\partial \mu_v} &= 0 \quad (v = 1, 2, \dots, l), \end{aligned} \quad (4.9)$$

where we use notation

$$F_{tj}(\mathbf{q}) := \frac{1}{(\alpha + \beta + m + 2)_t} \binom{m}{j} \sum_{i=0}^t \binom{t}{i} (\alpha + 1)_{t+m-i-j} (\beta + 1)_{i+j} q_i$$

with $\mathbf{q} = [q_0, q_1, \dots, q_t]$.

In the case of $k = l = 3$, we compute the partial derivatives of h th coordinates of the control points (3.1)–(3.6). We obtain:

$$\frac{\partial r_{ih}}{\partial \lambda_1} = \begin{cases} \frac{n}{m} \Delta p_{0h} & (i = 1), \\ 2 \frac{n}{m} \Delta p_{0h} + 2 \lambda_1 \frac{(n-1)_2}{(m-1)_2} \Delta^2 p_{0h} & (i = 2), \\ 3 \frac{n}{m} \Delta p_{0h} + \left[2 \lambda_1 + \frac{1}{m-2} \lambda_2 \right] 3 \frac{(n-1)_2}{(m-1)_2} \Delta^2 p_{0h} + 3 \lambda_1^2 \frac{(n-2)_3}{(m-2)_3} \Delta^3 p_{0h} & (i = 3), \\ 0 & (i = 0; m-3 \leq i \leq m), \end{cases} \quad (4.10)$$

$$\frac{\partial r_{ih}}{\partial \lambda_2} = \begin{cases} \frac{n}{(m-1)_2} \Delta p_{0h} & (i = 2), \\ 3 \frac{n}{(m-1)_2} \Delta p_{0h} + 3 \lambda_1 \frac{(n-1)_2}{(m-2)_3} \Delta^2 p_{0h} & (i = 3), \\ 0 & (i = 0, 1; m-3 \leq i \leq m), \end{cases} \quad (4.11)$$

$$\frac{\partial r_{ih}}{\partial \lambda_3} = \begin{cases} \frac{n}{(m-2)_3} \Delta p_{0h} & (i = 3), \\ 0 & (i = 0, 1, 2; m-3 \leq i \leq m), \end{cases} \quad (4.12)$$

$$\frac{\partial r_{ih}}{\partial \mu_1} = \begin{cases} -\frac{n}{m} \Delta p_{n-1,h} & (i = m-1), \\ -2 \frac{n}{m} \Delta p_{n-1,h} + 2 \mu_1 \frac{(n-1)_2}{(m-1)_2} \Delta^2 p_{n-2,h} & (i = m-2), \\ -3 \frac{n}{m} \Delta p_{n-1,h} + \left[2 \mu_1 - \frac{1}{m-2} \mu_2 \right] 3 \frac{(n-1)_2}{(m-1)_2} \Delta^2 p_{n-2,h} - 3 \mu_1^2 \frac{(n-2)_3}{(m-2)_3} \Delta^3 p_{n-3,h} & (i = m-3), \\ 0 & (0 \leq i \leq 3; i = m), \end{cases} \quad (4.13)$$

$$\frac{\partial r_{ih}}{\partial \mu_2} = \begin{cases} \frac{n}{(m-1)_2} \Delta p_{n-1,h} & (i = m-2), \\ 3 \frac{n}{(m-1)_2} \Delta p_{n-1,h} - 3 \mu_1 \frac{(n-1)_2}{(m-2)_3} \Delta^2 p_{n-2,h} & (i = m-3), \\ 0 & (0 \leq i \leq 3; i = m-1, m), \end{cases} \quad (4.14)$$

$$\frac{\partial r_{ih}}{\partial \mu_3} = \begin{cases} -\frac{n}{(m-2)_3} \Delta p_{n-1,h} & (i = m-3), \\ 0 & (0 \leq i \leq 3; m-2 \leq i \leq m). \end{cases} \quad (4.15)$$

Notice that the partial derivatives of h th coordinates of control points (4.3) depend on (4.10)–(4.15) in the following way:

$$\frac{\partial r_{ih}}{\partial \lambda_u} = - \sum_{j=0}^n \binom{n}{j}^{-1} \sum_{g=u}^k \binom{n-m}{j-g} \binom{m}{g} \phi_{ij} \frac{\partial r_{gh}}{\partial \lambda_u}, \quad (4.16)$$

$$\frac{\partial r_{ih}}{\partial \mu_v} = - \sum_{j=0}^n \binom{n}{j}^{-1} \sum_{g=m-l}^{m-v} \binom{n-m}{j-g} \binom{m}{g} \phi_{ij} \frac{\partial r_{gh}}{\partial \mu_v}. \quad (4.17)$$

One can easily see, that when $k, l \leq 3$, we compute $\frac{\partial r_{ih}}{\partial \lambda_u}$, $\frac{\partial r_{ih}}{\partial \mu_v}$ by (4.16), (4.17) if $k < i < m-l$, and by (4.10)–(4.15) otherwise. Finally, we put the expressions (4.10)–(4.17) into the equations of system (4.9).

Observe that for $k \geq 2$ or $l \geq 2$, system (4.9) is nonlinear, which makes it quite difficult to solve. Furthermore, from a practical point of view, we additionally require that $\lambda_1, \mu_1 > 0$, which results in the same directions of tangent vectors at the endpoints of curves (1.1) and (1.3). Therefore, to guarantee that these conditions will be satisfied, it is not enough just to solve the system (4.9).

Now, let us discuss two possible ways of determining the values of geometric continuity parameters.

4.2.1. Determining the $G^{k,l}$ parameters using optimization methods

It is easy to check that if $(k = 1 \text{ and } l \leq 1)$ or $(l = 1 \text{ and } k \leq 1)$, then the error (4.6) is a quadratic function of continuity parameters.

In the case of $(k = 2 \text{ and } l \leq 2)$ or $(l = 2 \text{ and } k \leq 2)$, the error (4.6) is a fourth-degree polynomial function of continuity parameters.

For $(k = 3 \text{ and } l \leq 3)$ or $(l = 3 \text{ and } k \leq 3)$, the error (4.6) is a sixth-degree polynomial function of continuity parameters.

To find the optimum values of parameters λ_1, μ_1 in the case of $G^{1,1}$ -constrained multi-degree reduction problem, assuming that $\alpha, \beta = 0$, Lu and Wang [10] solve the quadratic programming problem, subject to the constraints

$$\lambda_1 \geq d_0, \quad \mu_1 \geq d_1, \quad (4.18)$$

where d_0 and d_1 are positive lower bounds, usually prescribed to small values (they set 10^{-4} for both lower bounds in the examples section). Such approach can be used in the cases which result in the quadratic error function (4.6). One can solve the quadratic programming problem using, e.g., an *iterative active-set method*, which is implemented in many software libraries. The active-set mechanism used by standard quadratic solvers is described in [1, §6.5].

Analogously, one can observe that for $k = 2, 3$ or $l = 2, 3$, the problem of minimizing the error (4.6), subject to the constraints (4.18) is a nonlinear programming problem. To solve it, one can use, for instance, a *sequential quadratic programming (SQP) method* (see, e.g., [1, §15.1]), which is available in many software libraries.

4.2.2. Determining the $C^{p,q}/G^{k,l}$ parameters by solving a system of linear equations

In the case of $G^{2,2}$, Rababah and Mann [11] simplified the problem by considering $C^{1,1}$ -continuity at the endpoints, i.e., they set $\lambda_1 = \mu_1 := 1$. Later, this approach was also used by Lu [6]. In [12], the same idea was used to simplify the $G^{3,3}$ case, and the authors noted that such approach leads to a system of linear equations.

Now, we generalize the above-described approach for any k, l such that $-1 \leq k, l \leq 3$. If $k \geq 2$, we set $\lambda_1 := 1$, which implies C^1 -continuity at $t = 0$ and consequently, $G^{k,l}$ constraints become $C^{1,q}/G^{k,l}$ constraints, where $q \in \{-1, 1\}$. Similarly, when $l \geq 2$, we set $\mu_1 := 1$, which implies C^1 -continuity at $t = 1$ and consequently, $G^{k,l}$ constraints become $C^{p,1}/G^{k,l}$ constraints, where $p \in \{-1, 1\}$.

Notice that in the cases of $k = 2, 3$ or $l = 2, 3$, the above-described method leads to the linear system (4.9) and the error (4.6) is a quadratic function of the continuity parameters. However, in the cases of $k = 1$ or $l = 1$, there is no guarantee that the solution satisfies $\lambda_1 > 0$ or $\mu_1 > 0$, respectively. In the case of solution with nonpositive values of these parameters, our choice is to solve a quadratic programming problem, subject to the constraints with prescribed positive lower bounds for the parameters (see (4.18)).

Observe that this approach uses no simplifying assumptions for $k, l \leq 1$.

Let us denote the above-described approach to Problem 1.1 as $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction of Bézier curves.

5. Algorithms

In this section, we show the details of implementation of the proposed method of $G^{k,l}$ -constrained multi-degree reduction of Bézier curves. Moreover, we give a short description of $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction algorithm.

5.1. $G^{k,l}$ -constrained multi-degree reduction algorithm

Now, we give the method of solving Problem 1.1, summarized in the following two-phase algorithm.

Phase A of the algorithm consists in finding values of the parameters λ_i and μ_j to minimize the error (4.6), which—by the results given in Theorem 4.2—depends only on these parameters. The idea is based on solving the quadratic or nonlinear programming problem (see §4.2.1). Notice that when $k, l < 1$, we can compute the ϕ_{ij} coefficients (cf. (4.7)) and omit the remaining steps of Phase A, since there are no continuity parameters to determine. During Phase B, we use the results of Theorem 4.2 and the obtained values of continuity parameters to compute control points $r_0, r_1, \dots, r_m$. Most of the known algorithms solve a system of normal equations, to get the inner control points of multi-degree reduced curve (1.3). Such approach makes these points dependent on the inverse of a certain matrix. Our formulas do not require matrix inversion. What is more, the complexity of Phase B is $O(mn)$, which is significantly less than the cost of other known methods for this phase. The algorithm works for any k and l not exceeding 3.

Algorithm 5.1. [$G^{k,l}$ -constrained multi-degree reduction]

Data: α, β – parameters of the inner product (2.1);

$n, p_0, p_1, \dots, p_n$ – degree and the control points of the Bézier curve (1.1);

m – degree of the reduced Bézier curve (1.3);

k, l – orders of the G -continuity at the endpoints of the curve (1.3);

d_0, d_1 – lower bounds for the parameters λ_1 and μ_1 , respectively (cf. §4.2.1).

Assumptions: $n > m > 0$; $-1 \leq k, l \leq 3$; $k + l < m - 1$; $d_0, d_1 > 0$; $\alpha, \beta > -1$.

Result: control points $r_0, r_1, \dots, r_m$ of the $G^{k,l}$ -constrained multi-degree reduced Bézier curve (1.3).

Phase A

Step I Compute $\{\phi_{ij}\}$ ($i = k+1, k+2, \dots, m-l-1$; $j = 0, 1, \dots, n$) by [5, Algorithm 4.2] and formula (4.7) (see Remark 4.3).

Step II Check if the remaining steps of Phase A can be omitted:

If ($k < 1$) and ($l < 1$) **then** go to **Step VI**.

Step III Compute $E^{(\alpha, \beta)}(\lambda_1, \lambda_2, \dots, \lambda_k, \mu_1, \mu_2, \dots, \mu_l)$ by (4.6).

Step IV Determine set c of constraints:

$c := \{\lambda_1 \geq d_0, \mu_1 \geq d_1\}$;

If ($k < 1$) **then** $c := c \setminus \{\lambda_1 \geq d_0\}$;

If ($l < 1$) **then** $c := c \setminus \{\mu_1 \geq d_1\}$.

Step V

If ($k > 1$ or $l > 1$) **then**

obtain $\lambda_1, \lambda_2, \dots, \lambda_k$, and $\mu_1, \mu_2, \dots, \mu_l$ by solving the nonlinear programming problem of minimizing the error (4.6), subject to the constraints c ;

else

obtain $\lambda_1, \lambda_2, \dots, \lambda_k$, and $\mu_1, \mu_2, \dots, \mu_l$ by solving the quadratic programming problem of minimizing the error (4.6), subject to the constraints c .

Phase B

Step VI Compute

1. $r_0, r_1, \dots, r_k$ by (3.1)–(3.3);

2. $r_{m-l}, r_{m-l+1}, \dots, r_m$ by (3.4)–(3.6).

Step VII Compute $v_0, v_1, \dots, v_n$ by (4.5).

Step VIII Compute $r_{k+1}, r_{k+2}, \dots, r_{m-l-1}$ by (4.3).

Step IX Return the solution, i.e., the control points $r_0, r_1, \dots, r_m$ of the reduced Bézier curve (1.3).

5.2. $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction algorithm

Now, let us give the outline of the two-phase $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction algorithm.

Phase A of the algorithm implements the ideas discussed in §4.2.2, therefore, it solves the system of linear equations (4.9) to determine values of the continuity parameters. In the case of solution with nonpositive values of λ_1 or μ_1 , which can happen when $k = 1$ or $l = 1$, the algorithm solves a quadratic programming problem, subject to the constraints with prescribed positive lower bounds for the parameters (see (4.18)). An example of a resulting Bézier curve that does not satisfy the positive condition can be found in [6, Fig. 1(a)]. We performed more than 40 different tests (results of some of them are available in the next section). None of them caused such problem. Phase B is the same as for Algorithm 5.1. The algorithm works for any k and l not exceeding 3. For details, see our implementation in MapleTM13 available on the website <http://www.ii.uni.wroc.pl/~pgo/GDegRed.mws>.

Obviously, Algorithm 5.1 costs more, but also produces more accurate results, since for the $C^{p,q}/G^{k,l}$ -constrained approach we additionally assume that $\lambda_1 = 1$ when $k > 1$, and $\mu_1 = 1$ when $l > 1$.

6. Examples

This section provides of the application of our $G^{k,l}$ -constrained and $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction algorithms. In each case, we give the least squares error $E_2^{(\alpha,\beta)} := \sqrt{E^{(\alpha,\beta)}}$ and the maximum error

$$E_\infty := \max_{t \in D_N} \|P_n(t) - R_m(t)\| \approx \max_{t \in [0,1]} \|P_n(t) - R_m(t)\|,$$

where $D_N := \{0, 1/N, 2/N, \dots, 1\}$ with $N := 500$.

In our experiments, we consider the “natural” choices for the values of parameters α, β , i.e., $(\alpha, \beta) \in \{(0, 0), (\frac{1}{2}, \frac{1}{2}), (-\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, -\frac{1}{2}), (-\frac{1}{2}, -\frac{1}{2})\}$, and set the lower bounds d_0, d_1 of λ_1, μ_1 to 10^{-4} (see (4.18)). Taking into account the different types of continuity constraints, we compare the following cases:

- (i) $C^{k,l}$ -constrained case (see (1.5)), which can be solved by using Theorem 4.2;
- (ii) $C^{p,q}/G^{k,l}$ -constrained case, solved by the algorithm described in §5.2;
- (iii) $G^{k,l}$ -constrained case, solved by Algorithm 5.1.

Results of the experiments have been obtained on a computer with Intel Core i5-3337U 1.8GHz processor and 8GB of RAM, using 32-digit arithmetic. MapleTM13 worksheet containing implementation of the algorithms and tests is available on the website <http://www.ii.uni.wroc.pl/~pgo/GDegRed.mws>. We use MapleTM `fsolve` procedure, in the $C^{p,q}/G^{k,l}$ case, to solve the system of linear equations, and `QPSolve`, `NLPSolve` procedures, to solve the quadratic and nonlinear programming problems, respectively. `QPSolve` uses the iterative active-set method, and for `NLPSolve` we select `sqp` method. Initial points for both procedures correspond to the values of continuity parameters in the $C^{k,l}$ case.

Example 6.1. First, let us consider degree eleven Bézier curve which is an outline of the font “alpha” (for the control points, see [15, Example 6.1]). The results of multi-degree reduction are given in Table 1. Figs. 1a and 1b illustrate two of the considered cases. One can see, that when it comes to minimizing E_∞ error, usually a good choice is $\alpha = \beta = -\frac{1}{2}$. As expected, solution to the $G^{k,l}$ case is the most accurate, while $C^{p,q}/G^{k,l}$ approach gives less precise results. $C^{k,l}$ conditions seem to be too restrictive, especially for k or l exceeding 2.

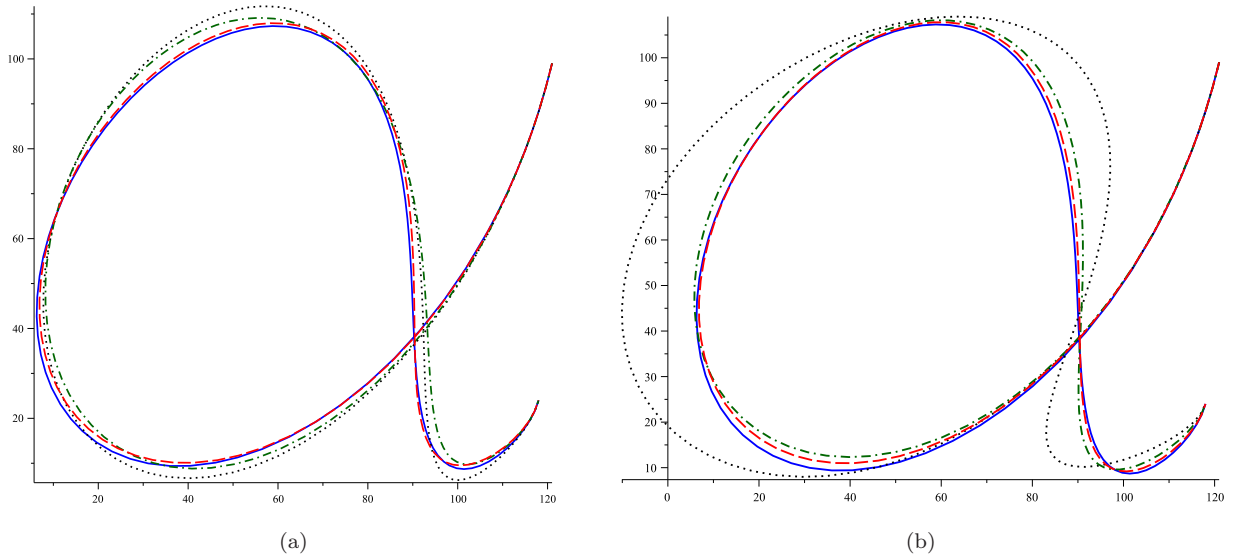


Figure 1: Multi-degree reduction of degree eleven curve (blue solid line) to degree seven with $C^{k,l}$ (black dotted line), $C^{p,q}/G^{k,l}$ (green dash-dotted line) and $G^{k,l}$ (red dashed line) continuity constraints; parameters: (a) $\alpha = \beta = -\frac{1}{2}$, $p = q = 1$, $k = l = 2$, and (b) $\alpha = \beta = -\frac{1}{2}$, $p = 1$, $q = -$, $k = 3$ and $l = 1$.

Parameters							$C^{k,l}$ solution		$C^{p,q}/G^{k,l}$ solution		$G^{k,l}$ solution	
m	k	l	p	q	α	β	$E_2^{(\alpha,\beta)}$	E_∞	$E_2^{(\alpha,\beta)}$	E_∞	$E_2^{(\alpha,\beta)}$	E_∞
7	2	2	1	1	0	0	3.73	5.97	2.83	5.27	0.93	2.26
					$-\frac{1}{2}$	$-\frac{1}{2}$	5.75	5.83	4.40	5.14	1.67	1.91
					$-\frac{1}{2}$	$\frac{1}{2}$	3.83	7.53	2.62	6.42	0.79	2.91
					$\frac{1}{2}$	$-\frac{1}{2}$	3.83	7.69	3.18	5.18	1.26	2.19
					$\frac{1}{2}$	$\frac{1}{2}$	2.43	6.10	1.83	5.40	0.53	2.54
7	3	1	1	—	0	0	9.13	16.24	2.51	5.11	1.02	2.41
					$-\frac{1}{2}$	$-\frac{1}{2}$	13.97	16.72	4.07	4.95	1.81	2.07
					$-\frac{1}{2}$	$\frac{1}{2}$	9.41	19.51	2.18	6.38	0.75	3.11
					$\frac{1}{2}$	$-\frac{1}{2}$	9.17	18.79	2.98	4.91	1.45	1.99
					$\frac{1}{2}$	$\frac{1}{2}$	6.00	15.82	1.56	5.25	0.59	2.69

Table 1: Least squares error and maximum error in multi-degree reduction of degree eleven Bézier “alpha” curve.

Example 6.2. Let us apply the algorithms to degree thirteen Bézier “heart” curve (for the control points, see [12, Appendix B]) and consider the case of $k = l = 2$. The results of experiments are given in Table 2. Notice that the case of $\alpha = \beta = 0$ was also considered in [12, §5.2] and [17, Example 4]. As in [17], we can clearly see that the solution to the $G^{2,2}$ case, in this paper obtained by Algorithm 5.1, is more accurate than the result given by the approach proposed in [12], which leads to the $C^{1,1}/G^{2,2}$ case (the same as for the algorithm discussed in §5.2). As our approach considers different weight functions, it can be seen that the best choice to minimize E_∞ error is $\alpha = \beta = -\frac{1}{2}$. Fig. 2 presents $\alpha = \beta = 0$ case.

Now, we focus on the running times. We have implemented $G^{1,1}$, $C^{1,1}/G^{2,2}$ and $C^{1,1}/G^{3,3}$ -constrained methods from [12], $G^{1,1}$ and $C^{1,1}/G^{2,2}$ -constrained methods from [6] as well as $G^{2,1}$ and $G^{2,2}$ -constrained methods from [17]. The methods of Rababah and Mann and of Lu solve the same problem and give the same results as our $C^{p,q}/G^{k,l}$ -constrained method (see §5.2). In Table 3, we compare the running times of these algorithms. Clearly, our approach is the fastest one. For the comparison of the $G^{k,l}$ -constrained algorithms, see Table 4. Notice that, in some cases, our $G^{k,l}$ -constrained approach is slightly faster than the methods from [17]. We use MapleTM `fsolve` procedure to solve the cubic equation [17, (23)] associated with the $G^{2,1}$ -constrained case. The implementation of $G^{2,2}$ -constrained method from [17] requires the *unconstrained nonlinear programming solver*. According to our experiments, the *nonlinear simplex method* (NLPSolve command with option `method = nonlinearsimplex` and the initial point $\lambda = \eta = 1$) is the fastest solver available in MapleTM13. Therefore, we use this solver for the purpose of the comparison. It is worth mentioning that Zhou et al. have omitted the constraints (4.18). Consequently, in some rare cases, the resulting curve may not preserve the original tangent directions at the endpoints. To avoid this issue, one can implement the improvements proposed by Lu [8].

Parameters			$C^{2,2}$ solution		$C^{1,1}/G^{2,2}$ solution		$G^{2,2}$ solution	
m	α	β	$E_2^{(\alpha,\beta)}$	E_∞	$E_2^{(\alpha,\beta)}$	E_∞	$E_2^{(\alpha,\beta)}$	E_∞
8	0	0	1.52	2.52	0.64	1.12	0.36	0.71
	$-\frac{1}{2}$	$-\frac{1}{2}$	2.37	2.39	1.05	1.00	0.62	0.53
	$-\frac{1}{2}$	$\frac{1}{2}$	1.58	3.34	0.64	1.55	0.42	0.94
	$\frac{1}{2}$	$-\frac{1}{2}$	1.52	3.48	0.74	1.31	0.37	0.89
	$\frac{1}{2}$	$\frac{1}{2}$	0.98	2.64	0.39	1.24	0.21	0.90

Table 2: Least squares error and maximum error in multi-degree reduction of degree thirteen Bézier “heart” curve.

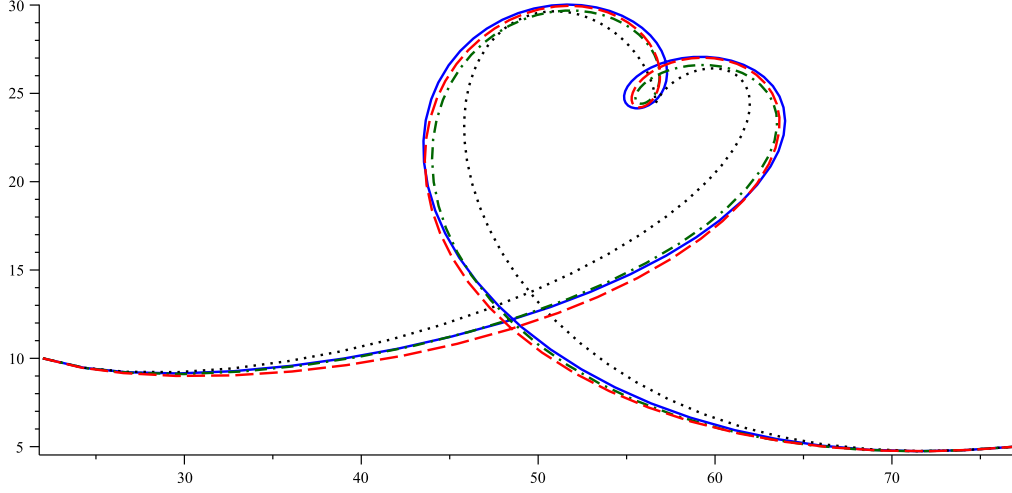


Figure 2: Multi-degree reduction of degree thirteen curve (blue solid line) to degree eight with $C^{2,2}$ (black dotted line), $C^{1,1}/G^{2,2}$ (green dash-dotted line) and $G^{2,2}$ (red dashed line) continuity constraints; parameters: $\alpha = \beta = 0$.

Parameters					Running times [ms]		
m	k	l	p	q	Our $C^{p,q}/G^{k,l}$ method	Rababah and Mann [12]	Lu [6]
8	1	1	—	—	32	62	63
10	1	1	—	—	31	94	63
12	1	1	—	—	47	94	62
8	2	2	1	1	16	63	47
10	2	2	1	1	31	63	62
12	2	2	1	1	31	78	78
8	3	3	1	1	15	78	—
10	3	3	1	1	31	109	—
12	3	3	1	1	63	110	—

Table 3: Running times of $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction of degree thirteen Bézier “heart” curve; parameters: $\alpha = \beta = 0$.

Parameters			Running times [ms]	
m	k	l	Our $G^{k,l}$ method	Zhou et al. [17]
8	2	1	92	108
10	2	1	121	137
12	2	1	168	166
8	2	2	153	204
10	2	2	298	248
12	2	2	290	292

Table 4: Running times of $G^{k,l}$ -constrained multi-degree reduction of degree thirteen Bézier “heart” curve; parameters: $\alpha = \beta = 0$.

7. Conclusions

In this paper, we propose efficient methods of solving the problems of $G^{k,l}$ -constrained and $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction of Bézier curves with respect to the least squares norm. We give two-phase algorithms of solving these problems.

The first phase of the algorithms consists in finding values of the geometric continuity parameters to minimize the error (4.2). In the case of $G^{k,l}$ -constrained problem, we solve the quadratic or nonlinear programming problem to obtain these values. For $C^{p,q}/G^{k,l}$ -constrained case, we use some simplifying assumptions, i.e., we impose constraints of C^1 -continuity at $t = 0$ when $k > 1$, and at $t = 1$ when $l > 1$. Therefore, by fixing some of the parameters, this approach leads to the system of linear equations (4.9). Assuming that $-1 \leq k, l \leq 3$ and including the hybrid cases, there are 37 continuity cases which require computation of the continuity parameters. Those variants of the problem differ, and we have not proven that in each case a unique solution exists.

During the second phase, which is the same for both approaches, we use the properties of constrained dual Bernstein basis polynomials to compute control points of the multi-degree reduced curve. The complexity of this phase is $O(mn)$, where n and m are the degrees of the input and output curves, respectively. This is significantly less than complexity of other algorithms. Moreover, our approach avoids matrix inversion.

As expected, solution to $G^{k,l}$ -constrained problem is the most accurate, while the one obtained by $C^{p,q}/G^{k,l}$ -constrained multi-degree reduction is less precise. $C^{k,l}$ conditions tend to be too restrictive, especially for k or l exceeding 2. Comparison of running times of our $C^{p,q}/G^{k,l}$ -constrained approach with analogous methods from [6, 12] shows advantage of our algorithm in practice. Furthermore, the experiments show that our $G^{k,l}$ -constrained approach is comparable to the methods of [17], even slightly faster in some cases.

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