

# New error bounds for linear complementarity problems of Nekrasov matrices and $B$ -Nekrasov matrices

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**Abstract** New error bounds for the linear complementarity problems are given respectively when the involved matrices are Nekrasov matrices and  $B$ -Nekrasov matrices. Numerical examples are given to show that the new bounds are better respectively than those provided by García-Esnaola and Peña in [15,16] in some cases.

**Keywords** Error bounds · Linear complementarity problem · Nekrasov matrices ·  $B$ -Nekrasov matrices ·  $P$ -matrices

## 1 Introduction

Linear complementarity problem  $\text{LCP}(M, q)$  is to find a vector  $x \in R^n$  such that

$$x \geq 0, Mx + q \geq 0, (Mx + q)^T x = 0 \quad (1)$$

or to show that no such vector  $x$  exists, where  $M = [m_{ij}] \in R^{n \times n}$  and  $q \in R^n$ . The  $\text{LCP}(M, q)$  has various applications in the Nash equilibrium point of a bimatrix game, the contact problem and the free boundary problem for journal bearing, for details, see [1, 5, 21].

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The LCP( $M, q$ ) has a unique solution for any  $q \in R^n$  if and only if  $M$  is a  $P$ -matrix [5]. We here say a matrix  $M \in R^{n,n}$  is a  $P$ -matrix if all its principal minors are positive. In [3], Chen and Xiang gave the following error bound of the LCP( $M, q$ ) when  $M$  is a  $P$ -matrix:

$$\|x - x^*\|_\infty \leq \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where  $x^*$  is the solution of the LCP( $M, q$ ),  $r(x) = \min\{x, Mx + q\}$ ,  $D = \text{diag}(d_i)$  with  $0 \leq d_i \leq 1$ ,  $d = [d_1, \dots, d_n]^T \in [0, 1]^n$  denotes  $0 \leq d_i \leq 1$  for each  $i \in N$ , and the min operator  $r(x)$  denotes the componentwise minimum of two vectors. Furthermore, if  $M$  is a certain structure matrix, such as an  $H$ -matrix with positive diagonals [3, 4, 12, 13, 15], a  $B$ -matrix [6, 11], a  $DB$ -matrix [7], an  $SB$ -matrix [8, 9], a  $B^S$ -matrix [14], an  $MB$ -matrix [2], and a  $B$ -Nekrasov matrix [16], then some corresponding results on the bound of  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$  can be derived; for details, see [2, 3, 4, 7, 8, 9, 10, 12, 13, 14, 15].

In this paper, we focus on the bound of  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ , and give its new bounds when  $M$  is a Nekrasov matrix with positive diagonals and a  $B$ -Nekrasov matrix, respectively. Numerical examples are given to show the new bounds are respectively better than those in [15] and [16] in some cases.

## 2 Error bounds for linear complementarity problems of Nekrasov matrices

García-Esnaola and Peña in [15] provided the following bound for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ , when  $M$  is a Nekrasov matrix with positive diagonals. Here, a matrix  $A = [a_{ij}] \in C^{n,n}$  is called a Nekrasov matrix [17, 18] if for each  $i \in N = \{1, 2, \dots, n\}$ ,

$$|a_{ii}| > h_i(A),$$

where  $h_1(A) = \sum_{j \neq 1} |a_{1j}|$  and  $h_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} h_j(A) + \sum_{j=i+1}^n |a_{ij}|$ ,  $i = 2, 3, \dots, n$ .

**Theorem 1** [15, Theorem 3] *Let  $M = [m_{ij}] \in R^{n,n}$  be a Nekrasov matrix with  $m_{ii} > 0$  for  $i \in N$  such that for each  $i = 1, 2, \dots, n-1$ ,  $m_{ij} \neq 0$  for some  $j > i$ . Let  $W = \text{diag}(w_1, \dots, w_n)$  with  $w_i = \frac{h_i(M)}{m_{ii}}$  for  $i = 1, 2, \dots, n-1$  and  $w_n = \frac{h_n(M)}{m_{nn}} + \varepsilon$ ,  $\varepsilon \in \left(0, 1 - \frac{h_n(M)}{m_{nn}}\right)$ . Then*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \max \left\{ \frac{\max_{i \in N} w_i}{\min_{i \in N} s_i}, \frac{\max_{i \in N} w_i}{\min_{i \in N} w_i} \right\}, \quad (2)$$

where for each  $i = 1, 2, \dots, n-1$ ,  $s_i = \sum_{j=i+1}^n |m_{ij}|(1 - w_j)$  and  $s_n = \varepsilon m_{nn}$ .

It is not difficult to see that when  $M = [m_{ij}] \in R^{n,n}$  is a Nekrasov matrix with  $m_{ij} = 0$  for any  $j > i$  and for some  $i \in \{1, 2, \dots, n-1\}$ , Theorem 1 cannot be used to estimate  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ , and that when  $\varepsilon \rightarrow 0$ ,

$$s_n = \varepsilon m_{nn} \rightarrow 0 \text{ and } \min_{i \in N} s_i \rightarrow 0,$$

which gives the bound

$$\max \left\{ \frac{\max_{i \in N} w_i}{\min_{i \in N} s_i}, \frac{\max_{i \in N} w_i}{\min_{i \in N} w_i} \right\} \rightarrow +\infty.$$

These facts show that in some cases the bound in Theorem 1 is not always effective to estimate  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$  when  $M$  is a Nekrasov matrix with positive diagonals. To conquer these two drawbacks, we next give a new bound which only depends on the entries of  $M$ . Before that, some results on Nekrasov matrices which will be used later are given as follows.

**Lemma 1** *Let  $M = [m_{ij}] \in C^{n,n}$  be a Nekrasov matrix with  $m_{ii} > 0$  for  $i \in N$  and let  $\tilde{M} = I - D + DM = [\tilde{m}_{ij}]$  where  $D = \text{diag}(d_i)$  with  $0 \leq d_i \leq 1$ . Then  $\tilde{M}$  is a Nekrasov matrix. Furthermore, for each  $i \in N$ ,*

$$\frac{h_i(\tilde{M})}{\tilde{m}_{ii}} \leq \frac{h_i(M)}{m_{ii}}. \quad (3)$$

*Proof* We prove that (3) holds by mathematical induction, and then (3) immediately implies that  $\tilde{M}$  is a Nekrasov matrix. Note that

$$\tilde{m}_{ij} = \begin{cases} 1 - d_i + d_i m_{ij}, & i = j, \\ d_i m_{ij}, & i \neq j. \end{cases}$$

Hence, for each  $i \in N$ ,

$$\frac{d_i}{\tilde{m}_{ii}} = \frac{d_i}{1 - d_i + d_i m_{ii}} \leq \frac{1}{m_{ii}}, \text{ for } 0 \leq d_i \leq 1, i \in N.$$

Then we have that for  $i = 1$ ,

$$\begin{aligned} \frac{h_1(\tilde{M})}{\tilde{m}_{11}} &= \frac{d_1 \sum_{j \neq 1} |m_{1j}|}{1 - d_1 + m_{11} d_1} \\ &\leq \frac{\sum_{j \neq 1} |m_{1j}|}{m_{11}} \\ &= \frac{h_1(M)}{m_{11}}. \end{aligned}$$

Now suppose that (3) holds for  $i = 2, 3, \dots, k$  and  $k < n$ . Since

$$\begin{aligned}
\frac{h_{k+1}(\tilde{M})}{\tilde{m}_{k+1,k+1}} &= \frac{\sum_{j=1}^k |\tilde{m}_{k+1,j}| \frac{h_j(\tilde{M})}{\tilde{m}_{jj}} + \sum_{j=k+2}^n |\tilde{m}_{k+1,j}|}{\tilde{m}_{k+1,k+1}} \\
&\leq \frac{\sum_{j=1}^k |\tilde{m}_{k+1,j}| \frac{h_j(M)}{m_{jj}} + \sum_{j=k+2}^n |\tilde{m}_{k+1,j}|}{\tilde{m}_{k+1,k+1}} \\
&= \frac{d_{k+1} \left( \sum_{j=1}^k |m_{k+1,j}| \frac{h_j(M)}{m_{jj}} + \sum_{j=k+2}^n |m_{k+1,j}| \right)}{1 - d_{k+1} + m_{k+1,k+1} d_{k+1}} \\
&\leq \frac{\sum_{j=1}^k |m_{k+1,j}| \frac{h_j(M)}{m_{jj}} + \sum_{j=k+2}^n |m_{k+1,j}|}{m_{k+1,k+1}} \\
&= \frac{h_{k+1}(M)}{m_{k+1,k+1}},
\end{aligned}$$

by mathematical induction we have that for each  $i \in N$ , (3) holds. Furthermore, the fact that  $M$  is a Nekrasov matrix yields

$$\frac{h_i(M)}{m_{ii}} < 1 \text{ for each } i \in N.$$

By (3) we can conclude that

$$\frac{h_i(\tilde{M})}{\tilde{m}_{ii}} < 1 \text{ for each } i \in N,$$

equivalently,  $|\tilde{m}_{ii}| > h_i(\tilde{M})$  for each  $i \in N$ , consequently,  $\tilde{M}$  is a Nekrasov matrix.

**Lemma 2** [19, Lemma 3] *Let  $\gamma > 0$  and  $\eta \geq 0$ . Then for any  $x \in [0, 1]$ ,*

$$\frac{1}{1 - x + \gamma x} \leq \frac{1}{\min\{\gamma, 1\}} \quad (4)$$

and

$$\frac{\eta x}{1 - x + \gamma x} \leq \frac{\eta}{\gamma}. \quad (5)$$

Lemma 2 will be used in the proofs of the following lemma and of Theorem 2.

**Lemma 3** *Let  $M = [m_{ij}] \in C^{n,n}$  be a Nekrasov matrix with  $m_{ii} > 0$  for  $i \in N$  and let  $\tilde{M} = I - D + DM = [\tilde{m}_{ij}]$  where  $D = \text{diag}(d_i)$  with  $0 \leq d_i \leq 1$ . Then*

$$z_i(\tilde{M}) \leq \eta_i(M) \quad (6)$$

and

$$\frac{z_i(\tilde{M})}{\tilde{m}_{ii}} \leq \frac{\eta_i(M)}{\min\{m_{ii}, 1\}}, \quad (7)$$

where  $z_1(\tilde{M}) = \eta_1(M) = 1$ ,  $z_i(\tilde{M}) = \sum_{j=1}^{i-1} \frac{|\tilde{m}_{ij}|}{|\tilde{m}_{jj}|} z_j(\tilde{M}) + 1$ , and

$$\eta_i(M) = \sum_{j=1}^{i-1} \frac{|m_{ij}|}{\min\{|m_{jj}|, 1\}} \eta_j(M) + 1, \quad i = 2, 3, \dots, n.$$

*Proof* We only prove (6), and (7) follows from the fact that

$$\frac{1}{\tilde{m}_{ii}} = \frac{1}{1 - d_i + d_i m_{ii}} \leq \frac{1}{\min\{m_{ii}, 1\}} \quad \text{for } i \in N.$$

Note that

$$z_1(\tilde{M}) \leq \eta_1(M).$$

We now suppose that (6) holds for  $i = 2, 3, \dots, k$  and  $k < n$ . Since

$$\begin{aligned} z_{k+1}(\tilde{M}) &= \sum_{j=1}^k |\tilde{m}_{k+1,j}| \frac{z_j(\tilde{M})}{|\tilde{m}_{jj}|} + 1 \\ &\leq \sum_{j=1}^k |\tilde{m}_{k+1,j}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1 \\ &= d_{k+1} \sum_{j=1}^k |m_{k+1,j}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1 \\ &\leq \sum_{j=1}^k |m_{k+1,j}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1 \\ &= \eta_{k+1}(M), \end{aligned}$$

by mathematical induction we have that for each  $i \in N$ , (6) holds.

**Lemma 4** [17, Theorem 2] *Let  $A = [a_{ij}] \in C^{n,n}$  be a Nekrasov matrix. Then*

$$\|A^{-1}\|_{\infty} \leq \max_{i \in N} \frac{z_i(A)}{|a_{ii}| - h_i(A)}, \quad (8)$$

where  $z_1(A) = 1$  and  $z_i(A) = \sum_{j=1}^{i-1} \frac{|a_{ij}|}{|a_{jj}|} z_j(A) + 1, i = 2, 3, \dots, n$ .

By Lemmas 1, 2, 3 and 4, we can obtain the following bound for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty}$  when  $M$  is a Nekrasov matrix.

**Theorem 2** Let  $M = [m_{ij}] \in R^{n,n}$  be a Nekrasov matrix with  $m_{ii} > 0$  for  $i \in N$  and let  $\tilde{M} = I - D + DM$  where  $D = \text{diag}(d_i)$  with  $0 \leq d_i \leq 1$ . Then

$$\max_{d \in [0,1]^n} \|\tilde{M}^{-1}\|_\infty \leq \max_{i \in N} \frac{\eta_i(M)}{\min\{m_{ii} - h_i(M), 1\}}, \quad (9)$$

where  $\eta_i(M)$  is defined in Lemma 3.

*Proof* Let  $\tilde{M} = I - D + DM = [\tilde{m}_{ij}]$ . By Lemma 1 and Lemma 4, we have that  $\tilde{M}$  is a Nekrasov matrix, and

$$\|\tilde{M}^{-1}\|_\infty \leq \max_{i \in N} \frac{z_i(\tilde{M})}{\tilde{m}_{ii} - h_i(\tilde{M})}. \quad (10)$$

Note that

$$\begin{aligned} \frac{z_1(\tilde{M})}{\tilde{m}_{11} - h_1(\tilde{M})} &= \frac{1}{\tilde{m}_{11} - \sum_{j=2}^n |\tilde{m}_{1j}|} \\ &= \frac{1}{1 - d_1 + m_{11}d_1 - \sum_{j=2}^n |m_{1j}|d_1} \\ &\leq \frac{1}{\min\{m_{11} - \sum_{j=2}^n |m_{1j}|, 1\}} \\ &= \frac{\eta_1(M)}{\min\{m_{11} - h_1(M), 1\}} \end{aligned}$$

and for  $i = 2, 3, \dots, n$ , we have by Lemma 3 and (3) that

$$\begin{aligned} \frac{z_i(\tilde{M})}{\tilde{m}_{ii} - h_i(\tilde{M})} &= \frac{\sum_{j=1}^{i-1} |\tilde{m}_{ij}| \frac{z_j(\tilde{M})}{\tilde{m}_{jj}} + 1}{\tilde{m}_{ii} - \left( \sum_{j=1}^{i-1} |\tilde{m}_{ij}| \frac{h_j(\tilde{M})}{\tilde{m}_{jj}} + \sum_{j=i+1}^n |\tilde{m}_{ij}| \right)} \\ &\leq \frac{\sum_{j=1}^{i-1} |\tilde{m}_{ij}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1}{\tilde{m}_{ii} - \left( \sum_{j=1}^{i-1} |\tilde{m}_{ij}| \frac{h_j(M)}{m_{jj}} + \sum_{j=i+1}^n |\tilde{m}_{ij}| \right)} \\ &= \frac{d_i \sum_{j=1}^{i-1} |m_{ij}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1}{1 - d_i + m_{ii}d_i - d_i \left( \sum_{j=1}^{i-1} |m_{ij}| \frac{h_j(M)}{m_{jj}} + \sum_{j=i+1}^n |m_{ij}| \right)} \end{aligned}$$

$$\begin{aligned}
& \sum_{j=1}^{i-1} |m_{ij}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1 \\
& \leq \frac{\sum_{j=1}^{i-1} |m_{ij}| \frac{\eta_j(M)}{\min\{m_{jj}, 1\}} + 1}{1 - d_i + m_{ii}d_i - d_i \left( \sum_{j=1}^{i-1} |m_{ij}| \frac{h_j(M)}{m_{jj}} + \sum_{j=i+1}^n |m_{ij}| \right)} \\
& \leq \frac{\eta_i(M)}{\min\{m_{ii} - h_i(M), 1\}}.
\end{aligned}$$

Therefore, by (10) we have

$$\|\tilde{M}^{-1}\|_{\infty} \leq \max_{i \in N} \frac{z_i(\tilde{M})}{\tilde{m}_{ii} - h_i(\tilde{M})} \leq \max_{i \in N} \frac{\eta_i(M)}{\min\{m_{ii} - h_i(M), 1\}}.$$

The conclusion follows.

Remark here that when  $m_{ii} = 1$  for all  $i \in N$  in Theorem 2, then

$$\min\{m_{ii} - h_i(M), 1\} = 1 - h_i(M),$$

which yields the following result.

**Corollary 1** *Let  $M = [m_{ij}] \in R^{n,n}$  be a Nekrasov matrix with  $m_{ii} = 1$  for  $i \in N$  and let  $\tilde{M} = I - D + DM$  where  $D = \text{diag}(d_i)$  with  $0 \leq d_i \leq 1$ . Then*

$$\max_{d \in [0,1]^n} \|\tilde{M}^{-1}\|_{\infty} \leq \max_{i \in N} \frac{\eta_i(M)}{1 - h_i(M)}.$$

*Example 1* Consider the following matrix

$$M = \begin{bmatrix} 5 & -\frac{1}{5} & -\frac{2}{5} & -\frac{1}{2} \\ -\frac{1}{10} & 2 & -\frac{1}{2} & -\frac{1}{10} \\ -\frac{1}{2} & -\frac{1}{10} & 1.5 & -\frac{1}{10} \\ -\frac{2}{5} & -\frac{2}{5} & -\frac{4}{5} & 1.2 \end{bmatrix}.$$

By computations,

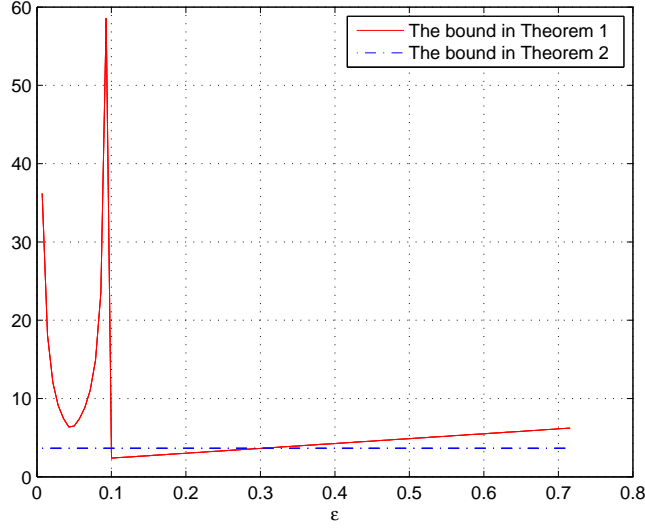
$$h_1(M) = 1.1000 < |m_{11}|, h_2(M) = 0.6220 < |m_{22}|,$$

$$h_3(M) = 0.2411 < |m_{33}|, \text{ and } h_4(M) = 0.3410 < |m_{44}|.$$

Hence,  $M$  is a Nekrasov matrix. The diagonal matrix  $W$  in Theorem 1 is given by

$$W = \text{diag}(0.2200, 0.3110, 0.1607, 0.2842 + \varepsilon)$$

with  $\varepsilon \in (0, 0.7158)$ . Hence, by Theorem 1 we can get the bound (2) involved with  $\varepsilon \in (0, 0.7158)$  for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty}$ , which is drawn in Figure 1. Furthermore, by Theorem 2, we can get that the bound (9) for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty}$  is 3.6414. It is easy to see from Figure 1 that the bound in Theorem 2 is smaller than that in Theorem 1 (Theorem 3 in [15]) in some cases.



**Fig. 1** The bounds in Theorems 1 and 2

*Example 2* Consider the following Nekrasov matrix

$$M = \begin{bmatrix} 1 & -\frac{2}{5} & -\frac{2}{5} & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{3}{5} & -\frac{2}{5} & 1 & 0 \\ -\frac{1}{5} & -\frac{3}{5} & -\frac{2}{5} & 1 \end{bmatrix}.$$

Since  $m_{34} = 0$ , we cannot use the bound (2) in Theorem 1. However, by Theorem 2, we have

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq 15.$$

### 3 Error bounds for linear complementarity problems of $B$ -Nekrasov matrices

The class of  $B$ -Nekrasov matrices is introduced by García-Esnaola and Peña [16] as a subclass of  $P$ -matrices. We say that  $M$  is a  $B$ -Nekrasov matrix if it can be written as

$$M = B^+ + C, \quad (11)$$

where

$$B^+ = [b_{ij}] = \begin{bmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{bmatrix}, \text{ and } C = \begin{bmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & & \vdots \\ r_n^+ & \cdots & r_n^+ \end{bmatrix}$$



with  $r_i^+ = \max\{0, m_{ij} | j \neq i\}$  and  $B^+$  is a Nekrasov matrix whose diagonal entries are all positive. Obviously,  $B^+$  is a  $Z$ -matrix and  $C$  is a nonnegative matrix of rank 1 [16]. Also in [16], García-Esnaola and Peña provided the following error bound for  $\text{LCP}(M, q)$  when  $M$  is a  $B$ -Nekrasov matrix.

**Theorem 3** [16, Theorem 2] *Let  $M = [m_{ij}] \in R^{n,n}$  be a  $B$ -Nekrasov matrix such that for each  $i = 1, 2, \dots, n-1$  there exists  $k > i$  with  $m_{ik} < \max\{0, m_{ij} | j \neq i\} = r_i^+$ , let  $B^+$  be the matrix of (11) and let  $W = \text{diag}(w_1, \dots, w_n)$  with  $w_i = \frac{h_i(B^+)}{m_{ii} - r_i^+}$  for  $i = 1, 2, \dots, n-1$  and  $w_n = \frac{h_n(B^+)}{m_{nn} - r_n^+} + \varepsilon$ ,  $\varepsilon \in (0, 1 - \frac{h_n(B^+)}{m_{nn} - r_n^+})$ , such that  $\bar{B} = B^+W = [\bar{b}_{ij}]$  is a strictly diagonally dominant  $Z$ -matrix. Let  $\beta_i = \bar{b}_{ii} - \sum_{j \neq i} |\bar{b}_{ij}|$  and  $\delta_i = \frac{\beta_i}{w_i}$  for  $i \in N$ , and  $\delta = \min_{i \in N} \delta_i$ . Then*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{(n-1) \max_{i \in N} w_i}{\min\{\delta, 1\} \min\{w_i\}}. \quad (12)$$

Remark here that the bound (12) in Theorem 3 has some drawbacks because it is involved with a parameter  $\varepsilon$  in the interval  $(0, 1 - \frac{h_n(B^+)}{m_{nn} - r_n^+})$  and it is not easy to decide the optimum value of  $\varepsilon$  in general. Based on the results obtained in Section 2, we next give a new bound, which only depends on the entries of  $M$ , for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$  when  $M$  is a  $B$ -Nekrasov matrix.

**Theorem 4** *Let  $M = [m_{ij}] \in R^{n \times n}$  be a  $B$ -Nekrasov matrix, and let  $B^+ = [b_{ij}]$  be the matrix of (11). Then*

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \max_{i \in N} \frac{(n-1)\eta_i(B^+)}{\min\{b_{ii} - h_i(B^+), 1\}}, \quad (13)$$

where  $\eta_1(B^+) = 1$ , and

$$\eta_i(B^+) = \sum_{j=1}^{i-1} \frac{|b_{ij}|}{\min\{b_{jj}, 1\}} \eta_j(B^+) + 1, \quad i = 2, 3, \dots, n.$$

*Proof* Since  $M$  is a  $B$ -Nekrasov matrix,  $M = B^+ + C$  as in (11), with  $B^+$  being a Nekrasov  $Z$ -matrix with positive diagonal entries. Given a diagonal matrix  $D = \text{diag}(d_i)$ , with  $0 \leq d_i \leq 1$ , we have  $\tilde{M} = I - D + DM = (I - D + DB^+) + DC = \tilde{B}^+ + \tilde{C}$ , where  $\tilde{B}^+ = I - D + DB^+$  and  $\tilde{C} = DC$ . By Theorem 2 in [16], we can easily have

$$\|\tilde{M}^{-1}\|_\infty \leq \|(I + (\tilde{B}^+)^{-1}\tilde{C})^{-1}\|_\infty \|(\tilde{B}^+)^{-1}\|_\infty \leq (n-1)\|(\tilde{B}^+)^{-1}\|_\infty. \quad (14)$$

Next, we give an upper bound for  $\|(\tilde{B}^+)^{-1}\|_\infty$ . Note that  $B^+$  is a Nekrasov matrix and  $\tilde{B}^+ = I - D + DB^+$ . By Lemma 1,  $\tilde{B}^+$  is also a Nekrasov matrix. By Theorem 2, we easily get

$$\|(\tilde{B}^+)^{-1}\|_\infty \leq \max_{i \in N} \frac{\eta_i(B^+)}{\min\{b_{ii} - h_i(B^+), 1\}}. \quad (15)$$

From (14) and (15), the conclusion follows.

*Example 3* Consider the following matrix

$$M = \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{5} & 1 & -\frac{2}{5} & \frac{1}{5} \\ -1 & 0 & 1 & -\frac{1}{6} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & 1 \end{bmatrix}.$$

It is not difficult to check that  $M$  is not an  $H$ -matrix, consequently, not a Nekrasov matrix, so we cannot use the bounds in [12], and bounds in Theorems 1 and 2. On the other hand,  $M$  can be written  $M = B^+ + C$  as in (11), with

$$B^+ = \begin{bmatrix} \frac{1}{2} & -\frac{1}{6} & -\frac{1}{6} & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} & 0 \\ -1 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

Obviously,  $B^+$  is not strictly diagonally dominant and  $M$  is not a  $B$ -matrix, so we cannot apply the bound in [11]. However,  $B^+$  is a Nekrasov matrix and so  $M$  is a  $B$ -Nekrasov matrix. The diagonal matrix  $W$  of Theorem 3 is given by

$$W = \text{diag}\left(\frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{5}{6} + \varepsilon\right)$$

with  $\varepsilon \in (0, \frac{1}{6})$ . Hence, by Theorem 3 we can get the bound (12) involved with  $\varepsilon \in (0, \frac{1}{6})$  for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ , which is drawn in Figure 2.

Meanwhile, by Theorem 4, we can get the bound (13) for  $\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty$ , is 126.0000. It is easy to see from Figures 2 and 3 that the bound in Theorem 4 is smaller than that in Theorem 3 (Theorem 2 in [16]).

*Example 4* Consider the following matrix

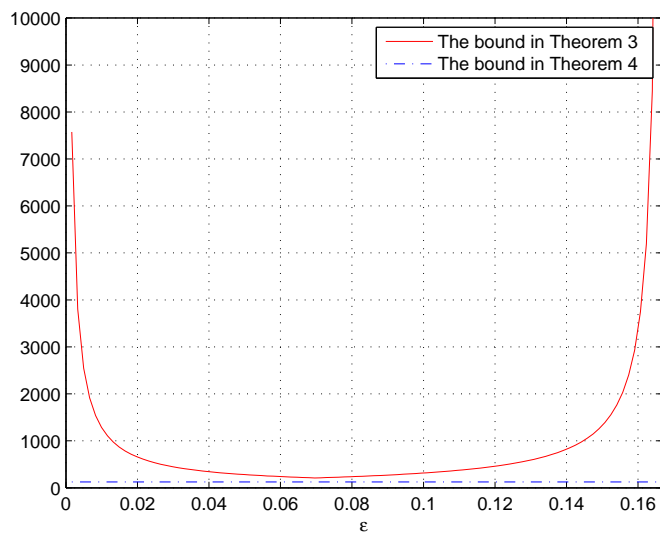
$$M = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & 1 & -\frac{2}{5} & \frac{1}{5} \\ -1 & 0 & 1 & -\frac{1}{6} \\ \frac{3}{4} & \frac{3}{4} & \frac{1}{2} & 1 \end{bmatrix}.$$

And  $M$  can be written  $M = B^+ + C$  as in (11), with

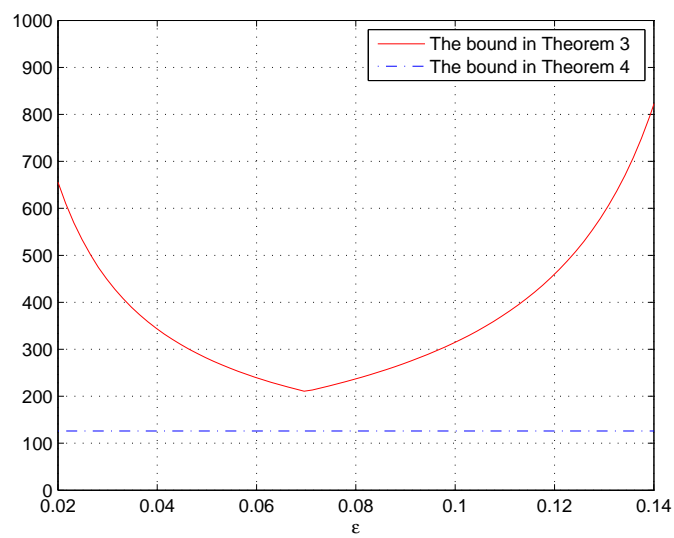
$$B^+ = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{4}{5} & -\frac{3}{5} & 0 \\ -1 & 0 & 1 & -\frac{1}{6} \\ 0 & 0 & -\frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & 0 & 0 & 0 \\ \frac{3}{4} & \frac{3}{4} & \frac{3}{4} & \frac{3}{4} \end{bmatrix}.$$

By computations,

$$h_1(B^+) = 0, h_2(B^+) = \frac{3}{5}, h_3(B^+) = \frac{1}{6}, h_4(B^+) = \frac{1}{24}.$$



**Fig. 2** The bounds in Theorems 3 and 4



**Fig. 3** The bounds in Theorems 3 and 4 with  $\varepsilon \in [0.02, 0.14]$

Obviously,  $B^+$  is a Nekrasov matrix and then  $M$  is a  $B$ -Nekrasov matrix. Since for any  $k > 1$ ,  $m_{1k} = r_1^+ = \frac{1}{2}$ , we cannot use the bound of Theorem 3 (Theorem 2 in [16]). However, by Theorem 4, we have

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_{\infty} \leq \frac{126}{5}.$$

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