AperTO - Archivio Istituzionale Open Access dell'Università di Torino

Trivariate near-best blending spline quasi-interpolation operators

This is the author's manuscript

Original Citation:

Availability:

This version is available http://hdl.handle.net/2318/1644988 since 2023-12-06T14:32:25Z

Published version:

DOI:10.1007/s11075-017-0373-2

Terms of use:

Open Access

Anyone can freely access the full text of works made available as "Open Access". Works made available under a Creative Commons license can be used according to the terms and conditions of said license. Use of all other works requires consent of the right holder (author or publisher) if not exempted from copyright protection by the applicable law.

(Article begins on next page)

26 April 2024

Trivariate near best blending spline quasi-interpolation operators

D. Barrera, C. Dagnino, M. J. Ibáñez, S. Remogna*

Abstract

A method to define trivariate spline quasi-interpolation operators (QIO) is developed by blending univariate and bivariate operators whose linear functionals allow oversampling. In this paper, we construct new operators based on univariate B-splines and bivariate box splines, exact on appropriate spaces of polynomials and having small infinity norms. An upper bound of the infinity norm for a general blending trivariate spline QIO is derived from the Bernstein-Bézier coefficients of the fundamental functions associated with the operators involved in the construction. The minimization of the resulting upper bound is then proposed and the existence of a solution is proved. The quadratic and quartic cases are completely worked out and their exact solutions are explicitly calculated.

Keywords: Blending quasi-interpolation, Minimal norm, Bernstein-Bézier coefficients

Mathematics Subject Classification (2010): 41A05, 41A15, 65D07

1 Introduction

The approximation of functions in one and several real variables is a ubiquitous problem that can be approached through a great variety of procedures. Interpolation is one of them and usually requires the resolution of linear systems to determine the expression of the interpolant. In many situations, only noisy data of the function to be approached are known, so it is more appropriate to use other constructive techniques. One of them is quasi-interpolation.

As quoted in [10, p. 63] "A quasi-interpolant Q for [a linear space] $\mathcal{S}$ is a linear map into $\mathcal{S}$ which is local, bounded (in some relevant norm), and reproduces some (nontrivial) polynomial space", we will refer to Q as quasi-interpolation operator (QIO) and to Qf as quasi-interpolant (QI) for the given function f provided by Q. From the first systematic study by I. J. Schoenberg in [28, 29] (see also [30]), intensive research has been carried out on quasi-interpolation. Books [10, 11] (and references therein) present results on the construction of QIOs from compact support functions, in particular box splines (see also [35]). They are operators of the type

$$Q : C(\mathbb{R}^s) \longrightarrow \mathcal{S}(\phi)$$

where $\mathcal{S}(\phi)$ is the space spanned by the integer translates of the compactly supported function ϕ. These operators are constructed to be exact on the space $\mathbb{P}(\phi)$ of polynomials of maximal total degree included in $\mathcal{S}(\phi)$. The QI associated with a given function f will have the form

$$Qf = \sum_{i\in\mathbb{Z}^s} \lambda(f(\cdot+i))\,\phi(\cdot-i),$$

λ being a general linear functional (see e.g. [10, p. 63]). Usually, λ is a point, derivative or integral linear functional. In the first case, λf is a finite linear combination of values of

*Department of Applied Mathematics, University of Granada, Campus de Fuentenueva s/n, 18071-Granada, Spain (dbarrera@ugr.es, mibanez@ugr.es) Department of Mathematics, University of Torino, via C. Alberto, 10 – 10123 Torino, Italy (catterina.dagnino@unito.it, sara.remogna@unito.it)

f at some points in some open set containing the support of ϕ. In the second case, λf is a finite linear combination of values of f and some of its partial derivatives at some points in a neighbourhood of ϕ. Finally, in the third case, λf is a finite linear combination of weighted mean values of f.

In the literature, there exist different methods to construct spline QIOs of the type indicated above giving the maximal approximation order. For instance, in [10, 11] Appel sequences, Neumann series or Fourier transform are used. In [24, 26], bivariate QIOs based on point and integral linear functionals are defined from the values of an H-spline on a three direction mesh by exploiting the relation between hexagonal sequences and central difference operators. In [25, 19], these kinds of operators have been defined from the values of an Ω-spline on a four direction mesh by exploiting the relation between lozenge sequences and central difference operators.

A completely different approach has been used in [31, 32, 33] to define QIOs for bivariate and trivariate splines of low polynomial degree. The quasi-interpolating splines are directly determined by setting the Bernstein-Bézier coefficients of the splines to appropriate combinations of the given data values. Also a method to increase the approximation order in the univariate case was proposed in [34] and extended to the multivariate setting in [18] (see also [7, 8]).

QIOs have been used to solve problems in many different areas, like science and engineering. Some applications of QIOs concern for example the computation of multivariate integrals, the solution of differential and integral equations (see e.g. [12, 13, 15, 16, 17]).

Approximating noisy data requires the use of adapted methods. Specific types of QIOs have been proposed in the literature to diminish as much as possible the increase of noise present in the data. They are based on the minimization of the infinity norm $\|Q\|_\infty$ of the operator Q. If λ is the point linear functional given by

$$\lambda f = \sum_{j \in J} c_j f\left(\cdot - j\right),$$

J being a finite subset of $\mathbb{Z}^s$, then

$$\|Q\|_\infty \leq \sum_{j \in J} |c_j| =: \|c\|_1,$$

where $c := \left(c_j\right)_{j \in J}$. Then, the upper bound $\|c\|_1$ is minimized instead of $\|Q\|_\infty$, subject to the linear constraints yielding the exactness of Q on $\mathbb{P}\left(\phi\right)$. The existence of solution of this minimization problem is guaranteed, but in general it is not unique. Every QIO associated with a solution of this minimization problem will be said to be a near best (NB) QIO with respect to the upper bound given by the 1-norm of the sequence of coefficients (type-1 NB QIO for short).

As far as the authors know, the first systematic study on the construction of type-1 NB QIO appears in [4], where univariate QIOs are based on a point or derivative linear functional (see [5] for the nonuniform case). The bivariate case was considered in [3, 6] by using a H-spline and a Ω-spline, respectively. In [1], the construction of a type-1 NB QIO into the linear space $S_4^2\left(\tau\right)$ spanned by the integer translates of three C^2-quartic B-splines on the four-direction mesh τ of the plane is presented. The extension to the three-dimensional case is done in [14, 21, 22, 23]. In [9], the construction of trivariate near-best quasi-interpolants based on C^2 quartic splines on type-6 tetrahedral partitions is addressed. We recall that a type-6 tetrahedral partition is a uniform partition of $\mathbb{R}^3$, obtained from a given cube partition of the space by subdividing each cube into 24 tetrahedra.

In order to obtain a better upper bound to be minimized it is possible to bound the Lebesgue function associated with Q from the BB-coefficients of ϕ. This approach has been considered in [2] to construct QIO based on box splines. Also in this case the existence is guaranteed, but not the uniqueness. The operator associated with a solution of the involved minimization problem will be said to be a type-2 NB QIO.

In this paper, we deal with the construction of trivariate NB QIOs by blending 1D and 2D QIOs and minimizing an objective function constructed from the BB-coefficients of the Lebesgue function of the resulting operator. In this way, we can study the problem by subdividing it into one and two dimensions, instead of addressing the problem directly in the three dimensional set, for example by using trivariate box splines.

The paper is organized as follows. In Section 2, the spline spaces are introduced as well as the quasi-interpolation operators and their trivariate extensions to define the blending trivariate QIO. In Section 2.1, an upper bound for the blending operator is established from the Bernstein-Bézier representations of the fundamental functions of the operators involved in the construction, and the minimization problem is proposed. In Sections 3 and 4, the quadratic and quartic cases are worked out, providing the explicit solutions of the corresponding minimization problems. Finally, in Section 5, some conclusions are presented.

2 Spline spaces and quasi-interpolation operators

We consider the uniform integer partition of the real line τ_1 and the uniform triangulation τ_2 of the plane generated by the four directions $d_1 = (1,0)$, $d_2 = (0,1)$, $d_3 = d_1 + d_2$, $d_4 = d_2 - d_1$ and called four-directional mesh.

Let $\mathbb{P}_k(\mathbb{R})$ be the space of univariate polynomials of degree at most k and $\mathbb{P}_k(\mathbb{R}^2)$ be the space of bivariate polynomials of total degree at most k. We denote by $S_k^l(\tau_\ell)$, $\ell = 1, 2$ the spaces of functions in $C^l(\mathbb{R}^\ell)$ whose restrictions to each element of τ_ℓ is in $\mathbb{P}_k(\mathbb{R}^\ell)$.

Let B_{d+1} be the univariate B-spline of degree d centered in the origin, with support $\left[-\frac{d+1}{2}, \frac{d+1}{2}\right]$ and belonging to the space $S_d^{d-1}(\tau_1)$ (see [11, Chap. 1]). Let $X_{i,j}$ be the set of directions given by

$$X_{i,j} := \{\underbrace{d_1,\ldots,d_1}_{i},\underbrace{d_2,\ldots,d_2}_{i},\underbrace{d_3,\ldots,d_3}_{j},\underbrace{d_4,\ldots,d_4}_{j}\},\ i,j>0,$$

and $M_{i,j}$ be the corresponding centered box splines (see e.g. [10, p. 10], [11, p. 17]) of degree $g = \#X_{i,j} - 2$, where $\#A$ denotes the cardinality of the set A, and belonging to the space $S_g^l(\tau_2)$. In particular, the box spline $M_{1,1} \in S_2^1(\tau_2)$, and $M_{k,k+1}$, $M_{k+1,k}$ are box splines in $S_{4k}^{3k-1}(\tau_2)$.

In the sequel, B is one of the univariate B-splines and M is one of the bivariate box splines. Moreover, we define $\mathcal{B}_1 := \operatorname{span}\{B(\cdot - k) : k \in \mathbb{Z}\}$, $\mathcal{B}_2 := \operatorname{span}\left\{M(\cdot - i_1, \cdot - i_2) : (i_1, i_2) \in \mathbb{Z}^2\right\}$ and $n := \max\left\{k : \mathbb{P}_k\left(\mathbb{R}^2\right) \subset \mathcal{B}_2\right\}$. It is well known (cf. [10, p. 53], [11, p. 19]) that $n = 2$ if $M = M_{1,1}$, and $n = 3k$ for the other two box splines considered.

Now we focus on univariate and bivariate QIOs in order to construct the trivariate ones. Consider the univariate Schoenberg QIO $\overline{S}$ defined by

$$\overline{S}f(z) := \sum_{k\in\mathbb{Z}} f(k)\, B(z-k).$$

and let $\overline{Q}$ be the one defined by

$$\overline{Q}f(z) := \sum_{k\in\mathbb{Z}} \left(\sum_{\ell\in\Lambda} a_\ell f(k-\ell)\right) B(z-k), \tag{2.1}$$

where Λ is a finite subset of $\mathbb{Z}$ and the coefficients a_ℓ are chosen to produce an operator exact on $\mathbb{P}_d(\mathbb{R})$. $\overline{S}$ is exact on $\mathbb{P}_1(\mathbb{R})$. The QI $\overline{Q}f$ can also be written by means of the integer translates of its fundamental function, i.e.

$$\overline{Q}f(z) = \sum_{k\in\mathbb{Z}} f(k)\, LB(z-k),$$

where the function LB is obtained as a linear combination of integer translates of the B-spline B, according to the coefficient functional expressions. Explicitly,

$$LB(z) = \sum_{\ell\in\Lambda} a_\ell B(\cdot - \ell).$$

Now, let S be the bivariate Schoenberg QIO defined by

$$Sf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, M(x-i_1, y-i_2),$$

and Q be one of the form

$$Qf(x,y) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1, i_2-j_2) \right) M(x-i_1, y-i_2), \qquad (2.2)$$

with J a finite subset of $\mathbb{Z}^2$ and $\{(c_{j_1,j_2}),\ (j_1,j_2)\in J\}$ a lozenge sequence [2] such that Q is exact on the space $\mathbb{P}_n\left(\mathbb{R}^2\right)$. S is exact on the space of bilinear polynomials. The spline Qf can also be written by means of the associated fundamental function, i.e.

$$Qf(x,y) = \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2)\, LM(x-i_1, y-i_2),$$

where LM is expressed as a linear combination of the integer translates of the box spline M, according to the coefficient functional expressions, namely

$$LM = \sum_{(i_1,i_2)\in J} c_{j_1,j_2} M(\cdot - j_1, \cdot - j_2).$$

Now we consider the trivariate extensions of these operators, given by

$$\begin{aligned}
\overline{S}f(x,y,z) &= \sum_{k\in\mathbb{Z}} f(x,y,k)\, B(z-k), \\
\overline{Q}f(x,y,z) &= \sum_{k\in\mathbb{Z}} \left(\sum_{\ell\in\Lambda} a_\ell f(x,y,k-\ell) \right) B(\cdot - k) = \sum_{k\in\mathbb{Z}} f(x,y,k)\, LB(z-k), \\
Sf(x,y,z) &= \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, M(x-i_1, y-i_2), \\
Qf(x,y,z) &= \sum_{(i_1,i_2)\in\mathbb{Z}^2} \left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1, i_2-j_2, z) \right) M(x-i_1, y-i_2) \\
&= \sum_{(i_1,i_2)\in\mathbb{Z}^2} f(i_1,i_2,z)\, LM(x-i_1, y-i_2).
\end{aligned}$$

Following a logical scheme similar to that one proposed in [23], we define the blending operator R as

$$R := S\overline{Q} + Q\overline{S} - S\overline{S}.$$

Since

$$\begin{aligned}
S\overline{S}f(x,y,z) &= S\left(\sum_{k\in\mathbb{Z}} f(\cdot,\cdot,k)\, B(z-k) \right)(x,y) \\
&= \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} f(i_1,i_2,k)\, M(x-i_1, y-i_2)\, B(z-k), \\
S\overline{Q}f(x,y,z) &= S\left(\sum_{k\in\mathbb{Z}} \left(\sum_{\ell\in\Lambda} a_\ell f(\cdot,\cdot,k-\ell) \right) B(z-k) \right)(x,y)
\end{aligned}$$

$$= \sum_{(i_1,i_2)\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}}\sum_{\ell\in\Lambda} a_\ell f(i_1,i_2,k-\ell)\, M(x-i_1,y-i_2)\, B(z-k),$$

$$Q\overline{S}f(x,y,z) = Q\left(\sum_{k\in\mathbb{Z}} f(\cdot,\cdot,k)\, B(z-k)\right)(x,y)$$

$$= \sum_{(i_1,i_2)\in\mathbb{Z}^2}\left(\sum_{(j_1,j_2)\in J} c_{j_1,j_2}\sum_{k\in\mathbb{Z}} f(i_1-j_1,i_2-j_2,k)\, B(z-k)\right)$$
$$\times M(x-i_1,y-i_2),$$

we get

$$Rf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}} \rho_{i_1,i_2,k,\ell}(f)\, M(x-i_1,y-i_2)\, B(z-k) \tag{2.3}$$

with

$$\rho_{i_1,i_2,k,\ell}(f) := \sum_{\ell\in\Lambda} a_\ell f(i_1,i_2,k-\ell) + \sum_{(j_1,j_2)\in J} c_{j_1,j_2} f(i_1-j_1,i_2-j_2,k)$$
$$- f(i_1,i_2,k).$$

By using the fundamental functions in $\overline{Q}$ and Q, we can also write Rf in the following form

$$Rf(x,y,z) = \sum_{(i_1,i_2)\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}} f(i_1,i_2,k)\, L(x-i_1,y-i_2,z-k), \tag{2.4}$$

with

$$L(x,y,z) := M(x,y)\, LB(z) + LM(x,y)\, B(z) - M(x,y)\, B(z). \tag{2.5}$$

Such an operator R is defined onto the tensor product spline space $S_g^l(\tau_2)\times S_d^{d-1}(\tau_1)$, spanned by the trivariate piecewise polynomial functions

$$\left\{M(\cdot-i_1,\cdot-i_2)\, B(\cdot-k),\ (i_1,i_2,k)\in\mathbb{Z}^3\right\}$$

of coordinate degree $g+d$, on the partition $\tau := \tau_2\times\tau_1$, obtained from the bivariate and the univariate ones.

2.1 Trivariate type-2 near best quasi-interpolation operators

From (2.3), for a function f such that $\|f\|_\infty \le 1$, it holds

$$|Rf(x,y,z)| \le \sum_{(i_1,i_2)\in\mathbb{Z}^2}\sum_{k\in\mathbb{Z}}\left(\sum_{\ell\in\Lambda}|a_\ell| + \sum_{(j_1,j_2)\in J}|c_{j_1,j_2}| + 1\right)$$
$$\times M(x-i_1,y-i_2)\, B(z-k)$$
$$= \sum_{\ell\in\Lambda}|a_\ell| + \sum_{(j_1,j_2)\in J}|c_{j_1,j_2}| + 1.$$

Therefore,

$$\|R\|_\infty \le \sum_{\ell\in\Lambda}|a_\ell| + \sum_{(j_1,j_2)\in J}|c_{j_1,j_2}| + 1$$

and the construction of type-1 NB trivariate blending operators by minimizing the above upper bound is equivalent to the construction of type-1 NB QIOs $\overline{Q}$ and Q by minimizing the upper bounds $\sum_{\ell\in\Lambda}|a_\ell|$ and $\sum_{(j_1,j_2)\in J}|c_{j_1,j_2}|$ to $\left\|\overline{Q}\right\|_\infty$ and $\|Q\|_\infty$, respectively.

It is possible to improve the upper bound to R taking into account the Bernstein-Bézier (BB-) coefficients of B and M. In order to do it, the following lemma is necessary.

Lemma 1 *Let R be the operator given in (2.4). Then*

$$\|R\|_\infty = \max_{(x,y,z)\in P} \mathcal{L}(x,y,z),$$

where

$$\mathcal{L}(x,y,z) := \sum_{(i_1,i_2)\in\mathbb{Z}^2} \sum_{k\in\mathbb{Z}} |L(x-i_1, y-i_2, z-k)|$$

is the Lebesgue function associated with R, P is the prism with triangular horizontal sections defined by $P = T\times I$ and

1. *T is the triangle with vertices $\left\{(0,0), \left(\frac{1}{2},-\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right)\right\}$ if $M = M_{1,1}$ or $M_{k,k+1}$ and $\left\{(1,0), \left(\frac{1}{2},\frac{1}{2}\right), (1,1)\right\}$ if $M = M_{k+1,k}$,*
2. *I is the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$ if d is even and $[0,1]$ if d is odd.*

We omit the proof of this lemma, because, taking into account the symmetries of the univariate B-splines and bivariate box splines, it is the natural extension of [2, Lemma 3].

The study of the real norm of R is rather complicated, therefore, we look for a good upper bound of $\|R\|_\infty$ and, from this lemma, it is natural to define, as upper bound of $\|R\|_\infty$, an upper bound of the piecewise polynomial function $\mathcal{L}$.

Consider the fundamental function L. It is a compactly supported piecewise polynomial function of coordinate degree $g+d$ defined on τ. Let Ω be a polyhedral domain including the support of L. Let $\mathcal{P}_\Omega := \{P_r : 1\le r\le m_\Omega\}$ be the collection of P-like prisms of τ included in Ω. Then we have that

$$P_r = T_s \times I_e, \quad 1\le s\le m_2, \quad 1\le e\le m_1,$$

where T_s is a T-like triangle, I_e is a I-like interval and $m_\Omega := m_1 m_2$. Now consider the following notations and results.

For every triangle T_s, $1\le s\le m_2$, with vertices $A_{s,v}$, $v=1,2,3$, let $\lambda_s := (\lambda_{s,1},\lambda_{s,2},\lambda_{s,3})$ be the barycentric coordinates of a point (x,y) with respect to T_s. Therefore, we have

$$(x,y) = \sum_{v=1}^{3} \lambda_{s,v} A_{s,v}, \ |\lambda_s| := \sum_{v=1}^{3}\lambda_{s,v} = 1.$$

Then, the BB-representations of M and LM on T_s are

$$M|_{T_s}(x,y) = \sum_{|\alpha|=g} b^M_{s,\alpha} BE^g_\alpha(\lambda_s) \quad \text{and} \quad LM|_{T_s}(x,y) = \sum_{|\alpha|=g} b^{LM}_{s,\alpha} BE^g_\alpha(\lambda_s) \qquad (2.6)$$

for some coefficients $b^M_{s,\alpha}$, $b^{LM}_{s,\alpha}\in\mathbb{R}$, where $\alpha := (\alpha_1,\alpha_2,\alpha_3)$, $\alpha\in\mathbb{N}_0^3$ and $BE^g_\alpha(\lambda) := \frac{g!}{\alpha!}\lambda^\alpha$, $|\alpha| = g$ are the bivariate Bernstein polynomials of degree g.

For every interval I_e, $1\le e\le m_1$, with endpoints $B_{e,w}$, $w=1,2$, let $\sigma_e := (\sigma_{e,1},\sigma_{e,2})$ be the barycentric coordinates of a point z with respect to I_e. Therefore, we have

$$z = \sum_{w=1}^{2}\sigma_{e,w}B_{e,w}, \ |\sigma_e| = \sum_{w=1}^{2}\sigma_{e,w} = 1.$$

Then, the BB-representations of B and LB on I_e are

$$B|_{I_e}(z) = \sum_{|\beta|=d} b^B_{e,\beta} BE^d_\beta(\sigma_e) \quad \text{and} \quad LB|_{I_e}(z) = \sum_{|\beta|=d} b^{LB}_{e,\beta} BE^d_\beta(\sigma_e), \qquad (2.7)$$

for some coefficients $b^B_{e,\beta}$, $b^{LB}_{e,\beta}\in\mathbb{R}$, where $\beta := (\beta_1,\beta_2)$, $\beta\in\mathbb{N}_0^2$ and $BE^d_\beta(\sigma) := \frac{d!}{\beta!}\sigma^\beta$, $|\beta| = d$ are the univariate Bernstein polynomials of degree d.

Lemma 2 *On the prism $P_r = T_s \times I_e$ the fundamental function L given in (2.5) can be expressed as*

$$L_{|P_r}(x,y,z) = \sum_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} b_{(s,e),(\alpha,\beta_2)} BE_\alpha^g(\lambda_s)\, BE_\beta^d(\sigma_e),$$

with $\beta = (\beta_1, \beta_2) = (d-\beta_2, \beta_2)$,

$$b_{(s,e),(\alpha,\beta_2)} := b^{M,LB}_{(s,e),(\alpha,\beta_2)} + b^{LM,B}_{(s,e),(\alpha,\beta_2)} - b^{M,B}_{(s,e),(\alpha,\beta_2)}$$

and

$$\begin{aligned} b^{M,LB}_{(s,e),(\alpha,\beta_2)} &= b^M_{s,\alpha} b^{LB}_{e,\beta} = b^M_{s,\alpha} b^{LB}_{e,(d-\beta_2,\beta_2)}, \\ b^{LM,B}_{(s,e),(\alpha,\beta_2)} &= b^{LM}_{s,\alpha} b^{B}_{e,\beta} = b^{LM}_{s,\alpha} b^{B}_{e,(d-\beta_2,\beta_2)}, \\ b^{M,B}_{(s,e),(\alpha,\beta_2)} &= b^M_{s,\alpha} b^{B}_{e,\beta} = b^M_{s,\alpha} b^{B}_{e,(d-\beta_2,\beta_2)}. \end{aligned}$$

Proof. Since

$$\begin{aligned} L_{|P_r}(x,y,z) &= M|_{T_s}(x,y)\, LB|_{I_e}(z) + LM|_{T_s}(x,y)\, B|_{I_e}(z) - M|_{T_s}(x,y)\, B|_{I_e}(z) \\ &= \sum_{|\alpha|=g} b^M_{s,\alpha} BE^g_\alpha(\lambda_s) \sum_{|\beta|=d} b^{LB}_{e,\beta} BE^d_\beta(\sigma_e) \\ &+ \sum_{|\alpha|=g} b^{LM}_{s,\alpha} BE^g_\alpha(\lambda_s) \sum_{|\beta|=d} b^{B}_{e,\beta} BE^d_\beta(\sigma_e) \\ &- \sum_{|\alpha|=g} b^{M}_{s,\alpha} BE^g_\alpha(\lambda_s) \sum_{|\beta|=d} b^{B}_{e,\beta} BE^d_\beta(\sigma_e), \end{aligned}$$

the claim follows.

We associate with L the matrix $\mathcal{F}_\Omega \in \mathbb{R}^{m_\Omega \times \frac{1}{2}(g+1)(g+2)d}$, whose rth row contains the BB-coefficients $b_{(s,e),(\alpha,\beta_2)}$, $|\alpha| = g$, $0 \le \beta_2 \le d$. We call it the matrix of BB-coefficients of L with respect to Ω and P.

Thus, on the prism P, we get

$$\begin{aligned} \mathcal{L}(\cdot) &= \sum_{r=1}^{m_\Omega} |L(\cdot - r)| \\ &= \sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left| \sum_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} b_{(s,e),(\alpha,\beta_2)} BE^g_\alpha(\cdot)\, BE^d_\beta(\cdot) \right| \\ &\le \sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left(\sum_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} \left| b_{(s,e),(\alpha,\beta_2)} \right| \right) BE^g_\alpha(\cdot)\, BE^d_\beta(\cdot) \\ &= \sum_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} \left(\sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left| b_{(s,e),(\alpha,\beta_2)} \right| \right) BE^g_\alpha(\cdot)\, BE^d_\beta(\cdot) \\ &\le \max_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} \left(\sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left| b_{(s,e),(\alpha,\beta_2)} \right| \right). \end{aligned} \tag{2.8}$$

Therefore, we get an upper bound for $\|R\|_\infty$ and it can be expressed in matrix form as

$$\|R\|_\infty \le \max_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} \left(\sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left| b_{(s,e),(\alpha,\beta_2)} \right| \right) =: \|\mathcal{F}_\Omega\|_1.$$

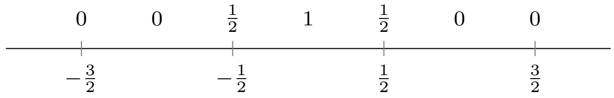


Figure 1: BB-coefficients of the quadratic B-spline B_3.

Definition 3 *The value* $\|\mathcal{F}_\Omega\|_1 := \max_{\substack{|\alpha|=g \\ 0\le\beta_2\le d}} \left(\sum_{\substack{1\le s\le m_2 \\ 1\le e\le m_1}} \left|b_{(s,e),(\alpha,\beta_2)}\right|\right)$ *is said to be a type-2 upper bound to the infinity norm of the trivariate blending spline QIO R.*

We remark that $\|\mathcal{F}_\Omega\|_1$ is independent of Ω since the BB-coefficients of L relative to a P-like prism contained in $\Omega \setminus \operatorname{supp} L$ are zero. Moreover, $\|\mathcal{F}_\Omega\|_1$ is independent of the ordering of the rows.

The entries of $\mathcal{F}_\Omega$ depend linearly on the coefficients $c := \left\{(c_{j_1,j_2})_{(j_1,j_2)\in J}\right\}$, and $a := \left\{(a_\ell)_{\ell\in\Lambda}\right\}$. So, we can consider the objective function

$$\mu(c,a) := \|\mathcal{F}_\Omega\|_1 \tag{2.9}$$

defined on

$$\mathcal{A}_{J,\Lambda} := \left\{c \mid Q \text{ is exact on } \mathbb{P}_n\left(\mathbb{R}^2\right)\right\} \times \left\{a \mid \overline{Q} \text{ is exact on } \mathbb{P}_d\left(\mathbb{R}\right)\right\}, \tag{2.10}$$

and we can state the following minimization problem.

Problem 4 *Given J and Λ, find (c,a) in $\mathcal{A}_{J,\Lambda}$ to minimize the function $\mu(c,a)$ given by (2.9).*

Since the function μ given by (2.9) is a convex function on $\mathcal{A}_{J,\Lambda}$, the existence of a solution for Problem 4 is guaranteed (see e.g. [36]). In general the solution is not unique.

Each solution of this constrained minimization problem produces a QIO. If (c,a) is a solution of Problem 4, the corresponding operator will said to be a NB QIO with respect to μ or type-2 NB QIO.

3 Trivariate type-2 near best quasi-interpolation operators based on quadratic B-splines and box splines

In this section, we want to construct particular QIOs of kind (2.4), with degrees $g = 2$ and $d = 2$. Therefore, $B = B_3$ is the univariate C^1 quadratic B-spline with support $\left[-\frac{3}{2}, \frac{3}{2}\right]$, belonging to the space $S_2^1(\tau_1)$ and $M = M_{1,1}$ is the bivariate C^1 quadratic box spline in $S_2^1(\tau_2)$.

Figure 1 shows the BB-coefficients of the B-spline B_3 in each subinterval of its support.

Figure 2 provides the BB-coefficients of $8 \cdot M_{1,1}$ in the triangles of τ_2 that determine the region with vertices $(0,0)$, $(1,-1)$ and $\left(\frac{3}{2},-\frac{1}{2}\right)$, $\left(\frac{3}{2},\frac{1}{2}\right)$ and $(1,1)$. The BB-coeficients in the other triangles contained in the support of $M_{1,1}$ are determined taking into account the symmetries of the octagon.

In order to construct a trivariate type-2 NB QIO R of kind (2.4), we have to fix Λ and J in (2.1) and (2.2), respectively, allowing oversampling. Let $\Lambda = \{-2,-1,0,1,2\}$, J be the subset of $\mathbb{Z}^2$ formed by the gridpoints lying in the rhombus with vertices $(\pm 2, 0)$ and $(0, \pm 2)$, and impose the conditions

$$\begin{aligned}
&a_{-2} = a_2,\ a_{-1} = a_1,\\
&c_{0,1} = c_{-1,0} = c_{0,-1} = c_{1,0},\ c_{0,2} = c_{-2,0} = c_{0,-2} = c_{2,0},\\
&c_{-1,1} = c_{-1,-1} = c_{1,-1} = c_{1,1}.
\end{aligned}$$

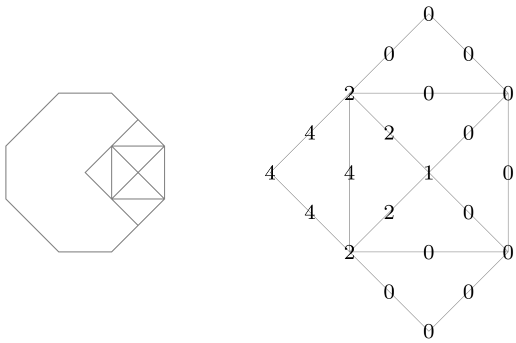


Figure 2: BB-coefficients of the box spline $8 \cdot M_{1,1}$.

Therefore, taking into account the symmetries, we require that $\overline{Q}$, defined in (2.1), has the form (see Fig. 3(a))

$$\overline{Q}f(z) = \sum_{k\in\mathbb{Z}} \left(a_0 f(k) + a_1 f(k \pm 1) + a_2 f(k \pm 2)\right) B\left(z - k\right),$$

where $f(k \pm \ell) := f(k+\ell) + f(k-\ell)$, and therefore the fundamental function LB is

$$LB(z) = a_0 B(z) + a_1(B(z-1) + B(z+1)) + a_2(B(z-2) + B(z+2)).$$

For the operator $\overline{Q}$, we impose the exactness on the polynomial space $\mathbb{P}_2(\mathbb{R})$. This leads to the system of equations

$$a_0 + 2a_1 + 2a_2 = 1 \quad \text{and} \quad 2a_1 + 8a_2 = -\frac{1}{4}. \tag{3.1}$$

We remark that the exactness on linear polynomials is automatically satisfied by the choice of symmetric coefficients $a_{-2} = a_2$, $a_{-1} = a_1$.

For the bivariate operator Q (see e.g. [2]), defined in (2.2), we choose the following expression for its linear functional (see Fig. 3(b)):

$$\begin{aligned}\lambda\left(f\right) = {} & c_{0,0} f\left(0,0\right) + c_{1,0}\left(f\left(\pm 1, 0\right) + f\left(0, \pm 1\right)\right) \\ & + c_{2,0}\left(f\left(\pm 2, 0\right) + f\left(0, \pm 2\right) + c_{1,1} f\left(\pm 1, 1\right) + f\left(\pm 1, -1\right)\right).\end{aligned}$$

Consequently, the fundamental function LM is given by

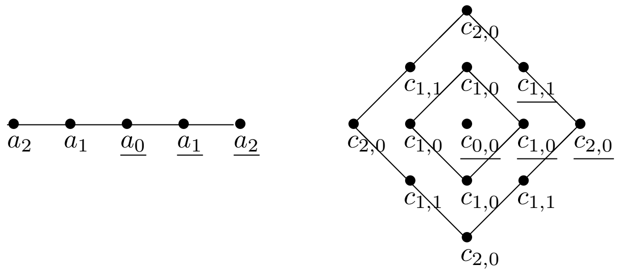
$$\begin{aligned}LM\left(x,y\right) = {} & c_{0,0} M\left(x,y\right) + c_{1,0}\left(M\left(x \pm 1, y\right) + M\left(x, y \pm 1\right)\right) \\ & + c_{2,0}\left(M\left(x \pm 2, y\right) + M\left(x, y \pm 2\right)\right) \\ & + c_{1,1}\left(M\left(x \pm 1, y - 1\right) + M\left(x - 1, y \pm 1\right)\right).\end{aligned}$$

In this case, the requirement $Qp = p$ for all $p \in \mathbb{P}_2(\mathbb{R}^2)$ leads to the system of equations

$$c_{0,0} + 4c_{1,0} + 4c_{2,0} + 4c_{1,1} = 1 \quad \text{and} \quad 2c_{1,0} + 8c_{2,0} + 4c_{1,1} = -\frac{1}{4}. \tag{3.2}$$

From the expressions of the fundamental functions associated with $\overline{Q}$ and Q, we are able to define the trivariate fundamental function L given in (2.5), defining the operator R in (2.4). The support of L is shown in Fig. 4, where we can identify the supports of $M\left(x,y\right) LB\left(z\right)$, $LM\left(x,y\right) B\left(z\right)$ and the support of $M\left(x,y\right) B\left(z\right)$.

Now, before solving Problem 4, we have to define the prism $P = T \times I$ of Lemma 1. In the quadratic case, T is the triangle of vertices $(0,0)$,$\left(\frac{1}{2}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$, and I the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (in Fig. 4 we can see the prism P in yellow). Moreover, we can define the polyhedral domain Ω, including the support of L, as the cube $\left[-\frac{7}{2}, \frac{7}{2}\right] \times \left[-\frac{7}{2}, \frac{7}{2}\right] \times \left[-\frac{7}{2}, \frac{7}{2}\right]$ (see Fig. 4). Therefore, the number of P-like prisms of τ included in Ω is 343.



(a) (b)

Figure 3: (a) Coefficient sequence defining the linear functional λ of $\overline{Q}$ and (b) lozenge sequence defining the linear functional λ corresponding to Q.

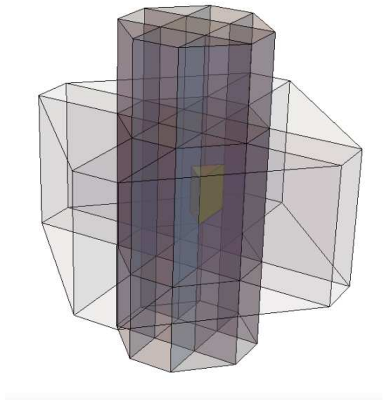


Figure 4: The support of the fundamental function L.

For a generic P-like prism $P_r = T_s \times I_e$, the structure of the BB-coefficients of L on P_r can be represented as shown in Fig. 5.

Therefore, we can define $c = (c_{0,0}, c_{1,0}, c_{2,0}, c_{1,1})$, $a = (a_0, a_1, a_2)$, and the objective function

$$\mu(c, a) = \max_{\substack{|\alpha|=2 \\ 0\le\beta_2\le 2}} \left(\sum_{\substack{1\le s\le 49 \\ 1\le e\le 7}} \left| b_{(s,e),(\alpha,\beta_2)} \right| \right) \tag{3.3}$$

in (2.9) and, taking into account (3.1) and (3.2), it is possible to define $\mathcal{A}_{J,\Lambda}$ in (2.10). Then, Problem 4 is specified.

The general solution of (3.1) and (3.2) can be written as

$$a_0 = -\tfrac{3}{2}x_1 + \tfrac{17}{16},\ a_2 = -\tfrac{1}{4}x_1 - \tfrac{1}{32},\ c_{0,0} = -3x_2 - 2x_3 + \tfrac{9}{8},\ c_{2,0} = -\tfrac{1}{4}x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{32}, \tag{3.4}$$

where $x_1 := a_1$, $x_2 := c_{1,0}$, $x_3 := c_{1,1}$. The substitution of the values (3.4) into (3.3) results in the minimization of the objective function given by

$$F(x) := \max_{1\le i\le 8} \|A_i x - b_i\|_1, \quad x = (x_1, x_2, x_3)^T,$$

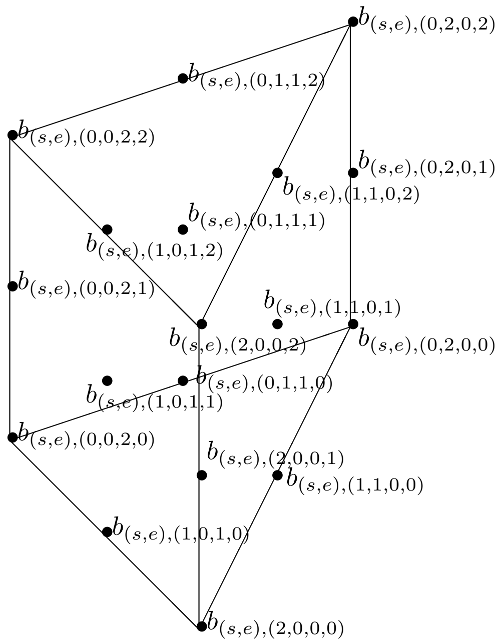


Figure 5: Representation of the BB-coefficients of L on a prism $P_r = T_s \times I_e$.

with

$$A_1 := \frac{1}{8}\begin{pmatrix} 6 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 8 & 8 \\ 0 & 2 & -4 \\ 2 & -3 & 2 \\ 0 & 8 & 16 \\ 0 & 1 & 2 \end{pmatrix}, A_2 := \frac{1}{8}\begin{pmatrix} 6 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 8 & 8 \\ 2 & 4 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 8 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \\ 0 & 2 & -4 \\ 0 & 4 & 8 \\ 0 & 7 & 2 \end{pmatrix}, A_3 := \frac{1}{4}\begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 4 & 4 \\ 0 & 0 & 4 \\ 0 & 2 & 4 \\ 0 & 6 & 4 \end{pmatrix}, A_4 := \frac{1}{4}\begin{pmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 8 & 8 \\ 0 & 0 & 8 \\ 0 & 8 & 8 \\ 0 & 3 & -2 \\ 0 & 3 & -6 \end{pmatrix},$$

$$A_5 := \frac{1}{8}\begin{pmatrix} 16 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 8 & 8 \\ 0 & 2 & -4 \\ 6 & -3 & 2 \\ 0 & 8 & 16 \\ 0 & 0 & 8 \\ 0 & 1 & 2 \end{pmatrix}, A_6 := \frac{1}{8}\begin{pmatrix} 16 & 0 & 0 \\ 4 & 0 & 0 \\ 6 & 8 & 8 \\ 6 & 4 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 8 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \\ 0 & 2 & -4 \\ 0 & 4 & 8 \\ 0 & 7 & 2 \end{pmatrix}, A_7 := \frac{1}{2}\begin{pmatrix} 4 & 0 & 0 \\ 1 & 0 & 0 \\ 3 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 3 & 2 \end{pmatrix}, A_8 := \frac{1}{4}\begin{pmatrix} 8 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & 8 & 8 \\ 0 & 0 & 8 \\ 0 & 8 & 8 \\ 0 & 3 & -2 \\ 0 & 3 & 6 \end{pmatrix},$$

and

$$\begin{aligned} b_1^T &= \frac{1}{64}(2,-2,38,-2,-37,0,-4,-1), & b_2^T &= \frac{1}{64}(2,-2,38,38,-1,0,0,0,2,-4,1), \\ b_3^T &= \frac{1}{32}(1,-1,19,0,-2,2), & b_4^T &= \frac{1}{32}(1,-1,38,0,0,1,-3), \\ b_5^T &= \frac{1}{64}(0,-4,38,-2,37,0,-4,-1), & b_6^T &= \frac{1}{64}(0,-4,38,38,-1,0,0,0,2,-4,1), \\ b_7^T &= \frac{1}{16}(0,-1,19,0,-1,1), & b_8^T &= \frac{1}{32}(0,-2,38,0,0,1,-3). \end{aligned}$$

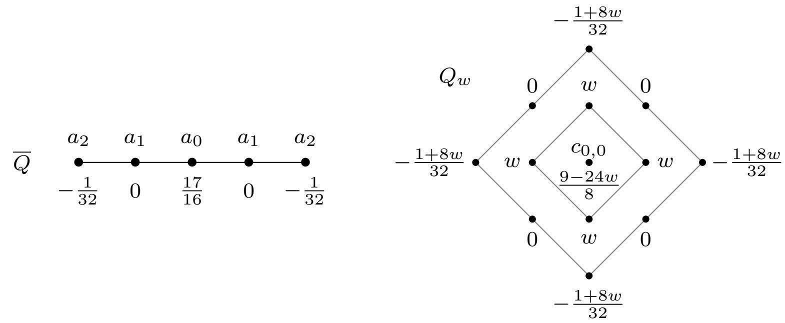


Figure 6: Coefficients of the linear functionals of the QIOs $\overline{Q}$ and Q_w.

The minimum value of F is equal to $\frac{11}{8} = 1.375$. It is attained at the points of the form $(0, w, 0)$, $0 \leq w \leq \frac{1}{24}$. Hence, the type-2 NB trivariate blending QIOs are obtained by combining the operators $\overline{Q}$ and Q_w with linear functionals whose coefficients appear in Figure 6.

Notice that the operator $\overline{Q}$ we have obtained is of type-1 NB type, i. e. the coefficients $\left\{\frac{17}{16}, 0, -\frac{1}{32}\right\}$ minimize the upper bound $\sum_{\ell\in\Lambda} |a_\ell|$ to the infinity norm of $\overline{Q}$. However, Q_w is not NB except for $w = 0$, i.e. the coefficients $\left\{\frac{1}{8}(9-24w), w, -\frac{1}{32}(1+8w), 0\right\}$ do not minimize the upper bound $\sum_{(j_1,j_2)\in J} |c_{j_1,j_2}|$ to $\|Q\|_\infty$ (see [6]).

Since the infinity norm of the blending operator R_w constructed from $\overline{Q}$ and Q_w is a continuous function of w on $\left[0, \frac{1}{24}\right]$, there exists a value w^* at which $\|R_w\|_\infty$ attains its minimum value. It is not possible to compute exactly w^*, but it can be approximated. For instance, since

$$\|R_w\|_\infty = \max_{(x,y,z)\in P} \mathcal{L}(x, y, z),$$

we can approximate $\|R_w\|_\infty$ from the maximum $F(w)$ of the absolute values obtained in evaluating $\mathcal{L}(x, y, z)$ at the points of a grid of P. A convex piecewise linear function is obtained. In dividing the prism $\left[-\frac{1}{2}, \frac{1}{2}\right]^3$ into $40\times40\times40$ equal parts, then the associated convex function $F_{40,40,40}(w)$ attains its minimum value at $w = \frac{1}{88}$, and this will be an approximation to w^*. Therefore,

$$\|R_{w^*}\|_\infty \simeq \left\|R_{\frac{1}{88}}\right\|_\infty \geq \frac{119}{88} \simeq 1.35227.$$

A value very close to the upper bound value has been obtained. The coefficients $c_{i,j}$ associated with this particular choice are

$$c_{00} = \frac{12}{11},\ c_{1,0} = \frac{1}{88},\ c_{1,1} = 0,\ c_{2,0} = -\frac{3}{88}.$$

The results we have obtained must be compared with those provided by the blending of classical QIOs $\overline{Q}_{\mathrm{c}}$ and Q_{c} or type-1 NB QIOs $\overline{Q}_{\mathrm{nb}}$ and Q_{nb} quadratic QIOs. Figure 7 shows the coefficients of the linear forms of such classical and type-1 NB quadratic QIOs based on B_3 and M_{11}.

Let $R_{\mathrm{c,c}}$ be the blending operator defined from $\overline{Q}_{\mathrm{c}}$ and Q_{c}. Then, $\|R_{\mathrm{c,c}}\|_\infty \leq F\left(-\frac{1}{8}, 0, 0\right) = \frac{7}{4}$ and it is straighforward to prove that $\|R_{\mathrm{c,c}}\|_\infty = \mathcal{L}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{7}{4}$. Similarly, let $R_{\mathrm{nb,nb}}$ be the operator provided by blending $\overline{Q}_{\mathrm{nb}}$ and Q_{nb}. Then, $\|R_{\mathrm{nb,nb}}\|_\infty \leq F(0,0,0) = \frac{11}{8}$ and $\|R_{\mathrm{nb,nb}}\|_\infty = \mathcal{L}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{11}{8}$.

In Figure 8 the plot of $F_{40,40,40}(w)$ is compared with the value of the infinity norm of the blending QIO associated with $\overline{Q}_{\mathrm{nb}}$ and Q_{nb}. We can deduce that the construction based on the minimization of the objective function gives better results in the quadratic case than the use of classical QIOs and it is comparable with type-1 NB QIOs, although a slightly better value for the infinity norm of the blending operator is expected.

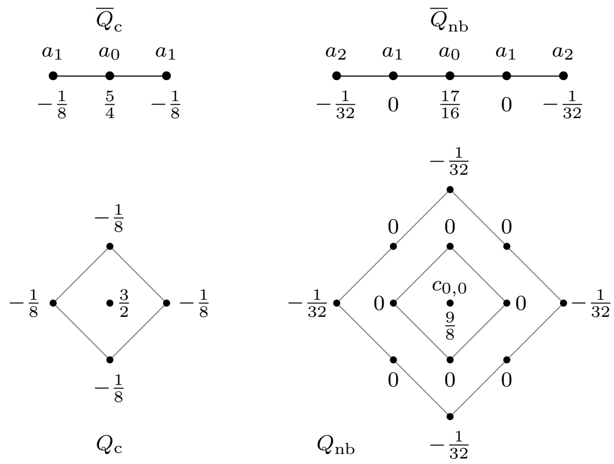


Figure 7: Coefficients of the linear functionals of the 1D and 2D classical (on the left) and NB type-1 QIOs on B_3 and M_{11}

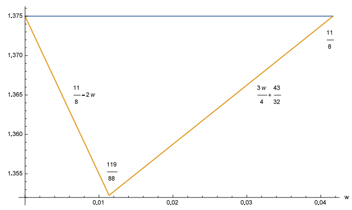


Figure 8: Plot of $F_{40,40,40}$. The minimum value is attained at $w = \frac{1}{88}$.

4 Trivariate type-2 near best quasi-interpolation operators based on quartic B-splines and box splines

In this section we construct particular QIOs for the degrees $g = 4$, $d = 4$. We use the quartic B-spline B_5 centered at the origin, whose BB-coefficients are shown in Fig. 9, and the box splines $M_{1,2}$ and $M_{2,1}$.

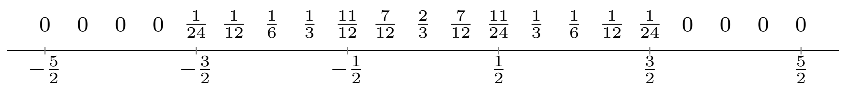


Figure 9: BB-coefficients of the quartic B-spline B_5.

Fig. 10 shows the BB-coefficients structure for the reference prism.

4.1 Trivariate case based on the box spline $M_{1,2}$

Firstly, we consider the box spline $M_{1,2}$, whose support and BB-coefficients are shown in Fig. 11.

The coefficients $a = (a_0, a_1, a_2, a_3)$ and $c = (c_{0,0}, c_{1,0}, c_{1,1}, c_{2,0})$ defining $\overline{Q}$ and Q must satisfy the following constraints to be exact on the space $\mathbb{P}_3$ of the univariate and bivariate cubic polynomials, respectively:

$$a_0 + 2a_1 + 2a_2 + 2a_3 = 1, \; a_1 + 4a_2 + 9a_3 = -\frac{5}{24}, \; a_1 + 16a_2 + 81a_3 = \frac{9}{32},$$

$$c_{0,0} + 4c_{1,0} + 4c_{1,1} + 4c_{2,0} = 1, \; c_{1,0} + 2c_{1,1} + 4c_{2,0} = -\frac{5}{24}.$$

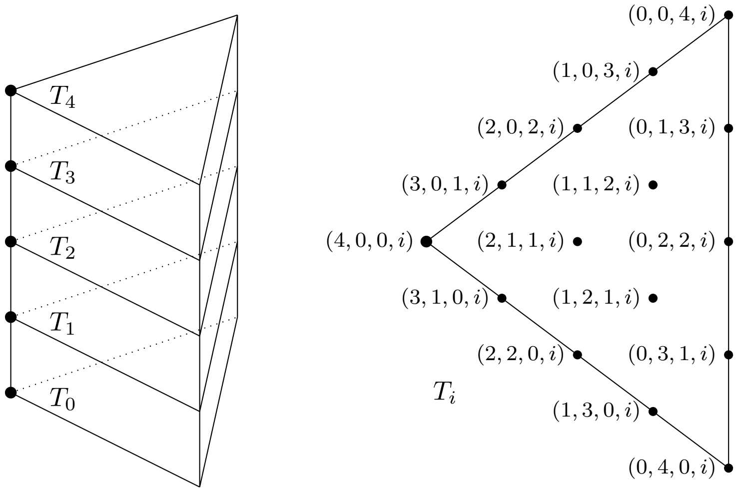


Figure 10: On the left, the five levels of the prism $P_r = T_s \times I_e$ in the quartic cases. On the right, indexes of the BB-coefficients $b_{(s,e),(\alpha,\beta_2}$ associated with the triangle T_i of P_r.

The objective function of the minimization problem depends on the variables a_1, $c_{1,0}$ and $c_{1,1}$. It has a unique solution, which has been exactly computed after converting it into a linear programming problem. The unique solution is given by

$$a_1 = -\tfrac{11209995503}{50649374592},\ c_{1,0} = \tfrac{40005773}{1489687488},\ c_{1,1} = \tfrac{2103011}{16883124864}.$$

Therefore, the coefficients providing the operators to be blended are

$$a_0 = \tfrac{37004143357}{25324687296},\ a_1 = -\tfrac{11209995503}{50649374592},\ a_2 = -\tfrac{976637497}{50649374592},\ a_3 = \tfrac{507176939}{50649374592},$$

and

$$c_{0,0} = \tfrac{1189752515}{1055195304},\ c_{1,0} = \tfrac{40005773}{1489687488},\ c_{2,0} = -\tfrac{2981191847}{50649374592},\ c_{1,1} = \tfrac{2103011}{16883124864}.$$

The upper bound is equal to

$$F\left(-\frac{11209995503}{50649374592}, \frac{40005773}{1489687488}, \frac{2103011}{16883124864}\right) = \frac{2788509053303}{1823377485312} \simeq 1.52931.$$

The infinity norm of the obtained type-2 NB QIO can be estimated by using a method similar to the one used in the quadratic case. We get the following result:

$$\|R\|_\infty \simeq \frac{103212192661}{67532499456} \simeq 1.52833.$$

The infinity norm is very close to the upper bound, so the method consisting in combining blending to define the QIO and BB-coefficients to bound the infinity norm provides a good upper bound to the infinity norm.

Now, we compare such results with those obtained when 1D and 2D *classical* or type-1 NB QIOs are blended. The first possibility (see Fig. 12) consists in blending the 1D QIO given by [27]

$$\overline{Q}_5 f = \sum_{i\in\mathbb{Z}} \left(\frac{319}{192} f(i) - \frac{107}{288} f(i\pm 1) + \frac{47}{1152} f(i\pm 2)\right) B_5(\cdot - i)$$

and the operator $P_{(1,2),a}$ defined as [25]

$$P_{(1,2),a} f = \sum_{i\in\mathbb{Z}^2} \lambda_{(1,2),a}(f(\cdot + i))\, M_{1,2}(\cdot - i),$$

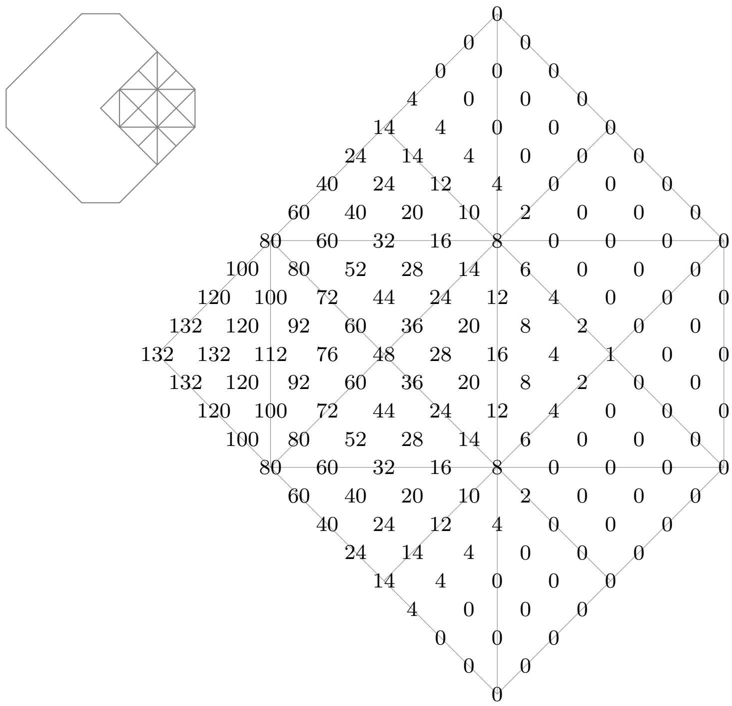


Figure 11: BB-coefficients of the box spline 384 $M_{1,2}$.

with

$$\lambda_{(1,2),a} g := \frac{97}{48} g(0) - \frac{13}{48}\left(g(\pm d_1) + g(\pm d_2)\right) + \frac{1}{64}\left(g(\pm d_1) + g(\pm d_2)\right).$$

The upper bound of the blending QIO in this case is equal to $\frac{128477}{73728} \simeq 1.74258$, and the estimated infinity norm is approximately equal to 1.68172.

If

$$\lambda_{(1,2),b} g := \frac{41}{24} g(0) - \frac{7}{48}\left(g(\pm d_1) + g(\pm d_2)\right) - \frac{1}{32}\left(g(\pm d_3) + g(i \pm d_4)\right)$$

and the operator

$$P_{(1,2),b} f = \sum_{i \in \mathbb{Z}^2} \lambda_{(1,2),b}\left(f(\cdot + i)\right) M_{1,2}(\cdot - i)$$

is used, then the upper bound is $\frac{5887}{3456} \simeq 1.70341$ and the estimated norm is approximately equal to 1.64000.

Notice that in both cases the type-2 QIO yields better results not only with respect to the upper bound (as expected by construction) but also regarding (the estimation of) the value of the infinity norm.

Finally, we can blend classical type-1 NB QIOs (see Fig. 13). They are given by the following expressions [4, 6]:

$$\overline{Q}_{\mathrm{nb}} f = \sum_{i \in \mathbb{Z}} \left(\frac{2015}{1728} f(i) - \frac{69}{640} f(i \pm 2) + \frac{107}{4320} f(i \pm 3)\right) B_5(\cdot - i),$$

$$Q_{\mathrm{nb}} f = \sum_{i \in \mathbb{Z}^2} \left(\frac{29}{24} f(i) - \frac{5}{96}\left(f(i \pm 2d_1) + f(i \pm 2d_2)\right)\right) M_{2,1}(\cdot - i).$$

The infinity norm of the resulting blending QIO is bounded by $\frac{8239}{5184} \simeq 1.58931$, and its estimated value is 1.55635. Also in this case both values are worse that the corresponding

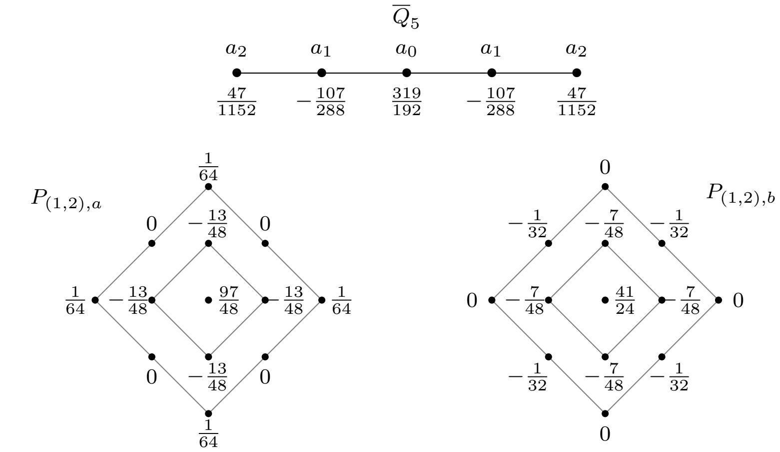


Figure 12: (Top) Coefficients of the linear functional of the classical 1D quartic QIO exact on $\mathbb{P}_4$. (Bottom) Coefficients of the linear functionals of the classical QIOs $P_{(1,2),a}$ and $P_{(1,2),b}$ defined by exploiting the relation between lozenge sequences and central difference operators.

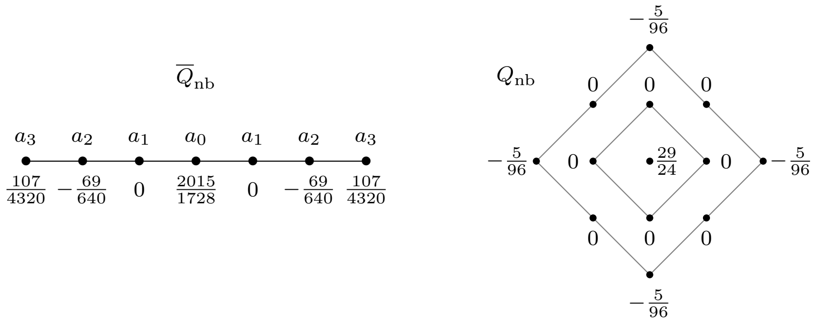


Figure 13: Coefficients of the linear functionals of the type-1 NB QIOs $\overline{Q}_{\mathrm{nb}}$ and Q_{nb} associated with the box spline $M_{1,2}$

ones obtained by minimizing the objective function, although they are better than the associated with $P_{(1,2),a}$ and $P_{(1,2),b}$.

As a conclusion, blending type-1 NB QIOs are better than blending *classical* operators. Moreover, blending type-2 NB QIOs are better than blending type-1 NB QIOs; therefore to blend general operators and minimize the upper bound to the infinity norm after imposing the exactness on $\mathbb{P}_3$ is a good method to construct QIOs.

4.2 Trivariate case based on the box spline $M_{2,1}$

In this section we present the results for the quartic box spline $M_{2,1}$. Fig. 14 shows the support of $M_{2,1}$ as well as its BB-coefficients.

The exactness of the operator $\overline{Q}$ and Q based on B_5 and $M_{2,1}$ on the spaces of cubic polynomials in one and two variables, respectively, is equivalent to the linear constraints

$$a_0 + 2a_1 + 2a_2 + 2a_3 = 1,\ a_1 + 4a_2 + 9a_3 = -\frac{5}{24},\ a_1 + 16a_2 + 81a_3 = \frac{9}{32},$$
$$c_{0,0} + 4c_{1,0} + 4c_{1,1} + 4c_{2,0} = 1,\ c_{1,0} + 2c_{1,1} + 4c_{2,0} = -\frac{1}{6}.$$

Also in this case the unconstrained minimization problem has been exactly solved. The variables of the objective function are a_3, $c_{1,0}$ and $c_{1,1}$. The minimum value is

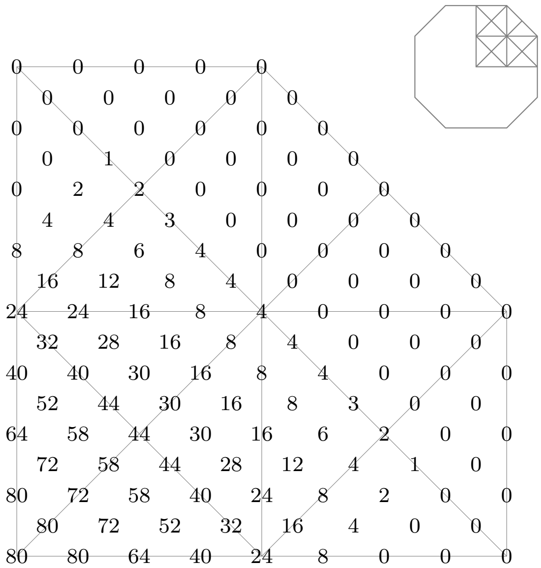


Figure 14: BB-coefficients of the box spline 384 $M_{2,1}$.

reached at

$$a_3 = \frac{21519}{1971520},\ c_{1,0} = \frac{9}{202},\ c_{1,1} = -\frac{4}{303},$$

providing for the coefficients involved in $\overline{Q}$ and Q the values

$$a_0 = \frac{1707131}{1182912},\ a_1 = -\frac{737441}{3548736},\ a_2 = -\frac{876217}{35487360},\ a_3 = \frac{21519}{1971520}$$

and

$$c_{0,0} = \frac{107}{101},\ c_{1,0} = \frac{9}{202},\ c_{2,0} = -\frac{14}{303},\ c_{1,1} = -\frac{4}{303}.$$

The upper bound for the infinity norm of the blending operator is

$$F\left(\frac{21519}{1971520}, \frac{9}{202}, -\frac{4}{303}\right) = \frac{80221363}{53231040} \simeq 1.50704,$$

and for its infinity norm we have estimated the value

$$\|R\|_\infty \simeq \frac{10228846717}{6813573120} \simeq 1.50125.$$

Again, the infinity norm is very close to the upper bound.

Also in this case we compare these results with those obtained when 1D and 2D classical or type-1 NB QIOs are blended. Fig. 15 shows the values of the linear functionals defining the QIOs $P_{(2,1),a}$ and $P_{(2,1),b}$ proposed in [25]. The values for $\overline{Q}_5$, defined in [27], appear in Fig. 12.

The upper bound of the blending operator associated with $P_{(2,1),a}$ and $\overline{Q}_5$ is equal to $\frac{1043}{576} \simeq 1.81076$, and its estimated norm is 1.74555. When $P_{(2,1),b}$ and $\overline{Q}_5$ are blended, the upper bound and estimated infinity norm are $\frac{751}{432} \simeq 1.73843$ and 1.63157, respectively. In both cases, the values we have obtained are worse than the corresponding ones associated with the type-2 NB QIO.

It only remains to blend 1D and 2D type-1 NB QIOs to compare their results with the values computed for the type-2 NB QIO. Fig. 16 shows the coefficients of the linear functionals of the type-1 NB QIOs $\overline{Q}_{\mathrm{nb}}$ and Q_{nb} associated with $M_{2,1}$ (see [4] and [6], respectively).

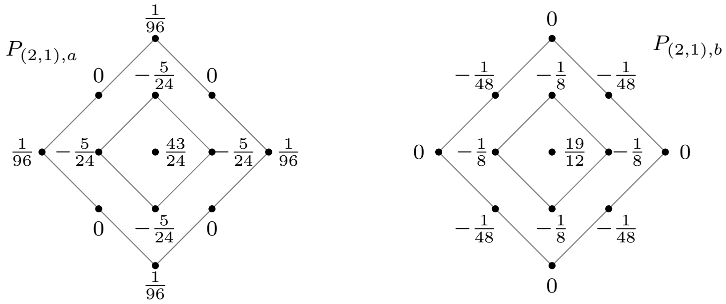


Figure 15: Coefficients of the linear functionals of the classical QIOs $P_{(2,1),a}$ and $P_{(2,1),b}$

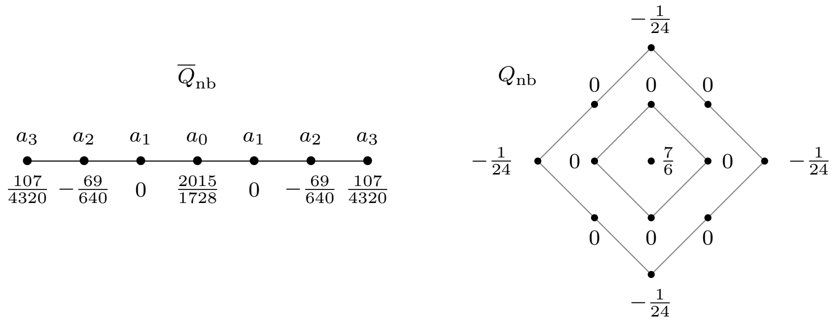


Figure 16: Coefficients of the linear functionals of the type-1 NB QIOs $\overline{Q}_{\mathrm{nb}}$ and Q_{nb} associated with $M_{2,1}$.

The upper bound of the blending QIO is equal to $\frac{8203}{5184} \simeq 1.58237$ and the estimated infinity norm is equal to 1.51479. We conclude that, also in this case, it is better to blend the operators and minimize the upper bound to the infinity norm than blending classical operators, i.e. blending type-2 NB QIOs are better than blending type-1 NB QIOs or classical QIOs.

In order to summarize, Table 1 shows the results we have obtained when the quartic box splines $M_{1,2}$ and $M_{2,1}$ are used to define blending QIOs. UB and eIN stand for Upper Bound and estimated Infinity Norm, respectively. Columns 2, 3 and contains the results obtained by trivariate type-2 and type-1 NB QIOs, respectively, and Columns 5, 6 show the results obtained when $\overline{Q}_5$ and classical bivariate QIOs are blended. For $M_{1,2}$ (resp. $M_{2,1}$) they correspond to $P_{(1,2),b}$ and $P_{(1,2),a}$ (resp. $P_{(2,1),b}$ and $P_{(2,1),a}$).

The box spline $M_{2,1}$ produces better operators than $M_{1,2}$ with respect to the UB and eIN. It also gives better results when type-1 NB QIOs or classical QIOs are blended.

box spline		type-2 NB	type-1 NB	classical b	classical a
$M_{1,2}$	UB	1.52931	1.58931	1.70341	1.74258
	eIN	1.52833	1.55635	1.64000	1.68172
$M_{2,1}$	UB	1.50704	1.58237	1.73843	1.81076
	eIN	1.50125	1.51479	1.63157	1.74555

Table 1: Comparison of upper bounds and estimated infinity norms for the different QIOs constructed from $M_{1,2}$ and $M_{2,1}$.

5 Conclusions

We have proposed a method for constructing trivariate quasi-interpolation operators based on the blending of bivariate and univariate operators. Such operators could be used for the construction of non-discrete models from given discrete data on volumetric grid, that is an important problem in many applications, such as scientific visualization, medical imaging and computer graphics. The coefficients of the corresponding linear functionals are computed to produce operators exact on appropriate spaces of polynomials and having small infinity norms. Instead of minimizing the true value of the infinity norm, an upper bound is minimized. The upper bound is established from the BB-coefficients of the fundamental functions associated with the operators involved in the construction. The proposed problem has always a solution, but it is not unique in general. We have completely worked out the quadratic and quartic cases. In the quadratic case, we have obtained infinitely many solutions depending of one parameter. Some numerical tests show the existence of a specific value yielding an excellent result. In the quartic cases, a unique solution exists for each one of the box splines considered. The quartic box spline $M_{2,1}$ produces better results than $M_{1,2}$ with respect to both the upper bound of the infinity norm and the infinity norm itself. Moreover, in all cases the upper bound of the infinity norm is very close to the value of the corresponding upper bound, so it seems that the proposed method provides a good upper bound of the infinity norm of the trivariate blending operator.

Acknowledgements

Work partially realized during the visiting of the third author to the Department of Mathematics, University of Torino. This work has been partially supported by the program "Progetti di Ricerca 2016" of the Gruppo Nazionale per il Calcolo Scientifico (GNCS) - INdAM.

Moreover, the authors thank the University of Torino for its support to their research. First and third authors also thank the Research Group FQM-191 for its support to this research.

Finally, the authors wish to thank the anonymous referees for their comments which helped them to improve the original manuscript.

References

[1] E.B. Ameur, D. Barrera, M.J. Ibáñez, and D. Sbibih. Near-best operators based on a C^2 quartic spline on the uniform four-directional mesh. Mathematics and Computers in Simulation, **77**:151–160, 2008.

[2] D. Barrera, M.J. Ibáñez. Bernstein-Bézier representation and near-minimally normed discrete quasi-interpolation operators. Applied Numerical Mathematics 58:59-68, 2008.

[3] D. Barrera, M.J. Ibáñez, P. Sablonnière, and D. Sbibih. Near-best quasi-interpolants associated with H-splines on a three-direction mesh. J. Comput. Appl. Math. 183:133–152, 2005.

[4] D. Barrera, M.J. Ibáñez, P. Sablonnière, and D. Sbibih. Near minimally normed spline quasi-interpolants on uniform partitions. J. Comput. Appl. Math. 181:211–233, 2005.

[5] D. Barrera, M.J. Ibáñez, P. Sablonnière, and D. Sbibih. Near-best univariate spline discrete quasi-interpolants on non-uniform partitions. Constr. Approx. 28:237–251, 2008.

[6] D. Barrera, M.J. Ibáñez, P. Sablonnière, and D. Sbibih. On near-best discrete quasi-interpolation on a four-directional mesh. J. Comput. Appl. Math. 233:1470–1477, 2010.

[7] D. Barrera, A. Guessab, M.J. Ibáñez, and O. Nouisser. Optimal bivariate C^1 cubic quasi-interpolation on a type-2 triangulation. J. Comput. Appl. Math. 234:1188–1199, 2010.

[8] D. Barrera, A. Guessab, M.J. Ibáñez, and O. Nouisser. Construction techniques for multivariate modified quasi-interpolants with high approximation order. Computers & Mathematics with Applications 65:29–41, 2013.

[9] D. Barrera, M.J. Ibáñez, and S. Remogna. On the construction of trivariate near-best quasi-interpolants based on C^2 quartic splines on type-6 tetrahedral partitions. J. Comput. Appl. Math. 311:252–261, 2017.

[10] C. de Boor, K. Höllig, S. Riemenschneider. *Box Splines*. Springer-Verlag, New York, 1993.

[11] C.K. Chui. *Multivariate Splines*. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 54, SIAM, Philadelphia, 1988.

[12] I. Cravero, C. Dagnino, S. Remogna. NURBS on Criss-cross Triangulations and Applications. Advances in Analysis 1:95-113, 2016.

[13] C. Dagnino, P. Lamberti. Numerical integration of 2-D integrals based on local bivariate C^1 quasi-interpolating splines. Advances in Computational Mathematics 8:19-31, 1998.

[14] C. Dagnino, P. Lamberti, and S. Remogna. Near-best C^2 quartic spline quasi-interpolants on type-6 tetrahedral partitions of bounded domains. Calcolo 52:475–494, 2015.

[15] C. Dagnino, P. Lamberti, and S. Remogna. Numerical integration based on trivariate C^2 quartic spline quasi-interpolants. BIT 53:873–896, 2013.

[16] C. Dagnino, S. Remogna. Quasi-interpolation based on the ZP-element for the numerical solution of integral equations on surfaces in $\mathbb{R}^3$, BIT DOI 10.1007/s10543-016-0633-x, 2016.

[17] C. Dagnino, S. Remogna, and P. Sablonnière. On the solution of Fredholm integral equations based on spline quasi-interpolating projectors, BIT 54:979–1008, 2014.

[18] A. Guessab, O. Nouisser, and G. Schmeisser. Multivariate approximation by a combination of modified Taylor polynomials. J. Comput. Appl. Math. 196:162–179, 2006.

[19] O. Nouisser. Construction de B-splines et quasi-interpolants associés sur un réseau quadridirectionnel uniforme du plan. Ph.D. Dissertation, Université Mohamed I, Maroc, 2002.

[20] S. Remogna. Constructing Good Coefficient Functionals for Bivariate C^1 Quadratic Spline Quasi-Interpolants. In: M. Daehlen & al. (eds.) Mathematical Methods for Curves and Surfaces, LNCS 5862, pp. 329–346. Springer-Verlag, Berlin Heidelberg, 2010.

[21] S. Remogna. Quasi-interpolation operators based on the trivariate seven-direction C^2 quartic box spline. BIT 51(3):757–776, 2011.

[22] S. Remogna. Bivariate C^2 cubic spline quasi-interpolants on uniform Powell-Sabin triangulations of a rectangular domain. Advances in Computational Mathematics 36:39–65, 2012.

[23] S. Remogna and P. Sablonnière. On trivariate blending sums of univariate and bivariate quadratic spline quasi-interpolants on bounded domains. Computer Aided Geometric Design 28:9–101, 2011.

[24] P. Sablonnière. Quasi-interpolants associated with H-splines on a three-direction mesh. J. Comput. Appl. Math. 66:433–442, 1996.

[25] P. Sablonnière. Quasi-interpolants associated with B-splines on the uniform four-directional mesh of the plane. Workshop Multivariate Approximation and Interpolation with Applications in CAGD, Signal and Image Processing. Eilat, Israel, 7-11 Sept. 1998.

[26] P. Sablonnière. H-splines and quasi-interpolants on a three directional mesh. In: M.D. Buhmann, D.H. Mache (Eds.), Advanced Problems in Constructive Approximation, International Series of Numerical Mathematics, vol. 142, pp. 187–201. Birkhäuser, 2002.

[27] P. Sablonnière. Univariate spline quasi-interpolants and applications to numerical analysis. Rend. Sem. Mat. Univ. Pol. Torino - Vol. 63:211-222, 2005.

[28] I. J. Schoenberg. Contributions to the problem of approximation of equidistant data by analytic functions, Part A: On the problem of smoothing or graduation, a first class of analytic approximation formulas. Quart. Appl. Math. vol. IV, No. 1:45–99, 1946.

[29] I. J. Schoenberg, Contributions to the problem of approximation of equidistant data by analytic functions, Part B: On the problem of osculatory interpolation, a second class of analytic approximation formulae, Quart. Appl. Math. vol. IV, No. 2:112–141, 1946.

[30] I. J. Schoenberg. *Cardinal Spline Interpolation*. CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 1973.

[31] T. Sorokina and F. Zeilfelder. Optimal quasi-interpolation by quadratic C^1 splines on four-directional meshes. In: C. Chui et al. (Eds.), Approximation Theory, vol. XI, Gatlinburg 2004, pp. 423–438. Nashboro Press, Brentwood, 2005.

[32] T. Sorokina and F. Zeilfelder. Local quasi-interpolation by cubic C^1 splines on type-6 tetrahedral partitions. IMA J. Numerical Analysis 27:74–101, 2007.

[33] T. Sorokina and F. Zeilfelder. An explicit quasi-interpolation scheme based on C^1 quartic splines on type-1 triangulations. Comput. Aided Geom. Design 25:1–13, 2008.

[34] H. Xuli. Multi-node higher order expansions of a function. Journal of Approximation Theory 124:242–253, 2003.

[35] R. H. Wang. *Multivariate Spline Functions and their Applications*. Dordrecht, Kluwer, 2001.

[36] G.A. Watson. *Approximation Theory and Numerical Methods*. Wiley, Chichester, 1980.