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On the asymptotic optimality of error bounds for some linear complementarity problems

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Abstract We introduce strong B-matrices and strong B-Nekrasov matrices, for which some error bounds for linear complementarity problems are analyzed. In particular, it is proved that the bounds of [5] and of [8] are asymptotically optimal for strong B-matrices and strong B-Nekrasov matrices, respectively. Other comparisons with a bound of [11] are performed.

Keywords Error bounds · Linear complementarity problems · B-matrices · B-Nekrasov matrices · P-matrices

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1 Introduction

The linear complementarity problem (LCP(M,q)) looks for a vector $x \in \mathbf{R}^n$ such that

$$Mx+q \geq 0, \quad x \geq 0, \quad x^T(Mx+q) = 0, \tag{1}$$

where M is an $n \times n$ real matrix and $q \in \mathbf{R}^n$.

It is well known that this problem has a unique solution if and only if M has positive principal minors (i.e., M is a P-matrix). Important applications of this problem can be seen in [2].

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Error bounds for LCP of P-matrices have been studied (cf., [1], [12]). For particular subclasses of P-matrices, the bounds can be refined: see [1] and [6] for the subclass of H-matrices with positive diagonal entries or [5] and [11] for the subclass of B-matrices. For classes of matrices containing B-matrices, error bounds for the LCP have also been obtained ([4], [7], [8], [9]). Among these classes of matrices we can mention the B-Nekrasov matrices, which will be also considered in this paper.

In some examples, the bound of [5] for B-matrices was improved by the bounds of [11]. We present and characterize in Section 2 a subclass of B-matrices called *strong B-matrices*, for which the bound of [5] is linear and asymptotically optimal (see Theorem 1) and for which the bound of [11] is worse than or equal to quadratic (see Theorem 3). At the end of Section 2, we also include a family of matrices that are simultaneously strong B-matrices and H-matrices and for which our bound is 1 and the bound of formula (2.4) of [1] (valid for H-matrices with positive diagonal entries) is arbitrarily large. A final example in Section 3 shows that our bound of [5] can improve that of [11] even for B-matrices that are not strong B-matrices. Finally, Section 4 introduces the class of strong B-Nekrasov matrices, which contains strong B-matrices and is contained in the class of B-Nekrasov matrices. We provide an error bound for the LCP of a strong B-Nekrasov matrix that is asymptotically optimal.

2 A class of B-matrices with an asymptotically optimal bound

The class of B-matrices is a subclass of P-matrices presented in [13] and has been applied to eigenvalues localization problems ([13], [14]) and to linear complementarity problems ([5], [11]). We recall the definition of a B-matrix.

Definition 1 A square real matrix $M = (m_{ij})_{1\le i,j\le n}$ is a B- matrix if it has positive row sums and all its off-diagonal entries are bounded above by the corresponding row means, that is, for each $i = 1,\dots,n$,

$$\sum_{j=1}^{n} m_{ij} > 0, \quad \text{and} \quad \frac{1}{n}\left(\sum_{k=1}^{n} m_{ik}\right) > m_{ij} \quad \forall\, j \neq i.$$

Given a matrix $M = (m_{ij})_{1\le i,j\le n}$, we define for each $i = 1,\dots,n$, $r_i^+ := \max\{0, m_{ij} | j \neq i\}$ and we can decompose M into the form $M = B^+ + C$, where

$$B^+ = \begin{pmatrix} m_{11} - r_1^+ & \cdots & m_{1n} - r_1^+ \\ \vdots & & \vdots \\ \vdots & & \vdots \\ m_{n1} - r_n^+ & \cdots & m_{nn} - r_n^+ \end{pmatrix} \text{ and } C = \begin{pmatrix} r_1^+ & \cdots & r_1^+ \\ \vdots & & \vdots \\ \vdots & & \vdots \\ r_n^+ & \cdots & r_n^+ \end{pmatrix}. \tag{2}$$

Then by Proposition 2.1 of [5], M is a B-matrix if and only if $B^+ =: (b_{ij})_{1\le i,j\le n}$ is a strictly diagonally dominant matrix ($|b_{ii}| > \sum_{j\neq i} |b_{ij}|, i = 1,\dots,n$) with positive diagonal entries. In this paper we introduce the following subclass of B-matrices by requiring a stronger diagonal dominant property to B^+.

Definition 2 Let M be a B-matrix and let us consider $M = B^+ + C$ as in (2). Given $B^+ =: (b_{ij})_{1\le i,j\le n}$, we define, for each $i = 1,\dots,n$, $\beta_i := b_{ii} - \sum_{j\neq i} |b_{ij}|$. Then we say that M is a strong B-matrix if $\beta_i > 1$ for all $i = 1,\dots,n$.

Given a complex matrix $M=(m_{ij})_{1\le i,j\le n}$, its *comparison* matrix $\tilde{M}=(\tilde{m}_{ij})_{1\le i,j\le n}$ is given by $\tilde{m}_{ij}:=-|m_{ij}|$ if $i\neq j$ and $\tilde{m}_{ii}:=|m_{ii}|$ for $i=1,\dots,n$. Let us recall that M is an *H-matrix* if $\tilde{M}$ is a nonsingular M-matrix, that is, if $\tilde{M}^{-1}$ is nonnegative. Error bounds for LCP with H-matrices cannot be applied to LCP with strong B-matrices because a strong B-matrix is not necessarily an H-matrix, as the following example shows.

Example 1 Let us consider the matrix

$$M=\begin{pmatrix} 61.1 & 30 & 20 & 10\\ -20 & 37.5 & 0 & -16\\ 0 & -40 & 51.5 & -10\\ 50 & 50 & 10 & 91.5 \end{pmatrix}.$$

The decomposition $M=B^{+}+C$ with B^{+} and C given in (2) leads to

$$B^{+}=\begin{pmatrix} 31.1 & 0 & -10 & -20\\ -20 & 37.5 & 0 & -16\\ 0 & -40 & 51.5 & -10\\ 0 & 0 & -40 & 41.5 \end{pmatrix}\quad\text{and}\quad C=\begin{pmatrix} 30 & 30 & 30 & 30\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 50 & 50 & 50 & 50 \end{pmatrix}.$$

Observe that M is a B-matrix and even a strong B-matrix. However, M is not an H-matrix because its comparison matrix $\tilde{M}$ has an inverse with nonpositive entries.

The following result characterizes strong B-matrices.

Proposition 1 *$M:=(m_{ij})_{1\le i,j\le n}$ is a strong B-matrix if and only if for each $i=1,\dots,n$,*

$$\sum_{j=1}^{n} m_{ij}>1 \quad \textit{and} \quad \frac{1}{n}\left(\sum_{j=1}^{n} m_{ij}-1\right)>m_{ik} \quad \forall k\neq i. \tag{3}$$

Proof Let us assume first that M is a strong B-matrix. Then $\beta_i=b_{ii}-\sum_{j\neq i}|b_{ij}|>1$, for all $i=1,\dots,n$, and $B^{+}=(b_{ij})_{1\le i,j\le n}$ is given in (2). Taking into account that $r_i=\max\{0,m_{ij}|j\neq i\}$, we have that $r_i^{+}\ge m_{ij}$ for all $j\neq i$, and we derive from the previous formula

$$1<m_{ii}-r_i^{+}-\sum_{j\neq i}(r_i^{+}-m_{ij})=\sum_{j=1}^{n} m_{ij}-nr_i^{+} \tag{4}$$

and so we conclude that $\sum_{j=1}^{n} m_{ij}>1+nr_i^{+}\ge 1$. From (4) we also have $\sum_{j=1}^{n} m_{ij}-1> nr_i^{+}\ge nm_{ik}$ for all $k\neq i$ and (3) holds.

Let us now assume that (3) holds. Clearly, (3) implies that M is a B-matrix. Since $r_i^{+}\ge m_{ij}$ for all $j\neq i$,

$$\beta_i=m_{ii}-r_i^{+}-\sum_{j\neq i}(r_i^{+}-m_{ij})=\sum_{j=1}^{n} m_{ij}-nr_i^{+}. \tag{5}$$

If $r_i^{+}=0$, then by (5) and (3), $\beta_i=\sum_{j=1}^{n} m_{ij}>1$. Finally, if $r_i^{+}\neq 0$, then there exists $k\neq i$ such that $r_i^{+}=m_{ik}>0$. Then, by (5) and (3) we obtain $\beta_i=\sum_{j=1}^{n} m_{ij}-nm_{ik}=n(\frac{1}{n}\sum_{j=1}^{n} m_{ij}-m_{ik})>n(\frac{1}{n})=1$ and M is a strong B-matrix.

Example 2 Now we present some examples arising in practical applications. Given the tridiagonal $n \times n$ matrix

$$M = \begin{pmatrix} b+\alpha\sin\left(\frac{1}{n}\right) & c & & & \\ a & b+\alpha\sin\left(\frac{2}{n}\right) & c & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b+\alpha\sin(1) \end{pmatrix}, \tag{6}$$

the LCP(M,q) with various q in an interval vector arises from the finite difference method for free boundary problems (cf. [15], [1]). Observe that, for $b > 3$, $a = c = -1$, $\alpha = 0$, the corresponding family is formed by strong B-matrices and also for $b = 3$, $a = c = -1$, $\alpha > 0$. In fact, the decomposition (2) for these matrices is of the form $M = B^+ + 0$ and B^+ satisfies the properties of Definition 2. Besides, for any $k_1, \dots, k_n \geq 0$, we can form matrices

$$C = \begin{pmatrix} k_1 & \cdots & k_1 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ k_n & \cdots & k_n \end{pmatrix}$$

and $M + C$ is again a strong B-matrix by Definition 2.

In this section we shall prove that the bound of Theorem 2.2 of [5] is asymptotically optimal for the class of strong B-matrices. For this purpose, we consider the following family of $n \times n$ strong B-matrices ($n \geq 2$),

$$M_m = \begin{pmatrix} m+k & m & \cdots & m \\ 1 & 1+k & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1+k \end{pmatrix}, \tag{7}$$

where m is a positive integer and $k > 1$. Observe that, if $k \in (0,1]$, then M_m is a B-matrix but it is not a strong B-matrix.

First, we recall some notations for the linear complementarity problem (1). Its solution x^* is unique if and only if M is a P-matrix. In this case, by Theorem 2.3 of [1],

$$\|x - x^*\|_\infty \leq \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty,$$

where I is the identity matrix, D is the diagonal matrix $\operatorname{diag}(d_i)$ with $0 \leq d_i \leq 1$ for all $i = 1, \dots, n$, and $r(x) := \min(x, Mx+q)$, where the min operator denotes the componentwise minimum of two vectors. If M is a B-matrix and β_i, $i = 1, \dots, n$, are defined as in Definition 2, let us denote by

$$\beta := \min_{i\in\{1,\dots,n\}}\{\beta_i\}. \tag{8}$$

Then, by Theorem 2.2 of [5],

$$\max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \leq \frac{n-1}{\min\{\beta, 1\}}. \tag{9}$$

Theorem 1 *For strong B-matrices, the bound (9) is asymptotically optimal and it is equal to $n-1$.*

Proof First observe that for strong B-matrices the bound (9) is equivalent to

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \le n-1, \tag{10}$$

because of (8) and $\beta > 1$. Let us consider the family of matrices M_m given by (7), with m a positive integer and $k>1$, and the particular choice in the left side of (9) given by $d=(1,\ldots,1)^T$, which corresponds to the diagonal matrix $D=I$. So, with this choice, we have the following inequality

$$\max_{d\in[0,1]^n}\|(I-D+DM_m)^{-1}\|_\infty \ge \|M_m^{-1}\|_\infty. \tag{11}$$

Observe that M_m can be written in the form (2) as $M_m = K+ue^T$ where

$$K := \begin{pmatrix} k & 0 & \cdots & 0 \\ 0 & k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & k \end{pmatrix}, \quad u := \begin{pmatrix} m \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}, \quad e := \begin{pmatrix} 1 \\ 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}.$$

Then we have

$$M_m = K(I+K^{-1}ue^T) = K(I+u_ke^T), \tag{12}$$

where $u_k := (m/k, 1/k, \ldots, 1/k)^T$. Then, since $e^Tu_k \neq -1$, by the Sherman-Morrison formula (see formula (2.1.5) of page 65 of [10]), we obtain from (12)

$$M_m^{-1} = (I+u_ke^T)^{-1}K^{-1} = (I-\frac{u_ke^T}{1+e^Tu_k})K^{-1}$$

and so

$$M_m^{-1} = \begin{pmatrix} 1-\frac{m}{m+n+k-1} & \frac{-m}{m+n+k-1} & \cdots & \frac{-m}{m+n+k-1} \\ \frac{-1}{m+n+k-1} & 1-\frac{1}{m+n+k-1} & \cdots & \frac{-1}{m+n+k-1} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \frac{-1}{m+n+k-1} & \frac{-1}{m+n+k-1} & \cdots & 1-\frac{1}{m+n+k-1} \end{pmatrix}.$$

Then $\|M_m^{-1}\|_\infty = 1-\frac{m}{m+n+k-1}+\frac{(n-1)m}{m+n+k-1} = \frac{n+(k-1)+(n-1)m}{m+n+k-1}$ and we derive

$$\lim_{m\to\infty}\|M_m^{-1}\| = n-1. \tag{13}$$

By (11) and (13), $\lim_{m\to\infty}\max_{d\in[0,1]^n}\|(I-D+DM_m)^{-1}\|_\infty = n-1$ and (10) is asymptotically optimal.

Let us illustrate the asymptotic optimality of the previous result with some particular values of n,k,m for the $n\times n$ matrix M_m of (7). A lower bound for $\max_{d\in[0,1]^n}\|(I-D+DM_m)^{-1}\|_\infty$ is $\|M_m^{-1}\|_\infty$ by (11) and in the line previous to (13) we have seen that $\|M_m^{-1}\|_\infty = 1-\frac{m}{m+n+k-1}+\frac{(n-1)m}{m+n+k-1} = \frac{n+(k-1)+(n-1)m}{m+n+k-1}$. For $n=10$ and $k=2$, we obtain that this lower bound is $\frac{20}{12}$ for $m=1$, $\frac{101}{21}$ for $m=10$, $\frac{911}{111}$ for $m=100$ and $\frac{9011}{1011}$ for $m=1000$, which shows the approximation to our upper bound $n-1=9$ of (10).

We now present a family of 2×2 matrices of Example 2 that are simultaneously strong B-matrices and H-matrices. For these matrices our bound (10) is 1 and the bound of formula (2.4) of [1] (valid for H-matrices with positive diagonal entries) is arbitrarily large.

Example 3 Let us consider 2×2 matrices of Example 2 with $b=4$, $a=c=-1$, $\alpha=0$ and $k_1=k_2=k>1$. So our matrices are of the form

$$M=\begin{pmatrix} 4+k & k-1 \\ k-1 & 4+k \end{pmatrix}.$$

These strong B-matrices are also H-matrices because the comparison matrix

$$\tilde{M}=\begin{pmatrix} 4+k & -k+1 \\ -k+1 & 4+k \end{pmatrix}$$

has nonnegative inverse

$$\tilde{M}^{-1}=\begin{pmatrix} \frac{4+k}{5(3+2k)} & \frac{k-1}{5(3+2k)} \\ \frac{k-1}{5(3+2k)} & \frac{4+k}{5(3+2k)} \end{pmatrix}.$$

Our bound (10) is $n-1=1$. Let us now consider the bound of (2.4) of [1]:

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \le \|\tilde{M}^{-1}\max(\Lambda,I)\|_\infty,$$

where Λ is the diagonal part of M ($\Lambda := \operatorname{diag}(m_{ii})$) and $\max(\Lambda,I) := \operatorname{diag}(\max\{m_{11},1\},\dots,\max\{m_{nn},1\})$. For our matrices M, $\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty = \frac{4}{5}+\frac{k}{5}$, which is greater than 1 and can be arbitrarily large.

3 Comparisons with another recent bound

In [11] the authors provided for a B-matrix M an upper bound for $\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty$ different from (9) and they showed some examples where this bound (presented in Theorem 4 of [11]) improves (9). We are now going to prove that for strong B-matrices the bound (9) (or its equivalent form (10) as shown in Theorem 1) is better than the bound of Theorem 4 of [11] . We shall prove in Theorem 2 that the bound of Theorem 4 of [11] is worse than or equal to quadratic $n(n-1)$, in contrast with our linear bound $n-1$ of (10). Previously we recall Theorem 4 of [11].

Theorem 2 *Let $M=(m_{ij})_{1\le i,j\le n}$ be a B- matrix with the form $M=B^+ + C$, where $B^+ := (b_{ij})_{1\le i,j\le n}$ and C are given in (2). Then*

$$\max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty \le \frac{n-1}{\min\{\bar{\beta}_1,1\}} + \sum_{i=2}^{n}\frac{n-1}{\min\{\bar{\beta}_i,1\}}\prod_{j=1}^{i-1}(1+\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|) =: b_n, \tag{14}$$

where for each $i=1,\dots,n$,

$$\bar{\beta}_i := b_{ii} - l_i(B^+)\sum_{j=i+1}^{n}|b_{ij}| \quad \text{and} \quad l_i(B^+) := \max_{i\le k\le n}\{\frac{1}{|b_{kk}|}\sum_{j=i,j\ne k}^{n}|b_{kj}|\}.$$

Let us now provide a lower bound for the bound of Theorem 2 in the case of strong B-matrices.

Theorem 3 *Let $M=(m_{ij})_{1\le i,j\le n}$ be a strong B-matrix. Then the bound b_n of (14) satisfies*

$$b_n \ge n(n-1) \tag{15}$$

Proof With the notations of Theorem 2, let us observe that for each $i = 1,\dots,n$, $l_i(B^+) < 1$ (see (11) of [11]). Therefore $\bar{\beta}_i \geq \beta_i$, for each $i = 1,\dots,n$. Since M is a strong B-matrix, for $i = 1,\dots,n$, $\beta_i > 1$, which implies that $\bar{\beta}_i > 1$ and so, $\min\{\bar{\beta}_i, 1\} = 1$. Then, since $\prod_{j=1}^{i-1}\left(1+\frac{1}{\bar{\beta}_j}\sum_{k=j+1}^{n}|b_{jk}|\right) \geq 1$, we can deduce that $b_n \geq n-1+\sum_{i=2}^{n}(n-1).1 = n(n-1)$.

Although theorems 1 and 3 only hold for strong B-matrices, the bound (9) can be sharp and even better than (14), also for B-matrices that are not strong B-matrices. In the following example we consider a B-matrix that is not strong and we shall compare the bound (9) with (14). Let us consider the matrix

$$M = \begin{pmatrix} 60.5 & 30 & 20 & 10 \\ -20 & 40 & 0 & -16 \\ 0 & -40 & 51 & -10 \\ 50 & 50 & 10 & 91 \end{pmatrix}.$$

The decomposition $M = B^+ + C$ with B^+ and C given in (2) leads to

$$B^+ = \begin{pmatrix} 30.5 & 0 & -10 & -20 \\ -20 & 40 & 0 & -16 \\ 0 & -40 & 51 & -10 \\ 0 & 0 & -40 & 41 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 30 & 30 & 30 & 30 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 50 & 50 & 50 & 50 \end{pmatrix}.$$

Observe that M is a B-matrix (because B^+ is strictly diagonally dominant with positive diagonal entries) and that it is not a strong B-matrix (because $\beta = \beta_1 = 0.5$). One can check that the bound of Theorem 4 of [11] (that is, the bound b_n of (14)) is in this case

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \leq b_n = 445.321.$$

Let us now compute our bound (9). Taking into account that $n = 4$ and $\beta_1 = 0.5$, $\beta_2 = 4$, $\beta_3 = \beta_4 = 1$, (and so, $\beta = 0.5$ by (8)), we deduce that (9) is now $3/0.5 = 6$.

4 Strong B-Nekrasov matrices and asymptotically optimal bounds

Let us recall the definition of a Nekrasov matrix. For this purpose, we need some notations. If $M = (m_{ij})_{1\leq i,j\leq n}$ is a complex matrix with $m_{ii} \neq 0$ for all $i = 1,\dots,n$, let us define

$$h_1(M) := \sum_{j\neq 1}|m_{1j}|,\ h_i(M) := \sum_{j=1}^{i-1}|m_{ij}|\frac{h_j(M)}{|m_{jj}|} + \sum_{j=i+1}^{n}|m_{ij}|, \quad i = 2,\dots,n. \qquad (16)$$

Then M is a *Nekrasov matrix* if $|m_{ii}| > h_i(M)$ for all $i = 1,\dots,n$ (cf. [16]). It is well-known that Nekrasov matrices are nonsingular matrices.

The following definition recalls the concept of B-Nekrasov matrix, which was introduced in [8], and introduces the new definition of strong B-Nekrasov matrices.

Definition 3 A real matrix M is a B-Nekrasov matrix if $M = B^+ + C$, where B^+ and C are given in (2) and B^+ is a Nekrasov Z-matrix with all diagonal entries positive. If $B^+ - I$ is a Nekrasov Z-matrix with all diagonal entries positive, then we say that M is a strong B-Nekrasov matrix.

By Remark 1 of [8], B-matrices are B-Nekrasov matrices and, by Proposition 1 of [8], B-Nekrasov matrices are P-matrices. Since a strong B-matrix M can be written $M = B^+ + C$ as in (2) and satisfies that $B^+ - I$ is a strictly diagonally dominant matrix with positive diagonal entries, we can deduce that a strong B-matrix is a strong B-Nekrasov matrix.

The following result shows that a strong B-Nekrasov matrix is also a B-Nekrasov matrix.

Proposition 2 *If M is a strong B-Nekrasov matrix, then M is also B-Nekrasov.*

Proof It is sufficient to prove that if $B^+ - I$ is a Nekrasov Z-matrix with positive diagonal entries, then $B^+ = (b_{ij})_{1\le i,j\le n}$ is a Nekrasov Z-matrix with positive diagonal entries. So, we assume that

$$b_{ii} - 1 > h_i(B^+ - I), \quad i = 1,\dots,n, \tag{17}$$

and let us prove that

$$b_{ii} > h_i(B^+), \quad i = 1,\dots,n. \tag{18}$$

Since $b_{ii} > b_{ii} - 1$ for all $i = 1,\dots,n$, in order to prove (18) from (17), we shall prove that

$$h_i(B^+ - I) \ge h_i(B^+), \quad i = 1,\dots,n. \tag{19}$$

Let us prove (19) by induction on i. Since $h_1(B^+ - I) = \sum_{j\ne 1} |b_{1j}| = h_1(B^+)$, (19) holds for $i = 1$. Let us now assume that (19) holds for all $j < i$ and let us prove it for i. Then

$$h_i(B^+ - I) = \sum_{j=1}^{i-1} |b_{ij}| \frac{h_j(B^+ - I)}{b_{jj} - 1} + \sum_{j=i+1}^{n} |b_{ij}|.$$

Taking into account the induction hypothesis and that $b_{jj} - 1 < b_{jj}$, we obtain

$$h_i(B^+ - I) \ge \sum_{j=1}^{i-1} |b_{ij}| \frac{h_j(B^+)}{b_{jj}} + \sum_{j=i+1}^{n} |b_{ij}| = h_i(B^+)$$

and the induction holds.

The converse of the previous proposition does not hold as the following example shows. Any matrix

$$M_k = \begin{pmatrix} k & -k+1 \\ -k+1 & k \end{pmatrix}, \quad k \ge 2,$$

is B-Nekrasov because $M_k = B_k^+ + C_k$ with $B_k^+ = M_k$ and $C_k = 0$ and it can be checked that $B_k = M_k$ is Nekrasov. However, M_k is not strong B-Nekrasov because

$$B_k^+ - I = \begin{pmatrix} k-1 & -k+1 \\ -k+1 & k-1 \end{pmatrix}$$

is singular and so, $B_k^+ - I$ can not be Nekrasov.

It is well–known that a complex matrix A is an H-matrix if there exists a diagonal matrix W =diag(w_i) such that AW is strictly diagonal dominant. Observe that the diagonal matrix W can be taken with positive diagonal entries (it is sufficient using diag$(|w_i|)$). It is known that a Nekrasov matrix is an H-matrix (see [16] and p. 5021 of [3]). So, given a B-Nekrasov matrix M, there exists a diagonal matrix W with positive diagonal entries such that B^+W is a strictly diagonally dominant Z-matrix (where B^+ is given by (2)). Then this holds, in particular, for strong B-Nekrasov matrices. Since a strong B-matrix satisfies that $B^+ - I$ is a strictly diagonally dominant matrix with positive diagonal entries, we can deduce that a strong B-matrix is a strong B-Nekrasov matrix.

In Theorem 2 of [8], we obtained an error bound for the LCP of a B-Nekrasov matrix $A = B^+ + C$ satisfying certain hypotheses that allowed us to construct a particular diagonal matrix W such that B^+W is a strictly diagonally dominant Z-matrix. As we have recalled in the previous paragraph, for any B-Nekrasov matrix $A(= B^+ + C)$, there exists a diagonal matrix W with positive diagonal entries such that B^+W is a strictly diagonally dominant Z-matrix. The same proof of Theorem 2 of [8] can be used to prove the following result, which does not require any additional hypothesis on the B-Nekrasov matrices.

Theorem 4 *Let $M = (m_{ij})_{1\le i,j\le n}$ be a B-Nekrasov matrix, let B^+ be the matrix of (2) and let $W = diag(w_i)$ be the diagonal matrix with positive diagonal entries such that $\bar{B} := B^+W = (\bar{b}_{ij})_{1\le i,j\le n}$ is a strictly diagonally dominant Z-matrix. Let $\beta_i := \bar{b}_{ii} - \sum_{j\neq i} |\bar{b}_{ij}|$ and $\delta_i := \frac{\beta_i}{w_i}$, for $i = 1,\dots,n$ and $\delta = min_i\{\delta_i\}$. Then*

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \le \frac{(n-1)\max_i\{w_i\}}{\min\{1,\delta\}\min_i\{w_i\}}. \tag{20}$$

Now, we shall provide an error bound for the LCP of a strong B-Nekrasov matrix M.

Theorem 5 *Let $M = (m_{ij})_{1\le i,j\le n}$ be a strong B-Nekrasov matrix, let B^+ be the matrix of (2) and let $W = diag(w_i)$ be the diagonal matrix with positive diagonal entries such that $(B^+ - I)W$ is a strictly diagonally dominant Z-matrix. Then*

$$\max_{d\in[0,1]^n} \|(I-D+DM)^{-1}\|_\infty \le \frac{(n-1)\max_i\{w_i\}}{\min_i\{w_i\}}. \tag{21}$$

Moreover, the bound (21) is asymptotically optimal.

Proof Since M is a strong B-matrix, $B^+ - I$ is a Nekrasov matrix and so it is an H-matrix. Therefore there exists a diagonal matrix W with positive diagonal entries such that $(B^+ - I)W$ is a strictly diagonally dominant Z-matrix. Then B^+W is also a strictly diagonally dominant Z-matrix and, since a strong B-Nekrasov matrix is a B-Nekrasov matrix, we can apply Theorem 4 with $\bar{B} := B^+W$ and (20) holds. Since $(B^+ - I)W = \bar{B} - W$ is also a strictly diagonally dominant Z-matrix, $\beta_i - w_i > 0$ for all $i = 1,\dots,n$. This implies that $\delta_i := \frac{\beta_i}{w_i} > 1$ for all $i = 1,\dots,n$. If $\delta = \min_i\{\delta_i\}$, then $\min\{1,\delta\} = 1$ and so (20) becomes (21).

Now, let us consider the family of strong B-Nekrasov matrices M_m given by (7) with m a positive integer and $k > 1$, and the particular choice in the left side of (21) given by $d = (1,\dots,1)^T$, which corresponds to the diagonal matrix $D = I$. So, with this choice we have (11). Observe that if we write $M = B^+ + C$ as in (2), then B^+ is the diagonal matrix whose diagonal entries are equal to k. So, $B^+ - I$ is a strictly diagonally dominant Z-matrix and we can choose the matrix $W = I$ to obtain the formula (21). In this case the right side of (21) is $n-1$. Finally, since (11) and (13) hold, we see that (21) is asymptotically optimal.

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