# Three-step alternating iterations for index one matrices 

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## Abstract

Iterative methods based on matrix splittings are useful in solving large sparse linear systems. In this direction, proper splittings and its several extensions are used to deal with singular and rectangular linear systems. In this article, we introduce a new iteration scheme called three-step alternating iterations using proper splittings and group inverses to find an approximate solution of singular linear systems, iteratively. A preconditioned alternating iterative scheme is also proposed to relax some sufficient conditions and to obtain faster convergence as well. We then show that our scheme converges faster than the existing one. The theoretical findings are then validated numerically.

## 1. Introduction

A real square matrix $A$ is called a $Z$-matrix if the off-diagonal entries of $A$ are non-positive. A $Z$-matrix $A$ can be written as $A=s I-B$, where $s \geq 0$ and $B \geq 0$. Here $B \geq 0$ means all the entries of $B$ are non-negative. A $Z$-matrix $A$ is called an $M$-matrix if $s \geq \rho(B)$, where $\rho(B)$ denotes the spectral radius of $B$ and is the maximum of the moduli of the eigenvalues of $B$. If $s>\rho(B)$, it follows that $A^{-1}$ exists and $A^{-1} \geq 0$. Many interesting characterizations of nonsingular $M$-matrices can be found in the book by Berman and Plemmons [4]. The set of nonsingular $M$-matrices are one of the most important subclass of monotone matrices. A real $n \times n$ matrix $A$ is called monotone if $A x \geq 0 \Rightarrow x \geq 0$. The book by Collatz [7] has discussed the natural occurrence of monotone matrices in finite difference approximation methods for certain type of partial differential equations. This class of matrices also arises in linear complementary problems in operations research, input-output production and growth models in economics and Markov processes in probability and statistics, to name a few. Singular $M$-matrices (when $s=\rho(B)$ ) very often appear in the same context as nonsingular $M$-matrices, in particular in the study of Markov processes (see Meyer [19]). These matrices
also arise in finite difference methods for solving certain partial differential equations such as the Neumann problem and Poissons equation on a sphere (see Plemmons [29]). The books by Berman and Plemmons [4] and Varga 32] give an excellent account of many characterizations of the notion of monotonicity to singular and rectangular matrices. In this article, we focus on the convergence of iterative methods for solving singular linear systems using group (generalized) inverses. This study will help us to find an approximate solution of a singular linear system of the form

$$
\begin{equation*}
A x=b, \tag{1}
\end{equation*}
$$

where $A$ is a real $n \times n$ matrix of index 1 and $x, b$ are real $n$-vectors. For a real square matrix $A$, the index of $A$ is defined as the smallest non-negative integer $k$, which satisfies $\operatorname{rank}\left(A^{k}\right)=\operatorname{rank}\left(A^{k+1}\right)$. We call a singular linear system $A x=b$ of index 1 if index of $A$ is 1. The group (generalized) inverse of a matrix $A \in \mathbb{R}^{n \times n}$, denoted by $A^{\#}$ (if it exists), is the unique matrix $X$ satisfying $A=A X A, X=X A X$ and $A X=X A$. For index 1 matrices, it always exists. A group invertible matrix $A$ is called group monotone if $A^{\#} \geq 0$.

Wei [33] showed that for a singular linear system $A x=b$ of index 1 , the iteration scheme:

$$
\begin{equation*}
x^{i+1}=U^{\#} V x^{i}+U^{\#} b \tag{2}
\end{equation*}
$$

converges to $A^{\#} b$ if and only if $\rho\left(U^{\#} V\right)<1$ (see Corollary 3.2, [33]) by using proper splitting $A=U-V$. A splitting $A=U-V$ of $A \in \mathbb{R}^{n \times n}$ is called a proper splitting [3] if $R(U)=R(A)$ and $N(U)=N(A)$, where $R(B)$ and $N(B)$ stand for the range space and the null space of a matrix $B$, respectively. Thereafter, he studied the convergence of the above iteration scheme for different sub-classes of proper splittings (see Theorem $4.1 \& 4.2$, [33]).

However, the iteration scheme (2) converges very slow in many practical cases. To overcome this, several comparison results are proposed in the literature (see [9], 13], 14], [15] and [34] and the references cited therein). In case of a matrix having many proper splittings, comparison results are not so useful to find the best splitting (in the sense that the iteration matrix arising from a matrix splitting has the smallest spectral radius). To deal with this case, we propose a three-step alternating iteration scheme by extending the idea of Benzi and Szyld [2] who proposed the concept of two-step alternating iteration method.

The rest of the paper is sectioned as follows. In the next section, we introduce our notations, definitions and some preliminary results which are basics for defining our problem. The notion of proper G-regular and proper G-weak regular splitting along with some perquisite results are proved in section 3. Section 4 contains the main results which discuss convergence criteria for the proposed alternating iteration scheme. It also provides an
algorithm for the three-step alternating iteration scheme with a little emphasis on preconditioning technique. The theoretical results are then validated through computation and are shown in section 5 . The last one is about concluding remarks.

## 2. Preliminaries

In the subsequent sections, $\mathbb{R}^{n}$ means an $n$-dimensional Euclidean space while $\mathbb{R}^{n \times n}$ denotes the set of all real square matrices of order $n$. Assume that $S$ and $T$ are complementary subspaces of $\mathbb{R}^{n}$. Then $P_{S, T}$ is the projection on $S$ along $T$. So, $P_{S, T} A=A$ if and only if $R(A) \subseteq S$ and $A P_{S, T}=A$ if and only if $N(A) \supseteq T$. We next produce the definitions of three important generalized inverses. The Drazin inverse of a matrix $A \in \mathbb{R}^{n \times n}$ is the unique solution $X \in \mathbb{R}^{n \times n}$ satisfying the equations: $A^{k}=A^{k} X A, X=X A X$ and $A X=X A$, where $k$ is the index of $A$. It is denoted by $A^{D}$. When $k=1$, then Drazin inverse is said to be the group inverse of $A$. For $A \in \mathbb{R}^{m \times n}$, the unique matrix $Z \in \mathbb{R}^{n \times m}$ satisfying the following four equations known as Penrose equations: $A Z A=A, Z A Z=Z,(A Z)^{t}=A Z$ and $(Z A)^{t}=$ $Z A$ is called the Moore-Penrose inverse of $A$, where $B^{t}$ denotes the transpose of $B$. It always exists, and is denoted by $A^{\dagger}$. When the matrix $A$ is nonsingular, then $A^{D}=A^{\#}=$ $A^{\dagger}=A^{-1}$. The criteria 'index 1 ' for the existence of the group inverse is also equivalent to $N(A)=N\left(A^{2}\right)$ or $R(A)=R\left(A^{2}\right)$ or $R(A) \oplus N(A)=\mathbb{R}^{n}$. A few basic properties which will be frequently used are: $R(A)=R\left(A^{\#}\right) ; N(A)=N\left(A^{\#}\right) ; A A^{\#}=P_{R(A), N(A)}=A^{\#} A$. In particular, if an element $x \in R(A)$, then $x=A^{\#} A x$.

The computation of the group inverse of an index one matrix is shown in Algorithm [1, and the same method can be found in [18].

```
Algorithm 1 Computation of the Group Inverse
    procedure GINV \((A)\)
        if \(\operatorname{rank}(A)=\operatorname{rank}\left(A^{2}\right)\) then
            \(r=\operatorname{rank}(A)\)
            \(Q=\left[\begin{array}{ll}B_{R(A)} & B_{N(A)}\end{array}\right]\)
            \(P=Q^{-1} A Q\)
            denote \(C=\) Top \(r \times r\) sub matrix of \(P\)
            \(D=\left[\begin{array}{c|c}C^{-1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0}\end{array}\right]_{n \times n}\)
            return \(A^{\#}=Q D Q^{-1}\)
        else
            "The matrix is not of index 1 "
        end if
    end procedure
```

The remaining results are collected in the next two subsections.

### 2.1. Non-negative matrices

We call $A \in \mathbb{R}^{n \times n}$ as non-negative (positive) if $A \geq 0$, $(A>0)$. We write $B \geq C$ if $B-C \geq 0$. The same notation and nomenclature are also used for vectors. The next results deal with the non-negativity of a matrix and the spectral radius.

Theorem 2.1 (Theorem 2.20, [32]). Let $A \in \mathbb{R}^{n \times n}$ and $A \geq 0$. Then
(i) A has a non-negative real eigenvalue equal to its spectral radius.
(ii) There exists a non-negative eigenvector for its spectral radius.

Theorem 2.2 (Theorem 2.1.11, [4]). Let $B \in \mathbb{R}^{n \times n}, B \geq 0, x \geq 0(x \neq 0)$ and $\alpha$ is a positive scalar.
(i) If $\alpha x \leq B x$, then $\alpha \leq \rho(B)$.
(ii) If $B x-\alpha x \leq 0, x>0$, then $\rho(B) \leq \alpha$.

The last result is a special case of Theorem 3.16, 32].
Theorem 2.3. Let $X \in \mathbb{R}^{n \times n}$ and $X \geq 0$. Then $\rho(X)<1$ if and only if $(I-X)^{-1}$ exists and $(I-X)^{-1}=\sum_{k=0}^{\infty} X^{k} \geq 0$.

### 2.2. Proper Splittings

The notion of proper splitting introduced by Berman and Plemmons [3] plays a key role in the study of the convergence of iterative methods to find an approximate solution of real large singular and rectangular linear systems. It is extended to index splitting by Wei [33] and index-proper splitting by Chen and Chen [6] to find the approximate iterative solution of $A^{D} b$ which is helpful in the study of singular differential and difference equations (see Chapter 9, [5]). A method of construction of proper splitting can be found in [27] while its uniqueness is shown very recently in [28]. The result produced below is a combination of Theorem 5.2, [25] and Theorem 4.1, [27], and is also a special case of Theorem $3.2 \& 3.3$, [14] and Theorem 3.1, [33] when index 1 matrices are considered.

Theorem 2.4. Let $A=U-V$ be a proper splitting of $A \in \mathbb{R}^{n \times n}$. Suppose that $A^{\#}$ exists. Then
(a) $U^{\#}$ exists.
(b) $A A^{\#}=U U^{\#} ; A^{\#} A=U^{\#} U$.
(c) $A=U\left(I-U^{\#} V\right)=\left(I-V U^{\#}\right) U$.
(d) $I-U^{\#} V$ and $I-V U^{\#}$ are nonsingular.
(e) $A^{\#}=\left(I-U^{\#} V\right)^{-1} U^{\#}=U^{\#}\left(I-V U^{\#}\right)^{-1}$.

## 3. Proper G-regular \& Proper G-weak regular Splitting

In this section, we recall first the definition of proper G-regular splittings and proper G-weak regular splittings, and then present some new results for index 1 matrices. Definition 2.1 and 2.2, [13] reduce to the following two definitions, respectively when we use the group inverse in the place of the Drazin inverse.

Definition 3.1. Let $A=U-V$ be a proper splitting of $A \in \mathbb{R}^{n \times n}$. Then the splitting is called proper $G$-regular splitting if $U^{\#}$ exists, $U^{\#} \geq 0$ and $V \geq 0$.

Definition 3.2. Let $A=U-V$ be a proper splitting of $A \in \mathbb{R}^{n \times n}$. Then the splitting is called a proper $G$-weak regular splitting if $U^{\#}$ exists, $U^{\#} \geq 0$ and $U^{\#} V \geq 0$.

Below is an algorithm which we have used for computing proper G-weak regular splittings in this article.

```
Algorithm 2 Generation of Proper G-weak regular splittings
    procedure PROP G-WEAK REG( \(A\) )
        Generate \(B=\{K: R(A)=R(K) \& N(A)=N(K)\}\)
        while (true) do
            \(U=\operatorname{Random}(B)\)
            if \(\left(U^{\#} \geq 0 \& U^{\#}(U-A) \geq 0\right)\) then
                return \(U\)
            end if
        end while
    end procedure
```

The next example shows that a proper G-weak regular splitting does not imply a proper G-regular splitting.

Example 3.1. Let $A=\left[\begin{array}{ccc}1 & -1 & 3 \\ -1 & 10 & -3 \\ 3 & -3 & 9\end{array}\right]=\left[\begin{array}{ccc}2 & -1 & 6 \\ -2 & 10 & -6 \\ 6 & -3 & 18\end{array}\right]-\left[\begin{array}{ccc}1 & 0 & 3 \\ -\mathbf{1} & 0 & -\mathbf{3} \\ 3 & 0 & 9\end{array}\right]=U-V$. Then $R(U)=R(A), N(U)=N(A), U^{\#}=\left[\begin{array}{ccc}0.0056 & 0.0056 & 0.0167 \\ 0.0112 & 0.1111 & 0.0335 \\ 0.0167 & 0.0167 & 0.05\end{array}\right] \geq 0$ and $U^{\#} V=$ $\left[\begin{array}{ccc}0.5 & 0 & 0.15 \\ 0.0004 & 0 & 0.0012 \\ 0.15 & 0 & 0.45\end{array}\right] \geq 0$. Hence, the splitting $A=U-V$ is a proper G-weak regular splitting but not a proper G-regular splitting since $V \nsupseteq 0$.

From the above example, it is clear that the class of proper G-weak regular splittings contains the class of proper G-regular splittings. We next recall convergence results for both of these class of matrices which also characterize the notion of group monotonicity. The first one concerns a proper G-regular splitting of a matrix, and a particular case of Theorem 3.2 and Theorem 3.4 of [1].

Theorem 3.1. Let $A=U-V$ be a proper $G$-regular splitting of $A \in \mathbb{R}^{n \times n}$. Then $A^{\#} \geq 0$ if and only if $\rho\left(U^{\#} V\right)<1$.

The next one is about the convergence of proper G-weak regular splittings. It follows from Theorem 3.8, [1] and Theorem 4.2, [33].

Theorem 3.2. Let $A=U-V$ be a proper $G$-weak regular splitting of $A \in \mathbb{R}^{n \times n}$. Then $A^{\#} \geq 0$ if and only if $\rho\left(U^{\#} V\right)<1$.

The rate of convergence of the scheme (22) depends upon $\rho\left(U^{\#} V\right)$. Therefore, the smaller spectral radius of the iteration matrix yields the faster convergence rate of the iterative scheme (2) to solve the system (11). The next result helps us to choose an iteration scheme having the faster convergence rate if $A$ has two different subclasses of proper splitting which leads to two different iteration schemes.

Theorem 3.3. Let $A=B-C$ be a proper $G$-weak regular splitting and $A=U-V$ be a proper $G$-regular splitting of a group monotone matrix $A \in \mathbb{R}^{n \times n}$. If $B^{\#} \geq U^{\#}$, then $\rho\left(B^{\#} C\right) \leq \rho\left(U^{\#} V\right)<1$.

Proof. By Theorem 3.1 and Theorem 3.2, we have $\rho\left(U^{\#} V\right)<1$ and $\rho\left(B^{\#} C\right)<1$, respectively. Since $B^{\#} C \geq 0$, there exists an eigenvector $x \geq 0$ such that $x^{t} B^{\#} C=\rho\left(B^{\#} C\right) x^{t}$, by Theorem 2.1. Hence $x \in R\left(C^{t}\right) \subseteq R\left(B^{t}\right)=R\left(A^{t}\right)$. Now, the condition $B^{\#} \geq U^{\#}$ yields $\left(I-B^{\#} C\right) A^{\#} \geq A^{\#}\left(I-V U^{\#}\right)$ by using Theorem 2.4 (e). This implies $B^{\#} C A^{\#} \leq$ $A^{\#} V U^{\#}$. Pre-multiplying $x^{t}$ to $B^{\#} C A^{\#} \leq A^{\#} V U^{\#}$, we obtain $x^{t} B^{\#} C A^{\#} \leq x^{t} A^{\#} V U^{\#}$, i.e., $\rho\left(B^{\#} C\right) x^{t} A^{\#} \leq x^{t} A^{\#} V U^{\#}$. Setting $x^{t} A^{\#}=z^{t}$ and taking transpose both sides, we get $\rho\left(B^{\#} C\right) z \leq\left(V U^{\#}\right)^{t} z$. Therefore, by Theorem 2.2 (i), we have $\rho\left(B^{\#} C\right) \leq \rho\left(V U^{\#}\right)^{t}=$ $\rho\left(V U^{\#}\right)=\rho\left(U^{\#} V\right)<1$ as $z \geq 0$ and $z \neq 0$ which is shown below by the method of contradiction. Suppose that $z=0$. Then $\left(A^{t}\right)^{\#} x=\left(A^{\#}\right)^{t} x=\left(x^{t} A^{\#}\right)^{t}=z^{t}=0$. So $x=P_{R\left(A^{t}\right), N\left(A^{t}\right)} x=A^{t}\left(A^{t}\right)^{\#} x=0$ as $x \in R\left(A^{t}\right)$ which is a contradiction.

We remark that the problems mentioned to be open in the concluding section of [26] can be easily now solved by using the Moore-Penrose inverse version of the above result which is Theorem 2.8, [28]. The proof of the above result follows analogous technique as in the proof of Theorem 2.8, [28]. However, these ideas are completely different from [8], where the author proved the above result in the nonsingular matrix setting. The present proof is much simple than Elsener [8]'s one. One may refer part (c) of a Lemma proved in section 3 of [8] for the same, and the same is produced below.
Corollary 3.1. [8] Let $A=B-C$ be a weak regular splitting and $A=U-V$ be a regular splitting of $A \in \mathbb{R}^{n \times n}$. If $B^{-1} \geq U^{-1}$ and $A^{-1} \geq 0$, then $\rho\left(B^{-1} C\right) \leq \rho\left(U^{-1} V\right)<1$.

## 4. Three-step Alternating Iterations

Throughout this section, we consider the co-efficient matrix $A$ in (11) as of index 1 unless otherwise mentioned. Let $A=K-L=U-V=X-Y$ be three proper splittings of $A \in \mathbb{R}^{n \times n}$. Let us consider the following iterative schemes:

$$
\begin{gather*}
x^{k+1 / 3}=K^{\#} L x^{k}+K^{\#} b  \tag{3}\\
x^{k+1 / 2}=U^{\#} V x^{k+1 / 3}+U^{\#} b  \tag{4}\\
x^{k+1}=X^{\#} Y x^{k+1 / 2}+X^{\#} b \tag{5}
\end{gather*}
$$

to introduce the three-step alternating iteration scheme. We form a single iteration scheme by eliminating $x^{k+1 / 3}$ and $x^{k+1 / 2}$ from (3), (4) and (5) to do computation. So, we get

$$
\begin{equation*}
x^{k+1}=X^{\#} Y U^{\#} V K^{\#} L x^{k}+X^{\#}\left(Y U^{\#} V K^{\#}+Y U^{\#}+I\right) b, \quad k=0,1,2, \cdots, \tag{6}
\end{equation*}
$$

where $H=X^{\#} Y U^{\#} V K^{\#} L$ is the iteration matrix of the new iterative scheme (6) called as the three-step alternating iteration scheme. The convergence of the individual splitting need not imply the convergence of the three-step alternating iteration scheme (6) which is shown in the following example.
Example 4.1. Let $A=\left[\begin{array}{ccc}4 & 4 & 10 \\ 7 & -29 & 31 \\ -1 & 11 & -7\end{array}\right], K=\left[\begin{array}{ccc}17.6 & 0.8 & 50.3 \\ -41.2 & -41.8 & -102.775 \\ 19.6 & 14.2 & 51.025\end{array}\right]$,

$$
U=\left[\begin{array}{ccc}
2.4 & 15.2 & 1.2 \\
18.6 & -31 & 65.1 \\
-5.4 & 15.4 & -21.3
\end{array}\right], X=\left[\begin{array}{ccc}
5.6 & 2.4 & 15.2 \\
-91 & -111 & -220 \\
32.2 & 37.8 & 78.4
\end{array}\right]
$$

Here $A=K-L=U-V=X-Y$ are three proper splittings. Also $\rho\left(K^{\#} L\right)=0.6835<1$, $\rho\left(U^{\#} V\right)=0.5957<1, \rho\left(X^{\#} Y\right)=0.8452<1$ but $\rho(H)=1.3579 \nless 1$.

Note that all the computations in this paper are made in fractions but for the presentation point of view, we have rounded to 4 decimal places. So, there might be a little rounding error. The algorithm for the three-step alternating iterations is produced next, and the same is also used in Example 4.1.

Example 4.1 motivates further to study the convergence criteria of the three-step alternating iterations, and the next result is in the same direction.
Theorem 4.1. If $A=K-L=U-V=X-Y$ are three proper $G$-weak regular splittings of a group monotone matrix $A$, then $\rho\left(X^{\#} Y U^{\#} V K^{\#} L\right)<1$.
Proof. We have $H=X^{\#} Y U^{\#} V K^{\#} L \geq 0$ as $A=K-L=U-V=X-Y$ are three proper G-weak regular splittings. By Theorem [2.4 (b), $A^{\#} A=K^{\#} K=U^{\#} U=X^{\#} X$. Since

$$
\begin{aligned}
X^{\#} A U^{\#} A K^{\#} A & =X^{\#}(X-Y) U^{\#}(U-V) K^{\#} A \\
& =X^{\#} X U^{\#} U K^{\#} A-X^{\#} X U^{\#} V K^{\#} A-X^{\#} Y U^{\#} U K^{\#} A+X^{\#} Y U^{\#} V K^{\#} A \\
& =K^{\#} A-U^{\#} V K^{\#} A-X^{\#} Y K^{\#} A+X^{\#} Y U^{\#} V K^{\#} A
\end{aligned}
$$

so the iteration matrix $H$ is expressed as

$$
\begin{aligned}
H= & X^{\#} Y U^{\#} V K^{\#} L \\
= & X^{\#}(X-A) U^{\#}(U-A) K^{\#}(K-A) \\
= & \left(U^{\#} U-U^{\#} A-X^{\#} A+X^{\#} A U^{\#} A\right)\left(K^{\#} K-K^{\#} A\right) \\
= & A^{\#} A-U^{\#} A-X^{\#} A+X^{\#} A U^{\#} A-K^{\#} A \\
& +U^{\#} A K^{\#} A+X^{\#} A K^{\#} A-X^{\#} A U^{\#} A K^{\#} A . \\
= & A^{\#} A-U^{\#} A-X^{\#} A+X^{\#} A U^{\#} A-K^{\#} A+U^{\#} A K^{\#} A+X^{\#} A K^{\#} A \\
& -K^{\#} A+U^{\#} V K^{\#} A+X^{\#} Y K^{\#} A-X^{\#} Y U^{\#} V K^{\#} A .
\end{aligned}
$$

This implies

$$
\begin{aligned}
H A^{\#}= & A^{\#}-U^{\#}-X^{\#}+X^{\#} A U^{\#}-K^{\#}+U^{\#} A K^{\#} \\
& +X^{\#} A K^{\#}-K^{\#}+U^{\#} V K^{\#}+X^{\#} Y K^{\#}-X^{\#} Y U^{\#} V K^{\#},
\end{aligned}
$$

and hence

$$
\begin{aligned}
(I-H) A^{\#}= & U^{\#}+X^{\#}-X^{\#} A U^{\#}+K^{\#}-U^{\#} A K^{\#}-X^{\#} A K^{\#} \\
& +K^{\#}-U^{\#} V K^{\#}-X^{\#} Y K^{\#}+X^{\#} Y U^{\#} V K^{\#} \\
= & U^{\#}+X^{\#}-U^{\#}(A+V) K^{\#}-X^{\#}(A+Y) K^{\#}-X^{\#} A U^{\#} \\
& +K^{\#}+X^{\#} Y U^{\#} V K^{\#}+K^{\#} \\
= & X^{\#} X U^{\#}-X^{\#} A U^{\#}+X^{\#}+X^{\#} Y U^{\#} V K^{\#} \\
= & X^{\#} Y U^{\#}+X^{\#}+X^{\#} Y U^{\#} V K^{\#} \geq 0 .
\end{aligned}
$$

Now, $0 \leq\left(I+H+H^{2}+\cdots+H^{m}\right)(I-H) A^{\#} \leq A^{\#}$ for each non-negative integer $m$.
Therefore, the partial sums of the series $\sum_{m=0}^{\infty} H^{m}$ remain uniformly bounded in norm. Hence $\rho(H)=\rho\left(X^{\#} Y U^{\#} V K^{\#} L\right)<1$.

We have the following result in case of nonsingular $A$.
Corollary 4.1. If $A=K-L=U-V=X-Y$ are three weak regular splittings of $a$ monotone matrix $A$, then $\rho\left(X^{-1} Y U^{-1} V K^{-1} L\right)<1$.

The above one extends the convergence criteria of two-step alternating iteration scheme proved by Benzi and Szyld in the first part of Theorem 3.2, [2]. The same is produced next as a corollary.

Corollary 4.2. (Theorem 3.2, [2]) If $A=U-V=X-Y$ are two weak regular splittings of a monotone matrix $A$, then $\rho\left(X^{-1} Y U^{-1} V\right)<1$.

The next example shows that the converse of Theorem 4.1 is not true.
Example 4.2. Let $A=\left[\begin{array}{ccc}-11 & 4 & 15 \\ 12 & 2 & 9 \\ 23 & -2 & -6\end{array}\right]$. Now

$$
\begin{aligned}
A & =\left[\begin{array}{ccc}
-33.5 & 20 & 76.8429 \\
62 & 10 & 45.1714 \\
95.5 & -10 & -31.6714
\end{array}\right]-\left[\begin{array}{ccc}
-22.5 & 16 & 61.8429 \\
50 & 8 & 36.1714 \\
72.5 & -8 & -25.6714
\end{array}\right]=K-L \\
& =\left[\begin{array}{ccc}
-58 & 53 & 206.271 \\
41 & -26 & -100.114 \\
99 & -79 & -306.386
\end{array}\right]-\left[\begin{array}{ccc}
-47 & 49 & 191.271 \\
29 & -28 & -109.114 \\
76 & -77 & -300.386
\end{array}\right]=U-V \\
& =\left[\begin{array}{ccc}
-53 & 39.5 & 152.893 \\
61 & -24 & -90.4286 \\
114 & -63.5 & -243.321
\end{array}\right]-\left[\begin{array}{ccc}
-42 & 35.5 & 137.893 \\
49 & -26 & -99.4286 \\
91 & -61.5 & -237.321
\end{array}\right]=X-Y
\end{aligned}
$$

are three proper splittings of $A$. Then $\rho(H)=\rho\left(X^{\#} Y U^{\#} V K^{\#} L\right)=0.4938<1$. Now, the individual splitting have the following property:

$$
\begin{aligned}
& K^{\#}=\left[\begin{array}{lll}
0.0029 & 0.0025 & 0.0101 \\
0.0138 & 0.0026 & 0.0117 \\
0.0109 & 0.0002 & 0.0016
\end{array}\right] \geq 0 \text { and } K^{\#} L=\left[\begin{array}{ccc}
0.7883 & -0.0140 & 0.0118 \\
0.6658 & 0.1493 & 0.6520 \\
-0.1224 & 0.1633 & 0.6402
\end{array}\right] \nsupseteq 0, \\
& U^{\#}=\left[\begin{array}{lll}
0.1041 & 0.0142 & 0.0654 \\
0.1100 & 0.0144 & 0.0667 \\
0.0059 & 0.0002 & 0.0013
\end{array}\right] \geq 0 \text { and } U^{\#} V=\left[\begin{array}{ccc}
0.4884 & -0.3314 & -1.2792 \\
0.3166 & -0.1488 & -0.5661 \\
-0.1719 & 0.1826 & 0.7131
\end{array}\right] \nsupseteq 0, \\
& X^{\#}=\left[\begin{array}{lll}
0.0490 & 0.0061 & 0.0284 \\
0.0577 & 0.0064 & 0.0305 \\
0.0087 & 0.0003 & 0.0021
\end{array}\right] \geq 0 \text { and } X^{\#} Y=\left[\begin{array}{ccc}
0.8291 & -0.1687 & -0.6011 \\
0.6690 & 0.0040 & 0.0734 \\
-0.1601 & 0.1727 & 0.6745
\end{array}\right] \nsupseteq 0 .
\end{aligned}
$$

Hence $A=K-L=U-V=X-Y$ are not G-weak regular splittings of $A$.
The following result shows that the iteration matrix $H$ of the three-step alternating iterations induces a unique proper G-weak regular splitting.

Theorem 4.2. Let $A=K-L=U-V=X-Y$ be three proper $G$-weak regular splittings of a group monotone matrix $A$. If $R(A)=R\left(K+X-A+Y U^{\#} L\right)$ and $N(A)=N(K+X-$
$A+Y U^{\#} L$ ), then there exists a unique proper $G$-weak regular splitting $A=B-C$ induced by $H$ with $B=K\left(K+X-A+Y U^{\#} L\right)^{\#} X$.

Proof. It is specified that $A=K-L=U-V=X-Y$ are three proper splittings of $A$. So, by Theorem 2.4(b), $A^{\#} A=K^{\#} K=U^{\#} U=X^{\#} X$ and $A A^{\#}=K K^{\#}=U U^{\#}=X X^{\#}$. Equation (6) yields

$$
\begin{aligned}
B^{\#}= & X^{\#}\left(Y U^{\#} V K^{\#}+Y U^{\#}+I\right) \\
= & X^{\#}(X-A) U^{\#}(U-A) K^{\#}+X^{\#}(X-A) U^{\#}+X^{\#} \\
= & X^{\#} X U^{\#} U K^{\#}-X^{\#} X U^{\#} A K^{\#}-X^{\#} A U^{\#} U K^{\#} \\
& +X^{\#} A U^{\#} A K^{\#}+X^{\#} X U^{\#}-X^{\#} A U^{\#}+X^{\#} X X^{\#} \\
= & X^{\#} X K^{\#}-X^{\#} X U^{\#} A K^{\#}-X^{\#} A K^{\#}+X^{\#} A U^{\#} A K^{\#} \\
& +X^{\#} X U^{\#} K K^{\#}-X^{\#} A U^{\#} K K^{\#}+X^{\#} K K^{\#} \\
= & X^{\#}\left(X-X U^{\#} A-A+A U^{\#} A+X U^{\#} K-A U^{\#} K+K\right) K^{\#} .
\end{aligned}
$$

On further simplification of $A U^{\#} A-X U^{\#} A-A U^{\#} K+X U^{\#} K$, we get $A U^{\#} A-X U^{\#} A-$ $A U^{\#} K+X U^{\#} K=(A-X)\left(U^{\#} A-U^{\#} K\right)=Y U^{\#} L$. Therefore,

$$
\begin{equation*}
B^{\#}=X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} . \tag{7}
\end{equation*}
$$

Since $R\left(K+X-A+Y U^{\#} L\right)=R(A)$ and $N\left(K+X-A+Y U^{\#} L\right)=N(A)$, we have $\left(K+X-A+Y U^{\#} L\right)^{\#}\left(K+X-A+Y U^{\#} L\right)=\left(K+X-A+Y U^{\#} L\right)\left(K+X-A+Y U^{\#} L\right)^{\#}$. Let $G=K\left(K+X-A+Y U^{\#} L\right)^{\#} X$, then $B^{\#} G B^{\#}=X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} K(K+$ $\left.X-A+Y U^{\#} L\right)^{\#} X X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#}=B^{\#}$ and $G B^{\#} G=K(K+X-$ $\left.A+Y U^{\#} L\right)^{\#} X X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} K\left(K+X-A+Y U^{\#} L\right)^{\#} X=G$. Also $B^{\#} G=X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} K\left(K+X-A+Y U^{\#} L\right)^{\#} X=X^{\#} X=X X^{\#}=$ $K\left(K+X-A+Y U^{\#} L\right)^{\#} X X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#}=G B^{\#}$. Hence $G=\left(B^{\#}\right)^{\#}=B$. Next, we prove that $A=B-C$ is a proper splitting. First, we show that $N(A)=N(B)$. Clearly, $N(X) \subseteq N(B)$ since $B=K\left(K+X-A+Y U^{\#} L\right)^{\#} X$. Let $B x=0$ which implies $K\left(K+X-A+Y U^{\#} L\right)^{\#} X x=0$. Pre-multiplying by $K^{\#}$, we get $\left(K+X-A+Y U^{\#} L\right)^{\#} X x=$ 0.

Again, pre-multiplying the last equation by $K+X-A+Y U^{\#} L$, we have $X x=0$. So
$N(B) \subseteq N(X)$. Hence $N(A)=N(B)$. From (7), we have

$$
\begin{aligned}
B^{\#}= & X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} \\
= & X^{\#}+K^{\#}-X^{\#} A K^{\#}+X^{\#}(X-A) U^{\#}(K-A) K^{\#} \\
= & X^{\#}+K^{\#}-X^{\#} A K^{\#}+U^{\#}-U^{\#} A K^{\#}-X^{\#} A U^{\#}+X^{\#} A U^{\#} A K^{\#} \\
= & A^{\#}-\left(A^{\#}-U^{\#}-X^{\#}+X^{\#} A U^{\#}-K^{\#}+U^{\#} A K^{\#}+X^{\#} A K^{\#}-X^{\#} A U^{\#} A K^{\#}\right) \\
= & A^{\#}-\left(A^{\#} A A^{\#}-U^{\#} A A^{\#}-X^{\#} A A^{\#}+X^{\#} A U^{\#} A A^{\#}-K^{\#} A A^{\#}\right. \\
& \left.+U^{\#} A K^{\#} A A^{\#}+X^{\#} A K^{\#} A A^{\#}-X^{\#} A U^{\#} A K^{\#} A A^{\#}\right) \\
= & A^{\#}-\left(U^{\#} U-U^{\#} A-X^{\#} A+X^{\#} A U^{\#} A\right)\left(K^{\#} K-K^{\#} A\right) A^{\#} \\
= & A^{\#}-X^{\#} Y U^{\#} V K^{\#} L A^{\#}=(I-H) A^{\#} .
\end{aligned}
$$

But, we have $\rho(H)<1$ by Theorem 4.1, So, $I-H$ is nonsingular by Theorem [2.3, Let $G_{1}=B=A(I-H)^{-1}$. Now $B^{\#} G_{1} B^{\#}=(I-H) A^{\#} A(I-H)^{-1}(I-H) A^{\#}=B^{\#}$. Similarly, $G_{1} B^{\#} G_{1}=G_{1}$. Again, $B^{\#} G_{1}=(I-H) A^{\#} A(I-H)^{-1}=\left(A^{\#} A-H\right)(I-H)^{-1}=$ $\left(A^{\#} A-A^{\#} A H\right)(I-H)^{-1}=A^{\#} A=A A^{\#}=G_{1} B^{\#}$. Therefore, $B=A(I-H)^{-1}$. Hence $A=B(I-H)$, and thus $R(A)=R(B)$. Therefore, $A=B-C$ is a proper splitting. Next, to show uniqueness of proper splitting $A=B-C$. Suppose there exists another induced splitting $A=B_{1}-C_{1}$ such that $H=B_{1}^{\#} C_{1}$. Then $B_{1} H=B_{1} B_{1}^{\#} C_{1}=C_{1}=B_{1}-A$. So, we get $B_{1}(I-H)=A$ and thus $B_{1}=A(I-H)^{-1}=B$. Finally, $B^{\#}=X^{\#}\left(Y U^{\#} V K^{\#}+Y U^{\#}+\right.$ $I)=X^{\#} Y U^{\#} V K^{\#}+X^{\#} Y U^{\#}+X^{\#} \geq 0$ and $B^{\#} C=X^{\#} Y U^{\#} V K^{\#} L \geq 0$. Therefore, $A=B-C$ is a unique proper G-weak regular splitting.

The above result in case of a nonsingular monotone matrix is stated by the following corollary.

Corollary 4.3. Let $A=K-L=U-V=X-Y$ be three weak regular splittings of $a$ monotone matrix $A$. Then there exists a unique weak regular splitting $A=B-C$ induced by $H$ with $B=K\left(K+X-A+Y U^{-1} L\right)^{-1} X$.

We also remark that this extends jointly Theorem 3.2 and 3.4 of [2]. To support Theorem 4.2, we have the following example.

Example 4.3. Let $A=\left[\begin{array}{ccc}9 & -3 & 6 \\ -3 & 5 & -2 \\ 6 & -2 & 4\end{array}\right], K=\left[\begin{array}{ccc}9.9 & -3.3 & 6.6 \\ -3.3 & 5.5 & -2.2 \\ 6.6 & -2.2 & 4.4\end{array}\right], U=\left[\begin{array}{ccc}13.5 & -4.5 & 9 \\ -4.5 & 7.5 & -3 \\ 9 & -3 & 6\end{array}\right]$,
$X=\left[\begin{array}{ccc}12.6 & -4.2 & 8.4 \\ -4.2 & 7 & -2.8 \\ 8.4 & -2.8 & 5.6\end{array}\right]$. Now $A^{\#}=\left[\begin{array}{lll}0.0666 & 0.0577 & 0.0444 \\ 0.0577 & 0.2500 & 0.0385 \\ 0.0444 & 0.0385 & 0.0296\end{array}\right] \geq 0$,

$$
\begin{aligned}
& K^{\#}=\left[\begin{array}{lll}
0.0605 & 0.0524 & 0.0403 \\
0.0524 & 0.2273 & 0.0350 \\
0.0403 & 0.0350 & 0.0269
\end{array}\right] \geq 0, K^{\#} L=\left[\begin{array}{ccc}
0.0629 & 0 & 0.0420 \\
0 & 0.0909 & 0 \\
0.0420 & 0 & 0.0280
\end{array}\right] \geq 0, \\
& U^{\#}=\left[\begin{array}{lll}
0.0444 & 0.0385 & 0.0296 \\
0.0385 & 0.1667 & 0.0256 \\
0.0296 & 0.0256 & 0.0197
\end{array}\right] \geq 0, U^{\#} V=\left[\begin{array}{ccc}
0.2308 & 0 & 0.1538 \\
0 & 0.3333 & 0 \\
0.1538 & 0 & 0.1026
\end{array}\right] \geq 0, \\
& X^{\#}=\left[\begin{array}{lll}
0.0475 & 0.0412 & 0.0317 \\
0.0412 & 0.1786 & 0.0275 \\
0.0317 & 0.0275 & 0.0211
\end{array}\right] \geq 0 \text { and } X^{\#} Y=\left[\begin{array}{ccc}
0.1978 & 0 & 0.1319 \\
0 & 0.2857 & 0 \\
0.1319 & 0 & 0.0879
\end{array}\right] \geq 0 .
\end{aligned}
$$

Then $A=K-L=U-V=X-Y$ are three proper G-weak regular splittings of $A$. Also $R\left(K+X-A+Y U^{\#} L\right)=R(A)$ and $N\left(K+X-A+Y U^{\#} L\right)=N(A)$. So,

$$
B=K\left(K+X-A+Y U^{\#} L\right)^{\#} X=\left[\begin{array}{ccc}
9.0786 & -3.0262 & 6.0524 \\
-3.0262 & 5.0437 & -2.0175 \\
6.0524 & -2.0175 & 4.0349
\end{array}\right]
$$

and

$$
C=B-A=\left[\begin{array}{ccc}
0.0786 & -0.0262 & 0.0524 \\
-0.0262 & 0.0437 & -0.0175 \\
0.0524 & -0.0175 & 0.0349
\end{array}\right]
$$

Now $B^{\#}=\left[\begin{array}{lll}0.0660 & 0.0572 & 0.0440 \\ 0.0572 & 0.2478 & 0.0381 \\ 0.0440 & 0.0381 & 0.0293\end{array}\right] \geq 0$ and $B^{\#} C=\left[\begin{array}{ccc}0.0060 & 0 & 0.0040 \\ 0 & 0.0087 & 0 \\ 0.0040 & 0 & 0.0027\end{array}\right] \geq 0$.
Therefore, the splitting $A=B-C$ induced by $H$ is a proper G-weak regular splitting.
The next result confirms that the proposed alternating iterative scheme converges faster than (2) under suitable assumptions.

Theorem 4.3. Suppose $A=K-L=U-V=X-Y$ are three proper $G$-regular splittings of a group monotone matrix $A$ with $R(A)=R\left(K+X-A+Y U^{\#} L\right)$ and $N(A)=N(K+$ $\left.X-A+Y U^{\#} L\right)$. Then $\rho(H) \leq \min \left\{\rho\left(K^{\#} L\right), \rho\left(U^{\#} V\right), \rho\left(X^{\#} Y\right)\right\}<1$.

Proof. By Theorem 4.2, $A=B-C$ is a proper G-weak regular splitting induced by $H$, and from (6),

$$
B^{\#}=X^{\#}\left(Y U^{\#} V K^{\#}+Y U^{\#}+I\right)=X^{\#} Y U^{\#} V K^{\#}+X^{\#} Y U^{\#}+X^{\#} \geq X^{\#}
$$

Again,

$$
\begin{aligned}
B^{\#} & =X^{\#} Y U^{\#} V K^{\#}+X^{\#} Y U^{\#}+X^{\#} \\
& =X^{\#} Y U^{\#} V K^{\#}+X^{\#} X U^{\#}-X^{\#} A U^{\#}+X^{\#} U U^{\#} \\
& =X^{\#} Y U^{\#} V K^{\#}+U^{\#}+X^{\#}(U-A) U^{\#} \\
& =X^{\#} Y U^{\#} V K^{\#}+U^{\#}+X^{\#} V U^{\#} \geq U^{\#} .
\end{aligned}
$$

Also,

$$
\begin{aligned}
B^{\#} & =X^{\#}\left(K+X-A+Y U^{\#} L\right) K^{\#} \\
& =X^{\#} K K^{\#}+X^{\#} X K^{\#}-X^{\#}(K-L) K^{\#}+X^{\#} Y U^{\#} L K^{\#} \\
& =X^{\#}+K^{\#}-X^{\#} K K^{\#}+X^{\#} L K^{\#}+X^{\#} Y U^{\#} L K^{\#} \\
& =X^{\#}+K^{\#}-X^{\#}+X^{\#} L K^{\#}+X^{\#} Y U^{\#} L K^{\#} \\
& =K^{\#}+X^{\#} L K^{\#}+X^{\#} Y U^{\#} L K^{\#} \geq K^{\#} .
\end{aligned}
$$

Applying Theorem 3.3 to the pair of the splittings $A=B-C$ and $A=K-L, A=B-C$ and $A=U-V$, and $A=B-C$ and $A=X-Y$, we have $\rho(H) \leq \rho\left(K^{\#} L\right)<$ $1, \rho(H) \leq \rho\left(U^{\#} V\right)<1$ and $\rho(H) \leq \rho\left(X^{\#} Y\right)<1$, respectively. Therefore, $\rho(H) \leq$ $\min \left\{\rho\left(K^{\#} L\right), \rho\left(U^{\#} V\right), \rho\left(X^{\#} Y\right)\right\}<1$.

The result below is the case when $A$ is nonsingular.
Corollary 4.4. Let $A=K-L=U-V=X-Y$ be three regular splittings of a monotone matrix $A$. Then $\rho(H) \leq \min \left\{\rho\left(K^{-1} L\right), \rho\left(U^{-1} V\right), \rho\left(X^{-1} Y\right)\right\}<1$.

Again, we have the following corollary when two splitting are considered, and is proved in [2].
Corollary 4.5. (Theorem 4.1, [2]) Let $A=U-V=X-Y$ be two regular splittings of $a$ monotone matrix $A$. Then $\rho(H) \leq \min \left\{\rho\left(U^{-1} V\right), \rho\left(X^{-1} Y\right)\right\}<1$.

The converse of Theorem 4.3 does not hold. The next example justifies the claim.
Example 4.4. Let $A=\left[\begin{array}{ccc}-1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 2 & 4\end{array}\right], K=\left[\begin{array}{ccc}-2 & 0 & -6 \\ 0 & 1 & 2 \\ 0 & 2 & 4\end{array}\right], U=\left[\begin{array}{ccc}-3 & 0 & -9 \\ 0 & 1 & 2 \\ 0 & 2 & 4\end{array}\right]$,
$X=\left[\begin{array}{ccc}-4 & 0 & -12 \\ 0 & 1 & 2 \\ 0 & 2 & 4\end{array}\right]$. Then $A=K-L=U-V=X-Y$ are three proper split-
tings with $\rho(H)=\rho\left(K^{\#} L U^{\#} V X^{\#} Y\right)=0.25<1$. But $A=K-L=U-V=X-Y$
are not proper G-regular splittings as $K^{\#}=\left[\begin{array}{ccc}-0.5000 & 0.3600 & -0.7800 \\ 0 & 0.0400 & 0.0800 \\ 0 & 0.0800 & 0.1600\end{array}\right] \nsupseteq 0, L=$ $\left[\begin{array}{ccc}-1 & 0 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right] \nsupseteq 0, U^{\#}=\left[\begin{array}{ccc}-0.3333 & 0.1600 & -0.6800 \\ 0 & 0.0400 & 0.0800 \\ 0 & 0.0800 & 0.1600\end{array}\right] \nsupseteq 0, V=\left[\begin{array}{ccc}-2 & 0 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right] \nsupseteq 0$, $X^{\#}=\left[\begin{array}{ccc}-0.2500 & 0.0600 & -0.6300 \\ 0 & 0.0400 & 0.0800 \\ 0 & 0.0800 & 0.1600\end{array}\right] \nsupseteq 0$ and $Y=\left[\begin{array}{ccc}-3 & 0 & -9 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right] \nsupseteq 0$.

The next example shows that the condition of G-regular cannot be dropped.
Example 4.5. Let

$$
\begin{gathered}
A=\left[\begin{array}{ccc}
25 & -6 & 1 \\
-7 & 4 & 0 \\
4 & 6 & 1
\end{array}\right], A^{\#}=\left[\begin{array}{lll}
0.0428 & 0.0200 & 0.0054 \\
0.0539 & 0.2218 & 0.0305 \\
0.2044 & 0.6854 & 0.0968
\end{array}\right] \geq 0, \\
K=\left[\begin{array}{ccc}
-8.75 & 30.5 & 3.0776 \\
8.25 & -15.5 & -1.3017 \\
16 & -16 & -0.8276
\end{array}\right], U=\left[\begin{array}{ccc}
-17.75 & 43 & 3.9655 \\
14.25 & -19 & -1.3103 \\
25 & -14 & 0.0345
\end{array}\right], \\
X=\left[\begin{array}{ccc}
-58.5 & 64.75 & 3.7802 \\
24.5 & -11.75 & 0.2716 \\
15 & 29.5 & 4.5948
\end{array}\right] . \text { Now } X^{\#}=\left[\begin{array}{ccc}
0.0911 & 0.1619 & 0.0258 \\
0.0370 & 0.0195 & 0.0049 \\
0.2020 & 0.2203 & 0.0405
\end{array}\right], \\
K^{\#}=\left[\begin{array}{ccc}
0.0436 & 0.0956 & 0.0145 \\
0.0283 & 0.0214 & 0.0045 \\
0.1285 & 0.1597 & 0.0281
\end{array}\right] \text { and } U^{\#}=\left[\begin{array}{lll}
0.0031 & 0.0430 & 0.0054 \\
0.0147 & 0.0290 & 0.0045 \\
0.0473 & 0.1299 & 0.0189
\end{array}\right] .
\end{gathered}
$$

Then $A=K-L=U-V=X-Y$ are three proper splittings with $R(A)=R(K+X-$ $\left.A+Y U^{\#} L\right)$ and $N(A)=N\left(K+X-A+Y U^{\#} L\right)$. But $A=K-L=U-V=X-Y$ are not G-regular splittings as

$$
L=\left[\begin{array}{ccc}
-33.75 & 36.5 & 2.0776 \\
15.25 & -19.5 & -1.3017 \\
12 & -22 & -1.8276
\end{array}\right] \nsupseteq 0, V=\left[\begin{array}{ccc}
-42.75 & 49 & 2.9655 \\
21.25 & -23 & -1.3103 \\
21 & -20 & -0.9655
\end{array}\right] \nsupseteq 0
$$

and

$$
\begin{gathered}
Y=\left[\begin{array}{ccc}
-83.5 & 70.75 & 2.7802 \\
31.5 & -15.75 & 0.2716 \\
11 & 23.5 & 3.5948
\end{array}\right] \nsupseteq 0 \text {. Therefore, } \\
\rho(H)=1.7746 \not \leq \min \left\{\rho\left(K^{\#} L\right)=1.2987, \rho\left(U^{\#} V\right)=1.2530, \rho\left(X^{\#} Y\right)=1.2975\right\} \not \leq 1 .
\end{gathered}
$$

Theorem 4.1 shows that the assumption of group monotonicity of $A$ guarantees the convergence of the three-step alternating iteration scheme. If we drop this assumption, then the proposed theory may fail. To overcome this, the concept of a preconditioned matrix is introduced next. In such a case, we consider the following system

$$
\begin{equation*}
Q A x=Q b \tag{8}
\end{equation*}
$$

where $Q$ is a nonsingular matrix called preconditioned matrix. Milaszewicz [23], used the iteration matrix $T$ which is irreducible and non-negative to improve the convergence rate of the Gauss-Seidel and the Jacobi method. Gunawardena et al. [12] proposed the preconditioned matrix $P_{c}=I+S$, (where $S$ is the matrix shown in remark 3.3 [12]). Kohno et al. [16] and Kotakemori et al. [17] extended the upper triangular approach by considering a parametric preconditioned matrix $P_{c}=I+S(\alpha)$ to obtain faster convergence in the iterative schemes which used for solving consistent linear systems. In case of a singular linear system, we discuss the system (8) converges to $A^{\#} b$ under suitable choice of $Q$. The iterative scheme of the modified system (8) is defined by,

$$
\begin{equation*}
x^{k+1}=K_{q}^{\#} L_{q} x^{k}+K_{q}^{\#} Q b, \tag{9}
\end{equation*}
$$

where $Q A=K_{q}-L_{q}$ be a proper splitting of the matrix $Q A \in \mathbb{R}^{n \times n}$, will converge to $A^{\#} b$ for any initial guess $x^{0}$ if and only if $\rho\left(K_{q}^{\#} L_{q}\right)<1$.

Next, we discuss the existence of preconditioned matrix for some particular cases as well as the convergence of the iterative scheme for proper G-weak regular splittings.

Lemma 4.4. If there exists a nonsingular matrix $Q \in \mathbb{R}^{n \times n}$ such that $Q A=A Q$, then $(Q A)^{\#}=A^{\#} Q^{-1}=Q^{-1} A^{\#}$.

Proof. The assumption $Q A=A Q$ yields $A=Q^{-1} A Q$. Now $A^{\#}=Q^{-1} A^{\#} Q$ is the group inverse of $A$ which can be easily verified by the definition of group inverse. Pre-multiplying $Q$ in $A^{\#}=Q^{-1} A^{\#} Q$ we obtain $Q A^{\#}=A^{\#} Q$. Let $B=Q A$ and $X=A^{\#} Q^{-1}$. By the definition of the group inverse: $B X B=Q A A^{\#} Q^{-1} Q A=B, X B X=A^{\#} Q^{-1} Q A A^{\#} Q^{-1}=X$ and $B X=Q A A^{\#} Q^{-1}=A Q A^{\#} Q^{-1}=A A^{\#} Q Q^{-1}=A^{\#} A=A^{\#} Q^{-1} Q A=X B$. Now postmultiplying $Q^{-1}$ in $A^{\#}=Q^{-1} A^{\#} Q$ we have $A^{\#} Q^{-1}=Q^{-1} A^{\#}$.

Remark 4.1. For $A^{\#} \leq 0$, if we choose $Q=-c I,(c>0)$ and for $A^{\#} \geq 0$, if we choose $Q=c I,(c>0)$ then Lemma 4.4 along with $(Q A)^{\#} \geq 0$ holds.

Lemma 4.5. Let $A^{\#} \nsupseteq 0$. If there exists a nonsingular matrix $Q \in \mathbb{R}^{n \times n}$ such that $Q A=$ $A Q, A^{\#} Q^{-1} \geq 0$, and $Q A=K_{q}-L_{q}$ is a proper $G$-weak regular splittings of $Q A$, then the iterative scheme (9) converges to $A^{\#} b$.

Proof. Since $Q A=A Q$ and $A^{\#} Q^{-1} \geq 0$, we obtain $(Q A)^{\#}=A^{\#} Q^{-1} \geq 0$. As $Q A=K_{q}-L_{q}$ is a proper G-weak regular splittings of a group monotone matrix $Q A$. Then, the iterative scheme (9) converges to $(Q A)^{\#} Q b=(Q A)^{\#} Q b=A^{\#} Q^{-1} Q b=A^{\#} b$, by Theorem 3.2.

Remark 4.2. If $A^{\#}$ is neither non-positive nor non-negative, i.e., some elements of $A^{\#}$ are positive and some are negative, then the construction of such $Q$ seems to be an open problem.

Remark 4.3. Under the same assumptions as in Lemma 4.5, and if $Q A=K_{q}-L_{q}=$ $U_{q}-V_{q}=X_{q}-Y_{q}$ are three proper G-weak regular splittings of $Q A$. Then, by Theorem 4.1 the alternating iterative scheme generated from the splittings of $Q A$, i.e.,

$$
\begin{equation*}
x^{k+1}=X_{q}^{\#} Y_{q} U_{q}^{\#} V_{q} K_{q}^{\#} L_{q} x^{k}+X_{q}^{\#}\left(Y_{q} U_{q}^{\#} V_{q} K_{q}^{\#}+Y_{q} U_{q}^{\#}+I\right) b \tag{10}
\end{equation*}
$$

will converge to $A^{\#} b$, for any initial value $x^{0}$. The numerical implementation of the iterative scheme (10) and its comparison are discussed in the next section.

The next result shows that the preconditioned approach is also more preferable even for group monotone matrices.

Theorem 4.6. Let $A=K-L$ be a proper $G$-weak regular splitting of a group monotone matrix A. Assume that there exists a nonsingular matrix $Q$ such that $Q A=A Q$ and $A^{\#} Q^{-1} \geq 0$. If $Q A=K_{q}-L_{q}$ is a proper $G$-regular splitting of $Q A$ and $Q K_{q}^{\#} \geq K^{\#}$, then $\rho\left(K_{q}^{\#} L_{q}\right) \leq \rho\left(K^{\#} L\right)<1$.
Proof. By Theorem 3.2 and Theorem 3.1, we have $\rho\left(K_{q}^{\#} L_{q}\right)<1$ and $\rho\left(K^{\#} L\right)<1$, respectively. Since $L_{q} K_{q}^{\#} \geq 0$, there exists an eigenvector $x \geq 0$ such that

$$
\begin{equation*}
L_{q} K_{q}^{\#} x=\rho\left(K_{q}^{\#} L_{q}\right) x \tag{11}
\end{equation*}
$$

by Theorem 2.1. This implies $x \in R\left(L_{q} K_{q}^{\#}\right) \subseteq R(A)$. Since $N\left(L_{q}\right) \supseteq N\left(K_{q}\right)$, so $Q A=$ $\left(I-L_{q} K_{q}^{\#}\right) K_{q}$. Therefore, $A^{\#}=Q K_{q}^{\#}\left(I-L_{q} K_{q}^{\#}\right)^{-1}$ by Theorem 2.4 (e). Now $Q K_{q}^{\#} \geq K^{\#}$ implies $A^{\#}\left(I-L_{q} K_{q}^{\#}\right) \geq\left(I-K^{\#} L\right) A^{\#}$. Further simplification yields

$$
\begin{equation*}
A^{\#} L_{q} K_{q}^{\#} \leq K^{\#} L A^{\#} \tag{12}
\end{equation*}
$$

Post-multiplying $x$ in (12) and using equation (11), we have $\rho\left(K_{q}^{\#} L_{q}\right) A^{\#} x \leq K^{\#} L A^{\#} x$. Let us assume $z=A^{\#} x$. We obtain $z \geq 0$ and $z \neq 0$ ( $z=0$ leads to $x \in R(A) \cap N(A)$ which is not possible). Hence, by Theorem 2.2 (i) the required result follows.

## 5. Numerical Examples

In this section, we discuss a few examples and its numerical implementation for the proposed theory in the previous section. The performance measures calculated are the number of iterations (IT), the mean processing time in seconds (MT) and the estimation of error bounds. All the numerical examples are worked out by using Mathematica 10.0 (for examples 5.1-5.5) and MATLAB R2017a (for examples 5.6-5.7) on an $\operatorname{Intel}(\mathrm{R}) \operatorname{Core}(\mathrm{TM}) \mathrm{i} 5,2.5 \mathrm{GHz}$, 4GBRAM, which runs on the operating system: Mac OS X El Capitan Version 10.11.6. We use the following stopping criterion to terminate the process: The iteration is terminated if $\left\|x_{k}-x_{k-1}\right\|_{2} \leq \epsilon$ or it reaches to the maximum allowed iterations 2000.
Example 5.1. Let us consider a linear system $A x=b$ with $A=\left[\begin{array}{ccc}3 & 1 & 2 \\ 1 & -12 & 13 \\ 2 & 13 & -11\end{array}\right]$ and $b=(1,1,0)^{t}$. Then $A^{\#}=\left[\begin{array}{lll}0.1471 & 0.0691 & 0.0781 \\ 0.0691 & 0.0120 & 0.0571 \\ 0.0781 & 0.0571 & 0.0210\end{array}\right] \geq 0$. Consider the following three proper G-weak regular splittings of $A$ as

$$
\begin{aligned}
A & =\left[\begin{array}{ccc}
4.75 & 2.5 & 2.25 \\
1.5833 & -11.5 & 13.0833 \\
3.1667 & 14 & -10.8333
\end{array}\right]-\left[\begin{array}{ccc}
1.75 & 1.5 & 0.25 \\
0.5833 & 0.5 & 0.0833 \\
1.1667 & 1 & 0.1667
\end{array}\right]=K-L=\text { (Splitting 1) } \\
& =\left[\begin{array}{ccc}
5 & 2 & 3 \\
2 & -12 & 14 \\
3 & 14 & -11
\end{array}\right]-\left[\begin{array}{lll}
2 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right]=U-V=\text { (Splitting 2) } \\
& =\left[\begin{array}{ccc}
5.2083 & 2.9583 & 2.25 \\
2.25 & -10.8333 & 13.0833 \\
2.9583 & 13.7917 & -10.8333
\end{array}\right]-\left[\begin{array}{ccc}
2.2083 & 1.9583 & 0.25 \\
1.25 & 1.1667 & 0.0833 \\
0.9583 & 0.7917 & 0.1667
\end{array}\right]=X-Y=\text { (Splitting 3) }
\end{aligned}
$$

$$
\text { with } K^{\#}=\left[\begin{array}{lll}
0.0937 & 0.0476 & 0.0462 \\
0.0424 & 0.0013 & 0.0411 \\
0.0514 & 0.0463 & 0.0051
\end{array}\right] \geq 0, K^{\#} L=\left[\begin{array}{lll}
0.2456 & 0.2105 & 0.0351 \\
0.1228 & 0.1053 & 0.0175 \\
0.1228 & 0.1053 & 0.0175
\end{array}\right] \geq 0 \text {, }
$$

$$
\begin{gathered}
U^{\#}=\left[\begin{array}{ccc}
0.0885 & 0.0417 & 0.0469 \\
0.0417 & 0 & 0.0417 \\
0.0469 & 0.0417 & 0.0052
\end{array}\right] \geq 0, U^{\#} V=\left[\begin{array}{lll}
0.2656 & 0.1354 & 0.1302 \\
0.1250 & 0.0833 & 0.0417 \\
0.1406 & 0.0521 & 0.0885
\end{array}\right] \geq 0, \\
X^{\#}=\left[\begin{array}{lll}
0.0855 & 0.0446 & 0.0409 \\
0.0409 & 0.0007 & 0.0401 \\
0.0446 & 0.0439 & 0.0007
\end{array}\right] \geq 0 \text { and } X^{\#} Y=\left[\begin{array}{lll}
0.2838 & 0.2519 & 0.0319 \\
0.1297 & 0.1127 & 0.0170 \\
0.1541 & 0.1392 & 0.0149
\end{array}\right] \geq 0 .
\end{gathered}
$$

Therefore, $\rho(H)=0.0614 \leq \min \left\{\rho\left(K^{\#} L\right)=0.3684, \rho\left(U^{\#} V\right)=0.3983, \rho\left(X^{\#} Y\right)=0.4163\right\}<$ 1. The numerical results for the convergence analysis is provided in Table and comparison results discussed in Table .

The next example shows the importance of the study of the alternating iteration scheme in the group inverse setting. Note that existing theory in the literature uses the nonnegativity of the Moore-Penrose inverse, see [10, 24, 26] which fails here.
Example 5.2. Let us consider another system $A x=b$ with $A=\left[\begin{array}{ccc}10 & -4 & 17 \\ 54 & -42 & 77 \\ -12 & 15 & -13\end{array}\right]$ and $b=(-1,-11,4)^{t}$. The matrix $A$ has non-negative group inverse but does not have nonnegative Moore-Penrose inverse. Since

$$
A^{\dagger}=\left[\begin{array}{ccc}
-0.0028 & 0.0043 & -0.0064 \\
0.0577 & 0.0033 & 0.0849 \\
0.0363 & 0.0109 & 0.0489
\end{array}\right] \nsupseteq 0 \text { but } A^{\#}=\left[\begin{array}{lll}
0.0242 & 0.0113 & 0.0565 \\
0.0548 & 0.0102 & 0.1164 \\
0.0090 & 0.0119 & 0.0265
\end{array}\right] \geq 0 .
$$

Now consider $A$ as $A=K-L=U-V=X-Y$, where $K=\left[\begin{array}{ccc}14.8681 & -0.1590 & 29.4750 \\ 73.5383 & -42.9540 & 115.1930 \\ -14.4671 & 21.2385 & -13.3840\end{array}\right]$,

$$
U=\left[\begin{array}{ccc}
16.1942 & -0.9500 & 31.5405 \\
76.0650 & -35.7555 & 125.4440 \\
-13.7411 & 16.4528 & -15.4113
\end{array}\right], X=\left[\begin{array}{ccc}
16.9186 & -2.1315 & 32.1250 \\
76.7119 & -39.0705 & 124.3265 \\
-12.9780 & 16.3380 & -13.9758
\end{array}\right]
$$

Clearly,

$$
K^{\#}=\left[\begin{array}{lll}
0.0098 & 0.0049 & 0.0230 \\
0.0269 & 0.0012 & 0.0544 \\
0.0012 & 0.0068 & 0.0073
\end{array}\right] \geq 0, K^{\#} L=\left[\begin{array}{lll}
0.0874 & 0.1763 & 0.3018 \\
0.0197 & 0.4414 & 0.3595 \\
0.1212 & 0.0438 & 0.2729
\end{array}\right] \geq 0
$$

$$
\begin{aligned}
& U^{\#}=\left[\begin{array}{lll}
0.0122 & 0.0047 & 0.0277 \\
0.0357 & 0.0015 & 0.0722 \\
0.0004 & 0.0063 & 0.0054
\end{array}\right] \geq 0, U^{\#} V=\left[\begin{array}{lll}
0.1313 & 0.1067 & 0.3388 \\
0.1283 & 0.2230 & 0.4170 \\
0.1328 & 0.0486 & 0.2996
\end{array}\right] \geq 0, \\
& X^{\#}=\left[\begin{array}{lll}
0.0129 & 0.0043 & 0.0288 \\
0.0365 & 0.0004 & 0.0730 \\
0.0011 & 0.0063 & 0.0068
\end{array}\right] \geq 0, X^{\#} Y=\left[\begin{array}{lll}
0.1593 & 0.0754 & 0.3718 \\
0.1899 & 0.1670 & 0.4992 \\
0.1440 & 0.0296 & 0.3081
\end{array}\right] \geq 0 .
\end{aligned}
$$

The spectral radius of the individual splittings and the alternating iteration matrix are as follows: $\rho\left(K^{\#} L\right)=0.5841, \rho\left(U^{\#} V\right)=0.5515, \rho\left(X^{\#} Y\right)=0.5541, \rho(H)=0.1728<1$. The convergence analysis is provided in the Table ??.
Example 5.3. Let us consider a system $A x=b$ with $A=\left[\begin{array}{ccc}3 & -1 & -9 \\ -5 & -5 & -12 \\ -18 & -14 & -27\end{array}\right]$ and $b=$ $(-18,-4,6)^{t}$. Here $A^{\#}=\left[\begin{array}{ccc}0.0972 & 0.0294 & -0.0413 \\ 0.0099 & 0.0007 & -0.0137 \\ -0.0674 & -0.0274 & 0.0001\end{array}\right] \nsupseteq 0$. We find a nonsingular matrix

$$
Q=\left[\begin{array}{ccc}
4.9000 & -1.8600 & -0.5300 \\
-2.0371 & 7.5984 & -2.8996 \\
-1.8670 & -2.7776 & 0.9755
\end{array}\right]
$$

such that with $Q A=A Q$ and $(Q A)^{\#} \geq 0$. Now consider the new system $Q A x=Q b$, where $Q A$ has three different proper G-weak regular splittings i.e., $Q A=K_{q}-L_{q}=U_{q}-V_{q}=$ $X_{q}-Y_{q}$. Now $K_{q}=\left[\begin{array}{ccc}36.0660 & 12.5447 & -8.7030 \\ 9.8863 & 6.1737 & 8.6910 \\ -6.4071 & 5.9764 & 34.7760\end{array}\right]$,

$$
U_{q}=\left[\begin{array}{ccc}
35.6316 & 12.4668 & -8.3015 \\
9.4460 & 5.8062 & 7.9290 \\
-7.2936 & 4.9516 & 32.0885
\end{array}\right], X_{q}=\left[\begin{array}{ccc}
34.9083 & 12.2843 & -7.8472 \\
8.7488 & 5.3427 & 7.2025 \\
-8.6617 & 3.7439 & 29.4545
\end{array}\right] .
$$

Clearly,

$$
K_{q}^{\#}=\left[\begin{array}{lll}
0.0222 & 0.0091 & 0.0001 \\
0.0077 & 0.0052 & 0.0084 \\
0.0008 & 0.0066 & 0.0254
\end{array}\right] \geq 0, K_{q}^{\#} L_{q}=\left[\begin{array}{lll}
0.0725 & 0.0303 & 0.0030 \\
0.0530 & 0.0465 & 0.1007 \\
0.0866 & 0.1091 & 0.2990
\end{array}\right] \geq 0
$$

$$
\begin{aligned}
& U_{q}^{\#}=\left[\begin{array}{lll}
0.0225 & 0.0092 & 0.0001 \\
0.0081 & 0.0056 & 0.0092 \\
0.0018 & 0.0075 & 0.0277
\end{array}\right] \geq 0, U_{q}^{\#} V_{q}=\left[\begin{array}{lll}
0.0597 & 0.0255 & 0.0048 \\
0.0428 & 0.0381 & 0.0838 \\
0.0686 & 0.0889 & 0.2466
\end{array}\right] \geq 0, \\
& X_{q}^{\#}=\left[\begin{array}{lll}
0.0231 & 0.0094 & 0.0001 \\
0.0088 & 0.0061 & 0.0102 \\
0.0034 & 0.0089 & 0.0304
\end{array}\right] \geq 0, X_{q}^{\#} Y_{q}=\left[\begin{array}{lll}
0.0379 & 0.0175 & 0.0083 \\
0.0223 & 0.0251 & 0.0649 \\
0.0290 & 0.0578 & 0.1863
\end{array}\right] \geq 0 .
\end{aligned}
$$

The spectral radius of the individual splittings and the alternating iteration matrix are as follows: $\rho\left(K_{q}^{\#} L_{q}\right)=0.3417, \rho\left(U_{q}^{\#} V_{q}\right)=0.2823, \rho\left(X_{q}^{\#} Y_{q}\right)=0.2097, \rho(H)=0.0203<1$. The convergence analysis is shown in the Table.
Example 5.4. Let us consider a system $A x=b$ with $A=\left[\begin{array}{ccc}47 & -9 & -5 \\ 5 & 0 & 4 \\ -14 & 3 & 3\end{array}\right]$ and $b=$ $(6,3,-1)^{t}$. Clearly $A$ is group monotone matrix since $A^{\#}=\left[\begin{array}{lll}0.0843 & 0.0311 & 0.2145 \\ 0.2801 & 0.1526 & 0.9462 \\ 0.0653 & 0.0405 & 0.2439\end{array}\right] \geq 0$. Let

$$
A=K-L=\left[\begin{array}{ccc}
52.2707 & -9.3666 & -2.5190 \\
7.1598 & 1.8711 & 14.5844 \\
-15.0370 & 3.7459 & 5.7011
\end{array}\right]-\left[\begin{array}{ccc}
5.2707 & -0.3666 & 2.4810 \\
2.1598 & 1.8711 & 10.5844 \\
-1.0370 & 0.7459 & 2.7011
\end{array}\right]
$$

is a proper G-weak regular splitting of $A$ since

$$
K^{\#}=\left[\begin{array}{lll}
0.0380 & 0.0065 & 0.0610 \\
0.0884 & 0.0449 & 0.2834 \\
0.0168 & 0.0128 & 0.0741
\end{array}\right] \geq 0 \text { and } K^{\#} L=\left[\begin{array}{lll}
0.1510 & 0.0436 & 0.3274 \\
0.2693 & 0.2630 & 1.4604 \\
0.0394 & 0.0731 & 0.3777
\end{array}\right] \geq 0
$$

Then there exists a nonsingular matrix

$$
Q=\left[\begin{array}{ccc}
12.1426 & 2.4576 & 7.0308 \\
7.1770 & 22.2770 & 26.4823 \\
4.3098 & 0.6414 & 24.3790
\end{array}\right], Q A=\left[\begin{array}{ccc}
484.5610 & -88.1914 & -29.7904 \\
77.9525 & 14.8537 & 132.6700 \\
-135.536 & 34.3484 & 54.1534
\end{array}\right]
$$

such that $Q A=A Q$ and $(Q A)^{\#}=\left[\begin{array}{lll}0.0041 & 0.0007 & 0.0068 \\ 0.0092 & 0.0049 & 0.0308 \\ 0.0017 & 0.0014 & 0.0080\end{array}\right] \geq 0$. Now let us consider a new system $Q A x=Q b$, where $Q A$ has a proper G-regular splitting i.e., $Q A=K_{q}-L_{q}$.

Now

$$
\begin{gathered}
K_{q}=\left[\begin{array}{ccc}
493.1640 & -76.7488 & 31.2533 \\
89.3695 & 28.7235 & 207.4530 \\
-134.5980 & 35.1574 & 58.7334
\end{array}\right] \text { and } L_{q}=\left[\begin{array}{ccc}
8.6028 & 11.4425 & 61.0437 \\
11.4170 & 13.8697 & 74.7837 \\
0.9386 & 0.8091 & 4.5800
\end{array}\right] \\
K_{q}^{\#}=\left[\begin{array}{ccc}
0.0032 & 0.0002 & 0.0035 \\
0.0063 & 0.0032 & 0.0202 \\
0.0010 & 0.0010 & 0.0056
\end{array}\right] \geq 0 \text { and } L_{q} \geq 0 . \\
\text { Since } Q K_{q}^{\#}-K^{\#}=\left[\begin{array}{lll}
0.0235 & 0.0109 & 0.0703 \\
0.1029 & 0.0543 & 0.3393 \\
0.0265 & 0.0145 & 0.0897
\end{array}\right] \geq 0 . \text { Therefore, } \rho\left(K_{q}^{\#} L_{q}\right)=0.3318 \leq
\end{gathered}
$$ $0.6993=\rho\left(K^{\#} L\right)<1$. The numerical results for comparison is discussed in Table.

The concept of the three-step alternating iteration scheme and comparison results can be applied to nonsingular system. The validation of the proposed approach is explained in the next example.

Example 5.5. Let us consider the nonsingular $M$-matrix
$A=\left[\begin{array}{ccccccccc}10.8654 & -0.3333 & -1.4444 & -1.2222 & -0.6667 & -0.1111 & -1.3333 & -2 & -0.5556 \\ -1.6667 & 9.0877 & -2 & -1.3333 & -0.8889 & -2 & -0.2222 & -0.5556 & -0.3333 \\ -1.6667 & -1.5556 & 9.8654 & -1.1111 & -1.2222 & -1.3333 & -1.5556 & -1.8889 & -2.2222 \\ -0.7778 & -0.8889 & -2.2222 & 9.1988 & -1.8889 & -1 & -0.1111 & -0.5556 & -0.5556 \\ -1.4444 & -0.4444 & -1.2222 & -1.2222 & 10.6432 & -0.1111 & -0.1111 & -1.8889 & -2.1111 \\ -1.5556 & -0.5556 & -0.4444 & -0.3333 & -1.7778 & 9.9765 & -1.5556 & -1.1111 & -2 \\ -1.8889 & -1.1111 & -0.3333 & -0.5556 & -2.1111 & -1.5556 & 9.6432 & -1.8889 & -2.1111 \\ -0.8889 & -2 & -0.1111 & -0.1111 & -0.5556 & -0.3333 & -0.2222 & 10.8654 & -0.3333 \\ -0.6667 & -0.6667 & -1.3333 & -1.4444 & -1.5556 & -1.4444 & -1.6667 & -2.2222 & 9.5321\end{array}\right]$
$=K-L=U-V=X-Y$ are three weak regular splitting of $A$ such that $A^{-1} \geq 0$. In
this case
$K=\left[\begin{array}{ccccccccc}11.1529 & 0.0167 & -1.5694 & -1.0347 & -0.3042 & 0.3014 & -1.2333 & -1.6875 & -0.5931 \\ -1.3417 & 8.9502 & -1.7125 & -1.1333 & -0.4639 & -2.2750 & 0.1278 & -0.7556 & -0.7083 \\ -1.7292 & -1.2306 & 10.0779 & -1.2611 & -1.0097 & -1.5833 & -1.4556 & -1.9889 & -2.2722 \\ -0.5778 & -1.0139 & -2.2972 & 9.2488 & -1.4139 & -1 & -0.4111 & -0.3806 & -0.2681 \\ -1.8069 & -0.0694 & -1.3347 & -1.2597 & 10.7057 & 0.2889 & -0.0986 & -1.4389 & -1.9986 \\ -1.1431 & -0.7056 & -0.1819 & -0.1458 & -1.3653 & 9.8390 & -1.1056 & -1.2236 & -1.6375 \\ -1.7139 & -0.8986 & -0.0208 & -0.1806 & -2.4361 & -1.2056 & 9.8182 & -1.7514 & -2.2486 \\ -0.4889 & -1.95 & -0.2361 & -0.0236 & -0.2431 & 0.1292 & -0.0972 & 11.1279 & 0.0417 \\ -0.2792 & -0.3542 & -1.0458 & -1.7069 & -1.7181 & -1.0819 & -1.8167 & -1.9347 & 9.7696\end{array}\right]$,
$U=\left[\begin{array}{ccccccccc}11.0404 & -0.0458 & -1.3569 & -1.4097 & -0.5542 & 0.0389 & -1.5333 & -1.7125 & -1.0181 \\ -1.1667 & 9.1002 & -1.9250 & -0.8958 & -0.7389 & -2.2750 & -0.3097 & -0.3681 & -0.1083 \\ -1.1792 & -1.5806 & 9.5654 & -0.8736 & -0.9972 & -0.9833 & -1.3806 & -1.7764 & -2.0347 \\ -1.0653 & -0.6639 & -1.8472 & 9.1113 & -1.6139 & -0.8250 & 0.0014 & -0.5806 & -0.1681 \\ -1.6944 & -0.1319 & -1.0097 & -1.1597 & 10.6557 & 0.1139 & 0.1014 & -1.6139 & -1.8986 \\ -1.2181 & -0.8931 & -0.0319 & -0.2458 & -1.5903 & 10.2015 & -1.6181 & -0.8486 & -1.5 \\ -1.5014 & -0.9986 & -0.3458 & -0.2056 & -2.1111 & -1.5806 & 10.1432 & -1.6264 & -1.6236 \\ -0.6514 & -1.5375 & -0.1611 & 0.2139 & -0.0556 & -0.1458 & 0.1778 & 11.2529 & 0.0917 \\ -0.8292 & -0.7042 & -0.8833 & -1.4694 & -1.7056 & -1.1319 & -1.4167 & -2.1722 & 9.1446\end{array}\right]$,
$X=\left[\begin{array}{ccccccccc}11.1779 & -0.0833 & -1.6069 & -1.3972 & -1.1167 & 0.3764 & -0.8333 & -1.5875 & -0.8556 \\ -2.1292 & 9.5502 & -2.0250 & -1.3833 & -0.5389 & -1.7375 & 0.2778 & -0.1806 & 0.0667 \\ -1.2417 & -1.4681 & 9.9029 & -1.2236 & -1.2347 & -0.8958 & -1.9431 & -2.1639 & -2.3222 \\ -0.5278 & -1.1139 & -2.4097 & 9.6363 & -1.8264 & -1.0750 & -0.2611 & -0.5806 & -0.4931 \\ -1.0319 & -0.1319 & -0.9597 & -0.7347 & 11.0307 & 0.0264 & 0.2514 & -1.7139 & -2.4361 \\ -1.3556 & -0.5181 & -0.2694 & -0.6708 & -1.6278 & 9.6890 & -1.3931 & -0.6611 & -1.9250 \\ -2.1389 & -1.0111 & -0.1833 & -0.0931 & -1.8361 & -1.5181 & 9.5807 & -1.8014 & -1.8236 \\ -0.7014 & -2.2125 & -0.1236 & 0.2764 & -0.0681 & -0.2458 & -0.4472 & 10.7029 & 0.0292 \\ -0.9417 & -0.7667 & -1.0708 & -1.2569 & -1.7056 & -1.0319 & -1.5792 & -1.9347 & 10.0071\end{array}\right]$.
Here $\rho\left(X^{-1} Y U^{-1} V K^{-1} L\right)=0.1513 \leq 0.3038=\rho\left(U^{-1} V K^{-1} L\right) \leq 0.5346=\rho\left(K^{-1} L\right)$. The comparison analysis of one step, two-step and three-step alternating iteration scheme is provided in the table. It also contains the same analysis for two random nonsingular matrices of order 1000 and 2000.

## 6. Conclusion

We have introduced the three-step alternating iterations for singular linear systems of index 1 and studied its convergence criteria. Three algorithms are also provided for numerical computation and complexity. Finally, a comparison result is proved which guarantees the fact that the three-step alternating iterations converges faster than the usual one, and is also shown through examples. The authors of [10], [24] and [26] studied the two-step alternating iterations for rectangular matrices using the Moore-Penrose inverses, very recently. However, their works lack computational implementation which is addressed in this paper. Finally, we conclude the paper with the comparative analysis of one step, two step and three step iterations.

The iterative methods (i.e., matrix splitting methods) and semi-iterative methods are among many methods that have been suggested in the literature to solve real singular linear systems. A matrix is called an EP matrix if $R(A)=R\left(A^{t}\right)$. If $A$ is an EP matrix, then the proposed scheme will converge to the least squares solution of minimum norm. Migallón et al. [21] studied alternating two-stage methods for consistent linear systems to obtain the parallel solution of Markov chains, recently. The same authors further extended the same notion in [22]. Further applications of this theory to compute the PageRank of a google matrix can also be found in the recent article [11]. Hence, we conclude this article with the hope that our work may help to deal with singular linear systems which appear in different areas of mathematics as mentioned above and in the introduction part. We hope that this work will provide useful insights into extending this approach and thus help in solving rectangular linear systems in a faster way.

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