Directly Indecomposables in Semidegenerate Varieties of Connected po-Groupoids

Pedro Sánchez Terraf*

Abstract

We study varieties with a term-definable poset structure, *po-groupoids*. It is known that connected posets have the *strict refinement property* (SRP). In [7] it is proved that semidegenerate varieties with the SRP have definable factor congruences and if the similarity type is finite, directly indecomposables are axiomatizable by a set of first-order sentences. We obtain such a set for semidegenerate varieties of connected po-groupoids and show its quantifier complexity is bounded in general.

1 Introduction and Basic Definitions

Definition 1. A *po-groupoid* is a groupoid $\langle A, \cdot \rangle$ such that the relation defined by

 $x \leq y$ if and only if $x \cdot y = x$

is a partial order on A, the order related to $\langle A, \cdot \rangle$. A variety of po-groupoids is an equational class \mathcal{V} of algebras with binary term \cdot in the language of \mathcal{V} such that for every $A \in \mathcal{V}, \langle A, \cdot^A \rangle$ is a po-groupoid.

For every poset $\langle A, \preceq \rangle$ one can define a po-groupoid operation * on A setting

$$x * y := \begin{cases} x & \text{if } x \preceq y \\ y & \text{if } x \not\preceq y. \end{cases}$$

such that \leq is the order related to $\langle A, * \rangle$.

^{*}Supported by Conicet.

Keywords: connected poset, strict refinement property, semidegenerate variety, definable factor congruences. *MSC 2000:* 06A12, 20M10.

Po-groupoids are obviously idempotent, but need not be associative nor commutative. Examples of po-groupoids are semilattices and, more generally, the variety axiomatized by the following identities:

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

$$x \cdot x \approx x$$

$$x \cdot y \cdot x \approx y \cdot x.$$
(1)

This variety is exactly the class of associative po-groupoids, *po-semigroups* for short (see Claim 11). J. Gerhard [2] proved it is not residually small.

A poset $\langle A, \preceq \rangle$ is said to be *connected* if the associated graph is. Equivalently, $\langle A, \preceq \rangle$ is connected if for all $x, y \in A$, there exists a positive integer n and elements m_1, \ldots, m_{2n-1} such that

$$x \succeq m_1 \preceq m_2 \succeq \ldots \preceq m_{2n-2} \succeq m_{2n-1} \preceq y \tag{2}$$

where \succeq is the converse relation to \preceq . A po-groupoid will be called connected if the related order is.

A variety \mathcal{V} is *semidegenerate* if no non trivial member has a trivial subalgebra. Equivalently, if every $A \in \mathcal{V}$ has a compact universal congruence (Kollar [3]). In this work we will consider semidegenerate varieties of connected po-groupoids over a finite language.

It is known [6, Section 5.6] that connected posets have the *strict refinement property* (SRP). In a joint work with D. Vaggione [7] it is proved that semidegenerate varieties with the SRP have definable factor congruences and if the similarity type is finite, directly indecomposables are axiomatizable by a set of first-order sentences. Our main result is an application of [7] and a result of R. Willard [11].

Theorem 2. Let \mathcal{V} be a semidegenerate variety of connected po-groupoids over a finite language. Then the class of directly indecomposable algebras of \mathcal{V} is axiomatizable by a Π_6 (i.e., $\forall \exists \forall \exists \forall \exists)$) first-order sentence plus axioms for \mathcal{V} .

2 Results

We first state two auxiliary results that provide a useful Mal'cev condition for semidegeneracy. If $\vec{a}, \vec{b} \in A^n$, $\operatorname{Cg}^A(\vec{a}, \vec{b})$ will stand for the A-congruence generated by $(a_1, b_1), \ldots, (a_n, b_n)$.

Lemma 3 (Mal'cev). Let A be any algebra and let $a, b \in A, \vec{a}, \vec{b} \in A^n$. Then $(a, b) \in Cg^A(\vec{a}, \vec{b})$ if and only if there exist (n + m)-ary terms $p_1(\vec{x}, \vec{u}), \ldots, p_k(\vec{x}, \vec{u})$, with k odd and, $\vec{u} \in A^m$ such that:

$$\begin{split} a &= p_1(\vec{a}, \vec{u}) \\ p_i(\vec{b}, \vec{u}) &= p_{i+1}(\vec{b}, \vec{u}), \ i \ odd \\ p_i(\vec{a}, \vec{u}) &= p_{i+1}(\vec{a}, \vec{u}), \ i \ even \\ p_k(\vec{b}, \vec{u}) &= b \end{split}$$

Lemma 4. If \mathcal{V} is semidegenerate, there exist positive integers l and k (with k odd), unary terms $0_1(w), \ldots, 0_l(w), 1_1(w), \ldots, 1_l(w)$ and (2+l)-ary terms $U_i(x, y, \vec{z}), i = 1, \ldots, k$, such that the following identities hold in \mathcal{V} :

$$x \approx U_{1}(x, y, \vec{0})$$

$$U_{i}(x, y, \vec{1}) \approx U_{i+1}(x, y, \vec{1}) \text{ with } i \text{ odd}$$

$$U_{i}(x, y, \vec{0}) \approx U_{i+1}(x, y, \vec{0}) \text{ with } i \text{ even}$$

$$U_{k}(x, y, \vec{1}) \approx y$$
(3)

where w, x and y are distinct variables, $\vec{0} = (0_1(w), \dots, 0_l(w))$ and $\vec{1} = (1_1(w), \dots, 1_l(w))$.

Proof. By Kollar [3] every algebra in a semidegenerate variety has a compact universal congruence. By Lemma 3 in Vaggione [9], there are terms $0_1(w), \ldots, 0_l(w), 1_1(w), \ldots, 1_l(w)$ such that $\operatorname{Cg}^A(\vec{0}, \vec{1})$ is the universal congruence in each $A \in \mathcal{V}$. Now apply Lemma 3.

Throughout this paper we will assume that we may find closed terms $\vec{0}$ and $\vec{1}$ for \mathcal{V} . Of course, this can be achieved when the language has a constant symbol and we will make this assumption in order to clarify our treatment. The proofs remain valid in the general case.

The next lemma ensures that we have a uniform way to witness connection.

Lemma 5. For every variety \mathcal{V} of connected po-groupoids, there exist a positive integer n and binary terms $m_i(x, y)$, $i = 1, \ldots, 2n - 1$ in the language of \mathcal{V} such that the following identities hold in \mathcal{V} :

$$m_1(x, y) \cdot x \approx m_1(x, y)$$

$$m_i(x, y) \cdot m_{i\pm 1}(x, y) \approx m_i(x, y) \qquad \text{if } i < 2n - 2 \text{ is odd}$$

$$m_{2n-1}(x, y) \cdot y \approx m_{2n-1}(x, y).$$

Proof. It is enough to consider the \mathcal{V} -free algebra freely generated by $\{x, y\}$. The elements m_i that connect x and y are binary terms, and the desired equations are equivalent to the assertions in (2).

For the rest of this section, \mathcal{V} will denote a semidegenerate variety of po-groupoids with k terms U_i and 2n - 1 terms m_i as in the previous lemmas. In the following we define formulas ψ , ϕ and π codifying the fact that connected po-groupoids have the SRP. They appear under the same names in Willard [11, Section 5], though the key improvement is a radical simplification of the first one.

A formula is called \mathcal{V} -factorable (factorable for short) if it belongs to the smallest set **F** containing every atomic formula that is closed under conjunction, existential and universal quantification, and the following rule:

if
$$\alpha(\vec{x}), \beta(\vec{x}, \vec{y}), \gamma(\vec{x}, \vec{y}) \in \mathbf{F}$$
 and $\mathcal{V} \models \forall \vec{x} (\alpha(\vec{x}) \to \exists \vec{y} \beta(\vec{x}, \vec{y}))$ then $\forall \vec{y} (\beta(\vec{x}, \vec{y}) \to \gamma(\vec{x}, \vec{y})) \in \mathbf{F}$.

Factorable formulas are preserved by direct factors and direct products (see [11, Section 1]).

Lemma 6. There exists a factorable Π_2 formula $\psi(x, y, z)$ such that:

- 1. $\mathcal{V} \models \psi(x, y, x)$.
- 2. $\mathcal{V} \models \psi(x, y, y)$.
- 3. $\mathcal{V} \models \psi(x, x, z) \rightarrow x \preceq z$.

Proof. Let $\psi(x, y, z)$ be the following formula:

$$\forall u_1, \dots, u_{2n-1}$$

$$u_1 \cdot x = u_1 \cdot u_2 \land \bigwedge_{i=2}^{n-1} u_{2i-1} \cdot u_{2i-2} = u_{2i-1} \cdot u_{2i} \land u_{2n-1} \cdot u_{2n-2} = u_{2n-1} \cdot y \rightarrow$$

$$\Rightarrow \exists v_1, \dots, v_{n-1} : u_1 \cdot x = u_1 \cdot v_1 \land \bigwedge_{i=2}^{n-1} u_{2i-1} \cdot v_{i-1} = u_{2i-1} \cdot v_i \land u_{2n-1} \cdot v_{n-1} = u_{2n-1} \cdot z$$

We may see that ψ is factorable observing that $u_i := m_i(x, y)$ satisfies the antecedent for any choice of x and y.

To prove 1, simply take $v_i := x$ for all *i*. For 2, it suffices to assign $v_i := u_{2i}$ for all *i*. Finally, suppose $\psi(x, x, z)$ holds. Take $u_i := x$. With this choice, the antecedent holds. The consequent turns to:

$$x \cdot x = x \cdot v_1 \ \land \ \bigwedge_{i=2}^{n-1} x \cdot v_{i-1} = x \cdot v_i \ \land \ x \cdot v_{n-1} = x \cdot z,$$

from which we conclude $x = x \cdot z$, and we have proved 3.

4

Lemma 7. There exists a factorable Π_3 formula $\pi(x, y, z, w)$ such that:

1. $\mathcal{V} \models \pi(x, x, z, w)$

2.
$$\mathcal{V} \models \pi(x, y, x, y)$$

3. $\mathcal{V} \models \pi(x, y, z, z) \rightarrow x = y$

Proof. We first define $\phi(x, y, w_1, w_2)$ to be $\psi(x, w_1, y) \wedge \psi(y, w_2, x)$. Take $\pi(x, y, z, w)$ to be $\forall w_1, w_2 : \phi(z, w, w_1, w_2) \rightarrow \phi(x, y, w_1, w_2)$.

It is immediate that 2 holds. Property 1 holds thanks to Lemma 6(1). And we can check 3 by taking $w_1 := y$ and $w_2 := x$.

Finally, π is factorable since $\exists w_1, w_2 : \phi(z, w, w_1, w_2)$ holds in \mathcal{V} due to Lemma 6(2). \Box

The formula Φ appearing in the next lemma is a first-order definition of factor congruences in \mathcal{V} using central elements. This concept (in its full generality) is due to Vaggione [8].

If $\vec{a} \in A^l$ and $\vec{b} \in B^l$, we will write $[\vec{a}, \vec{b}]$ in place of $((a_1, b_1), \ldots, (a_l, b_l)) \in (A \times B)^l$. If $A \in \mathcal{V}$, we say that $\vec{e} \in A^l$ is a *central element* of A if there exists an isomorphism $A \to A_1 \times A_2$ such that

 $\vec{e} \mapsto [\vec{0}, \vec{1}].$

The set of all central elements of A will be called the *center* of A. Two central elements \vec{e}, \vec{f} will be called *complementary* if there exists an isomorphism $A \to A_1 \times A_2$ such that $\vec{e} \mapsto [\vec{0}, \vec{1}]$ and $\vec{f} \mapsto [\vec{1}, \vec{0}]$. It is immediate that an algebra is directly indecomposable if and only if it has exactly two central elements, namely $\vec{0}$ and $\vec{1}$.

Lemma 8. There exists a factorable Σ_4 formula $\Phi(x, y, \vec{z})$ such that for all $A, B \in \mathcal{V}$, and $a, c \in A, b, d \in B$,

$$A \times B \models \Phi(\langle a, b \rangle, \langle c, d \rangle, [\vec{0}, \vec{1}])$$
 if and only if $a = c$.

Proof. Take $\Phi(x, y, \vec{z})$ to be

$$\exists a_1, \dots, a_{n-1} : \pi(x, a_1, U_1(x, y, \vec{z}), U_1(x, y, \vec{w})) \land \bigwedge_{i \text{ odd}} \pi(a_i, a_{i+1}, U_{i+1}(x, y, \vec{w}), U_{i+1}(x, y, \vec{z})) \land \bigwedge_{i \text{ even}} \pi(a_i, a_{i+1}, U_{i+1}(x, y, \vec{z}), U_{i+1}(x, y, \vec{w})) \land \pi(a_{n-1}, y, U_k(x, y, \vec{z}), U_k(x, y, \vec{w})).$$

Suppose that $A \times B \models \Phi(\langle a, b \rangle, \langle c, d \rangle, [\vec{0}, \vec{1}])$. Looking at the first coordinate, we obtain

$$\pi(a, a_{1}^{1}, U_{1}(a, c, \vec{0}), U_{1}(a, c, \vec{0})) \wedge \bigwedge_{i \text{ odd}} \pi(a_{i}^{1}, a_{i+1}^{1}, U_{i+1}(a, c, \vec{0}), U_{i+1}(a, c, \vec{0})) \wedge \\ \wedge \bigwedge_{i \text{ even}} \pi(a_{i}^{1}, a_{i+1}^{1}, U_{i+1}(a, c, \vec{0}), U_{i+1}(a, c, \vec{0})) \wedge \pi(a_{n-1}^{1}, c, U_{k}(a, c, \vec{0}), U_{k}(a, c, \vec{0})),$$

where $a_i = \langle a_i^1, a_i^2 \rangle$. Applying Lemma 7(3) we obtain a = c.

Finally, to show that

$$A \times B \models \Phi(\langle a, b \rangle, \langle a, d \rangle, [0, 1])$$

it suffices to take $a_i := \langle a, U_i(b, d, \mathbf{i}) \rangle$, where \mathbf{i} is $\vec{0}$ if i is even, otherwise $\vec{1}$. This is routine.

Now we define the center of algebras in \mathcal{V} in first-order logic.

Lemma 9. There is a Π_5 formula $\zeta(\vec{z}, \vec{w})$ such that for all $A \in \mathcal{V}$ and $\vec{e}, \vec{f} \in A^l$ we have that \vec{e} and \vec{f} are complementary central elements if and only if $A \models \zeta(\vec{e}, \vec{f})$.

Proof. The following formulas in the language of \mathcal{V} will assert the properties needed to force $\Phi(\cdot, \cdot, \vec{z})$ and $\Phi(\cdot, \cdot, \vec{w})$ to define the pair of complementary factor congruences associated with \vec{z} and \vec{w} . The names are almost self-explanatory.

• $CAN(\vec{z}, \vec{w}) = \bigwedge_{i=1}^{l} \Phi(0_i, z_i, \vec{z}) \land \bigwedge_{i=1}^{l} \Phi(1_i, w_i, \vec{z})$

•
$$PROD(\vec{z}, \vec{w}) = \forall x, y \exists z \left(\Phi(x, z, \vec{z}) \land \Phi(z, y, \vec{w}) \right)$$

- $INT(\vec{z}, \vec{w}) = \forall x, y \ \left(\Phi(x, y, \vec{z}) \land \Phi(x, y, \vec{w}) \to x = y \right)$
- $REF(\vec{z}, \vec{w}) = \forall x \ \Phi(x, x, \vec{z})$

•
$$SYM(\vec{z}, \vec{w}) = \forall x, y, z \left(\Phi(x, y, \vec{z}) \land \Phi(y, z, \vec{z}) \land \Phi(z, x, \vec{w}) \rightarrow z = x \right)$$

•
$$TRANS(\vec{z}, \vec{w}) = \forall x, y, z, u \left(\Phi(x, y, \vec{z}) \land \Phi(y, z, \vec{z}) \land \Phi(x, u, \vec{z}) \land \Phi(u, z, \vec{w}) \rightarrow u = z \right)$$

• For each m-ary function symbol F:

$$PRES_F(\vec{z}, \vec{w}) = \forall u_1, v_1, \dots, u_m, v_m$$
$$\left(\bigwedge_j \Phi(u_j, v_j, \vec{z})\right) \land \Phi(F(u_1, \dots, u_m), z, \vec{z}) \land \Phi(z, F(v_1, \dots, v_m), \vec{w}) \rightarrow$$
$$\rightarrow z = F(v_1, \dots, v_m)$$

Now take CAN', REF', SYM', TRANS' and $PRES'_F$ to be the result of interchanging \vec{z} with \vec{w} in CAN, REF, SYM, TRANS and $PRES_F$, respectively, and let ζ be the conjunction of:

$$\bigwedge \{CAN, PROD, INT, REF, SYM, TRANS, CAN', REF', SYM', TRANS'\} \\ \bigwedge \{PRES_F, PRES'_F : F \text{ a function symbol}\}.$$

Details can be found in [7, Lemma 4.1].

6

Proof of Theorem 2. The formula

$$\vec{0} \neq \vec{1} \ \land \ \forall \vec{e}, \vec{f} : \zeta(\vec{e}, \vec{f}) \ \rightarrow \ \left((\vec{e} = \vec{0} \land \vec{f} = \vec{1}) \phi(\vec{e} = \vec{1} \land \vec{f} = \vec{0}) \right)$$

together with axioms for \mathcal{V} defines the subclass of directly indecomposables.

3 Examples

We first mention that we cannot eliminate the semidegeneracy hypothesis, since even the class of directly indecomposable lattices with 0 is not axiomatizable in first-order logic (see Willard [10]). Also, in [7, Section 6] it is shown that semidegeneracy by itself does not ensure definability of directly indecomposables. By considering in this last case a trivial (antichain) po-groupoid structure (for instance, defining $x \cdot y := y$ for all x, y) we deduce that an arbitrary semidegenerate variety of po-groupoids may not have a first-order-axiomatizable class of indecomposables; hence we cannot drop connectedness.

We will now consider the following variety \mathcal{R} :

$$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$$

$$x \cdot x \approx x$$

$$(4)$$

$$x \cdot y \cdot z \approx y \cdot x \cdot z.$$

This is obviously a variety of po-groupoids. Moreover, the variety of po-semigroups (defined by equations (1)) covers \mathcal{R} in the lattice of equational classes of idempotent semigroups (see Gerhard [1]).

From the partial-order point of view, this groupoids are "relative meet-semilattices": whenever $A \in \mathcal{V}$, every bounded subalgebra of A is a semilattice. Actually, the third axiom is equivalent to this property.

Lemma 10. Let \mathcal{V} be a variety of associative po-groupoids. The following are equivalent:

- 1. $\mathcal{V} \models x \leq z \land y \leq z \rightarrow x \cdot y = y \cdot x$.
- 2. $\mathcal{V} \models x \cdot y \cdot z \approx y \cdot x \cdot z$.

Proof. We first need

Claim 11. Every associative po-groupoid satisfies $x \cdot y \cdot x \approx y \cdot x$.

Proof. We now that $x \cdot y \preceq y$. We obtain immediately that $x \cdot y \cdot x \preceq y \cdot x$ and $y \cdot x = y \cdot x \cdot y \cdot x \preceq x \cdot y \cdot x$. By antisymmetry we get $x \cdot y \cdot x = y \cdot x$.

Assume (1). Note that $x \cdot y \cdot z \preceq z$ and $y \cdot x \cdot z \preceq z$, hence $x \cdot y \cdot z \cdot y \cdot x \cdot z = y \cdot x \cdot z \cdot x \cdot y \cdot z$. We may simplify this expression using the Claim to obtain (2).

Now suppose (2) holds, and assume $x, y \leq z$. Hence

$$x \cdot y = x \cdot y \cdot z = y \cdot x \cdot z = y \cdot x,$$

and we have (1).

Lemma 12. Let \mathcal{V} will be a variety of connected po-groupoids that satisfies the defining identities of \mathcal{R} , and let $x_1, \ldots, x_j \in A \in \mathcal{V}$. Then $A \models \exists u : \bigwedge_{i=2}^{j} u \cdot x_1 = u \cdot x_i$.

Proof. We only prove the case j = 2, from which the rest can be easily derived. We will prove by induction on n:

For all $m_1, \ldots, m_{2n-1} \in A$ such that

$$m_1 \preceq m_2 \succeq \ldots \preceq m_{2n-2} \succeq m_{2n-1}$$

we have $m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1} = m_{2n-1} \cdot \ldots \cdot m_3 \cdot m_1$.

If n = 2, we have $m_1, m_3 \leq m_2$, hence by Lemma 10 $m_1 \cdot m_3 = m_3 \cdot m_1$. Now suppose the assertion holds for n, and assume $m_1, \ldots, m_{2n+1} \in A$ satisfy:

$$m_1 \preceq m_2 \succeq \ldots \preceq m_{2n-2} \succeq m_{2n-1} \preceq m_{2n} \succeq m_{2n+1}$$

We may apply the inductive hypothesis and obtain

$$m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1} \cdot m_{2n+1} = m_1 \cdot m_{2n+1} \cdot m_{2n-1} \cdot \ldots \cdot m_3$$

This last term equals $m_{2n+1} \cdot m_{2n-1} \cdot \ldots \cdot m_1 \cdot m_3$ (by the third axiom of \mathcal{R}) and by the case n = 2 we may commute m_1 and m_3 , obtaining:

$$m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1} \cdot m_{2n+1} = m_{2n+1} \cdot m_{2n-1} \cdot \ldots \cdot m_3 \cdot m_1. \tag{5}$$

Once we have this, we may take $u := m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1}$, where

$$x_1 \succeq m_1 \preceq m_2 \succeq \ldots \preceq m_{2n-2} \succeq m_{2n-1} \preceq x_2$$

since $u \cdot x_2 = m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1} \cdot x_2 = m_1 \cdot m_3 \cdot \ldots \cdot m_{2n-1} = m_{2n-1} \cdot \ldots \cdot m_3 \cdot m_1 = m_{2n-1} \cdot \ldots \cdot m_3 \cdot m_1 \cdot x_1 = u \cdot x_1$

The previous lemmas were discovered using the *Prover9/Mace4* program bundle by W. McCune [5, 4].

Lemma 13. Let \mathcal{V} will be a semidegenerate variety of connected po-groupoids that satisfies the defining identities of \mathcal{R} . There exists a factorable Π_1 formula $\Phi(x, y, \vec{z})$ such that for all $A, B \in \mathcal{V}$, and $a, c \in A$, $b, d \in B$,

$$A \times B \models \Phi(\langle a, b \rangle, \langle c, d \rangle, [\vec{0}, \vec{1}])$$
 if and only if $a = c$.

Proof. To fix notation, we assume again there are k terms U_i and 2n - 1 terms m_i that witness semidegeneracy and connection, respectively (see Lemmas 4 and 5).

Take $\Phi(x, y, \vec{z})$ to be

$$\forall u : \bigwedge_{i=1}^{k} \left(u \cdot U_i(x, y, \vec{0}) = u \cdot U_i(x, y, \vec{z}) \right) \longrightarrow u \cdot x = u \cdot y.$$

This formula is factorable by Lemma 12. Let $A, B \in \mathcal{V}$, and $a \in A, b, d \in B$. First we prove that

$$A \times B \models \Phi(\langle a, b \rangle, \langle a, d \rangle, [\vec{0}, \vec{1}])$$

Suppose that for some $\langle u, v \rangle$ we have

$$A \times B \models \bigwedge_{i=1}^{\kappa} \langle u, v \rangle \cdot U_i(\langle a, b \rangle, \langle a, d \rangle, [\vec{0}, \vec{0}]) = \langle u, v \rangle \cdot U_i(\langle a, b \rangle, \langle a, d \rangle, [\vec{0}, \vec{1}]).$$

Then

$$B \models \bigwedge_{i=1}^{k} v \cdot U_i(b, d, \vec{0}) = v \cdot U_i(b, d, \vec{1}).$$

But the above equations in combination with (3) produce

$$v \cdot b = v \cdot d$$

and hence

$$\langle u, v \rangle \cdot \langle a, b \rangle = \langle u, v \rangle \cdot \langle a, d \rangle.$$

Now suppose

$$A \times B \models \Phi(\langle a, b \rangle, \langle c, d \rangle, [\vec{0}, \vec{1}]).$$

Since Φ is preserved by direct factors, we obtain $A \models \Phi(a, c, \vec{0})$ and by inspection this is equivalent to $\forall u : u \cdot a = u \cdot c$. Using this equation for u = a, c we obtain $a \leq c$ and $c \leq a$, therefore a = c.

Using this new definition of factor congruences, we obtain:

Theorem 14. Let \mathcal{V} be a semidegenerate variety of connected po-groupoids over a finite language that satisfy (4). Then the class of directly indecomposable algebras of \mathcal{V} is axiomatizable by a Π_4 sentence plus axioms for \mathcal{V} .

References

- J. A. GERHARD, The lattice of equational classes of idempotent semigroups, J. Algebra 15 (1970): 195-224.
- [2] J. A. GERHARD, Subdirectly irreducible idempotent semigroups, Pacific J. Math. **39** (1971): 669–676.
- [3] J. KOLLAR, Congruences and one element subalgebras, Algebra univers. 9 (1979): 266–267.
- [4] W. MCCUNE, Mace4 Reference Manual and Guide, Tech. Memo ANL/MCS-TM-264, Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, IL, August 2003.
- [5] W. MCCUNE, Prover9 and Mace4 Webpage, http://www.cs.unm.edu/~mccune/prover9/
- [6] R. MCKENZIE, G. MCNULTY AND W. TAYLOR, Algebras, Lattices, Varieties, Volume 1, The Wadsworth & Brooks/Cole Math. Series, Monterey, California (1987).
- [7] P. SÁNCHEZ TERRAF AND D. VAGGIONE, Varieties with Definable Factor Congruences, Trans. Amer. Math. Soc., to appear.
- [8] D. VAGGIONE, \mathcal{V} with factorable congruences and $\mathcal{V} = \mathbf{I}\Gamma^{a}(\mathcal{V}_{DI})$ imply \mathcal{V} is a discriminator variety. Acta Sci. Math. **62** (1996): 359–368.
- [9] D. VAGGIONE, Varieties of shells, Algebra univers. 36 (1996): 483–487.
- [10] R. WILLARD, A note on indecomposable lattices, Algebra univers. 26 (1989): 257–258.
- [11] R. WILLARD, Varieties Having Boolean Factor Congruences, J. Algebra, 132 (1990): 130–153.

CIEM — Facultad de Matemática, Astronomía y Física (Fa.M.A.F.) Universidad Nacional de Córdoba - Ciudad Universitaria Córdoba 5000. Argentina. sterraf@famaf.unc.edu.ar