# Directly Indecomposables in Semidegenerate Varieties of Connected po-Groupoids 

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#### Abstract

We study varieties with a term-definable poset structure, po-groupoids. It is known that connected posets have the strict refinement property (SRP). In [7 it is proved that semidegenerate varieties with the SRP have definable factor congruences and if the similarity type is finite, directly indecomposables are axiomatizable by a set of first-order sentences. We obtain such a set for semidegenerate varieties of connected po-groupoids and show its quantifier complexity is bounded in general.


## 1 Introduction and Basic Definitions

Definition 1. A po-groupoid is a groupoid $\langle A, \cdot\rangle$ such that the relation defined by

$$
x \preceq y \text { if and only if } x \cdot y=x
$$

is a partial order on $A$, the order related to $\langle A, \cdot\rangle$. A variety of po-groupoids is an equational class $\mathcal{V}$ of algebras with binary term $\cdot$ in the language of $\mathcal{V}$ such that for every $A \in \mathcal{V},\left\langle A,{ }^{A}\right\rangle$ is a po-groupoid.

For every poset $\langle A, \preceq\rangle$ one can define a po-groupoid operation $*$ on $A$ setting

$$
x * y:= \begin{cases}x & \text { if } x \preceq y \\ y & \text { if } x \preceq y .\end{cases}
$$

such that $\preceq$ is the order related to $\langle A, *\rangle$.

[^0]Po-groupoids are obviously idempotent, but need not be associative nor commutative. Examples of po-groupoids are semilattices and, more generally, the variety axiomatized by the following identities:

$$
\begin{align*}
(x \cdot y) \cdot z & \approx x \cdot(y \cdot z) \\
x \cdot x & \approx x  \tag{1}\\
x \cdot y \cdot x & \approx y \cdot x .
\end{align*}
$$

This variety is exactly the class of associative po-groupoids, po-semigroups for short (see Claim (11). J. Gerhard [2] proved it is not residually small.

A poset $\langle A, \preceq\rangle$ is said to be connected if the associated graph is. Equivalently, $\langle A, \preceq\rangle$ is connected if for all $x, y \in A$, there exists a positive integer $n$ and elements $m_{1}, \ldots, m_{2 n-1}$ such that

$$
\begin{equation*}
x \succeq m_{1} \preceq m_{2} \succeq \ldots \preceq m_{2 n-2} \succeq m_{2 n-1} \preceq y \tag{2}
\end{equation*}
$$

where $\succeq$ is the converse relation to $\preceq$. A po-groupoid will be called connected if the related order is.

A variety $\mathcal{V}$ is semidegenerate if no non trivial member has a trivial subalgebra. Equivalently, if every $A \in \mathcal{V}$ has a compact universal congruence (Kollar [3]). In this work we will consider semidegenerate varieties of connected po-groupoids over a finite language.

It is known [6, Section 5.6] that connected posets have the strict refinement property (SRP). In a joint work with D. Vaggione [7] it is proved that semidegenerate varieties with the SRP have definable factor congruences and if the similarity type is finite, directly indecomposables are axiomatizable by a set of first-order sentences. Our main result is an application of [7] and a result of R. Willard [11].

Theorem 2. Let $\mathcal{V}$ be a semidegenerate variety of connected po-groupoids over a finite language. Then the class of directly indecomposable algebras of $\mathcal{V}$ is axiomatizable by $a \Pi_{6}$ (i.e., $\forall \exists \forall \exists \forall \exists$ ) first-order sentence plus axioms for $\mathcal{V}$.

## 2 Results

We first state two auxiliary results that provide a useful Mal'cev condition for semidegeneracy. If $\vec{a}, \vec{b} \in A^{n}, \mathrm{Cg}^{A}(\vec{a}, \vec{b})$ will stand for the $A$-congruence generated by $\left(a_{1}, b_{1}\right), \ldots,\left(a_{n}, b_{n}\right)$.

Lemma 3 (Mal'cev). Let $A$ be any algebra and let $a, b \in A, \vec{a}, \vec{b} \in A^{n}$. Then $(a, b) \in \operatorname{Cg}^{A}(\vec{a}, \vec{b})$ if and only if there exist $(n+m)$-ary terms $p_{1}(\vec{x}, \vec{u}), \ldots, p_{k}(\vec{x}, \vec{u})$, with $k$ odd and, $\vec{u} \in A^{m}$
such that:

$$
\begin{aligned}
a & =p_{1}(\vec{a}, \vec{u}) \\
p_{i}(\vec{b}, \vec{u}) & =p_{i+1}(\vec{b}, \vec{u}), i \text { odd } \\
p_{i}(\vec{a}, \vec{u}) & =p_{i+1}(\vec{a}, \vec{u}), i \text { even } \\
p_{k}(\vec{b}, \vec{u}) & =b
\end{aligned}
$$

Lemma 4. If $\mathcal{V}$ is semidegenerate, there exist positive integers $l$ and $k$ (with $k$ odd), unary terms $0_{1}(w), \ldots, 0_{l}(w), 1_{1}(w), \ldots, 1_{l}(w)$ and $(2+l)$-ary terms $U_{i}(x, y, \vec{z}), i=1, \ldots, k$, such that the following identities hold in $\mathcal{V}$ :

$$
\begin{align*}
x & \approx U_{1}(x, y, \overrightarrow{0}) \\
U_{i}(x, y, \overrightarrow{1}) & \approx U_{i+1}(x, y, \overrightarrow{1}) \text { with } i \text { odd } \\
U_{i}(x, y, \overrightarrow{0}) & \approx U_{i+1}(x, y, \overrightarrow{0}) \text { with } i \text { even }  \tag{3}\\
U_{k}(x, y, \overrightarrow{1}) & \approx y
\end{align*}
$$

where $w, x$ and $y$ are distinct variables, $\overrightarrow{0}=\left(0_{1}(w), \ldots, 0_{l}(w)\right)$ and $\overrightarrow{1}=\left(1_{1}(w), \ldots, 1_{l}(w)\right)$.
Proof. By Kollar [3] every algebra in a semidegenerate variety has a compact universal congruence. By Lemma 3 in Vaggione [9], there are terms $0_{1}(w), \ldots, 0_{l}(w), 1_{1}(w), \ldots, 1_{l}(w)$ such that $\operatorname{Cg}^{A}(\overrightarrow{0}, \overrightarrow{1})$ is the universal congruence in each $A \in \mathcal{V}$. Now apply Lemma 3,

Throughout this paper we will assume that we may find closed terms $\overrightarrow{0}$ and $\overrightarrow{1}$ for $\mathcal{V}$. Of course, this can be achieved when the language has a constant symbol and we will make this assumption in order to clarify our treatment. The proofs remain valid in the general case.

The next lemma ensures that we have a uniform way to witness connection.
Lemma 5. For every variety $\mathcal{V}$ of connected po-groupoids, there exist a positive integer $n$ and binary terms $m_{i}(x, y), i=1, \ldots, 2 n-1$ in the language of $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$ :

$$
\begin{array}{rlr}
m_{1}(x, y) \cdot x & \approx m_{1}(x, y) & \\
m_{i}(x, y) \cdot m_{i \pm 1}(x, y) & \approx m_{i}(x, y) \\
m_{2 n-1}(x, y) \cdot y & \approx m_{2 n-1}(x, y) . & \text { if } i<2 n-2 \text { is odd } \\
\end{array}
$$

Proof. It is enough to consider the $\mathcal{V}$-free algebra freely generated by $\{x, y\}$. The elements $m_{i}$ that connect $x$ and $y$ are binary terms, and the desired equations are equivalent to the assertions in (2).

For the rest of this section, $\mathcal{V}$ will denote a semidegenerate variety of po-groupoids with $k$ terms $U_{i}$ and $2 n-1$ terms $m_{i}$ as in the previous lemmas. In the following we define formulas $\psi, \phi$ and $\pi$ codifying the fact that connected po-groupoids have the SRP. They appear under the same names in Willard [11, Section 5], though the key improvement is a radical simplification of the first one.

A formula is called $\mathcal{V}$-factorable (factorable for short) if it belongs to the smallest set $\mathbf{F}$ containing every atomic formula that is closed under conjunction, existential and universal quantification, and the following rule:

$$
\begin{aligned}
& \text { if } \alpha(\vec{x}), \beta(\vec{x}, \vec{y}), \gamma(\vec{x}, \vec{y}) \in \mathbf{F} \text { and } \mathcal{V} \models \forall \vec{x}(\alpha(\vec{x}) \rightarrow \exists \vec{y} \beta(\vec{x}, \vec{y})) \text { then } \forall \vec{y}(\beta(\vec{x}, \vec{y}) \rightarrow \\
& \gamma(\vec{x}, \vec{y})) \in \mathbf{F} \text {. }
\end{aligned}
$$

Factorable formulas are preserved by direct factors and direct products (see [11, Section 1]).
Lemma 6. There exists a factorable $\Pi_{2}$ formula $\psi(x, y, z)$ such that:

1. $\mathcal{V} \models \psi(x, y, x)$.
2. $\mathcal{V} \models \psi(x, y, y)$.
3. $\mathcal{V} \models \psi(x, x, z) \rightarrow x \preceq z$.

Proof. Let $\psi(x, y, z)$ be the following formula:

$$
\begin{aligned}
& \forall u_{1}, \ldots, u_{2 n-1} \\
& \qquad u_{1} \cdot x=u_{1} \cdot u_{2} \wedge \bigwedge_{i=2}^{n-1} u_{2 i-1} \cdot u_{2 i-2}=u_{2 i-1} \cdot u_{2 i} \wedge u_{2 n-1} \cdot u_{2 n-2}=u_{2 n-1} \cdot y \rightarrow \\
& \rightarrow \exists v_{1}, \ldots, v_{n-1}: u_{1} \cdot x=u_{1} \cdot v_{1} \wedge \bigwedge_{i=2}^{n-1} u_{2 i-1} \cdot v_{i-1}=u_{2 i-1} \cdot v_{i} \wedge u_{2 n-1} \cdot v_{n-1}=u_{2 n-1} \cdot z
\end{aligned}
$$

We may see that $\psi$ is factorable observing that $u_{i}:=m_{i}(x, y)$ satisfies the antecedent for any choice of $x$ and $y$.

To prove 1. simply take $v_{i}:=x$ for all $i$. For 2, it suffices to assign $v_{i}:=u_{2 i}$ for all $i$. Finally, suppose $\psi(x, x, z)$ holds. Take $u_{i}:=x$. With this choice, the antecedent holds. The consequent turns to:

$$
x \cdot x=x \cdot v_{1} \wedge \bigwedge_{i=2}^{n-1} x \cdot v_{i-1}=x \cdot v_{i} \wedge x \cdot v_{n-1}=x \cdot z
$$

from which we conclude $x=x \cdot z$, and we have proved 3.

Lemma 7. There exists a factorable $\Pi_{3}$ formula $\pi(x, y, z, w)$ such that:

1. $\mathcal{V} \models \pi(x, x, z, w)$
2. $\mathcal{V} \models \pi(x, y, x, y)$
3. $\mathcal{V} \models \pi(x, y, z, z) \rightarrow x=y$

Proof. We first define $\phi\left(x, y, w_{1}, w_{2}\right)$ to be $\psi\left(x, w_{1}, y\right) \wedge \psi\left(y, w_{2}, x\right)$. Take $\pi(x, y, z, w)$ to be $\forall w_{1}, w_{2}: \phi\left(z, w, w_{1}, w_{2}\right) \rightarrow \phi\left(x, y, w_{1}, w_{2}\right)$.

It is immediate that 2 holds. Property holds thanks to Lemma (1). And we can check 3 by taking $w_{1}:=y$ and $w_{2}:=x$.

Finally, $\pi$ is factorable since $\exists w_{1}, w_{2}: \phi\left(z, w, w_{1}, w_{2}\right)$ holds in $\mathcal{V}$ due to Lemma (6(2).
The formula $\Phi$ appearing in the next lemma is a first-order definition of factor congruences in $\mathcal{V}$ using central elements. This concept (in its full generality) is due to Vaggione [8].

If $\vec{a} \in A^{l}$ and $\vec{b} \in B^{l}$, we will write $[\vec{a}, \vec{b}]$ in place of $\left(\left(a_{1}, b_{1}\right), \ldots,\left(a_{l}, b_{l}\right)\right) \in(A \times B)^{l}$. If $A \in \mathcal{V}$, we say that $\vec{e} \in A^{l}$ is a central element of $A$ if there exists an isomorphism $A \rightarrow A_{1} \times A_{2}$ such that

$$
\vec{e} \mapsto[\overrightarrow{0}, \overrightarrow{1}] .
$$

The set of all central elements of $A$ will be called the center of $A$. Two central elements $\vec{e}, \vec{f}$ will be called complementary if there exists an isomorphism $A \rightarrow A_{1} \times A_{2}$ such that $\vec{e} \mapsto[\overrightarrow{0}, \overrightarrow{1}]$ and $\vec{f} \mapsto[\overrightarrow{1}, \overrightarrow{0}]$. It is immediate that an algebra is directly indecomposable if and only if it has exactly two central elements, namely $\overrightarrow{0}$ and $\overrightarrow{1}$.
Lemma 8. There exists a factorable $\Sigma_{4}$ formula $\Phi(x, y, \vec{z})$ such that for all $A, B \in \mathcal{V}$, and $a, c \in A, b, d \in B$,

$$
A \times B \models \Phi(\langle a, b\rangle,\langle c, d\rangle,[\overrightarrow{0}, \overrightarrow{1}]) \quad \text { if and only if } \quad a=c .
$$

Proof. Take $\Phi(x, y, \vec{z})$ to be

$$
\begin{aligned}
& \exists a_{1}, \ldots, a_{n-1}: \pi\left(x, a_{1}, U_{1}(x, y, \vec{z}), U_{1}(x, y, \vec{w})\right) \wedge \bigwedge_{i \text { odd }} \pi\left(a_{i}, a_{i+1}, U_{i+1}(x, y, \vec{w}), U_{i+1}(x, y, \vec{z})\right) \wedge \\
& \wedge \bigwedge_{i \text { even }} \pi\left(a_{i}, a_{i+1}, U_{i+1}(x, y, \vec{z}), U_{i+1}(x, y, \vec{w})\right) \wedge \pi\left(a_{n-1}, y, U_{k}(x, y, \vec{z}), U_{k}(x, y, \vec{w})\right)
\end{aligned}
$$

Suppose that $A \times B \models \Phi(\langle a, b\rangle,\langle c, d\rangle,[\overrightarrow{0}, \overrightarrow{1}])$. Looking at the first coordinate, we obtain

$$
\begin{aligned}
& \pi\left(a, a_{1}^{1}, U_{1}(a, c, \overrightarrow{0}), U_{1}(a, c, \overrightarrow{0})\right) \wedge \bigwedge_{i \text { odd }} \pi\left(a_{i}^{1}, a_{i+1}^{1}, U_{i+1}(a, c, \overrightarrow{0}), U_{i+1}(a, c, \overrightarrow{0})\right) \wedge \\
& \wedge \bigwedge_{i \text { even }} \pi\left(a_{i}^{1}, a_{i+1}^{1}, U_{i+1}(a, c, \overrightarrow{0}), U_{i+1}(a, c, \overrightarrow{0})\right) \wedge \pi\left(a_{n-1}^{1}, c, U_{k}(a, c, \overrightarrow{0}), U_{k}(a, c, \overrightarrow{0})\right),
\end{aligned}
$$

where $a_{i}=\left\langle a_{i}^{1}, a_{i}^{2}\right\rangle$. Applying Lemma (7(3) we obtain $a=c$.
Finally, to show that

$$
A \times B \models \Phi(\langle a, b\rangle,\langle a, d\rangle,[\overrightarrow{0}, \overrightarrow{1}])
$$

it suffices to take $a_{i}:=\left\langle a, U_{i}(b, d, \mathbf{i})\right\rangle$, where $\mathbf{i}$ is $\overrightarrow{0}$ if $i$ is even, otherwise $\overrightarrow{1}$. This is routine.
Now we define the center of algebras in $\mathcal{V}$ in first-order logic.
Lemma 9. There is a $\Pi_{5}$ formula $\zeta(\vec{z}, \vec{w})$ such that for all $A \in \mathcal{V}$ and $\vec{e}, \vec{f} \in A^{l}$ we have that $\vec{e}$ and $\vec{f}$ are complementary central elements if and only if $A \models \zeta(\vec{e}, \vec{f})$.
Proof. The following formulas in the language of $\mathcal{V}$ will assert the properties needed to force $\Phi(\cdot, \cdot, \vec{z})$ and $\Phi(\cdot, \cdot, \vec{w})$ to define the pair of complementary factor congruences associated with $\vec{z}$ and $\vec{w}$. The names are almost self-explanatory.

- $\operatorname{CAN}(\vec{z}, \vec{w})=\bigwedge_{i=1}^{l} \Phi\left(0_{i}, z_{i}, \vec{z}\right) \wedge \bigwedge_{i=1}^{l} \Phi\left(1_{i}, w_{i}, \vec{z}\right)$
- $\operatorname{PROD}(\vec{z}, \vec{w})=\forall x, y \exists z(\Phi(x, z, \vec{z}) \wedge \Phi(z, y, \vec{w}))$
- $\operatorname{INT}(\vec{z}, \vec{w})=\forall x, y(\Phi(x, y, \vec{z}) \wedge \Phi(x, y, \vec{w}) \rightarrow x=y)$
- $\operatorname{REF}(\vec{z}, \vec{w})=\forall x \Phi(x, x, \vec{z})$
- $\operatorname{SYM}(\vec{z}, \vec{w})=\forall x, y, z(\Phi(x, y, \vec{z}) \wedge \Phi(y, z, \vec{z}) \wedge \Phi(z, x, \vec{w}) \rightarrow z=x)$
- $\operatorname{TRANS}(\vec{z}, \vec{w})=\forall x, y, z, u(\Phi(x, y, \vec{z}) \wedge \Phi(y, z, \vec{z}) \wedge \Phi(x, u, \vec{z}) \wedge \Phi(u, z, \vec{w}) \rightarrow u=z)$
- For each $m$-ary function symbol $F$ :

$$
\begin{aligned}
& \operatorname{PRES}_{F}(\vec{z}, \vec{w})=\forall u_{1}, v_{1}, \ldots, u_{m}, v_{m} \\
& \qquad \begin{array}{l}
\left(\bigwedge_{j} \Phi\left(u_{j}, v_{j}, \vec{z}\right)\right) \wedge \Phi\left(F\left(u_{1}, \ldots, u_{m}\right), z, \vec{z}\right) \wedge \Phi\left(z, F\left(v_{1}, \ldots, v_{m}\right), \vec{w}\right) \rightarrow \\
\\
\rightarrow z=F\left(v_{1}, \ldots, v_{m}\right)
\end{array}
\end{aligned}
$$

Now take $C A N^{\prime}, R E F^{\prime}, S Y M^{\prime}, T R A N S^{\prime}$ and $P R E S_{F}^{\prime}$ to be the result of interchanging $\vec{z}$ with $\vec{w}$ in $C A N, R E F, S Y M, T R A N S$ and $P R E S_{F}$, respectively, and let $\zeta$ be the conjunction of:

$$
\begin{gathered}
\bigwedge\left\{C A N, P R O D, I N T, R E F, S Y M, T R A N S, C A N^{\prime}, R E F^{\prime}, S Y M^{\prime}, T R A N S^{\prime}\right\} \\
\bigwedge\left\{P R E S_{F}, P R E S_{F}^{\prime}: F \text { a function symbol }\right\} .
\end{gathered}
$$

Details can be found in [7, Lemma 4.1].

Proof of Theorem 园. The formula

$$
\overrightarrow{0} \neq \overrightarrow{1} \wedge \forall \vec{e}, \vec{f}: \zeta(\vec{e}, \vec{f}) \rightarrow((\vec{e}=\overrightarrow{0} \wedge \vec{f}=\overrightarrow{1}) \varnothing(\vec{e}=\overrightarrow{1} \wedge \vec{f}=\overrightarrow{0}))
$$

together with axioms for $\mathcal{V}$ defines the subclass of directly indecomposables.

## 3 Examples

We first mention that we cannot eliminate the semidegeneracy hypothesis, since even the class of directly indecomposable lattices with 0 is not axiomatizable in first-order logic (see Willard [10]). Also, in [7, Section 6] it is shown that semidegeneracy by itself does not ensure definability of directly indecomposables. By considering in this last case a trivial (antichain) po-groupoid structure (for instance, defining $x \cdot y:=y$ for all $x, y$ ) we deduce that an arbitrary semidegenerate variety of po-groupoids may not have a first-order-axiomatizable class of indecomposables; hence we cannot drop connectedness.

We will now consider the following variety $\mathcal{R}$ :

$$
\begin{align*}
(x \cdot y) \cdot z & \approx x \cdot(y \cdot z) \\
x \cdot x & \approx x  \tag{4}\\
x \cdot y \cdot z & \approx y \cdot x \cdot z
\end{align*}
$$

This is obviously a variety of po-groupoids. Moreover, the variety of po-semigroups (defined by equations (1)) covers $\mathcal{R}$ in the lattice of equational classes of idempotent semigroups (see Gerhard [1]).

From the partial-order point of view, this groupoids are "relative meet-semilattices": whenever $A \in \mathcal{V}$, every bounded subalgebra of $A$ is a semilattice. Actually, the third axiom is equivalent to this property.

Lemma 10. Let $\mathcal{V}$ be a variety of associative po-groupoids. The following are equivalent:

1. $\mathcal{V} \models x \preceq z \wedge y \preceq z \rightarrow x \cdot y=y \cdot x$.
2. $\mathcal{V} \models x \cdot y \cdot z \approx y \cdot x \cdot z$.

Proof. We first need
Claim 11. Every associative po-groupoid satisfies $x \cdot y \cdot x \approx y \cdot x$.
Proof. We now that $x \cdot y \preceq y$. We obtain immediately that $x \cdot y \cdot x \preceq y \cdot x$ and $y \cdot x=$ $y \cdot x \cdot y \cdot x \preceq x \cdot y \cdot x$. By antisymmetry we get $x \cdot y \cdot x=y \cdot x$.

Assume (1). Note that $x \cdot y \cdot z \preceq z$ and $y \cdot x \cdot z \preceq z$, hence $x \cdot y \cdot z \cdot y \cdot x \cdot z=y \cdot x \cdot z \cdot x \cdot y \cdot z$. We may simplify this expression using the Claim to obtain (2).

Now suppose (2) holds, and assume $x, y \preceq z$. Hence

$$
x \cdot y=x \cdot y \cdot z=y \cdot x \cdot z=y \cdot x
$$

and we have (1).
Lemma 12. Let $\mathcal{V}$ will be a variety of connected po-groupoids that satisfies the defining identities of $\mathcal{R}$, and let $x_{1}, \ldots, x_{j} \in A \in \mathcal{V}$. Then $A \models \exists u: \bigwedge_{i=2}^{j} u \cdot x_{1}=u \cdot x_{i}$.

Proof. We only prove the case $j=2$, from which the rest can be easily derived. We will prove by induction on $n$ :

For all $m_{1}, \ldots, m_{2 n-1} \in A$ such that

$$
m_{1} \preceq m_{2} \succeq \ldots \preceq m_{2 n-2} \succeq m_{2 n-1}
$$

we have $m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1}=m_{2 n-1} \cdot \ldots \cdot m_{3} \cdot m_{1}$.
If $n=2$, we have $m_{1}, m_{3} \preceq m_{2}$, hence by Lemma $10 m_{1} \cdot m_{3}=m_{3} \cdot m_{1}$. Now suppose the assertion holds for $n$, and assume $m_{1}, \ldots, m_{2 n+1} \in A$ satisfy:

$$
m_{1} \preceq m_{2} \succeq \ldots \preceq m_{2 n-2} \succeq m_{2 n-1} \preceq m_{2 n} \succeq m_{2 n+1}
$$

We may apply the inductive hypothesis and obtain

$$
m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1} \cdot m_{2 n+1}=m_{1} \cdot m_{2 n+1} \cdot m_{2 n-1} \cdot \ldots \cdot m_{3}
$$

This last term equals $m_{2 n+1} \cdot m_{2 n-1} \cdot \ldots \cdot m_{1} \cdot m_{3}$ (by the third axiom of $\mathcal{R}$ ) and by the case $n=2$ we may commute $m_{1}$ and $m_{3}$, obtaining:

$$
\begin{equation*}
m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1} \cdot m_{2 n+1}=m_{2 n+1} \cdot m_{2 n-1} \cdot \ldots \cdot m_{3} \cdot m_{1} . \tag{5}
\end{equation*}
$$

Once we have this, we may take $u:=m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1}$, where

$$
x_{1} \succeq m_{1} \preceq m_{2} \succeq \ldots \preceq m_{2 n-2} \succeq m_{2 n-1} \preceq x_{2}
$$

since $u \cdot x_{2}=m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1} \cdot x_{2}=m_{1} \cdot m_{3} \cdot \ldots \cdot m_{2 n-1}=m_{2 n-1} \cdot \ldots \cdot m_{3} \cdot m_{1}=$ $m_{2 n-1} \cdot \ldots \cdot m_{3} \cdot m_{1} \cdot x_{1}=u \cdot x_{1}$

The previous lemmas were discovered using the Prover9/Mace 4 program bundle by W. McCune [5, 4].

Lemma 13. Let $\mathcal{V}$ will be a semidegenerate variety of connected po-groupoids that satisfies the defining identities of $\mathcal{R}$. There exists a factorable $\Pi_{1}$ formula $\Phi(x, y, \vec{z})$ such that for all $A, B \in \mathcal{V}$, and $a, c \in A, b, d \in B$,

$$
A \times B \models \Phi(\langle a, b\rangle,\langle c, d\rangle,[\overrightarrow{0}, \overrightarrow{1}]) \quad \text { if and only if } \quad a=c .
$$

Proof. To fix notation, we assume again there are $k$ terms $U_{i}$ and $2 n-1$ terms $m_{i}$ that witness semidegeneracy and connection, respectively (see Lemmas 4 and 5).

Take $\Phi(x, y, \vec{z})$ to be

$$
\forall u: \bigwedge_{i=1}^{k}\left(u \cdot U_{i}(x, y, \overrightarrow{0})=u \cdot U_{i}(x, y, \vec{z})\right) \longrightarrow u \cdot x=u \cdot y
$$

This formula is factorable by Lemma 12, Let $A, B \in \mathcal{V}$, and $a \in A, b, d \in B$. First we prove that

$$
A \times B \models \Phi(\langle a, b\rangle,\langle a, d\rangle,[\overrightarrow{0}, \overrightarrow{1}])
$$

Suppose that for some $\langle u, v\rangle$ we have

$$
A \times B \models \bigwedge_{i=1}^{k}\langle u, v\rangle \cdot U_{i}(\langle a, b\rangle,\langle a, d\rangle,[\overrightarrow{0}, \overrightarrow{0}])=\langle u, v\rangle \cdot U_{i}(\langle a, b\rangle,\langle a, d\rangle,[\overrightarrow{0}, \overrightarrow{1}])
$$

Then

$$
B \models \bigwedge_{i=1}^{k} v \cdot U_{i}(b, d, \overrightarrow{0})=v \cdot U_{i}(b, d, \overrightarrow{1}) .
$$

But the above equations in combination with (3) produce

$$
v \cdot b=v \cdot d
$$

and hence

$$
\langle u, v\rangle \cdot\langle a, b\rangle=\langle u, v\rangle \cdot\langle a, d\rangle .
$$

Now suppose

$$
A \times B \models \Phi(\langle a, b\rangle,\langle c, d\rangle,[\overrightarrow{0}, \overrightarrow{1}])
$$

Since $\Phi$ is preserved by direct factors, we obtain $A \models \Phi(a, c, \overrightarrow{0})$ and by inspection this is equivalent to $\forall u: u \cdot a=u \cdot c$. Using this equation for $u=a, c$ we obtain $a \preceq c$ and $c \preceq a$, therefore $a=c$.

Using this new definition of factor congruences, we obtain:
Theorem 14. Let $\mathcal{V}$ be a semidegenerate variety of connected po-groupoids over a finite language that satisfy (4). Then the class of directly indecomposable algebras of $\mathcal{V}$ is axiomatizable by a $\Pi_{4}$ sentence plus axioms for $\mathcal{V}$.

## References

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