

# SPECIAL ELEMENTS IN THE LATTICE OF OVERCOMMUTATIVE SEMIGROUP VARIETIES REVISITED

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ABSTRACT. We completely determine all distributive, codistributive, standard, costandard, and neutral elements in the lattice of overcommutative semigroup varieties, thus correcting a gap contained in [5].

## 1. INTRODUCTION

The class of all semigroup varieties forms a lattice under the following naturally defined operations: for varieties  $\mathcal{X}$  and  $\mathcal{Y}$ , their *join*  $\mathcal{X} \vee \mathcal{Y}$  is the variety generated by the set-theoretical union of  $\mathcal{X}$  and  $\mathcal{Y}$  (as classes of semigroups), while their *meet*  $\mathcal{X} \wedge \mathcal{Y}$  coincides with the class-theoretical intersection of  $\mathcal{X}$  and  $\mathcal{Y}$ . This lattice has been intensively studied for about four decades. A systematic overview of the material accumulated here is given in the recent survey [4].

It is a common knowledge that the lattice **SEM** of all semigroup varieties is divided into two large sublattices with essentially different properties: the coideal **OC** of all *overcommutative* varieties (that is, varieties containing the variety of all commutative semigroups) and the ideal of all *periodic* varieties (that is, varieties consisting of periodic semigroups).

The global structure of the lattice **OC** has been revealed by Volkov in [14]. It is proved there that this lattice decomposes into a subdirect product of its certain intervals and each of these intervals is anti-isomorphic to the congruence lattice of a certain unary algebra of a special type (namely, of a so-called  $G$ -set; a basic information about  $G$ -sets see in [3], for instance). The exact formulation of this result may be found also in [4, Theorem 5.1]. We do not reproduce this formulation here because we do not use it below.

There are several articles where special elements of different types in the lattice **SEM** have been examined (see [2, 6–13, 15]). We refer an interested reader to [4, Section 14] for an overview of the most part of results obtained in these articles.

Recall that an element  $x$  of a lattice  $\langle L; \vee, \wedge \rangle$  is called *distributive* if

$$\forall y, z \in L: \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z);$$

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*standard* if

$$\forall y, z \in L: \quad (x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z);$$

*neutral* if, for all  $y, z \in L$ , the sublattice of  $L$  generated by  $x$ ,  $y$ , and  $z$  is distributive. *Codistributive* [*costandard*] elements are defined dually to distributive [respectively standard] ones. An extensive information about elements of all these five types in abstract lattices may be found in [1, Section III.2], for instance. Note that any [co]standard element is [co]distributive, and an element is neutral if and only if it is standard and costandard simultaneously (see [1, Theorem III.2.5], for instance). On the other hand, a [co]distributive element may be not [co]standard, while a [co]standard element may be not neutral.

A complete description of neutral elements in the lattice **SEM** has been given in [15, Proposition 4.1] (see also [4, Theorem 14.2]). In [12], all distributive elements in **SEM** are completely determined. In [11], quite a strong necessary condition for semigroup varieties to be a codistributive element in **SEM** is obtained. In particular, all varieties with each of these three properties (except the trivial extreme case of the variety  $\mathcal{SEM}$  of all semigroups) turn out to be periodic varieties.

So, an examination of special elements of all the mentioned types in the lattice **SEM** gives no any information concerning the lattice **OC**. Aiming to obtain some new knowledge about this lattice, it is natural to investigate its special elements.

Such investigations have been started by the second author in [5]. Five types of special elements (namely, distributive, codistributive, standard, costandard, and neutral elements) in the lattice **OC** have been considered there. Unfortunately, it turns out that considerations in [5] contain a gap, and the main result of this article is incorrect. Namely, it was proved in [5] that, for an overcommutative semigroup variety, the properties of being a distributive element of **OC**, of being a codistributive element of **OC**, of being a standard element of **OC**, of being a costandard element of **OC**, and of being a neutral element of **OC** are equivalent. This result of [5] is true. But, besides that, the main result of [5] contains a list of all overcommutative varieties that possess the five mentioned properties. Unfortunately, this list turns out to be non-complete. All varieties from the list really have all the mentioned properties, but there are many other such varieties. The objective of this article is to give a correct description of distributive, codistributive, standard, costandard, and neutral elements in the lattice **OC**.

The article is structured as follows. In Section 2, we introduce a necessary notation and formulate the main result of the article (Theorem 2.2). In Section 3, we prove several auxiliary facts. Sections 4 and 5 are devoted to the proof of Theorem 2.2. In Section 6, we show that this theorem can not be improved, in a sense. Finally, in Section 7, we formulate some open problems.

## 2. PRELIMINARIES AND SUMMARY

We denote by  $F$  the free semigroup over a countably infinite alphabet  $\{x_1, x_2, \dots, x_n, \dots\}$ . As usual, elements of  $F$  are called *words*. By  $F^1$  we denote the semigroup  $F$  with the empty word ajoined. The symbol  $\equiv$  stands for the

equality relation on  $F$  and  $F^1$ . If  $u$  is a word, then  $\ell(u)$  denotes the length of  $u$ ,  $\ell_i(u)$  is the number of occurrences of the letter  $x_i$  in  $u$ ,  $c(u)$  stands for the set of all letters occurring in  $u$ , and  $n(u) = |c(u)|$  is the number of letters occurring in  $u$ . An identity  $u \approx v$  is called *balanced* if  $\ell_i(u) = \ell_i(v)$  for all  $i$ . It is a common knowledge that if an overcommutative variety satisfies some identity then this identity is balanced.

Let  $m$  and  $n$  be integers with  $2 \leq m \leq n$ . A *partition of the number  $n$  into  $m$  parts* is a sequence of positive integers  $(\ell_1, \ell_2, \dots, \ell_m)$  such that

$$\ell_1 \geq \ell_2 \geq \dots \geq \ell_m \quad \text{and} \quad \sum_{i=1}^m \ell_i = n.$$

The numbers  $\ell_1, \ell_2, \dots, \ell_m$  are called *components* of the partition  $\lambda$ . We denote by  $\Lambda_{n,m}$  the set of all partitions of the number  $n$  into  $m$  parts and by  $\Lambda$  the union of the sets  $\Lambda_{n,m}$  for all natural numbers  $m$  and  $n$  with  $2 \leq m \leq n$ . If  $\lambda \in \Lambda_{n,m}$  then we denote the numbers  $n$  and  $m$  by  $n(\lambda)$  and  $m(\lambda)$  respectively.

If  $u$  is a word then we denote by  $\text{part}(u)$  the partition of the number  $\ell(u)$  into  $n(u)$  parts consisting of integers  $\ell_i(u)$  for all  $i$  such that  $x_i \in c(u)$  (the numbers  $\ell_i(u)$  are placed in  $\text{part}(u)$  in non-increasing order). If  $u \approx v$  is a balanced identity then, obviously,  $\ell(u) = \ell(v)$ ,  $n(u) = n(v)$ , and  $\text{part}(u) = \text{part}(v)$ . We call the partition  $\text{part}(u)$  a *partition* of the identity  $u \approx v$ . We denote the numbers  $\ell(u) = \ell(v)$  and  $n(u) = n(v)$  by  $\ell(u \approx v)$  and  $n(u \approx v)$  respectively, and the partition  $\text{part}(u) = \text{part}(v)$  by  $\text{part}(u \approx v)$ .

Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$ . We denote by  $W_{n,m,\lambda}$ , or simply  $W_\lambda$ , the set of all words  $u$  such that  $\ell(u) = n$ ,  $c(u) = \{x_1, x_2, \dots, x_m\}$ ,  $\ell_i(u) \geq \ell_{i+1}(u)$  for all  $i = 1, 2, \dots, m-1$ , and  $\text{part}(u) = \lambda$ . It is evident that every balanced identity  $u \approx v$  with  $\ell(u \approx v) = n$ ,  $n(u \approx v) = m$ , and  $\text{part}(u \approx v) = \lambda$  is equivalent to some identity  $s \approx t$  where  $s, t \in W_{n,m,\lambda}$ .

We call sets of the kind  $W_{n,m,\lambda}$  *transversals*. We say that an overcommutative variety  $\mathcal{V}$  *reduces* [collapses] a transversal  $W_{n,m,\lambda}$  if  $\mathcal{V}$  satisfies some non-trivial identity [all identities] of the kind  $u \approx v$  with  $u, v \in W_{n,m,\lambda}$ . An overcommutative variety  $\mathcal{V}$  is said to be *greedy* if it collapses any transversal it reduces. The following assertion has been proved in [5].

**Proposition 2.1.** *An overcommutative semigroup variety is a distributive [co-distributive, standard, costandard, neutral] element of the lattice  $\mathbf{OC}$  if and only if it is greedy.*  $\square$

This assertion was not formulated in [5] explicitly but it directly follows from the proof of Theorem 2 in [5] (and the corresponding part of the proof in [5] is correct).

It is an appropriate place here to indicate the error made in [5]. Let  $m$  and  $n$  be positive integers with  $2 \leq m \leq n$  and  $\lambda \in \Lambda_{n,m}$ . A semigroup variety given

by an identity system  $\Sigma$  is denoted by  $\text{var } \Sigma$ . We put

$$\begin{aligned}\mathcal{X}_n &= \text{var} \{u \approx v \mid \text{the identity } u \approx v \text{ is balanced and } \ell(u \approx v) \geq n\}, \\ \mathcal{X}_{n,m} &= \mathcal{X}_{n+1} \wedge \text{var} \{u \approx v \mid \text{the identity } u \approx v \text{ is balanced, } \ell(u \approx v) = n, \\ &\quad \text{and } n(u \approx v) \leq m\}, \\ \mathcal{X}_{n,1} &= \mathcal{X}_{n+1}, \\ \mathcal{X}_{n,m,\lambda} &= \mathcal{X}_{n,m-1} \wedge \text{var} \{u \approx v \mid u, v \in W_{n,m,\lambda}\}.\end{aligned}$$

It is claimed in [5] without any proof that an overcommutative variety is greedy if and only if it coincides with one of the varieties  $\mathcal{SEM}$ ,  $\mathcal{X}_n$ ,  $\mathcal{X}_{n,m}$  or  $\mathcal{X}_{n,m,\lambda}$ . Combining this claim with Proposition 2.1, we obtain the main result of [5]: the varieties  $\mathcal{SEM}$ ,  $\mathcal{X}_n$ ,  $\mathcal{X}_{n,m}$ ,  $\mathcal{X}_{n,m,\lambda}$ , and only they are [co]distributive, [co]standard, and neutral elements in **OC**. In actual fact, it is true that all these varieties are elements of the mentioned types in **OC**. But the list of [co]distributive, [co]standard, and neutral elements in **OC** is not exhausted by the varieties  $\mathcal{SEM}$ ,  $\mathcal{X}_n$ ,  $\mathcal{X}_{n,m}$ , and  $\mathcal{X}_{n,m,\lambda}$ . There are many other varieties with such a property. Exactly this fact has been so unfortunately overseen in [5].

For a partition  $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$ , we define numbers  $q(\lambda)$ ,  $r(\lambda)$ , and  $s(\lambda)$  by the following way:

$$\begin{aligned}q(\lambda) &\text{ is the number of } \ell_i\text{'s with } \ell_i = 1 \text{ (if } \ell_m > 1 \text{ then } q(\lambda) = 0); \\ r(\lambda) &\text{ is the sum of all } \ell_i\text{'s with } \ell_i > 1 \text{ (if } \ell_1 = 1 \text{ then } r(\lambda) = 0); \\ s(\lambda) &= \max \{r(\lambda) - q(\lambda) - \delta, 0\}\end{aligned}$$

where

$$\delta = \begin{cases} 0 & \text{whenever } n = 3, m = 2, \text{ and } \lambda = (2, 1), \\ 1 & \text{otherwise.} \end{cases}$$

If  $k$  is a non-negative integer then  $\lambda^k$  stands for the following partition of  $n + k$  into  $m + k$  parts:

$$\lambda^k = (\ell_1, \ell_2, \dots, \ell_m, \underbrace{1, \dots, 1}_{k \text{ times}})$$

(in particular,  $\lambda^0 = \lambda$ ).

For a partition  $\lambda \in \Lambda_{n,m}$ , we put

$$\mathcal{W}_{n,m,\lambda} = \text{var} \{u \approx v \mid u, v \in W_{n,m,\lambda}\} \quad \text{and} \quad \mathcal{S}_\lambda = \bigwedge_{i=0}^{s(\lambda)} \mathcal{W}_{n+i,m+i,\lambda^i}.$$

Sometimes we will write  $\mathcal{W}_\lambda$  rather than  $\mathcal{W}_{n,m,\lambda}$ .

The main result of the article is the following

**Theorem 2.2.** *For an overcommutative semigroup variety  $\mathcal{V}$ , the following are equivalent:*

- (i)  $\mathcal{V}$  is a distributive element of the lattice **OC**;
- (ii)  $\mathcal{V}$  is a codistributive element of the lattice **OC**;
- (iii)  $\mathcal{V}$  is a standard element of the lattice **OC**;
- (iv)  $\mathcal{V}$  is a costandard element of the lattice **OC**;
- (v)  $\mathcal{V}$  is a neutral element of the lattice **OC**;

(vi) either  $\mathcal{V} = \mathcal{SEM}$  or  $\mathcal{V} = \bigwedge_{i=1}^k \mathcal{S}_{\lambda_i}$  for some partitions  $\lambda_1, \lambda_2, \dots, \lambda_k \in \Lambda$ .

The following claim was formulated in [5] as a corollary of the main result of that article. Theorem 2.2 shows that the claim is correct.

**Corollary 2.3.** *The set of all [co]distributive elements of the lattice **OC** is countably infinite.*  $\square$

This corollary is of some interest because the set of all overcommutative semigroup varieties is well known to be uncountably infinite. On the other hand, it is interesting to note that the set of all neutral elements in the lattice **OC** is infinite, while the set of all neutral elements in the lattice **SEM** consists of 5 varieties only [15, Proposition 4.1].

In view of Proposition 2.1, Theorem 2.2 is equivalent to the following

**Proposition 2.4.** *An overcommutative semigroup variety  $\mathcal{V}$  satisfies the condition (vi) of Theorem 2.2 if and only if it is greedy.*

It is this claim that will be verified in Sections 4 and 5 (in fact, we prove the ‘only if’ and ‘if’ parts of Proposition 2.4 in Sections 4 and 5 respectively). To prepare this proof, we introduce some order relation on the set  $\Lambda$  and consider some properties of this relation in Section 3.

### 3. AN ORDER RELATION ON THE SET $\Lambda$

Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$  where  $m \geq 3$  and  $1 \leq i < j \leq m$ . We denote by  $U_{i,j}(\lambda)$  the partition of the number  $n$  into  $m-1$  parts with the components  $\ell_1, \ell_2, \dots, \ell_{i-1}, \ell_{i+1}, \dots, \ell_{j-1}, \ell_{j+1}, \dots, \ell_m$ , and  $\ell_i + \ell_j$  (these components are written in  $U_{i,j}(\lambda)$  in non-increasing order). We will say that the partition  $U_{i,j}(\lambda)$  is obtained from  $\lambda$  by the *union of components*  $\ell_i$  and  $\ell_j$ . The partition obtained from  $\lambda$  by a finite (may be empty) set  $S$  of unions of components is denoted by  $U_S(\lambda)$ ; in particular,  $U_\emptyset(\lambda) = \lambda$ .

We introduce a binary relation  $\preceq$  on the set  $\Lambda$  by the following rule:

$$\lambda \preceq \mu \text{ if and only if } \mu = U_S(\lambda^k) \text{ for some } S \text{ and } k.$$

The principal property of the relation  $\preceq$  is given by the following

**Lemma 3.1.** *The relation  $\preceq$  is a partial order on the set  $\Lambda$ .*

*Proof.* Reflexivity of  $\preceq$  is evident because  $\lambda = U_\emptyset(\lambda^0)$ . The claim that  $\preceq$  is transitive also is evident because if  $\mu = U_S(\lambda^k)$  and  $\nu = U_T(\mu^\ell)$  then  $\nu = U_{S \cup T}(\lambda^{k+\ell})$ . To prove that  $\preceq$  is antisymmetric, we suppose that  $\lambda \preceq \mu$  and  $\mu \preceq \lambda$  for some  $\lambda, \mu \in \Lambda$ . Then  $\mu = U_S(\lambda^k)$  and  $\lambda = U_T(\mu^\ell)$  for some  $S, T, k$ , and  $\ell$ . Let  $n(\lambda) = n$  and  $n(\mu) = q$ . Then  $q = n + k$  and  $n = q + \ell$ . Therefore,  $q = q + k + \ell$ , whence  $k + \ell = 0$ . This means that  $k = \ell = 0$ . Thus,  $\mu = U_S(\lambda)$  and  $\lambda = U_T(\mu)$ . If  $S \neq \emptyset$  then  $r < m$ . But  $m \leq r$  because  $\lambda$  is obtained from  $\mu$  by unions of components. Therefore,  $S = \emptyset$  and  $\mu = U_\emptyset(\lambda) = \lambda$ .  $\square$

Now we are going to show that the partial order  $\preceq$  has some nice properties that will be played the crucial role in Section 5. The first such property is given by the following

**Lemma 3.2.** *The partially ordered set  $\langle \Lambda; \preceq \rangle$  satisfies the descending chain condition.*

*Proof.* Let  $\lambda, \mu \in \Lambda$  and  $\lambda \preceq \mu$ . Put  $n(\lambda) = n$  and  $n(\mu) = q$ . Then  $q \leq n$ . Evidently, the set

$$\bigcup_{\substack{q \leq n \\ 2 \leq r \leq q}} \Lambda_{q,r}$$

is finite. Thus, there exists finitely many partitions  $\mu$  with  $\mu \preceq \lambda$  only. This immediately implies the desirable conclusion.  $\square$

We define one more binary relation  $\trianglelefteq$  on the set  $\Lambda$  by the following rule. Let  $\lambda, \nu \in \Lambda$ ,  $\lambda = (\ell_1, \ell_2, \dots, \ell_m)$ , and  $\nu = (n_1, n_2, \dots, n_k)$ . Then  $\lambda \trianglelefteq \nu$  if and only if  $m \leq k$  and  $\ell_i \leq n_i$  for all  $i = 1, 2, \dots, m$ . It is evident that  $\trianglelefteq$  is a partial order on  $\Lambda$ . The following claim shows a relationship between orders  $\preceq$  and  $\trianglelefteq$ .

**Lemma 3.3.** *Let  $\lambda, \nu \in \Lambda$ . If  $\lambda \trianglelefteq \nu$  then  $\lambda \preceq \nu$ .*

*Proof.* Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m)$  and  $\nu = (n_1, n_2, \dots, n_k)$ . Then  $m \leq k$  and  $\ell_i \leq n_i$  for all  $i = 1, 2, \dots, m$ . Put  $s = n(\nu) - n(\lambda)$ . It is evident that  $s \geq 0$ . If  $s = 0$  then  $\lambda = \nu$  and we are done. Let now  $s > 0$ . By the trivial induction, it suffices to consider the case  $s = 1$ . Then either  $k = m + 1$ ,  $\ell_i = n_i$  for all  $i = 1, 2, \dots, m$ , and  $n_k = 1$  or  $k = m$ ,  $n_i = \ell_i + 1$  for some  $i \in \{1, 2, \dots, m\}$  and  $n_j = \ell_j$  for all  $j \neq i$ . It is evident that  $\nu = U_{\emptyset}(\lambda^1)$  in the former case, while  $\nu = U_{i, m+1}(\lambda^1)$  in the latter one. Thus,  $\lambda \preceq \nu$  in any case.  $\square$

The second important property of the relation  $\preceq$  is given by the following

**Lemma 3.4.** *The partially ordered set  $\langle \Lambda; \preceq \rangle$  does not contain infinite anti-chains.*

*Proof.* Arguing by contradiction, suppose that  $\Lambda$  contains an infinite anti-chain  $A_0$ . Put  $m_1 = \min \{m(\lambda) \mid \lambda \in A_0\}$ . Let us fix a partition  $\lambda_1 = (\ell_1^1, \ell_2^1, \dots, \ell_{m_1}^1) \in A_0$ . If  $\nu = (n_1, n_2, \dots, n_k)$  is an arbitrary partition from  $A_0$  then  $\lambda_1 \not\preceq \nu$ , whence  $\lambda_1 \not\trianglelefteq \nu$  by Lemma 3.3. Since  $m_1 \leq k$ , this means that  $\ell_i^1 > n_i$  for some  $i \in \{1, 2, \dots, m_1\}$ . The set  $A_0$  is infinite, while the index  $i$  runs over the finite set  $\{1, 2, \dots, m_1\}$ . Hence there is an index  $i_1 \leq m_1$  such that  $n_{i_1} < \ell_{i_1}^1$  for an infinite set of partitions  $A_1 \subseteq A_0$ . Put  $j_1 = \ell_{i_1}^1$ .

Put  $m_2 = \min \{m(\lambda) \mid \lambda \in A_1\}$ . Let us fix a partition  $\lambda_2 = (\ell_1^2, \ell_2^2, \dots, \ell_{m_2}^2) \in A_1$ . The same arguments as in the previous paragraph show that there is a number  $i_2 \leq m_2$  and an infinite set  $A_2 \subseteq A_1$  such that  $n_{i_2} < \ell_{i_2}^2$  for every  $\nu = (n_1, n_2, \dots, n_k) \in A_2$ . Put  $j_2 = \ell_{i_2}^2$ .

Continuing this process, we construct a sequence of infinite sets of partitions  $A_0 \supseteq A_1 \supseteq A_2 \supseteq \dots$ , a sequence of partitions  $\{\lambda_s = (\ell_1^s, \ell_2^s, \dots, \ell_{m_s}^s) \mid s \in \mathbb{N}\}$ , and two sequences of numbers  $\{i_s \mid s \in \mathbb{N}\}$  and  $\{j_s \mid s \in \mathbb{N}\}$  such that, for any  $s \in \mathbb{N}$ , the following holds:  $\lambda_s \in A_{s-1}$ ,  $i_s \leq m_s$ ,  $j_s = \ell_{i_s}^s$ , and  $n_{i_s} < j_s$  for any  $\nu = (n_1, n_2, \dots, n_k) \in A_s$ . The choice of the partitions  $\lambda_1, \lambda_2, \dots$  guarantees that if  $p > q$  then

$$\ell_{i_q}^p < \ell_{i_q}^q = j_q.$$

In particular, if  $p > q$  and  $i_p = i_q$  then

$$j_p = \ell_{i_p}^p = \ell_{i_q}^p < j_q.$$

This means that all pairs of the kind  $(i_s, j_s)$  are different. Furthermore, if  $i_p \geq i_q$  then  $\ell_{i_p}^p \leq \ell_{i_q}^p$  because all partitions  $\lambda_1, \lambda_2, \dots$  are non-increasing sequences of numbers. Therefore, if  $p > q$  and  $i_p \geq i_q$  then

$$j_p = \ell_{i_p}^p \leq \ell_{i_q}^p < j_q.$$

Put  $i_r = \min \{i_s \mid s \in \mathbb{N}\}$ ,  $j_t = \min \{j_s \mid s \in \mathbb{N}\}$ , and  $h = \max \{r, t\}$ . If  $s > h$  then  $i_s \geq i_r$  and  $j_s \geq j_t$ , whence  $j_s < j_r$  and  $i_s < i_t$ . We see that both the sequences  $\{i_s \mid s \in \mathbb{N}\}$  and  $\{j_s \mid s \in \mathbb{N}\}$  are bounded. But this is impossible because all pairs of the kind  $(i_s, j_s)$  are different. The contradiction completes the proof.  $\square$

#### 4. PROOF OF PROPOSITION 2.4: NECESSITY

Here we aim to verify that if an overcommutative variety satisfies the condition (vi) of Theorem 2.2 then it is greedy. We start with some new notation and several auxiliary facts.

For arbitrary words  $w_1, w_2$  and an identity system  $\Sigma$ , we write  $w_1 \xrightarrow{\Sigma} w_2$  if there exist  $a, b \in F^1$ ,  $s, t \in F$ , and an endomorphism  $\zeta$  on  $F$  such that  $w_1 \equiv a\zeta(s)b$ ,  $w_2 \equiv a\zeta(t)b$ , and the identity  $s \approx t$  belongs to  $\Sigma$ . It is a common knowledge that an identity  $u \approx v$  follows from a system  $\Sigma$  if and only if there exists a sequence of words  $w_0, w_1, \dots, w_\ell$  such that

$$(4.1) \quad u \equiv w_0 \xrightarrow{\Sigma} w_1 \xrightarrow{\Sigma} \dots \xrightarrow{\Sigma} w_\ell \equiv v.$$

This sequence is called a *deduction* of the identity  $u \approx v$  from  $\Sigma$ . Note that if  $\Sigma$  consists of balanced identities then  $\ell(w_i) = \ell(u \approx v)$ ,  $n(w_i) = n(u \approx v)$ , and  $\text{part}(w_i) = \text{part}(u \approx v)$  for all  $i = 0, 1, \dots, \ell$ .

**Lemma 4.1.** *Let  $u$  be a word and  $\xi$  an endomorphism on  $F$  such that  $\ell(\xi(u)) = \ell(u)$ . Then  $\text{part}(u) \preceq \text{part}(\xi(u))$ .*

*Proof.* Put  $\lambda = \text{part}(u)$ . It is clear that  $\xi(x)$  is a letter for every letter  $x$ . The requirement conclusion follows from the following evident observation:  $\text{part}(\xi(u)) = U_S(\lambda^0)$  where  $S$  is a finite (may be empty) set of unions of components of  $\lambda$  corresponding to letters from  $c(u)$  with the same image under  $\xi$ .  $\square$

**Lemma 4.2.** *Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$ . If a non-trivial identity  $u \approx v$  holds in the variety  $S_\lambda$  then  $\lambda \preceq \text{part}(u \approx v)$ .*

*Proof.* We put

$$\Sigma = \{f \approx g \mid \text{there is } i \in \{0, 1, \dots, s(\lambda)\} \text{ with } f, g \in W_{n+i, m+i, \lambda^i}\}.$$

Thus,  $S_\lambda = \text{var } \Sigma$ . Let (4.1) be a deduction of the identity  $u \approx v$  from  $\Sigma$ . Note that  $\ell \geq 1$  because the identity  $u \approx v$  is non-trivial. We have  $w_0 \equiv a\zeta(s)b$  and  $w_1 \equiv a\zeta(t)b$  for some homomorphism  $\zeta$  on  $F$ , some  $a, b \in F^1$ , and some  $s, t \in W_{n+i, m+i, \lambda^i}$  where  $i \in \{0, 1, \dots, s(\lambda)\}$ .

If  $\ell(a\zeta(s)b) = \ell(s)$  then the words  $a$  and  $b$  are empty and  $\ell(\zeta(s)) = \ell(s)$ . Here we may apply Lemma 4.1 and conclude that

$$\begin{aligned}\lambda \preceq U_{\emptyset}(\lambda^i) &= \lambda^i = \text{part}(s) \preceq \text{part}(\zeta(s)) = \\ &= \text{part}(a\zeta(s)b) = \text{part}(w_0) = \text{part}(u \approx v).\end{aligned}$$

Therefore,  $\lambda \preceq \text{part}(u \approx v)$ , and we are done.

Suppose now that  $\ell(a\zeta(s)b) > \ell(s)$ . For each  $j = 1, 2, \dots, m+i$ , we denote by  $y_j$  the first letter of the word  $\zeta(x_j)$ . Thus,  $\zeta(x_j) \equiv y_j u_j$  for some  $u_j \in F^1$ . We have

$$\begin{aligned}\text{part}(a\zeta(s)b) &= \text{part}(a(\zeta(x_1))^{\ell_1} \cdots (\zeta(x_m))^{\ell_m} \zeta(x_{m+1}) \cdots \zeta(x_{m+i})b) = \\ &= \text{part}(a(y_1 u_1)^{\ell_1} \cdots (y_m u_m)^{\ell_m} y_{m+1} u_{m+1} \cdots y_{m+i} u_{m+i} b) = \\ &= \text{part}(y_1^{\ell_1} \cdots y_m^{\ell_m} y_{m+1} \cdots y_{m+i} u_1^{\ell_1} \cdots u_m^{\ell_m} u_{m+1} \cdots u_{m+i} ab).\end{aligned}$$

Let  $k = \ell(a\zeta(s)b) - \ell(s)$ . Put

$$\begin{aligned}c_1 &\equiv y_1^{\ell_1} \cdots y_m^{\ell_m} y_{m+1} \cdots y_{m+i}; \\ c_2 &\equiv u_1^{\ell_1} u_2^{\ell_2} \cdots u_m^{\ell_m} u_{m+1} \cdots u_{m+i} ab; \\ c &\equiv c_1 c_2 \equiv y_1^{\ell_1} \cdots y_m^{\ell_m} y_{m+1} \cdots y_{m+i} u_1^{\ell_1} \cdots u_m^{\ell_m} u_{m+1} \cdots u_{m+i} ab; \\ d &\equiv x_1^{\ell_1} \cdots x_m^{\ell_m} x_{m+1} \cdots x_{m+i+k}.\end{aligned}$$

Since  $\text{part}(c) = \text{part}(a\zeta(s)b)$  and  $\text{part}(c_1) = \text{part}(s)$ , we have  $\ell(c) = \ell(a\zeta(s)b)$  and  $\ell(c_1) = \ell(s)$ . Besides that,  $\ell(c) = \ell(c_1) + \ell(c_2)$ . Therefore,

$$\ell(c_2) = \ell(c) - \ell(c_1) = \ell(a\zeta(s)b) - \ell(s) = k.$$

It is convenient for us to rewrite the word  $c_2$  in the form  $c_2 \equiv z_1 z_2 \dots z_k$  where  $z_1, z_2, \dots, z_k$  are (not necessarily different) letters. Let  $\xi$  be an endomorphism on  $F$  such that

$$\xi(x_j) \equiv \begin{cases} y_j & \text{whenever } 1 \leq j \leq m+i, \\ z_{j-m-i} & \text{whenever } m+i+1 \leq j \leq m+i+k. \end{cases}$$

Then  $c \equiv \xi(d)$ . It is clear that

$$\ell(d) = \ell(c_1) + k = \ell(s) + \ell(a\zeta(s)b) - \ell(s) = \ell(a\zeta(s)b) = \ell(c) = \ell(\xi(d)).$$

Now we may apply Lemma 4.1 and conclude that

$$\begin{aligned}\lambda \preceq U_{\emptyset}(\lambda^k) &= \lambda^k = \text{part}(d) \preceq \text{part}(\xi(d)) = \text{part}(c) = \\ &= \text{part}(a\zeta(s)b) = \text{part}(w_0) = \text{part}(u \approx v).\end{aligned}$$

Therefore,  $\lambda \preceq \text{part}(u \approx v)$ , and we are done.  $\square$

**Lemma 4.3.** *Let  $\lambda, \mu \in \Lambda$ . If  $\mu = U_S(\lambda)$  for some finite set  $S$  of unions of components then  $\mathcal{W}_\lambda \subseteq \mathcal{W}_\mu$ .*

*Proof.* Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m) \in \Lambda_{n,m}$ . By the trivial induction, it suffices to consider the case when  $\mu = U_{i,j}(\lambda)$  for some  $i$  and  $j$ . We have to verify that if  $u, v \in W_{n,m-1,\mu}$  then the identity  $u \approx v$  holds in  $\mathcal{W}_\lambda$ . Since  $\text{part}(u \approx v) = U_{i,j}(\lambda)$ , there is a letter  $x_k$  with  $\ell_k(u) = \ell_k(v) = \ell_i + \ell_j$ . Let  $x_p$  and  $x_q$  be some letters such that  $x_p, x_q \notin c(u)$ . One can change the first  $\ell_i$  occurrences of  $x_k$  in



$u$  and in  $v$  by  $x_p$ , while the last  $\ell_j$  occurrences of  $x_k$  in  $u$  and in  $v$  by  $x_q$ . We obtain some identity  $s \approx t$  with  $\text{part}(s \approx t) = \lambda$ . Hence the variety  $\mathcal{W}_{n,m,\lambda}$  satisfies  $s \approx t$ . If we substitute  $x_k$  for  $x_p$  and  $x_q$  in  $s \approx t$  then we return to the identity  $u \approx v$ . Therefore,  $u \approx v$  follows from  $s \approx t$ , whence  $u \approx v$  holds in  $\mathcal{W}_{n,m,\lambda}$ .  $\square$

Recall that a letter  $x_i$  is called *simple in a word*  $u$  if  $\ell_i(u) = 1$ .

**Lemma 4.4.** *If  $\lambda \in \Lambda_{n,m}$  and  $s(\lambda) = 0$  then  $\mathcal{W}_{n,m,\lambda} \subseteq \mathcal{W}_{n+k,m+k,\lambda^k}$  for any positive integer  $k$ .*

*Proof.* The definition of the number  $s(\lambda)$  immediately implies that if  $s(\lambda) = 0$  then  $s(\lambda^k) = 0$  as well for any  $k > 0$ . This observation implies that, by the trivial induction, it suffices to consider the case  $k = 1$ .

First of all, we note that  $\lambda \neq (2, 1)$  because  $s(\lambda) = 1$  otherwise. For brevity, put  $q = q(\lambda)$ ,  $r = r(\lambda)$ ,  $s = s(\lambda)$ , and  $t = m - q$ . Since  $\lambda \neq (2, 1)$ , we have  $\delta = 1$  and therefore,  $s = \max\{r - q - 1, 0\}$ . The equality  $s = 0$  implies now that  $r - q - 1 \leq 0$ , that is

$$(4.2) \quad r \leq q + 1.$$

Suppose that  $q \leq 1$ . Then  $r \leq 2$ . Let  $\lambda = (\ell_1, \ell_2, \dots, \ell_m)$ . The definition of the number  $r(\lambda)$  and the inequality  $r \leq 2$  imply that either  $\ell_1 = \ell_2 = \dots = \ell_m = 1$  or  $\ell_1 = 2$  and  $\ell_2 = \dots = \ell_m = 1$ . Since  $\lambda \neq (2, 1)$ , this implies that  $q \geq 2$ . We have a contradiction with the inequality  $q \leq 1$ . Therefore,  $q \geq 2$ . This implies that every word from the transversal  $W_{n+1,m+1,\lambda^1}$  has at least three simple letters.

We need to verify that any identity of the kind  $u \approx v$  with  $u, v \in W_{n+1,m+1,\lambda^1}$  holds in  $W_{n,m,\lambda}$ . It suffices to check that if  $u \in W_{n+1,m+1,\lambda^1}$  then the variety  $W_{n,m,\lambda}$  satisfies the identity

$$(4.3) \quad u \approx x_1^{\ell_1} \cdots x_t^{\ell_t} x_{t+1} \cdots x_{m+1}.$$

At the rest part of the proof of this lemma, the words ‘a simple letter’ mean ‘a simple in  $u$  letter’. One can note that one of the following three claims hold:

- 1) the word  $u$  ends with a simple letter;
- 2) the word  $u$  starts with a simple letter;
- 3) the word  $u$  contains a subword of the kind  $x_i x_j$  where  $x_i$  and  $x_j$  are simple letters.

Indeed, if all these three claims fail then

$$u \equiv w_1 y_1 w_2 y_2 \cdots w_{q+1} y_{q+1} w_{q+2}$$

where  $y_1, y_2, \dots, y_{q+1}$  are simple letters, while  $w_1, w_2, \dots, w_{q+2}$  are non-empty words such that the word  $w \equiv w_1 w_2 \cdots w_{q+2}$  does not contain simple letters.

Then  $r = \ell(w) = \sum_{i=1}^{q+2} \ell(w_i) \geq q + 2$ , contradicting the inequality (4.2).

Now we consider three cases corresponding to the claims 1)–3).

*Case 1:*  $u \equiv w x_i$  for some word  $w$  and some simple letter  $x_i$ . The identity

$$(4.4) \quad w \approx x_1^{\ell_1} \cdots x_t^{\ell_t} x_{t+1} \cdots x_{i-1} x_{i+1} \cdots x_{m+1}$$

has the partition  $\lambda$ , whence it holds in  $\mathcal{W}_{n,m,\lambda}$ . Multiplying (4.4) by  $x_i$  from the right, we have the identity

$$(4.5) \quad u \approx x_1^{\ell_1} \cdots x_t^{\ell_t} x_{t+1} \cdots x_{i-1} x_{i+1} \cdots x_{m+1} x_i$$

that also holds in  $\mathcal{W}_{n,m,\lambda}$ . If  $i = m + 1$  then (4.5) coincides with (4.3) and we are done. Let now  $i \leq m$ . Put

$$j = \begin{cases} m - 1 & \text{whenever } i = m, \\ m & \text{otherwise.} \end{cases}$$

Since  $u$  contains at least three simple letters, the letter  $x_j$  is simple. The identity (4.5) has the form

$$(4.6) \quad u \approx a x_j x_{m+1} x_i$$

for some  $a \in F^1$ . Let  $x_p$  be a letter with  $x_p \notin c(u)$ . The identity

$$(4.7) \quad a x_p x_i \approx a x_i x_p$$

has the partition  $\lambda$ , whence it holds in  $\mathcal{W}_{n,m,\lambda}$ . Substituting  $x_j x_{m+1}$  for  $x_p$  in (4.7), we obtain the identity

$$(4.8) \quad a x_j x_{m+1} x_i \approx a x_i x_j x_{m+1}$$

that holds in  $\mathcal{W}_{n,m,\lambda}$ . The identity

$$(4.9) \quad a x_i x_j \approx x_1^{\ell_1} \cdots x_t^{\ell_t} x_{t+1} \cdots x_m$$

has the partition  $\lambda$ , whence it holds in  $\mathcal{W}_{n,m,\lambda}$  too. Multiplying (4.9) on  $x_{m+1}$  from the right, we obtain the identity

$$(4.10) \quad a x_i x_j x_{m+1} \approx x_1^{\ell_1} \cdots x_t^{\ell_t} x_{t+1} \cdots x_{m+1}$$

that holds in  $\mathcal{W}_{n,m,\lambda}$  as well. Combining the identities (4.6), (4.8), and (4.10), we obtain the identity (4.3).

*Case 2:*  $u \equiv x_i w$  for some simple letter  $x_i$  and some word  $w$ . The word  $u$  contains at least three simple letters. Therefore, there is a simple letter  $x_j \in c(w)$ . Thus,  $w \equiv a x_j b$  for some  $a, b \in F^1$ . The identity

$$(4.11) \quad a x_j b \approx a b x_j$$

has the partition  $\lambda$ , whence it holds in  $\mathcal{W}_{n,m,\lambda}$ . Multiplying (4.11) on  $x_i$  from the left, we obtain the identity  $u \approx x_i a b x_j$  that also holds in  $\mathcal{W}_{n,m,\lambda}$ . We come to the situation considered in Case 1.

*Case 3:*  $u \equiv a x_i x_j b$  for some  $a, b \in F^1$  and some simple letters  $x_i$  and  $x_j$ . Let  $x_p$  be a letter with  $x_p \notin c(u)$ . The identity

$$(4.12) \quad a x_p b \approx a b x_p$$

has the partition  $\lambda$ , whence it holds in  $\mathcal{W}_{n,m,\lambda}$ . Substituting  $x_i x_j$  for  $x_p$  in (4.12), we obtain the identity  $u \approx a b x_i x_j$  that holds in  $\mathcal{W}_{n,m,\lambda}$ . We come to the situation considered in Case 1 again.  $\square$

**Corollary 4.5.** *If  $\lambda \in \Lambda$  then  $\mathcal{S}_\lambda \subseteq \mathcal{W}_{\lambda^k}$  for any  $k \geq 0$ .*

*Proof.* If  $k \leq s(\lambda)$  then the desired inclusion holds by the definition of the variety  $\mathcal{S}_\lambda$ . Let now  $k > s(\lambda)$ . It is easy to see that  $s(\lambda^{s(\lambda)}) = 0$ . Now we may apply Lemma 4.4 and conclude that  $\mathcal{S}_\lambda \subseteq \mathcal{W}_{\lambda^{s(\lambda)}} \subseteq \mathcal{W}_{\lambda^k}$ .  $\square$

**Proposition 4.6.** *If  $\lambda \in \Lambda$  then the variety  $\mathcal{S}_\lambda$  is greedy.*

*Proof.* Suppose that  $\mu \in \Lambda$  and the variety  $\mathcal{S}_\lambda$  reduces the transversal  $W_\mu$ , that is  $\mathcal{S}_\lambda$  satisfies some non-trivial identity  $u \approx v$  with  $u, v \in W_\mu$ . Lemma 4.2 implies that  $\lambda \preceq \mu$ , that is  $\mu = U_S(\lambda^k)$  for some  $S$  and  $k$ . Applying Corollary 4.5 and Lemma 4.3, we have  $\mathcal{S}_\lambda \subseteq \mathcal{W}_{\lambda^k} \subseteq \mathcal{W}_\mu$ . Thus, if  $s, t \in W_\mu$  then the identity  $s \approx t$  holds in  $\mathcal{S}_\lambda$ . This means that  $\mathcal{S}_\lambda$  collapses  $W_\mu$ . We see that the variety  $\mathcal{S}_\lambda$  collapses any transversal it reduces, that is  $\mathcal{S}_\lambda$  is greedy.  $\square$

Now we are well prepared to prove the ‘only if’ part of Proposition 2.4. Let an overcommutative variety  $\mathcal{V}$  satisfy the condition (vi) of Theorem 2.2. We need to verify that  $\mathcal{V}$  is greedy. It is evident that the variety  $\mathcal{SEM}$  is greedy because it does not reduce any transversal. Let now  $\mathcal{V} = \bigwedge_{i=1}^k \mathcal{S}_{\lambda_i}$  for some partitions  $\lambda_1, \lambda_2, \dots, \lambda_k$ . By Proposition 4.6, the varieties  $\mathcal{S}_{\lambda_1}, \mathcal{S}_{\lambda_2}, \dots, \mathcal{S}_{\lambda_k}$  are greedy. Proposition 2.1 implies now that all these varieties are neutral elements of the lattice  $\mathbf{OC}$ . It is well known that the set of all neutral elements of a lattice  $L$  forms a sublattice of  $L$  (see [1, Theorem III.2.9], for instance). Therefore,  $\mathcal{V}$  is a neutral element of  $\mathbf{OC}$ . Now we may apply Proposition 2.1 again and conclude that  $\mathcal{V}$  is greedy.

## 5. PROOF OF PROPOSITION 2.4: SUFFICIENCY

Here we are going to verify that a greedy overcommutative variety satisfies the condition (vi) of Theorem 2.2, thus completing the proof of Proposition 2.4 and therefore, of Theorem 2.2. We start with a few easy observations.

**Lemma 5.1.** *If an overcommutative semigroup variety  $\mathcal{V}$  reduces (in particular, collapses) a transversal  $W_\lambda$  then  $\mathcal{V}$  reduces transversals  $W_{\lambda^k}$  for all  $k \geq 0$ .*

*Proof.* The case  $k = 0$  is obvious because  $W_{\lambda^0} = W_\lambda$ . Let now  $k > 0$ . Suppose that  $\mathcal{V}$  satisfies a non-trivial identity of the kind  $u \approx v$  with  $u, v \in W_\lambda$ . Let  $y_1, \dots, y_k$  be letters with  $y_1, \dots, y_k \notin c(u)$ . The identity  $uy_1 \cdots y_k \approx vy_1 \cdots y_k$  is non-trivial and holds in  $\mathcal{V}$  because it follows from  $u \approx v$ . Since

$$\text{part}(uy_1 \cdots y_k \approx vy_1 \cdots y_k) = \lambda^k,$$

$\mathcal{V}$  reduces  $W_{\lambda^k}$ .  $\square$

**Lemma 5.2.** *Let  $\mathcal{V}$  be a greedy variety. If a non-trivial identity  $u \approx v$  holds in  $\mathcal{V}$  and  $\text{part}(u \approx v) = \lambda$  then  $\mathcal{V} \subseteq \mathcal{S}_\lambda$ .*

*Proof.* By Lemma 5.1,  $\mathcal{V}$  reduces transversals  $W_{\lambda^k}$  for all  $k = 0, 1, \dots, s(\lambda)$ . Being greedy,  $\mathcal{V}$  collapses all these transversals. Therefore,  $\mathcal{V} \subseteq \mathcal{S}_\lambda$ .  $\square$

**Corollary 5.3.** *Let  $\lambda, \mu \in \Lambda$ . Then  $\mathcal{S}_\lambda \subseteq \mathcal{S}_\mu$  if and only if  $\lambda \preceq \mu$ .*

*Proof. Necessity.* Suppose that  $\mathcal{S}_\lambda \subseteq \mathcal{S}_\mu$ . Let  $u \approx v$  be an identity with  $\text{part}(u \approx v) = \mu$ . Then  $u \approx v$  holds in  $\mathcal{S}_\mu$ , whence it holds in  $\mathcal{S}_\lambda$ . Now Lemma 4.2 applies with the conclusion that  $\lambda \preceq \mu$ .

*Sufficiency.* Let  $\lambda \preceq \mu$  and  $u \approx v$  an identity with  $\text{part}(u \approx v) = \lambda$ . Since  $\lambda \preceq \mu$ , there is an identity  $s \approx t$  such that  $u \approx v$  implies  $s \approx t$  and  $\text{part}(s \approx t) = \mu$ . The variety  $\mathcal{S}_\lambda$  satisfies the identity  $u \approx v$ . Hence  $s \approx t$  holds

in  $\mathcal{S}_\lambda$  as well. According to Proposition 4.6, the variety  $\mathcal{S}_\lambda$  is greedy. Now Lemma 5.2 successfully applies with the conclusion that  $\mathcal{S}_\lambda \subseteq \mathcal{S}_\mu$ .  $\square$

Now we are ready to prove the ‘if’ part of Proposition 2.4. Let  $\mathcal{V}$  be a greedy variety and  $\mathcal{V} \neq \mathcal{SEM}$ . The last inequality means that  $\mathcal{V}$  satisfies some non-trivial identity  $u \approx v$ . Put  $\lambda = \text{part}(u \approx v)$ . Lemma 5.2 shows that  $\mathcal{V} \subseteq \mathcal{S}_\lambda$ . Thus, the set  $\Gamma = \{\lambda \in \Lambda \mid \mathcal{V} \subseteq \mathcal{S}_\lambda\}$  is non-empty. Put  $\mathcal{X} = \bigwedge_{\lambda \in \Gamma} \mathcal{S}_\lambda$ . Clearly,  $\mathcal{V} \subseteq \mathcal{X}$ . Suppose that  $\mathcal{V} \neq \mathcal{X}$ . Then there is an identity  $u \approx v$  that holds in  $\mathcal{V}$  but fails in  $\mathcal{X}$ . Let  $\mu = \text{part}(u \approx v)$ . By Lemma 5.2,  $\mathcal{V} \subseteq \mathcal{S}_\mu$ . This means that  $\mu \in \Gamma$ , whence  $\mathcal{X} \subseteq \mathcal{S}_\mu$ . Since  $\mathcal{W}_\mu$  satisfies  $u \approx v$  and  $\mathcal{X} \subseteq \mathcal{S}_\mu \subseteq \mathcal{W}_\mu$ , we have that the identity  $u \approx v$  holds in  $\mathcal{X}$ . A contradiction shows that  $\mathcal{V} = \mathcal{X}$ .

Lemma 3.2 and Corollary 5.3 imply together that  $\mathcal{V} = \bigwedge_{\lambda \in \Gamma'} \mathcal{S}_\lambda$  where  $\Gamma'$  is the set of all minimal elements of the partially ordered set  $\langle \Gamma; \preceq \rangle$ . Since  $\Gamma'$  forms an anti-chain in  $\langle \Lambda; \preceq \rangle$ , Lemma 3.4 implies that the set  $\Gamma'$  is finite. Thus,  $\mathcal{V}$  satisfies the condition (vi) of Theorem 2.2.

Proposition 2.4 and Theorem 2.2 are proved.  $\square$

## 6. ADDITIONAL REMARKS

Here we are going to show that the description of the varieties under consideration given by Theorem 2.2 may not be improved, in a sense. Theorem 2.2 shows that the varieties of the kind  $\mathcal{S}_\lambda$  play the crucial role in the description of varieties we consider in this article. Recall that the variety  $\mathcal{S}_\lambda$  is defined as the intersection of the varieties  $\mathcal{W}_{\lambda^i}$  where  $i$  runs over the set  $\{0, 1, \dots, s(\lambda)\}$ . A natural question arises, whether or not the number  $s(\lambda)$  may be changed on some lesser number here.

For any  $\lambda \in \Lambda$  and  $k \in \{0, 1, \dots, s(\lambda)\}$ , we put

$$\mathcal{S}_\lambda^k = \bigwedge_{i=0}^k \mathcal{W}_{\lambda^i}.$$

In particular,  $\mathcal{S}_\lambda^0 = \mathcal{W}_\lambda$  and  $\mathcal{S}_\lambda^{s(\lambda)} = \mathcal{S}_\lambda$ . The crucial property of the variety  $\mathcal{S}_\lambda$  is given by Proposition 4.6: this variety is greedy. The following statement together with Lemma 5.1 show that varieties  $\mathcal{S}_\lambda^k$  with  $k < s(\lambda)$  does not have this property. Thus, the question posed in the previous paragraph is answered in negative.

**Proposition 6.1.** *Let  $\lambda \in \Lambda$ ,  $s(\lambda) > 0$ , and  $0 \leq k < s(\lambda)$ . Then the variety  $\mathcal{S}_\lambda^k$  does not collapse the transversals  $W_{\lambda^{k+1}}, W_{\lambda^{k+2}}, \dots, W_{\lambda^{s(\lambda)}}$ .*

*Proof.* Let  $i \in \{k+1, \dots, s(\lambda)\}$ . Suppose that  $\mathcal{S}_\lambda^k$  collapses the transversal  $W_{\lambda^i}$ . Further considerations are divided into two cases.

*Case 1:*  $\lambda \neq (2, 1)$ . The definition of the number  $s(\lambda)$  and the inequality  $s(\lambda) > 0$  imply that  $s(\lambda) = r(\lambda) - q(\lambda) - 1$  here. Since  $s(\lambda) \geq i$ , we have  $r(\lambda) - q(\lambda) - 1 \geq i$ . Evident equalities  $r(\lambda^i) = r(\lambda)$  and  $q(\lambda^i) = q(\lambda) + i$  then imply that  $r(\lambda^i) \geq q(\lambda^i) + 1$ . Hence the transversal  $W_{\lambda^i}$  contains a word  $u$  of the kind

$$u \equiv w_1 y_1 w_2 y_2 \cdots w_{q+i} y_{q+i} w_{q+i+1}$$

where  $y_1, y_2, \dots, y_{q+i}$  are simple in  $u$  letters, while  $w_1, w_2, \dots, w_{q+i+1}$  are non-empty words such that the word  $w_1 w_2 \dots w_{q+i+1}$  does not contain simple in  $u$  letters. Let  $v \in W_{\lambda^i}$  and  $v \neq u$ . Since  $\mathcal{S}_{\lambda}^k$  collapses  $W_{\lambda^i}$ , the identity  $u \approx v$  holds in  $\mathcal{S}_{\lambda}^k$ . Therefore, this identity follows from the identity system

$$\Sigma = \{g \approx h \mid \text{there is } j \in \{0, 1, \dots, k\} \text{ with } g, h \in W_{\lambda^j}\}.$$

Let (4.1) be a deduction of  $u \approx v$  from  $\Sigma$ . We have  $u \equiv a\zeta(s)b$  and  $w_1 \equiv a\zeta(t)b$  for some homomorphism  $\zeta$  on  $F$ , some  $a, b \in F^1$ , and some  $s, t \in W_{\lambda^j}$  where  $j \in \{0, 1, \dots, k\}$ . Furthermore,

$$(6.1) \quad r(\text{part}(s)) = r(\lambda^j) = r(\lambda^i) = r(\text{part}(u)) = r(\text{part}(a\zeta(s)b)).$$

On the other hand, it is evident that

$$(6.2) \quad r(\text{part}(s)) \leq r(\text{part}(\zeta(s))) \leq r(\text{part}(a\zeta(s)b)).$$

Combining (6.1) and (6.2), we have

$$(6.3) \quad r(\text{part}(s)) = r(\text{part}(\zeta(s))) = r(\text{part}(a\zeta(s)b)).$$

This implies that the subword  $\zeta(s)$  of the word  $a\zeta(s)b$  contains all occurrences of non-simple in  $a\zeta(s)b$  letters, whence all letters from  $c(ab)$  are simple in  $u$ . But the word  $a\zeta(s)b \equiv u$  starts and ends with non-simple in  $u$  letters. Therefore, the words  $a$  and  $b$  are empty. Thus,  $u \equiv \zeta(s)$ . If either there is a non-simple in  $s$  letter  $x$  with  $\ell(\zeta(x)) > 1$  or there is a simple in  $s$  letter  $y$  such that the word  $\zeta(y)$  contains some non-simple in  $u$  letter then  $r(\text{part}(\zeta(s))) > r(\text{part}(s))$ , contradicting (6.3). Hence  $\zeta$  maps every non-simple in  $s$  letter to a letter and maps every simple in  $s$  letter to a word consisting of simple letters. If  $y$  is a simple in  $s$  letter then  $\zeta(y)$  is a subword of  $u$ . But  $u$  does not contain subwords consisting of simple letters except subwords of length 1, that is letters. Thus,  $\zeta$  maps a simple in  $s$  letter to a simple in  $\zeta(s) \equiv u$  letter. We conclude that  $\ell(u) = \ell(\zeta(s)) = \ell(s)$ . But  $\ell(s) = n + j$  and  $\ell(u) = n + i$  where  $n = \ell(\lambda)$ . Therefore,  $j = i$ . But this is impossible because  $i \geq k + 1$ , while  $j \leq k$ .

*Case 2:*  $\lambda = (2, 1)$ . Here  $r(\lambda) = 2$ ,  $q(\lambda) = 1$ , and  $\delta = 0$ , whence  $s(\lambda) = 1$ . Therefore,  $k = 0$  and  $i = 1$ . This means that  $\mathcal{S}_{\lambda}^k = \mathcal{W}_{\lambda}$  and  $\lambda^i = (2, 1, 1)$ . Let  $u \equiv x^2 y z$  and  $v \equiv x^2 z y$ . Suppose that the identity  $u \approx v$  holds in  $\mathcal{W}_{\lambda}$ . Then it follows from the identity system

$$\Sigma = \{x^2 y \approx x y x \approx y x^2\}.$$

Let (4.1) be a deduction of  $u \approx v$  from  $\Sigma$ . Then there is  $j \in \{0, 1, \dots, \ell\}$  such that the first occurrence of  $z$  in the word  $w_j$  precedes the first occurrence of  $y$  in  $w_j$ . Let  $j$  be the least number with such a property. It is evident that  $j > 0$ . Thus, the following holds:

$$(6.4) \quad w_{j-1} \in \{x^2 y z, x y x z, x y z x, y x^2 z, y x z x, y z x^2\},$$

$$(6.5) \quad w_j \in \{x^2 z y, x z x y, x z y x, z x^2 y, z x y x, z y x^2\}.$$

Furthermore,  $w_{j-1} \equiv a\zeta(s)b$  and  $w_j \equiv a\zeta(t)b$  for some homomorphism  $\zeta$  on  $F$ , some  $a, b \in F^1$ , and some  $s, t \in \{x^2 y, x y x, y x^2\}$ . Repeating mutatis mutandi arguments from Case 1), we obtain that  $r(\text{part}(w_{j-1})) = r(\text{part}(w_j)) = 2$  and deduct from these equalities that  $x \notin c(ab)$ ,  $\zeta(x)$  is a letter, and  $\zeta(y) \in$

$\{y, z, yz, zy\}$ . If  $\zeta(x) \equiv e$  then  $e$  is a non-simple in  $w_{j-1}$  letter. In view of (6.4),  $\zeta(x) \equiv x$ .

If  $\zeta(y) \equiv yz$  then the word  $w_j$  contains the subword  $yz$ , contradicting (6.5). Analogously, if  $\zeta(y) \equiv zy$  then the word  $w_{j-1}$  contains the subword  $zy$ , contradicting (6.4).

Suppose now that  $\zeta(y) \equiv y$ . Then  $\zeta(s) \equiv s$  and  $\zeta(t) \equiv t$ . Since  $\zeta(s)$  is a subword in  $w_{j-1}$ , this means that one of the words  $x^2y$ ,  $xyx$  or  $yx^2$  is a subword in  $w_{j-1}$ . In view of (6.4), this means that  $w_{j-1}$  coincides with one of the words  $x^2yz$ ,  $xyxz$  or  $yx^2z$ . Thus, the word  $a$  is empty and therefore,  $w_j \equiv \zeta(t)b \equiv tb$ . Since  $z \notin c(t)$ , we have that the first occurrence of  $y$  in  $w_j$  precedes the first occurrence of  $z$  in  $w_j$ . But this contradicts the choice of the number  $j$ .

Finally, let  $\zeta(y) \equiv z$ . Then  $\zeta(s) \in \{x^2z, xzx, zx^2\}$ , whence  $w_{j-1}$  coincides with one of the words  $ax^2zb$ ,  $axzxb$  or  $azx^2b$ . In view of (6.4), this means that  $w_{j-1} \in \{yx^2z, yxzx, yzx^2\}$ . Therefore,  $a \equiv y$ . Thus, the word  $w_j$  starts with the letter  $y$ . As in the previous paragraph, we have that the first occurrence of  $y$  in  $w_j$  precedes the first occurrence of  $z$  in  $w_j$  that contradicts the choice of  $j$ .

We prove that the variety  $\mathcal{W}_\lambda = \mathcal{S}_\lambda^k$  does not satisfy the identity  $x^2yz \approx x^2zy$ . Since  $\text{part}(x^2yz \approx x^2zy) = (2, 1, 1) = \lambda^i$ , we have that  $\mathcal{S}_\lambda^k$  does not collapse the transversal  $W_{\lambda^i}$ .  $\square$

One can return to the definition of the variety  $\mathcal{S}_\lambda$ . It may be written in the form

$$(6.6) \quad \mathcal{S}_\lambda = \bigwedge_{\mu \in \Gamma} \mathcal{W}_\mu$$

where  $\Gamma = \{\lambda^k \mid k = 0, 1, \dots, s(\lambda)\}$ . The following assertion shows that the set  $\{\lambda^k \mid k = 0, 1, \dots, s(\lambda)\}$  is the least set of partitions  $\Gamma$  such that the equality (6.6) holds.

**Corollary 6.2.** *If the equality (6.6) holds for some  $\Gamma \subseteq \Lambda$  then  $\lambda^k \in \Gamma$  for all  $k = 0, 1, \dots, s(\lambda)$ .*

*Proof.* Suppose that  $\lambda^k \notin \Gamma$  for some  $k \in \{0, 1, \dots, s(\lambda)\}$ . Let  $u, v \in W_{\lambda^k}$ . The definition of the variety  $\mathcal{S}_\lambda$  implies that the identity  $u \approx v$  holds in  $\mathcal{S}_\lambda$ . Therefore, this identity follows from the identity system

$$\Sigma = \{g \approx h \mid \text{there is } \mu \in \Gamma \text{ such that } g, h \in W_\mu\}.$$

As usual, let (4.1) be a deduction of  $u \approx v$  from  $\Sigma$ . Let  $i \in \{0, 1, \dots, \ell - 1\}$ . Then the identity  $w_i \approx w_{i+1}$  follows from an identity of the kind  $s \approx t$  where  $s, t \in W_\mu$  for some  $\mu \in \Gamma$ . The identity  $s \approx t$  holds in the variety  $\mathcal{S}_\mu$ . Therefore,  $u \approx v$  holds in  $\mathcal{S}_\mu$  too. Then Lemma 4.2 implies that  $\mu \preceq \text{part}(u \approx v) = \lambda^k$ . Furthermore, the identity  $s \approx t$  holds in the variety  $\mathcal{S}_\lambda$  because  $\mathcal{S}_\lambda \subseteq \mathcal{W}_\mu$ . Applying Lemma 4.2 again, we have  $\lambda \preceq \mu$ . Therefore,  $\mu = U_S(\lambda^j)$  and  $\lambda^k = U_T(\mu^q)$  for some finite (may be empty) sets of partitions  $S$  and  $T$  and some non-negative integers  $j$  and  $q$ .

Let  $n(\lambda) = n$ . Then  $n(\mu) = n + j$ , while  $n(\lambda^k)$  equals both  $n + k$  (that is evident) and  $n + j + q$  (because  $n(\lambda^k) = n(\mu) + q$ ). Therefore,  $n + k = n + j + q$ , whence  $q = k - j$ . Thus,  $\mu = U_S(\lambda^j)$  and  $\lambda^k = U_T(\mu^{k-j})$ . Hence

$\lambda^k = U_{S \cup T}(\lambda^k)$  and therefore,  $S = T = \emptyset$ . In particular, this means that  $\mu = U_{\emptyset}(\lambda^j) = \lambda^j$ . We see that  $\lambda^j = \mu \preceq \lambda^k$ , whence  $j \leq k$ . But  $j \neq k$  because  $\lambda^k \notin \Gamma$ , while  $\lambda^j = \mu \in \Gamma$ . Hence  $j < k$ . Besides that, the equality  $\mu = \lambda^j$  implies that the identity  $w_i \approx w_{i+1}$  holds in the variety  $\mathcal{W}_{\lambda^j}$ . Since  $j \leq k-1$ , we have  $\mathcal{S}_{\lambda}^{k-1} \subseteq \mathcal{W}_{\lambda^j}$ . Thus,  $w_i \approx w_{i+1}$  holds in  $\mathcal{S}_{\lambda}^{k-1}$ . This is the case for all  $i = 0, 1, \dots, \ell-1$ . Therefore, the identity  $u \approx v$  holds in  $\mathcal{S}_{\lambda}^{k-1}$  too. This is valid for all  $u, v \in W_{\lambda^k}$ . Hence the variety  $\mathcal{S}_{\lambda}^{k-1}$  collapses the transversal  $W_{\lambda^k}$ , contradicting Proposition 6.1.  $\square$

## 7. OPEN PROBLEMS

Recall that an element  $x$  of a lattice  $L$  is called *modular* if

$$\forall y, z \in L: \quad y \leq z \longrightarrow (x \vee y) \wedge z = (x \wedge z) \vee y,$$

and *upper-modular* if

$$\forall y, z \in L: \quad y \leq x \longrightarrow x \wedge (y \vee z) = (x \wedge z) \vee y.$$

*Lower-modular* elements are defined dually to upper-modular ones.

**Problem 7.1.** *Describe*

- a) *modular*;
- b) *upper-modular*;
- c) *lower-modular*

*elements of the lattice OC.*

As we have already mentioned in Section 1, neutral elements of the lattice **SEM** are completely determined in [15], while distributive elements of this lattice are completely described in [12]. It is interesting to note that Proposition 2.1 plays an important role in the proof of the result of [12].

**Problem 7.2.** *Describe*

- a) *codistributive*;
- b) *standard*;
- c) *costandard*

*elements of the lattice SEM.*

Some particular results concerning Problem 7.2a) are obtained in [11]. The following two examples show that, in contrast with the overcommutative case, the lattice **SEM** contains distributive but not codistributive elements and codistributive but not distributive ones. In particular, there are [co]distributive but not neutral elements of **SEM**.

**Example 7.3.** The variety  $\mathcal{N} = \text{var}\{x^2y = xyx = yx^2 = 0\}$  is a distributive element of the lattice **SEM** by [12, Theorem 1.1]. But this variety is not a codistributive (and moreover not a neutral) element of **SEM** by [11, Theorem 1.1].

**Example 7.4.** The varieties  $\mathcal{A}_p = \text{var}\{x^py = y, xy = yx\}$  with any prime  $p$ ,  $\mathcal{LZ} = \text{var}\{xy = x\}$ , and  $\mathcal{RZ} = \text{var}\{xy = y\}$  are codistributive elements of the

lattice **SEM**. This follows from the well known facts that these varieties are atoms of **SEM** and **SEM** satisfies the condition

$$\forall x, y, z: \quad x \wedge z = y \wedge z = 0 \longrightarrow (x \vee y) \wedge z = 0$$

(see [4, Section 1], for instance). But  $\mathcal{A}_p$ ,  $\mathcal{LZ}$ , and  $\mathcal{RZ}$  are not distributive (and moreover not neutral) elements of **SEM** by [12, Theorem 1.1].

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