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On the Homomorphism Order of Labeled Posets

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Abstract Partially ordered sets labeled with k labels (k-posets) and their homomorphisms are examined. We give a representation of directed graphs by k-posets; this provides a new proof of the universality of the homomorphism order of k-posets. This universal order is a distributive lattice. We investigate some other properties, namely the infinite distributivity, the computation of infinite suprema and infima, and the complexity of certain decision problems involving the homomorphism order of k-posets. Sublattices are also examined.

Keywords Partial order · Labeled poset · Homomorphism

1 Introduction

A partially ordered set labeled with k labels (k-poset), also known as a partially ordered multiset (pomset) or a partial word, is an object $(P; \leq, c)$, where $(P; \leq)$ is a partially ordered set and c is a function that assigns to each element of P a label from the set $\{0, 1, \ldots, k-1\}$. A homomorphism between k-posets is a mapping $h\colon (P; \leq, c) \to (P'; \leq', c')$ that preserves both order and labels. A quasiorder,

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called the homomorphism quasiorder, can be defined on the set of all k-posets as follows: $(P; \leq, c) \leq (P'; \leq', c')$ if and only if there is a homomorphism of $(P; \leq, c)$ to $(P'; \leq', c')$.

Labeled posets have been used as a model of parallel processes (see Pratt [20]), and they can be viewed as a generalization of strings. Algebraic properties of labeled posets have been studied by Grabowski [6], Gischer [5], Bloom and Ésik [1], and Rensink [22]. Homomorphisms of k-posets were studied in the context of Boolean hierarchies of partitions by Kosub [12], Kosub and Wagner [13], and Selivanov [23]. Kuske [15] and Kudinov and Selivanov [14] studied the undecidability of the first-order theory of the homomorphism quasiorder of k-posets. The second author applied k-posets to analyse substitution instances of operations on finite sets when the inner functions are monotone functions (with respect to some fixed partial order on the base set) [16] and showed that for $k \geq 2$ and $\ell \geq 3$, the homomorphism order of finite k-posets and that of finite ℓ-lattices are distributive lattices which are universal in the sense that they admit an embedding of every countable poset [17]. The condition $k \geq 2$ is clearly necessary for universality, because all nonempty 1-posets are homomorphically equivalent to each other. The results of Kosub and Wagner [13] also show that the homomorphism order of 2-lattices is not universal. Moreover these homomorphism orders are not complete lattices.

The current paper continues the investigation of some properties and sublattices of the homomorphism order of k-posets. We establish a representation of directed graphs by k-posets, which gives rise to a new proof of the universality of the homomorphism order of k-posets and enables us to study the complexity of certain decision problems related to k-posets. We are also interested in computing with infinite suprema and infima. In particular we examine join-infinite distributivity (JID) and its dual, meet-infinite distributivity (MID); these are special cases of complete infinite distributivity (CID). These properties are defined by the identities below, with $I, J \neq \emptyset$.

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i), \tag{JID}$$

$$x \vee \bigwedge_{i \in I} x_i = \bigwedge_{i \in I} (x \vee x_i), \tag{MID}$$

$$\bigwedge_{i \in I} \bigvee_{j \in J} a_{ij} = \bigvee_{\varphi : I \to J} \bigwedge_{i \in I} a_{i\varphi(i)}. \tag{CID}$$

2 Labeled Posets and Homomorphisms

For a positive natural number k, a *partially ordered set labeled with k labels* (*k-poset*) is an object $(P; \leq, c)$, where $(P; \leq)$ is a partially ordered set and $c\colon P \to \{0, 1, \ldots, k-1\}$ is a *labeling* function. A *labeled poset* is a k-poset for some k. Every subset P' of a k-poset $(P; \leq, c)$ may be considered as a k-poset $(P'; \leq|_{P'}, c|_{P'})$, called a *$k$-subposet* of $(P; \leq, c)$. We often simplify these notations and write (P, c) or P instead of $(P; \leq, c)$, and we simply write c for the restriction $c|_S$ of c to any subset S of its domain. If the underlying poset of a k-poset is a lattice, chain, tree, or forest, then we refer to *k-lattices, k-chains, k-trees, k-forests,* and so on. For $k \leq \ell$, every

k-poset is also an ℓ-poset. Finite k-posets can be represented by Hasse diagrams with numbers designating the labels assigned to each element; see the various figures of this paper. For general background on partially ordered sets and lattices, see any textbook on the subject, e.g., [2, 3, 7].

A k-chain $a_1 < a_2 < \cdots < a_n$ with labeling c is *alternating,* if $c(a_i) \neq c(a_{i+1})$ for all $1 \leq i \leq n-1$. The *alternation number* of a k-poset (P, c), denoted $\mathrm{Alt}(P, c)$, is the cardinality of the longest alternating k-chain that is a k-subposet of (P, c).

We will adopt much of the terminology used for graphs and their homomorphisms (see [10]). (Recall that a graph *homomorphism* $h \colon G \to G'$ is an edge-preserving mapping between the vertex sets of graphs G and G'. A *core* is a graph that does not admit a homomorphism to any proper subgraph of itself.) Let (P, c) and (P', c') be k-posets. A mapping $h \colon P \to P'$ that preserves both ordering and labels (i.e., $h(x) \leq h(y)$ in P' whenever $x \leq y$ in P, and $c = c' \circ h$) is called a *homomorphism* of (P, c) to (P', c') and denoted $h \colon (P, c) \to (P', c')$. The composition of homomorphisms is again a homomorphism. An *endomorphism* of (P, c) is a homomorphism $h \colon (P, c) \to (P, c)$. If a homomorphism $h \colon (P, c) \to (P', c')$ is bijective and the inverse of h is a homomorphism of (P', c') to (P, c), then h is called an *isomorphism,* and (P, c) and (P', c') are said to be *isomorphic.*

We denote by $\mathcal{P}_k$ and $\mathcal{L}_k$ the classes of all finite k-posets and k-lattices, respectively. We define a quasiorder $\leq$ on $\mathcal{P}_k$ as follows: $(P, c) \leq (P', c')$ if and only if there is a homomorphism of (P, c) to (P', c'). Denote by $\equiv$ the equivalence relation on $\mathcal{P}_k$ induced by $\leq$. If $(P, c) \equiv (P', c')$, we say that (P, c) and (P', c') are *homomorphically equivalent.* We denote by $\tilde{\mathcal{P}}_k$ the quotient set $\mathcal{P}_k/{\equiv}$, and the partial order on $\tilde{\mathcal{P}}_k$ induced by the homomorphism quasiorder $\leq$ is also denoted by $\leq$. The quasiorder $\leq$ and the equivalence relation $\equiv$ can be restricted to $\mathcal{L}_k$, and we denote by $\tilde{\mathcal{L}}_k$ the quotient set $\mathcal{L}_k/{\equiv}$. We will refer to the partial orders $(\tilde{\mathcal{P}}_k, \leq)$ and $(\tilde{\mathcal{L}}_k, \leq)$ as the *homomorphism order of k-posets* and the *homomorphism order of k-lattices,* respectively.

The homomorphic equivalence class of $(P, c) \in \mathcal{P}_k$ is denoted by $[(P, c)] := \{(P', c') \in \mathcal{P}_k \mid (P, c) \equiv (P', c')\}$. We tend to identify the $\equiv$-classes by their representatives; that is, whenever we say that (P, c) is an element of $\tilde{\mathcal{P}}_k$, it is to be understood as referring to the $\equiv$-class $[(P, c)]$.

A finite k-poset (P, c) such that all endomorphisms of (P, c) are surjective (equivalently, (P, c) is not homomorphically equivalent to any k-poset of smaller cardinality) is called a *core.* Every finite k-poset is homomorphically equivalent to a core. Isomorphic k-posets are homomorphically equivalent by definition. Homomorphically equivalent k-posets are not necessarily isomorphic, but homomorphically equivalent cores are isomorphic. Thus we can choose non-isomorphic cores as the representatives of the homomorphic equivalence classes; the restriction of the quasiorder $\leq$ on $\mathcal{P}_k$ to this set of cores is isomorphic to $(\tilde{\mathcal{P}}_k, \leq)$.

Two elements a and b of a poset P are *connected,* if there exists a sequence $a_1, \ldots, a_n$ of elements of P such that $a_1 = a$, $a_n = b$, and for all $1 \leq i \leq n-1$ either $a_i \leq a_{i+1}$ or $a_i \geq a_{i+1}$. A nonempty poset is *connected* if all pairs of its elements are connected. A *connected component* of a poset P is a subposet $C \subseteq P$ that is connected and such that for every $x \in P \setminus C$ the subposet $C \cup \{x\}$ is not connected. It is easy to verify that all homomorphic images of a connected poset are connected. A k-poset is a core if and only if all its connected components are cores and pairwise incomparable under $\leq$.

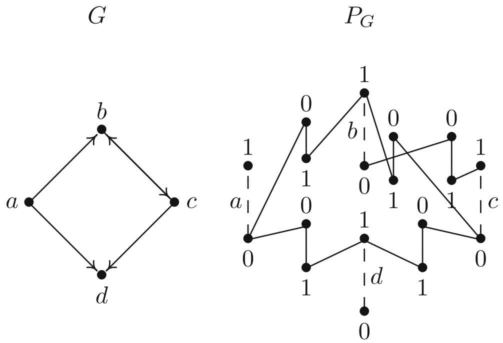

Fig. 1 Directed graph G and its representation by a 2-poset P_G. Each vertex of G is represented by a two-element chain with label 0 at its bottom and label 1 at its top (*dashed lines*). Each edge (x, y) of G is represented by a zig-zag from the bottom of the chain representing x to the top of the chain representing y (*solid lines*)

3 Representation of Directed Graphs by *k*-Posets

Let $G = (V, E)$ be a directed graph. We associate with G a 2-poset $P_G := (P; \leq, c)$, where $P := (V \cup E) \times \{0, 1\}$, and $c(a, b) = b$ for all $a \in V \cup E$, $b \in \{0, 1\}$, and the covering relations of $\leq$ are exactly the following:

- $(a, 0) < (a, 1)$ for all $a \in V$,
- $(a, 1) < (a, 0)$ for all $a \in E$,
- for each edge $(u, v) \in E$, $(u, 0) < ((u, v), 0)$ and $((u, v), 1) < (v, 1)$.

It is clear from the construction that if G is a subgraph of H, then P_G is a k-subposet of P_H. See Fig. 1 for an example of a directed graph and its representation by a 2-poset.

Proposition 3.1 *Let G and H be directed graphs. Then G is homomorphic to H if and only if P_G is homomorphic to P_H.*

Proof Let $h\colon G \to H$ be a graph homomorphism. Then the mapping $g\colon P_G \to P_H$ defined as $g(v, b) = (h(v), b)$ for all $v \in V(G)$, $b \in \{0, 1\}$; $g((u, v), b) = ((h(u), h(v)), b)$ for all $(u, v) \in E(G)$, $b \in \{0, 1\}$, is easily seen to be a homomorphism. Clearly g preserves the labels, and in order to show that $g(x) \leq g(y)$ in P_H whenever $x \leq y$ in P_G we have four cases to consider; recall that if $(u, v) \in E(G)$, then $(h(u), h(v)) \in E(H)$.

- If $x = (u, 0)$, $y = (u, 1)$ where $u \in V(G)$, then $g(x) = g(u, 0) = (h(u), 0) < (h(u), 1) = g(u, 1) = g(y)$.
- If $x = ((u, v), 1)$, $y = ((u, v), 0)$ where $u, v \in V(G)$ and $(u, v) \in E(G)$, then $g(x) = g((u, v), 1) = ((h(u), h(v)), 1) < ((h(u), h(v)), 0) = g((u, v), 0) = g(y)$.
- If $x = (u, 0)$, $y = ((u, v), 0)$ where $u, v \in V(G)$ and $(u, v) \in E(G)$, then $g(x) = g(u, 0) = (h(u), 0) < ((h(u), h(v)), 0) = g((u, v), 0) = g(y)$.
- If $x = ((u, v), 1)$, $y = (v, 1)$ where $u, v \in V(G)$ and $(u, v) \in E(G)$, then $g(x) = g((u, v), 1) = ((h(u), h(v)), 1) < (h(v), 1) = g(v, 1) = g(y)$.

Assume then that $g\colon P_G \to P_H$ is a homomorphism. Since alternating chains must be mapped to isomorphic alternating chains by homomorphisms, we have that there are mappings $h\colon V(G) \to V(H)$, $e\colon E(G) \to E(H)$ such that $g(v, b) = (h(v), b)$ and $g((u, v), b) = (e(u, v), b)$ for all $v \in V(G)$, $(u, v) \in E(G)$, $b \in \{0, 1\}$. Furthermore,

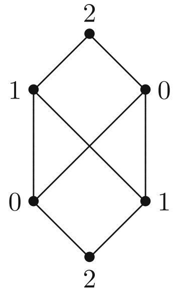

Fig. 2 The 3-poset representation of a loop

the comparabilities $(u, 0) < ((u, v), 0)$ and $((u, v), 1) < (v, 1)$ in P_G must be preserved by g for all edges $(u, v) \in E(G)$, that is, $(h(u), 0) = g(u, 0) < g((u, v), 0) = (e(u, v), 0)$ and $(e(u, v), 1) = g((u, v), 1) < g(v, 1) = (h(v), 1)$. Therefore, $e(u, v) \in E(H)$ equals $(h(u), h(v))$. We conclude that h is a homomorphism of G to H. □

Proposition 3.2 *Let G be a graph. Then P_G is a core if and only if G is a core.*

Proof If P_G is a core, then it is not homomorphic to any of its proper k-subposets. In particular, by Proposition 3.1, there is no proper subgraph H of G such that P_G is homomorphic to P_H. Thus, G does not retract to any proper subgraph, and hence G is a core.

If P_G is not a core, then there is a homomorphism $h\colon P_G \to P'$ for some proper k-subposet $P' = \operatorname{Im} h$ of P_G. It is clear from the proof of Proposition 3.1 that the homomorphic image P' of P_G is of the form P_H for some graph H. Then H is a proper subgraph and a retract of G, and so G is not a core. □

We describe a variant of the above representation of directed graphs by labeled posets. We associate with each directed graph G the 3-poset L_G, which is defined like P_G but with a greatest element and a least element adjoined. The two new elements have label 2. (For the empty graph $\emptyset$, we agree that $L_\emptyset$ is the empty 3-poset.) It is easy to see that L_G is a 3-lattice if and only if G is loopless. (A single loop gives rise to the 3-poset shown in Fig. 2, which is not a 3-lattice.)

Proposition 3.3 *Let G and H be directed graphs. Then G is homomorphic to H if and only if L_G is homomorphic to L_H.*

Proof The proof is similar to that of Proposition 3.1. We only need to observe that the greatest and least elements are the only elements with label 2, and every homomorphism must map the greatest and least elements to the greatest and least elements, respectively. Otherwise homomorphisms act as described in the proof of Proposition 3.1. □

Proposition 3.4 *Let G be a graph. Then L_G is a core if and only if G is a core.*

Proof The proof is similar to that of Proposition 3.2. □

A countable poset is *universal* if every countable poset can be embedded into it. We established in [17] that the posets $\tilde{\mathcal{P}}_k$ ($k \geq 2$) and $\tilde{\mathcal{L}}_k$ ($k \geq 3$) are universal. Our representation of directed graphs by 2-posets and that of loopless directed graphs by 3-lattices provides a new proof of this fact.

Theorem 3.5 *The posets $\tilde{\mathcal{P}}_k$ ($k \geq 2$) and $\tilde{\mathcal{L}}_k$ ($k \geq 3$) are universal.*

Proof It is a well-known fact that the homomorphism order of (loopless) directed graphs is universal (see [21]; see also Hubička and Nešetřil's [11] simpler proof). The claim then follows from Propositions 3.1 and 3.3. □

How hard is it to find homomorphisms between k-posets? The k-poset representation of directed graphs given above has the property that there is a homomorphism between two graphs if and only if there is homomorphism between their corresponding k-posets. This allows us to transfer some complexity results from directed graphs to k-posets. It is an easy exercise to show that the problem of deciding whether there exists a homomorphism between two k-posets (k-**HOM**) is NP-complete and the problem of deciding whether a k-poset is a core (k-**CORE**) is coNP-complete, using this representation of graphs by labeled posets and the well-known fact that the analogous problems on graphs are NP-complete and coNP-complete [8, 9].

Consider also the problem of deciding whether a k-poset is homomorphic to a fixed k-poset (Q, d) (k-(Q, d)-**HOM**). It is clear that k-(Q, d)-**HOM** is in NP for any k-poset (Q, d). It was shown by Hell and Nešetřil [8] that the analogous problem on graphs is NP-complete for any non-bipartite graph H, and it is polynomial-time solvable for any bipartite graph H. Thus, there are NP-complete cases of k-(Q, d)-**HOM**, e.g., the cases where $(Q, d) = P_G$ for some nonbipartite graph G. There are also polynomial-time solvable cases, e.g., the cases where the labeling d in (Q, d) is a constant function. It remains an open question whether there is a dichotomy between the polynomial-time solvable and NP-complete cases of k-(Q, d)-**HOM**.

4 Properties of the Homomorphism Order of k-Posets

The homomorphism order of k-posets forms a distributive lattice with disjoint union as join, and label-matching product as meet [17]. The *disjoint union* of a family $(S_i)_{i\in I}$ of sets is defined as the set

$$\dot{\bigcup_{i\in I}} S_i := \{(i, x) \mid i \in I,\ x \in S_i\}.$$

If $I = \{1, 2\}$, then we write $S_1 \dot{\cup} S_2$ for $\dot{\bigcup}_{i\in\{1,2\}} S_i$. The *disjoint union* of a family $(P_i, c_i)_{i\in I}$ of k-posets is defined to be the k-poset $\dot{\bigcup}_{i\in I}(P_i, c_i) = \dot{\bigcup}_{i\in I}(P_i, d)$, where $d(i, x) = c_i(x)$ for all $(i, x) \in \dot{\bigcup}_{i\in I} P_i$, and the order on $\dot{\bigcup}_{i\in I} P_i$ is defined as $(i, x) \leq (j, y)$ if and only if $i = j$ and $x \leq y$ in P_i.

The *label-matching product* of a family $(P_i, c_i)_{i \in I}$ of k-posets is defined to be the k-poset $\bigotimes_{i \in I}(P_i, c_i) := (Q, d)$, where

$$Q := \{(a_i)_{i \in I} \in \prod_{i \in I} P_i \mid c_i(a_i) = c_j(a_j) \text{ for all } i, j \in I\},$$

$(a_i)_{i \in I} \leq (b_i)_{i \in I}$ in Q if and only if $a_i \leq b_i$ in P_i for all $i \in I$, and the labeling is defined by $d((a_i)_{i \in I}) = c_i(a_i)$ for some $i \in I$ (the choice of i does not matter by the definition of Q). If $I = \{1, 2\}$, then we write $(P_1, c_1) \otimes (P_2, c_2)$ for $\bigotimes_{i \in \{1,2\}} (P_i, c_i)$.

It was shown in [17] that $(\tilde{\mathcal{P}}_k, \leq)$ is a distributive lattice with the lattice operations defined as follows:

$$(P, c) \vee (P', c') = (P, c) \,\dot\cup\, (P', c'), \qquad \text{and} \qquad (P, c) \wedge (P', c') = (P, c) \otimes (P', c').$$

Here the lattice operations are defined in terms of equivalence class representatives.

Proposition 4.1 *The join-irreducible elements of $(\tilde{\mathcal{P}}_k, \leq)$ are (the equivalence classes of) the cores with at most one connected component.*

Proof The empty k-poset is the smallest element of $\tilde{\mathcal{P}}_k$, so it is clearly join-irreducible. We can then assume that (P, c) is a nonempty core. Let $(P_1, c_1), \ldots, (P_n, c_n)$ be the connected components of (P, c). These connected component are cores and they are pairwise incomparable under $\leq$. If $n > 1$, then (P, c) is the disjoint union of its connected components and thus it is not join-irreducible.

Assume then that $n = 1$. Suppose, on the contrary, that (P, c) is not join-irreducible. Then there exist cores (Q_1, d_1) and (Q_2, d_2) that are not equivalent to (P, c) such that $(P, c) \equiv (Q_1, d_1) \,\dot\cup\, (Q_2, d_2)$. Thus there exist homomorphisms $h\colon (P, c) \to (Q_1, d_1) \,\dot\cup\, (Q_2, d_2)$ and $g\colon (Q_1, d_1) \,\dot\cup\, (Q_2, d_2) \to (P, d)$. Since (P, c) is connected, h is in fact a homomorphism of (P, c) to (Q_1, d_1) or to (Q_2, d_2). Furthermore, for $i = 1, 2$, the restriction of g to Q_i is a homomorphism of (Q_i, d_i) to (P, c). Thus, (P, c) is homomorphically equivalent to either (Q_1, d_1) or (Q_2, d_2), a contradiction. □

Denote by $\mathcal{J}_k$ the set of join-irreducible elements of the lattice $(\tilde{\mathcal{P}}_k, \leq)$, which we just showed to be the set of (homomorphic equivalence classes of) cores with at most one connected component. Since every finite core has only a finite number of connected components and is the supremum of its connected components, we conclude that every element of $\tilde{\mathcal{P}}_k$ is the join of a finite number of elements of $\mathcal{J}_k$. Hence $\mathcal{J}_k$ is a join-dense subset of $\tilde{\mathcal{P}}_k$. As we have mentioned already, $\tilde{\mathcal{P}}_k$ is not complete. The smallest complete poset (lattice) containing $\tilde{\mathcal{P}}_k$ is its Dedekind-MacNeille completion. One way to construct it is to take the set of normal ideals of $\tilde{\mathcal{P}}_k$ ordered by inclusion [18] or to take the concept lattices of the formal contexts $(\tilde{\mathcal{P}}_k, \tilde{\mathcal{P}}_k, \leq)$ or $(\mathcal{J}_k, \tilde{\mathcal{P}}_k, \leq)$ [4]. We denote by $\hat{\mathcal{P}}_k$ the Dedekind-MacNeille completion of $\tilde{\mathcal{P}}_k$. Note that $\tilde{\mathcal{P}}_k$ is join-dense and meet-dense in $\hat{\mathcal{P}}_k$. Then $\mathcal{J}_k$ is a join-dense subset of $\hat{\mathcal{P}}_k$. *Is $\hat{\mathcal{P}}_k$ an algebraic lattice?* More generally, *is the MacNeille completion of any* *compactly generated lattice*[1] *also compactly generated?* In this contribution, we call an element a of a lattice L *compact* if $a \leq \bigvee X$ (whenever $\bigvee X$ exists) for some $X \subseteq L$ implies that $a \leq \bigvee X_1$ for some finite $X_1 \subseteq X$ and we say that a lattice L is *compactly generated*[1] if every element is the join of compact elements. An *algebraic lattice* is a complete and compactly generated lattice.

We are looking for posets containing $\tilde{\mathcal{P}}_k$ as subposet in which we can compute all suprema and infima of elements of $\tilde{\mathcal{P}}_k$. Since $\tilde{\mathcal{P}}_k$ is countably infinite, each completion should contain at least the countable unions of finite k-posets. Since any countable union of finite sets is again countable, we will start by enlarging a bit the class $\tilde{\mathcal{P}}_k$. We denote by $\mathcal{P}_{k\omega}$ the class of countable k-posets. The homomorphism quasi-order on $\mathcal{P}_{k\omega}$ is defined in the same way as for finite k-posets and it induces a partial order on the quotient $\mathcal{P}_{k\omega}/{\equiv}$, which we will denote by $\tilde{\mathcal{P}}_{k\omega}$. A poset $(P, \leq)$ is called *ω-complete*[2] if the suprema and infima of countable subsets of P exist. For countable posets, completeness and ω-completeness coincide.

Lemma 4.2 *The poset $(P_{k\omega}, \leq)$ is ω-complete.*

Proof Suprema and infima will be constructed as in [17]. Let $(P_t, c_t)_{t\in T}$ be a countable family of elements of $\mathcal{P}_{k\omega}$. Define a k-poset $(\bar{P}, c)$ as the disjoint union of (P_t, c_t)'s, i.e.,

$$\bar{P} := \dot{\bigcup_{t\in T}} P_t \quad \text{and} \quad c(t, a) = c_t(a).$$

Then $\bar{P}$ is countable and $(\bar{P}, c)$ is in $\mathcal{P}_{k\omega}$. Moreover $(\bar{P}, c)$ is the supremum of $(P_t, c_t)_{t\in T}$. In fact, it is clear that each inclusion map $\tau_t\colon P_t \to \bar{P}, x \mapsto (t, x)$ is a homomorphism of k-posets; if $(P_t, c_t) \leq (Q, d)$, then there are k-poset homomorphisms $h_t\colon P_t \to Q$ for each $t \in T$; define $h\colon \bar{P} \to Q$ by $h(t, p) := h_t(p)$, for every $t \in T$ and $p \in P_t$. The mapping h is a k-poset homomorphism and thus $(\bar{P}, c) \leq (Q, d)$. Therefore $(\bar{P}, c)$ is the supremum of $(P_t, c_t)_{t\in T}$. For the infimum, consider the label-matching product $(\tilde{P}, \tilde{c})$ of $\big((P_t, c_t)\big)_{t\in T}$ given by:

$$\tilde{P} := \{a \in \prod_{t\in T} P_t \mid c_t(a_t) = c_s(a_s) \text{ for all } s, t \in T\} \qquad \text{and} \qquad \tilde{c}(a) := c_t(a_t).$$

$\tilde{P}$ keeps only the elements having the same label on all components and sets this as its label. Of course the projections $\pi_t\colon (\tilde{P}, \tilde{c}) \to (P_t, c_t), a \mapsto a_t$ $(t \in T)$ are k-poset homomorphisms; thus $(\tilde{P}, \tilde{c}) \leq (P_t, c_t)$ for all $t \in T$. If $(Q, d) \leq (P_t, c_t)$ for all $t \in T$, then there are k-poset homomorphisms $g_t\colon (Q, d) \to (P_t, c_t)$. Define $g\colon Q \to \tilde{P}$ by $g(q) := \big(g_t(q)\big)_{t\in T}$. Then g is a homomorphism of k-posets, and $(Q, d) \leq (\tilde{P}, \tilde{c})$. □

As an ω-complete poset, $(\tilde{\mathcal{P}}_{k\omega}, \leq)$ is a lattice containing $(\tilde{\mathcal{P}}_k, \leq)$ as a sublattice, in which all suprema and infima of $\tilde{\mathcal{P}}_k$ exist. An ω-complete poset $(P, \leq)$ is called

[1] We do not assume completeness (as it is usually the case) in the definition of "compactly generated lattices". We then distinguish "algebraic lattices" from "compactly generated" ones.

[2] This notion can be generalized to κ-completeness for any cardinal $\kappa \geq \omega$ as follows: a poset $(P, \leq)$ is κ-complete if the suprema and infima of subsets of cardinality at most κ exist in P.

ω-join-distributive (*ω-meet-distributive*) if for any index set T of cardinality at most ω, for any family $(a_t)_{t\in T}$ of elements of P and for any $b \in P$, we have

$$b \wedge \bigvee_{t\in T} a_t = \bigvee_{t\in T}(b \wedge a_t)$$

$$\left(\quad b \vee \bigwedge_{t\in T} a_t = \bigwedge_{t\in T}(b \vee a_t), \qquad \text{respectively}\right).$$

If an ω-complete poset is both ω-join- and ω-meet-distributive, we call it *ω-distributive*[3]. The ω-complete poset $(\tilde{\mathcal{P}}_{k\omega}, \leq)$ is ω-distributive as we can see from Lemmas 4.3 and 4.4.

Lemma 4.3 *The ω-complete poset $(\tilde{\mathcal{P}}_{k\omega}, \leq)$ is ω-join-distributive.*

Proof Let $b := (Q, d) \in \tilde{\mathcal{P}}_{k\omega}$ and $(P_t, c_t)_{t\in T}$ be a countable family of elements of $\tilde{\mathcal{P}}_{k\omega}$. We set $a_t := (P_t, c_t)$. To show that $(\tilde{\mathcal{P}}_{k\omega}, \leq)$ is ω-join-distributive, we observe that $(Q, d) \otimes \dot{\bigcup}_{t\in T}(P_t, c_t)$ and $\dot{\bigcup}_{t\in T}\big((Q, d) \otimes (P_t, c_t)\big)$ are homomorphically equivalent. In fact for any t, x and y, we have

$$\begin{aligned} (x, t, y) \in (Q, d) \otimes \dot{\bigcup}_{t\in T}(P_t, c_t) &\iff x \in Q,\ t \in T,\ a \in P_t \ \text{ and} \\ &\qquad d(x) = \bar{c}(t, y) = c_t(y) \\ &\iff (x, y) \in (Q, d) \otimes (P_t, c_t) \\ &\iff (t, x, y) \in \dot{\bigcup}_{t\in T}\big((Q, d) \otimes (P_t, c_t)\big); \end{aligned}$$

then $h\colon (x, t, y) \mapsto (t, x, y)$ indeed defines a k-poset isomorphism of $(Q, d) \otimes \dot{\bigcup}_{t\in T}(P_t, c_t)$ onto $\dot{\bigcup}_{t\in T}\big((Q, d) \otimes (P_t, c_t)\big)$. Note that the label of (x, t, y) in $(Q, d)\otimes \big(\dot{\bigcup}_{t\in T}(P_t, c_t)\big)$ is $c_t(y)$, which is also the label of (t, x, y) in $\dot{\bigcup}_{t\in T}\big((Q, d) \otimes (P_t, c_t)\big)$. Thus in $(\mathcal{P}_{k\omega}, \leq)$ we have

$$b \wedge \bigvee_{t\in T} a_t = (Q, d) \otimes \dot{\bigcup}_{t\in T}(P_t, c_t) = \dot{\bigcup}_{t\in T}\big((Q, d) \otimes (P_t, c_t)\big) = \bigvee_{t\in T}(b \wedge a_t).$$

□

Lemma 4.4 *The ω-complete poset $(\tilde{\mathcal{P}}_{k\omega}, \leq)$ is ω-meet-distributive.*

Proof We know that

$$b \vee \bigwedge_{t\in T} a_t \leq \bigwedge_{t\in T}(b \vee a_t)$$

[3]Replacing ω with an arbitrary cardinal $\kappa \geq 2$ gives κ-distributivity. This is a generalization of distributivity ($\kappa = 2$). For finite cardinals $\kappa \geq 2$, the notions of κ-join-distributivity, κ-meet-distributivity and distributivity are equivalent. This is unfortunately no longer true for $\kappa \geq \omega$. Lemmas 4.3 and 4.4 extend to κ-join-distributivity and κ-meet-distributivity, because the index set T occurring in their proofs can in fact have arbitrary cardinality.

always holds. Our aim is to find a k-poset homomorphism of $\bigotimes_{t\in T}\big((Q,d)\uplus(P_t,c_t)\big)$ to $(Q,d)\uplus\bigotimes_{t\in T}(P_t,c_t)$. Note that

$$(s,x)\in(Q,d)\uplus\bigotimes_{t\in T}(P_t,c_t) \iff s=1\ \&\ x\in Q \text{ or } s=2\ \&\ x\in\bigotimes_{t\in T}(P_t,c_t).$$

Now let $\mathfrak{X}\in\bigotimes_{t\in T}\big((Q,d)\uplus(P_t,c_t)\big)$. Then $\mathfrak{X}$ is a T-sequence of elements of $(Q,d)\uplus(P_t,c_t)$ whose components have the same label, say $\mathfrak{X}=(i_t,x_t)_{t\in T}$ with $i_t\in\{1,2\}$ and $x_t\in Q$ if $i_t=1$ and $x_t\in P_t$ if $i_t=2$, and $(d\uplus c_t)(i_t,x_t)=(d\uplus c_s)(i_s,x_s)$ for all $s,t\in T$. Define the map

$$h\colon\ \bigotimes_{t\in T}\big((Q,d)\uplus(P_t,c_t)\big)\to(Q,d)\uplus\bigotimes_{t\in T}(P_t,c_t)$$

as follows:

$$h((i_t,x_t)_{t\in T})=\begin{cases}(2,(x_t)_{t\in T}) & \text{if } i_t=2 \text{ for all } t\in T,\\ (1,x_j) & \text{if } S=\{t\in T\mid i_t=1\}\neq\emptyset \text{ and } j=\min S.\end{cases}$$

(We assume that T is well-ordered, and we take the minimum with respect to a fixed well-ordering.) We need to verify that h is a homomorphism. It is clear that h preserves labels. As regards preservation of order, let $\mathfrak{X}_\ell=(i_t^\ell,x_t^\ell)_{t\in T}$ ($\ell=1,2$), and assume that $\mathfrak{X}_1\le\mathfrak{X}_2$ in $\bigotimes_{t\in T}\big((Q,d)\uplus(P_t,c_t)\big)$. Then $(i_t^1,x_t^1)\le(i_t^2,x_t^2)$ in $(Q,d)\uplus(P_t,c_t)$ for all $t\in T$, which in turn implies that $i_t^1=i_t^2$ and $x_t^1\le x_t^2$ (in (Q,d) or in (P_t,c_t), depending on the value of i_t^1) for all $t\in T$. Thus the sets

$$S_\ell=\{t\in T\mid i_t^\ell=1\}\quad(\ell=1,2)$$

are equal. Hence either $h(\mathfrak{X}_\ell)=(2,(x_t^\ell)_{t\in T})$ for $\ell=1,2$ or $h(\mathfrak{X}_\ell)=(1,x_j^\ell)$ for $\ell=1,2$, where $j=\min S_1=\min S_2$. In both cases it is obvious that $h(\mathfrak{X}_1)\le h(\mathfrak{X}_2)$. □

Theorem 4.5 *Let $(a_t)_{t\in T}$ be a family of elements of $\tilde{\mathcal{P}}_k$, and let $b\in\tilde{\mathcal{P}}_k$. If $(a_t)_{t\in T}$ has a supremum in $\tilde{\mathcal{P}}_k$, then the family $(b\wedge a_t)_{t\in T}$ has a supremum in $\tilde{\mathcal{P}}_k$, and it holds that*

$$b\wedge\bigvee_{t\in T}a_t=\bigvee_{t\in T}(b\wedge a_t).$$

Similarly, if $(a_t)_{t\in T}$ has an infimum in $\tilde{\mathcal{P}}_k$, then the family $(b\wedge a_t)_{t\in T}$ has an infimum in $\tilde{\mathcal{P}}_k$, and it holds that

$$b\vee\bigwedge_{t\in T}a_t=\bigwedge_{t\in T}(b\vee a_t).$$

Proof The claim follows from Lemmas 4.3 and 4.4 and the fact that we are dealing with finite k-posets only. □

Corollary 4.6 *$(\tilde{\mathcal{P}}_{k\omega},\le)$ is a distributive lattice.*

Recall that a core is a *finite* k-poset (P, c) such that all endomorphisms of (P, c) are surjective.

Proposition 4.7 *The (equivalence classes of) cores are compact in* $(\tilde{\mathcal{P}}_{k\omega}, \leq)$. *The (equivalence classes of) cores with at most one connected components are prime in* $(\tilde{\mathcal{P}}_{k\omega}, \leq)$.

Proof Let a be a core, and let $X \subseteq \tilde{\mathcal{P}}_{k\omega}$ such that $a \leq \bigvee X$. As $\tilde{\mathcal{P}}_k$ is countable and join-dense in $\tilde{\mathcal{P}}_{k\omega}$, we can assume that X is countable. We are looking for a finite subset $X_1 \subseteq X$ such that $a \leq \bigvee X_1$. We have $a = a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\}$, by the ω-join-distributivity. Therefore there is a k-poset homomorphism $\varphi : a \to \biguplus\{a \otimes x \mid x \in X\}$. Since a is a disjoint union of finitely many connected components, say $a = a_1 \uplus \cdots \uplus a_n$, then for each $1 \leq i \leq n$, $\varphi(a_i)$ is also connected and there is an $x_i \in X$ such that $\varphi(a_i) \subseteq a \otimes x_i$. Thus φ is a k-poset homomorphism from a to $a \otimes x_1 \uplus \cdots \uplus a \otimes a_n$, i.e., $a \leq (a \wedge x_1) \vee \cdots \vee (a \wedge x_n) = a \wedge (x_1 \vee \cdots \vee x_n) \leq x_1 \vee \cdots \vee x_n$. Therefore we can set $X_1 := \{x_1, \ldots, x_n\}$, and we conclude that a is compact.

If a is a core with exactly one connected component, say $a = a_1$, then the above proof shows that $a \leq x_1$ and we have that a is prime. □

All elements of $\tilde{\mathcal{P}}_k$ are finite joins of elements of $\mathcal{J}_k$, and are hence compact in $\tilde{\mathcal{P}}_{k\omega}$. Are they also compact in the MacNeille completion $\hat{\mathcal{P}}_k$ of $\tilde{\mathcal{P}}_k$? This is still an open question, and seems to be intimately related with the distributivity of $\hat{\mathcal{P}}_k$. A positive answer will say that $\hat{\mathcal{P}}_k$ is an algebraic lattice.

In [19], gaps and dualities in various Heyting categories are investigated, where Heyting category stands for a category whose homomorphism order constitutes a Heyting algebra, and henceforth it is a distributive lattice. We do not know whether the class of finite k-posets is a Heyting category. Also the gaps and dualities of the homomorphism order of k-posets remain a topic of future research.

5 Bounded k-Posets with Fixed Labels at the Extreme Points

Recall that we denote by $\mathcal{L}_k$ the set of all k-lattices and we denote $\tilde{\mathcal{L}}_k = \mathcal{L}_k/\equiv$. $\tilde{\mathcal{L}}_k$ is clearly a subposet of $\tilde{\mathcal{P}}_k$, but it is not a sublattice of $\tilde{\mathcal{P}}_k$, for the simple reason that the disjoint union of two incomparable k-lattices is not (homomorphically equivalent to) a k-lattice. Even if we consider the subposet of $\tilde{\mathcal{P}}_k$ consisting of (the equivalence classes of) those k-posets whose connected components are lattices, we do not have a sublattice nor even a meet-subsemilattice of $\tilde{\mathcal{P}}_k$. This is due to the fact that the label-matching product of two k-lattices is not in general (homomorphically equivalent to) a k-lattice, as Fig. 3 illustrates. An identical argument shows that k-trees do not constitute a sublattice of $\tilde{\mathcal{P}}_k$, and neither do k-forests (k-posets whose connected components are k-trees).

In this section, we will consider families of bounded k-posets with fixed labels on their extreme points. These families constitute meet-subsemilattices of $\tilde{\mathcal{P}}_k$. We will describe the suprema within these families, and we establish that these families constitute universal distributive lattices under the homomorphism order.

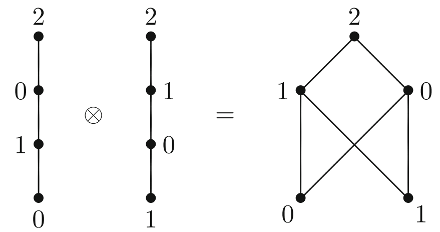

Fig. 3 The label-matching product of k-lattices is not in general a k-lattice

Let $k \geq 1$, and let $a, b \in \{0, 1, \dots, k-1\}$. Denote by $\mathcal{P}_k^{ab}$ the set of finite bounded k-posets (P, c) with a largest element $\top$ and a smallest element $\perp$ such that $c(\top) = a$ and $c(\perp) = b$. Denote $P^\dagger := P \setminus \{\top, \perp\}$. Again, denote by $\tilde{\mathcal{P}}_k^{ab}$ the quotient $\mathcal{P}_k^{ab}/\!\equiv$.

Let $(P, c), (P', c') \in \mathcal{P}_k^{ab}$. It is easy to verify that the label-matching product $(P, c) \otimes (P', c')$ is again in $\mathcal{P}_k^{ab}$, and hence $\tilde{\mathcal{P}}_k^{ab}$ is a meet-subsemilattice of $\tilde{\mathcal{P}}_k$. However, the core of the disjoint union $(P, c) \,\dot\cup\, (P', c')$ is not in general a bounded k-poset, and hence we need to verify if (P, c) and (P', c') have an infimum in $\tilde{\mathcal{P}}_k^{ab}$.

Define the binary operation $\uplus$ on $\mathcal{P}_k^{ab}$ as follows. For $i = 1, 2$, let $(P_i, c_i) \in \mathcal{P}_k^{ab}$, and let $\top_{P_i}$ and $\perp_{P_i}$ be the largest and smallest elements of P_i. We let $(P_1, c_1) \uplus (P_2, c_2) = (Q, d)$, where

$$Q = (P_1^\dagger \,\dot\cup\, P_2^\dagger) \cup \{\top_Q, \perp_Q\}$$

where $\top_Q, \perp_Q$ are new elements not occurring in P_1 nor P_2. The ordering of Q is defined as follows: $\top_Q$ and $\perp_Q$ are the largest and the smallest element of Q, respectively, and for $(i, x), (j, y) \in P_1^\dagger \,\dot\cup\, P_2^\dagger$, we have $(i, x) \leq (j, y)$ if and only if $i = j$ and $x \leq y$ in P_i. The labeling d of Q is defined by

$$d(x) = \begin{cases} a & \text{if } x = \top_Q, \\ b & \text{if } x = \perp_Q, \\ c_i(y) & \text{if } x = (i, y) \in P_1^\dagger \,\dot\cup\, P_2^\dagger. \end{cases}$$

Thus, we can think of $(P_1, c_1) \uplus (P_2, c_2)$ being obtained from the disjoint union $(P_1, c_1) \,\dot\cup\, (P_2, c_2)$ by gluing together the top and bottom elements of the connected components.

Lemma 5.1 $(P_1, c_1) \uplus (P_2, c_2)$ *is the supremum of* (P_1, c_1) *and* (P_2, c_2) *in* $\tilde{\mathcal{P}}_k^{ab}$.

Proof Denote $(Q, d) = (P_1, c_1) \uplus (P_2, c_2)$ For $i = 1, 2$, the mapping $h_i \colon (P_i, c_i) \to (Q, d)$ given by

$$h_i(x) = \begin{cases} \top_Q & \text{if } x = \top_{P_i}, \\ \perp_Q & \text{if } x = \perp_{P_i}, \\ (i, x) & \text{if } x \in P_i^\dagger \end{cases}$$

is easily seen to be a homomorphism.

Now, assume that $(P', c') \in \mathcal{P}_k^{ab}$ such that there exist homomorphisms $h_i\colon (P_i, c_i) \to (P', c')$ for $i = 1, 2$. Define a map $h\colon (Q, d) \to (P', c')$ by

$$h(x) = \begin{cases} \top_{P'} & \text{if } x = \top_Q, \\ \bot_{P'} & \text{if } x = \bot_Q, \\ h_i(y) & \text{if } x = (i, y) \in Q^\dagger. \end{cases}$$

It is straightforward to verify that h is a homomorphism. We conclude that $(P_1, c_1) \uplus (P_2, c_2)$ is the supremum of (P_1, c_1) and (P_2, c_2) in $\tilde{\mathcal{P}}_k^{ab}$. □

Proposition 5.2 $(\tilde{\mathcal{P}}_k^{ab}; \otimes, \uplus)$ *is a distributive lattice.*

Proof The claim that $(\tilde{\mathcal{P}}_k^{ab}; \otimes, \uplus)$ is a lattice follows from Lemma 5.1 and the discussion preceding it.

Let $(P_i, c_i) \in \mathcal{P}_k^{ab}$ for $i = 1, 2, 3$. We will verify that the distributive law

$$P_1 \otimes (P_2 \uplus P_3) \equiv (P_1 \otimes P_2) \uplus (P_1 \otimes P_3)$$

holds by showing that the k-posets on each side of the above equation are homomorphically equivalent.

First, define the map $h\colon P_1 \otimes (P_2 \uplus P_3) \to (P_1 \otimes P_2) \uplus (P_1 \otimes P_3)$ by

$$h(X, Y) = \begin{cases} \top & \text{if } X = \top_{P_1} \text{ or } Y = \top_{P_2 \uplus P_3}, \\ \bot & \text{if } X = \bot_{P_1} \text{ or } Y = \bot_{P_2 \uplus P_3}, \\ (i, (X, y)) & \text{if } X \in P_1^\dagger,\ Y = (i, y),\ y \in P_{i+1}^\dagger\ (i = 1, 2). \end{cases}$$

It is clear that h is label-preserving. We need to verify that h is also order-preserving. Thus, let $(X, Y) < (X', Y')$ in $P_1 \otimes (P_2 \uplus P_3)$. If $X = \bot_{P_1}$ or $Y = \bot_{P_2 \uplus P_3}$ or $X' = \top_{P_1}$ or $Y' = \top_{P_2 \uplus P_3}$, then it is clear that $h(X, Y) \leq h(X', Y')$. Otherwise $X, X' \in P_1^\dagger$, $Y, Y' \in (P_2 \uplus P_3)^\dagger$ and so $X \leq X'$ in P_1 and $Y \leq Y'$ in $P_2 \uplus P_3$. The latter condition implies that $Y = (i, y)$, $Y' = (i, y')$ for some $i \in \{1, 2\}$, $y, y' \in P_{i+1}$ and $y \leq y'$ in P_{i+1}. Thus,

$$h(X, Y) = (i, (X, y)) \leq (i, (X', y')) = h(X', Y') \quad \text{in } (P_1 \otimes P_2) \uplus (P_1 \otimes P_3).$$

Next, we define the map $g\colon (P_1 \otimes P_2) \uplus (P_1 \otimes P_3) \to P_1 \otimes (P_2 \uplus P_3)$ by

$$g(X) = \begin{cases} (\top_{P_1}, \top_{P_2 \uplus P_3}) & \text{if } X = \top, \\ (\bot_{P_1}, \bot_{P_2 \uplus P_3}) & \text{if } X = \bot, \\ (x, (i, y)) & \text{if } X = (i, (x, y)) \in \big((P_1 \otimes P_2) \uplus (P_1 \otimes P_3)\big)^\dagger. \end{cases}$$

It is clear that g is label-preserving. We need to verify that g is also order-preserving. Thus, let $X < X'$ in $(P_1 \otimes P_2) \uplus (P_1 \otimes P_3)$. If $X = \bot$ or $Y = \top$, then it is clear that $g(X) \leq g(X')$. Otherwise $X, X' \in \big((P_1 \otimes P_2) \uplus (P_1 \otimes P_3)\big)^\dagger$ and so $X = (i, (x, y))$, $X' = (i, (x', y'))$ for some $i \in \{1, 2\}$ and $x, x' \in P_1$, $y, y' \in P_{i+1}$ and $x \leq x'$ in P_1 and $y \leq y'$ in P_{i+1}. Thus

$$h(X) = (x, (i, y)) \leq (x', (i, y')) = h(X') \quad \text{in } P_1 \otimes (P_2 \uplus P_3).$$

Since both h and g are homomorphisms, we conclude that the claimed homomorphical equivalence holds. □

Theorem 5.3 *The posets $\tilde{\mathcal{P}}_k^{ab}$ and $\tilde{\mathcal{L}}_k^{ab}$ are universal for every $k \geq 3$, $a, b \in \{0, \ldots, k-1\}$.*

Proof The proof is a simple adaptation of the proof of the universality of $\tilde{\mathcal{L}}_k$ presented in [17, Theorem 4.6]. The k-posets $\mathcal{E}(A)$ used in the representation of an arbitrary countable poset are 3-lattices. We just need to adjoin new top and bottom elements $\top$ and $\perp$ with labels $c(\top) = a$ and $c(\perp) = b$. The resulting k-posets $\mathcal{E}'(A)$ are members of $\tilde{\mathcal{L}}_k^{ab}$, and it is clear that there exists a homomorphism from $\mathcal{E}'(A)$ to $\mathcal{E}'(B)$ if and only if there exists a homomorphism from $\mathcal{E}(A)$ to $\mathcal{E}(B)$. The claim thus follows. □

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