Counting maximal antichains and independent sets

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Abstract

Answering several questions of Duffus, Frankl and Rödl, we give asymptotics for the logarithms of (i) the number of maximal antichains in the n-dimensional Boolean algebra and (ii) the numbers of maximal independent sets in the covering graph of the n-dimensional hypercube and certain natural subgraphs thereof. The results in (ii) are implied by more general upper bounds on the numbers of maximal independent sets in regular and biregular graphs.

We also mention some stronger possibilities involving actual rather than logarithmic asymptotics.

1 Introduction

Write $\mathrm{ma}(P)$ for the number of maximal antichains of a poset P and $\mathrm{mis}(G)$ for the number of maximal independent sets in a graph G. (For "antichain" and "independent set" see e.g. [5] and [2] respectively.) Denote by $\mathcal{B}_n$ the Boolean algebra of order n (that is, the collection of subsets of $\{1,\ldots,n\}$ ordered by containment); by Q_n the "covering" (or "Hamming") graph of the n-cube (the graph with vertex set $\{0,1\}^n$ and two vertices adjacent if they differ in exactly one coordinate); and by $\mathcal{B}_{n,k}$ the subgraph of Q_n consisting of strings of weight k and $k+1$ (where, of course, *weight* means number of 1's).

We are interested in estimating the logarithms of the quantities $\mathrm{ma}(\mathcal{B}_n)$, $\mathrm{mis}(\mathcal{B}_{n,k})$ and $\mathrm{mis}(Q_n)$, all problems suggested by Duffus, Frankl and Rödl in [3] and [4]. As they observe (*cf.* the paragraph following Conjecture 5.1 below), it is not hard to see that (with $\log = \log_2$ throughout)

$$\log \mathrm{ma}(\mathcal{B}_n) \geq \binom{n-1}{\lfloor n/2 \rfloor}, \ \ \log \mathrm{mis}(\mathcal{B}_{n,k}) \geq \binom{n-1}{k} \ \text{ and } \ \log \mathrm{mis}(Q_n) \geq 2^{n-2}. \quad (1)$$

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On the other hand they show

$$\log \mathrm{ma}(\mathcal{B}_n) < (1+o(1))\tbinom{n}{\lfloor n/2\rfloor}, \tag{2}$$

$$\log \mathrm{mis}(Q_n) < (0.78+o(1))2^{n-1}$$

and

$$\log \mathrm{mis}(\mathcal{B}_{n,k}) < (1.3563+o(1))\tbinom{n-1}{k}.$$

Note that $\log \mathrm{mis}(Q_n) \le 2^{n-1}$ and $\log \mathrm{mis}(\mathcal{B}_{n,k}) \le \min\{\binom{n}{k}, \binom{n}{k+1}\}$ are trivial (see Proposition 2.1(a)), while (2) is Kleitman's celebrated bound [10] on the *total* number of antichains in $\mathcal{B}_n$. In particular (2) makes no use of maximality, and the authors of [4] say: "... the problem we are most interested in solving is this: show there exists $\alpha < 1$ such that $\log_2 \mathrm{ma}(n) \le \alpha\binom{n}{n/2}$" (where their $\mathrm{ma}(n)$ is our $\mathrm{ma}(\mathcal{B}_n)$).

Here we settle all these problems, showing that the lower bounds in (1) are asymptotically tight, *viz.*

Theorem 1.1. (a) $\log \mathrm{ma}(\mathcal{B}_n) = (1+o(1))\binom{n-1}{\lfloor n/2\rfloor}$*;*

(b) $\log \mathrm{mis}(Q_n) = (1+o(1))2^{n-2}$*;*

(c) $\log \mathrm{mis}(\mathcal{B}_{n,k}) = (1+o(1))\binom{n-1}{k}$

(where $o(1) \to 0$ as $n \to \infty$). Note that, with $\mathcal{B}_{n,k}$ regarded as the poset consisting of levels k and $k+1$ of $\mathcal{B}_n$, (c) also says $\log \mathrm{ma}(\mathcal{B}_{n,k}) = (1+o(1))\binom{n-1}{k}$.

The results giving (b) and (c) are actually far more general:

Theorem 1.2. (a) *For any d-regular, n-vertex graph G,*

$$\log \mathrm{mis}(G) < \begin{cases} (1+o(1))\frac{n}{4} & \text{if } G \text{ is triangle-free,} \\ (1+o(1))\frac{n\log 3}{6} & \text{in general,} \end{cases} \tag{3}$$

where $o(1) \to 0$ as $d \to \infty$.

(b) *For any (r,s)-biregular, n-vertex (bipartite) graph G,*

$$\log \mathrm{mis}(G) < (1+o(1))\frac{rsn}{(r+s)^2},$$

where $o(1) \to 0$ as $\max\{r,s\} \to \infty$.

The first and third bounds are easily seen to imply (respectively) parts (b) and (c) of Theorem 1.1. All three bounds are best possible at the level of the asymptotics of the logarithm. (This is shown, for example, by (1) for the first and third bounds, and by the graphs H_n described in Section 5 for the second.)

The rest of the paper is organized as follows. Section 2 recalls what little background we need, mainly a few known bounds on the parameter "mis" plus Shearer's entropy lemma. The proofs of Theorems 1.2 and 1.1(a) are given (in reverse order because the former is easier) in Sections 3 and 4 respectively. Finally, Section 5 suggests a strengthening of Theorem 1.1 and fills in examples—possibly extremal—for the second bound in Theorem 1.2(a).

The proofs of the theorems turn out not to involve too much work once one gets on the right track. In each case, seeking to identify an unknown (maximal independent set or antichain) I, we show that one can, at the cost of specifying a few small subsets of I and $\bar{I}$, reduce determination of I to determination of $I \cap Z$ for a relatively easily manageable subset Z of our universe (i.e. $V(G)$ or the ground set of $\mathcal{B}_n$). Since the specified sets are small, there are not many ways to choose them, so that possibilities for $I \cap Z$ contribute the main term (plus part of the error) in each of our bounds. Informally we tend to think of paying a small amount of "information" to reduce specification of I to specification of $I \cap Z$.

Our original arguments for Theorems 1.1 and 1.2 were based on a simple but (we think) novel combination of random sampling and entropy. The proof of Theorem 1.2 in Section 3 uses this approach. As we recently noticed, there is an even simpler idea, due to Sapozhenko [13], that can substitute for the sampling/entropy part of the original argument; this is explained at the end of Section 3. We have retained the original proof, in the case of Theorem 1.2 at least, because we think the approach is interesting and potentially useful elsewhere; but in the case of Theorem 1.1, for brevity's sake, we have omitted the original and given only the "Sapozhenko version."

2 Preliminaries

The assertions of Theorem 1.2 will reduce to the following known and/or easy facts.

Proposition 2.1. (a) *For G bipartite with bipartition $A \cup B$,*

$$\mathrm{mis}(G) \leq 2^{\min\{|A|,|B|\}}.$$

(b) *For any n-vertex graph G,*

$$\mathrm{mis}(G) \le 3^{n/3},$$

with equality iff G is the disjoint union of $n/3$ triangles.

(c) *For any n-vertex, triangle-free graph G,*

$$\mathrm{mis}(G) \le 2^{n/2},$$

with equality iff G is a perfect matching.

Here (a) is trivial; (b) was proved by Moon and Moser [12] in answer to a question of Erdős and Moser; and (c) is due to Hujter and Tuza [8]. We will not need the information about cases of equality. Note that the d-regularity (with d large) in Theorem 1.2(a) multiplies the exponents in the bounds directly implied by parts (b) and (c) of Proposition 2.1 by roughly 1/2.

We also need the following lemma of J. Shearer [1]. (See e.g. [11] for entropy basics or [9] for a quicker introduction and another application of Shearer's Lemma.) We use H for binary entropy and, for a random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ and $A \subset [n]$, set $\mathbf{X}_A = (\mathbf{X}_i : i \in A)$.

Lemma 2.2. *Let $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_n)$ be a discrete random vector and $\mathcal{A}$ a collection of subsets (possibly with repeats) of $[n]$, with each element of $[n]$ contained in at least m members of $\mathcal{A}$. Then*

$$H(\mathbf{X}) \le \frac{1}{m} \sum_{A \in \mathcal{A}} H(\mathbf{X}_A).$$

(The statement in [1] is less general, but its proof gives Lemma 2.2.)

Finally, we write $\binom{n}{<t}$ for $\sum_{j<t} \binom{n}{j}$, recalling that for $t \le n/2$ (see e.g. [6, Lemma 16.19])

$$\binom{n}{<t} \le 2^{H(t/n)n}. \tag{4}$$

3 Proof of Theorem 1.2

Notation. For G as in either part of the theorem, we use $\Delta = \Delta(G)$ for maximum degree, "$\sim$" for adjacency, N_x for the neighborhood of x, and $d(x)$ for $|N_x|$. For $S, T \subseteq V = V(G)$ and $x \in V$: $d_S(x) = |N_x \cap S|$; $\Gamma(S) = (\cup_{x \in S} N_x) \cup S$; $E(S)$ is the sets of edges contained in S; and $E(S, T)$

(used here only with $S \cap T = \emptyset$) is the set of edges having one end in each of S, T. We use $\mathcal{M}$ for the set of maximal independent sets of G.

Proof. We work with a parameter t to be specified below and set $k = t^2 \log \Delta$. For $I \in \mathcal{M}$, set $C_I = \{x \in V : d_I(x) \geq t \ln t\}$. We may associate with each $I \in \mathcal{M}$ some $I_0 \subseteq I$ of size $\lceil |I|/t \rceil$ with

$$|C_I \setminus \Gamma(I_0)| < n/t, \tag{5}$$

existence of such an I_0 being given by the observation that, for $\mathbf{I}_0$ chosen uniformly from the $\lceil |I|/t \rceil$-subsets of I,

$$\begin{aligned} \mathsf{E}|C_I \setminus \Gamma(\mathbf{I}_0)| &= \sum_{v \in C_I} \Pr(v \notin \Gamma(\mathbf{I}_0)) \\ &< n(1 - 1/t)^{t \ln t} < n/t. \end{aligned}$$

Let $X = V \setminus (C_I \cup \Gamma(I_0))$, $Y = \{v \in X : d_X(v) \geq k\}$, $J = I \cap Y$ and $Z = X \setminus (Y \cup \Gamma(J))$. We may choose $I \in \mathcal{M}$ by specifying: (i) I_0; (ii) $C_I \setminus \Gamma(I_0)$; (iii) J; and (iv) $I \cap Z$; so we just have to bound the numbers of choices for these steps. First, by (4), the number of choices in each of (i), (ii) is at most $\exp_2[H(1/t)n]$. Second, since $J \subseteq Y$ satisfies

$$|J \cap N_v| < t \ln t \quad \forall v \in X, \tag{6}$$

the log of the number of possibilities in (iii) is at most $H(\mathbf{J})$, where $\mathbf{J}$ is chosen uniformly from the collection of subsets J of Y satisfying (6). Here Shearer's Lemma (with $\mathbf{X}$ the indicator of $\mathbf{J}$ and $\mathcal{A} = \{N_v \cap Y : v \in X\}$) gives, using the definition of Y,

$$\begin{aligned} H(\mathbf{J}) \ &\leq \ k^{-1} \sum_{v \in X} H(\mathbf{J} \cap N_v) \ \leq \ k^{-1} \sum_{v \in X} \log \tbinom{\Delta}{< t \ln t} \\ &< \ (k^{-1} t \ln t \log \Delta) n \ = \ (n \ln t)/t. \end{aligned} \tag{7}$$

It remains to bound the number of possibilities in (iv). Note that $I \cap Z$ is a *maximal* independent subset of Z (since $Z \cap \Gamma(I \setminus Z) = \emptyset$). The (easy) point here is that the requirement

$$d_Z(v) < k \quad \forall v \in Z \tag{8}$$

(implied by $Z \subseteq X \setminus Y$) limits the size of Z (in (a)) or of the intersection of Z with one of the parts of the bipartition (in (b)). We now consider these cases separately.

(a) From (8) we have

$$d|Z| = \sum_{v\in Z} d(v) = 2|E(Z)| + |E(Z, V\setminus Z)| < k|Z| + d(n-|Z|),$$

whence (note we will have $k < d$)

$$|Z| < \frac{nd}{2d-k} = (1/2 + O(k/d))n.$$

Parts (c) and (b) of Proposition 2.1 thus bound the log of the number of possibilities for $I\cap Z$ by $(1+O(k/d))n/4$ if G is triangle-free, and by $(1+O(k/d))(\log 3)n/6$ in general. Combining this with our bounds for (i)-(iii) and setting $t = d^{1/3}$, we have Theorem 1.2(a) with $o(1) = O(\max\{(\log t)/t, (t^2\log d)/d\}) = O(d^{-1/3}\log d)$.

(b) Let the bipartition of G be $A\cup B$, with $d(x) = r$ for $x\in A$ and, w.l.o.g., $(\Delta =)\ r\geq s$. Notice to begin that Proposition 2.1(a) gives

$$\log \mathrm{mis}(G) \leq |A| = sn/(r+s) = (1+s/r)rsn/(r+s)^2. \tag{9}$$

We will prove the statement in (b) with $o(1) = O(\min\{r^{1/3}s^{-2/3}\log r, s/r\})$, so in view of (9) may assume (to deal with a *very* minor detail below) that

$$r^{4/3}\log r < s^{5/3}. \tag{10}$$

Let $U = Z\cap A$ and $W = Z\cap B$. We have

$$r|U| = \sum_{v\in U} d(v) = \sum_{v\in U} d_W(v) + |E(U, B\setminus W)| < k|U| + s(|B|-|W|),$$

whence $r(1-k/r)|U| + s|W| < rsn/(r+s)$, implying either $|U| \leq (1-k/r)^{-1}rsn/(r+s)^2 = (1+O(k/r))rsn/(r+s)^2$ or $|W| \leq rsn/(r+s)^2$ (where we used (10) to say $k<r$). Proposition 2.1(a) now bounds the number of possibilities for $I\cap Z$ by $\exp_2[(1+O(k/r))rsn/(r+s)^2]$.

Finally, combining this with (9) and our earlier bounds for (i)-(iii), and setting $t = r^{2/3}s^{-1/3}$, gives Theorem 1.2(b) with $o(1)$ on the order of

$$\min\left\{\max\left\{\frac{(\log t)(r+s)^2}{trs}, \frac{k}{r}\right\}, \frac{s}{r}\right\} = \Theta\left(\min\left\{\frac{r^{1/3}\log r}{s^{2/3}}, \frac{s}{r}\right\}\right).$$

∎

Sapozhenko's method. As promised, we next show how the first two paragraphs of the above argument (through (7)) can be replaced by a remarkably simple idea of A. Sapozhenko (see [13, Theorem 6] or, for another description, [7, Lemma 2.3]).

We again work with a parameter, say b, whose value will be specified below. For a given maximal independent set I, let $X_1 = V$ and repeat for $i = 1, \ldots$ until no longer possible: choose $x_i \in X_i \cap I$ with $d_{X_i}(x_i) \geq b$, and set $X_{i+1} = X_i \setminus (\{x_i\} \cup N_{x_i})$. Let $Y = X_{q+1}$ be the final X_i, and notice that $Y = V \setminus \Gamma(\{x_1, \ldots, x_q\})$ and

$$d_Y(x) < b \ \ \forall x \in Y \cap I.$$

Set $Z = \{x \in Y : d_Y(x) < b\}$ ($\supseteq Y \cap I$). We have

(i) $q < n/b$ (trivially);

(ii) $d_Z(x) < b \ \ \forall x \in Z$; and

(iii) $I \cap Z$ is a maximal independent subset of Z (since $I \cap (V \setminus Z) = \{x_1, \ldots, x_q\} \not\sim Z$).

Now (ii) corresponds to (8), and the discussion under (a) and (b) above (using (iii)) bounds the number of possibilities for $I \cap Z$ as before, with the k's replaced by b's. (For example, the main bound in (b) is now $\exp_2[(1 + O(b/r))rsn/(r+s)^2)]$.)

Essentially optimal values for b are then $(d \log d)^{1/2}$ and $rs^{-1/2}\sqrt{\log r}$ in (a) and (b) respectively, yielding $o(1)$'s (as in the statement of the theorem) on the order of $d^{-1/2}\sqrt{\log d}$ in (a), and, in (b),

$$\min\left\{\max\left\{\frac{(\log b)(r+s)^2}{brs}, \frac{b}{r}\right\}, \frac{s}{r}\right\} = \Theta\left(\min\left\{\sqrt{\frac{\log r}{s}}, \frac{s}{r}\right\}\right).$$

(So this also gives somewhat better error terms than the original argument, though the errors are in any case not likely to be close to the truth.)

4 Proof of Theorem 1.1(a)

This requires a little more care than the proof of Theorem 1.2, though the basic idea is similar. As noted earlier, we skip our original argument and just give the one based on "Sapozhenko's method."

Notation. We write $\mathcal{B}$ for $\mathcal{B}_n$ (and follow a common abuse in using the same symbol for a poset and its ground set). Elements of $\mathcal{B}$ (usually denoted

x, y, z) may be thought of as either binary strings or subsets of $[n]$ (so for $x \in \mathcal{B}$ thought of as a string, $|x|$ is the number of 1's in x). Set (for $i \in \{0\} \cup [n]$) $L_i = \{x \in \mathcal{B} : |x| = i\}$, $\ell_i = |L_i|$ $(= \binom{n}{i})$ and, for $S \subseteq \mathcal{B}$, $S_i = S \cap L_i$. We also set $N = 2^n$ and $M = \binom{n}{\lfloor n/2 \rfloor}$ $(= \Theta(n^{-1/2}N))$.

For $x \in \mathcal{B}$, N_x is the neighborhood of x in the *comparability graph*, say G, of $\mathcal{B}$ (that is, the graph with $x \sim y$ iff $x < y$ or $x > y$). Of course an antichain of $\mathcal{B}$ is just an independent set of G, but in this case Theorem 1.2 gives only a weak bound on $\mathrm{mis}(G) = \mathrm{ma}(\mathcal{B})$. We write (as usual) $x \gtrdot y$ (x *covers* y) if $x > y$ and there is no z with $x > z > y$, and set $d_S^+(y) = |\{x \in S : x \gtrdot y\}|$ $(S \subseteq \mathcal{B}, y \in \mathcal{B})$. Also for $S \subseteq \mathcal{B}$, we write $\Gamma^+(S)$ for $\{y \in \mathcal{B} \setminus S : \exists x \in S, x < y\}$ and $\mathrm{ma}(S)$ for the number of maximal antichains of S (more properly, of the restriction of $\mathcal{B}$ to S). Finally, we recall that $S \subseteq \mathcal{B}$ is *convex* if $x < y < z$ and $x, z \in S$ imply $y \in S$.

Proof. Set $b = n^{3/4}\sqrt{\log n}$. Given a maximal antichain I, let $X_1 = \mathcal{B}$ and repeat for $i = 1, \dots$ until no longer possible: choose $x_i \in I \cap X_i$ with $|N_{x_i} \cap X_i| \geq b$, and set $X_{i+1} = X_i \setminus (\{x_i\} \cup N_{x_i})$. Let $Y = X_{q+1}$ be the final X_i—so in particular $d_Y^+(y) < b$ $\forall y \in I \cap Y$—and set $Z = \{y \in Y : d_Y^+(y) < b\}$ $(\supseteq I \cap Y)$.

The number of possibilities for $\{x_1, \dots, x_q\}$ is at most $2^{H(1/b)N}$ (since $q < N/b$), and we have

$$d_Z^+(x) < b \ \ \forall x \in Z, \tag{11}$$

$$I \cap Z \textit{ is a maximal antichain of } Z$$

(since $I \cap (\mathcal{B} \setminus Z) = \{x_1, \dots, x_q\} \not\sim Z$), and, we assert,

$$Z \text{ is convex.} \tag{12}$$

Proof of (12). Since $Y = \mathcal{B} \setminus \bigcup_i (\{x_i\} \cup N_{x_i})$ is obviously convex, (12) follows from the observation that

$$Y \setminus Z \text{ is a downward-closed subset of } Y.$$

For suppose that—now regarding elements of $\mathcal{B}$ as subsets of $[n]$, for which we prefer capitals—$A \in Y$ and $A \subseteq B \in Y \setminus Z$. Since $B \notin Z$, there are distinct $i_1, \dots, i_b \in [n] \setminus B$ with $B \cup \{i_j\} \in Y$ $\forall j \in [b]$, whence, since Y is convex (and $A \in Y$), $A \cup \{i_j\} \in Y$ $\forall j \in [b]$. But then $A \in Y \setminus Z$.

$\blacksquare$

Theorem 1.1(a) (with $o(1)$ on the order of $n^{-1/4}\sqrt{\log n}$) thus follows from the next two assertions (and the fact that $M \le 2\binom{n-1}{\lfloor n/2 \rfloor}$).

Claim 1. If $Z \subseteq \mathcal{B}$ is convex and satisfies (11), then

$$|Z| < (1 + O(b/n))M. \tag{13}$$

Claim 2. If $Z \subseteq \mathcal{B}$ is convex, then $\mathrm{ma}(Z) \le 2^{|Z|/2}$.

(Note this does require convexity; e.g. it fails if Z is a 3-element chain.)

Proof of Claim 1. We may assume $Z \subseteq L_r \cup \cdots \cup L_s$, with $(r, s) = (.4n, .6n)$ (since the rest of $\mathcal{B}$ is too small to affect (13)). Let $F = \Gamma^+(Z)$, $f_i = |F_i|$ and $z_i = |Z_i|$. The degree assumption (11) implies that, for any $i \in \{r, \ldots, s\}$,

$$(i + 1)f_{i+1} \ge \partial(F_i \cup Z_i, F_{i+1}) \ge (n - i)f_i + (n - i - b)z_i$$

(where $\partial(S,T) := |\{(x, y) : x \in S, y \in T, x < y\}|$), or, since $(n-i)/(i+1) = \ell_{i+1}/\ell_i$,

$$f_{i+1} \ge \frac{\ell_{i+1}}{\ell_i} f_i + \left(\frac{\ell_{i+1}}{\ell_i} - \frac{b}{i+1}\right) z_i \ge \frac{\ell_{i+1}}{\ell_i}[f_i + (1 - \varepsilon)z_i], \tag{14}$$

with $\varepsilon = 2.5b/n$. Composing the inequalities (14) (for $i = r, \ldots, s$) gives

$$f_{s+1} \ge (1 - \varepsilon) \sum_{j=r}^{s} \frac{\ell_{s+1}}{\ell_j} z_j,$$

so that

$$\frac{|Z|}{M} \;\le\; \sum_{j=r}^{s} \frac{z_j}{\ell_j} \;\le\; (1 - \varepsilon)^{-1} \frac{f_{s+1}}{\ell_{s+1}} \;\le\; (1 - \varepsilon)^{-1}.$$

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Proof of Claim 2. This is an induction along the lines of (but easier than) the argument of [8]. Set $|Z| = m$. If Z does not contain a chain of length 3, then the comparability graph of Z is bipartite and we may apply Proposition 2.1(a). Otherwise there are $x, y \in Z$ with $x < y$ and $|y| \ge |x| + 2$, whence, since Z is convex, $|\Gamma^+(x) \cap Z| \ge 3$. We may, of course, further require that x be minimal in Z (so $Z \setminus \{x\}$ is convex), and then induction gives

$$\mathrm{ma}(Z) \le \mathrm{ma}(Z \setminus \{x\}) + \mathrm{ma}(Z \setminus (\{x\} \cup \Gamma^+(x))) \le 2^{(m-1)/2} + 2^{(m-4)/2} < 2^{m/2}.$$

(Notice that $\mathrm{ma}(Z \setminus (\{x\} \cup \Gamma^+(x)))$ bounds the number of maximal antichains of Z containing x, since for each such A, $A \setminus \{x\}$ is contained in some maximal antichain A' of $Z \setminus \{x\}$, and is recoverable from A' *via* $A = (A' \cup \{x\}) \setminus \{y \in Z : y \sim x\}$.)

∎

5 A stronger conjecture

In closing we would like to suggest that it might be possible to give the actual asymptotics (rather than just the asymptotics of the log) for the quantities considered in Theorem 1.1. This does not look easy, but we can at least guess what the truth should be:

Conjecture 5.1. (a) $\mathrm{ma}(\mathcal{B}_n) = \begin{cases} (1+o(1))\, n\, \exp_2[\binom{n-1}{(n-1)/2}] & \textit{if } n \textit{ is odd,} \\ (2+o(1))\, n\, \exp_2[\binom{n-1}{n/2}] & \textit{if } n \textit{ is even;} \end{cases}$

(b) $\mathrm{mis}(Q_n) = (2+o(1))\, n\, \exp_2[2^{n-2}]$*;*

(c) $\mathrm{mis}(\mathcal{B}_{n,k}) = (1+o(1))\, n\, \exp_2[\binom{n-1}{k}]$

(where $o(1) \to 0$ as $n \to \infty$).

The easy lower bounds are based on the observation (from [3]) that for any graph G and induced matching M of G, each of the $2^{|M|}$ sets consisting of one vertex from each edge of M extends to at least one maximal independent set, and these extensions are all different. For example, the lower bound for n odd in Conjecture 5.1(a) (other cases are similar) is obtained by noting that, for each $i \in [n]$, the set of pairs

$$\{\{x, x^i\} : x \in \{0,1\}^n, x_i = 0, |x| = (n-1)/2\}$$

(where x^i is gotten by flipping the ith coordinate of x) is an induced matching in the comparability graph of $\mathcal{B}_n$, and that there is only an insignificant amount of repetition in the corresponding list of at least $n \exp_2[\binom{n-1}{(n-1)/2}]$ maximal independent sets.

Finally we give the promised construction for the second bound in Theorem 1.2(a). For $d \equiv 2 \pmod 3$, let T be the disjoint union of $(d-2)/3$ triangles; let H consist of two disjoint copies of T plus all edges between them; and, for n divisible by $2(d-2)$, let H_n be the union of $\frac{n}{2(d-2)}$ disjoint copies of H. Then H_n is d-regular with $\mathrm{mis}(H_n) = 2^{n/(2(d-2))}3^{n/6}$, and it seems not impossible that this is extremal:

Conjecture 5.2. *For any d-regular, n-vertex G,* $\mathrm{mis}(G) \le 2^{n/(2(d-2))}3^{n/6}$*.*

This would be analogous to the fact—recently proved in spectacularly simple fashion by Yufei Zhao [14] (but by reducing to the bipartite case proved a little less simply in [9])—that the *total* number of independent sets in such a G is at most $(2^{d+1}-1)^{n/(2d)}$, a value achieved by a disjoint union of K_{dd}'s whenever $2d|n$. One would, of course, hope to also have analogues for the other parts of Theorem 1.2; but we don't see good candidates for these, and suspect that they do not have clean answers.

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