On infinite-finite duality pairs of directed graphs

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Abstract

The $(\mathcal{A}, \mathcal{D})$ duality pairs play a crucial role in the theory of general relational structures and in Constraint Satisfaction Problems. The case where both sides are finite is fully characterized. The case where both sides are infinite seems to be very complex. It is also known that no finite-infinite duality pair is possible if we make the additional restriction that both classes are antichains. In this paper (which is the first one of a series) we start the detailed study of the infinite-finite case.

Here we concentrate on directed graphs. We prove some elementary properties of the infinite-finite duality pairs, including lower and upper bounds on the size of $\mathcal{D}$, and show that the elements of $\mathcal{A}$ must be equivalent to forests if $\mathcal{A}$ is an antichain. Then we construct instructive examples, where the elements of $\mathcal{A}$ are paths or trees. Note that the existence of infinite-finite antichain dualities was not previously known.

Keywords: graph homomorphism; duality pairs; general relational structures; Constraint Satisfaction Problem; regular languages; nondeterministic finite automaton;

1. Introduction

In this paper we consider *directed graphs*, *homomorphisms* between them, and especially *duality pairs*. We start with the definitions.

A directed graph G is a pair (V, E) with $V = V(G)$ the set of vertices and $E = E(G) \subseteq V^2$ the set of (directed) edges. Unless stated otherwise "graph" refers to finite directed graphs in this paper. Forgetting about the orientation of the edges one

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gets the *underlying undirected graph*. For simplicity we drop the term "oriented" when referring to (oriented) *paths*, (oriented) *trees* and (oriented) *forests*, these are (directed) graphs whose underlying undirected graphs are path, trees, respectively forests in the traditional sense. In particular, paths, trees and forests have no loops and no pair of vertices is connected in both directions. Similarly, when we call a graph *connected*, refer to the *connected components* or the *girth* of a graph or to a *cycle* in a graph we mean the corresponding notion in the underlying undirected graph.

A *homomorphism* f between graphs G and H is a map $f : V(G) \to V(H)$ satisfying that for every edge $(x, y) \in E(G)$ we have $(f(x), f(y)) \in E(H)$. We write $f : G \to H$ to express that f is a homomorphism from G to H and we write $G \to H$ to express that such a homomorphism exists. This is clearly a transitive and reflexive relation. We write $G \not\to H$ if no homomorphism from G to H exists and call a family of graphs an *antichain* if no homomorphism exists between any two distinct members.

If both $G \to H$ and $H \to G$ hold for a pair of graphs we say G and H are *equivalent*. This is clearly an equivalence relation. In any equivalence class the graph with the fewest vertices is unique up to isomorphism. We call such a graph a *core* and also the *core of* any graph in its equivalence class. It is easy to see that a graph G is a core if and only if every homomorphism $f : G \to G$ is an isomorphism.

We say a graph G is *minimal* in a family $\mathcal{A}$ of graphs if $G \in \mathcal{A}$ and any graph $H \in \mathcal{A}$ satisfying $H \to G$ is equivalent to G. We define the dual notion of *maximal* in a family of graphs similarly, but with the homomorphism condition reversed. Note that there are two-way infinite chains of graphs, so infinite classes do not always have minimal or maximal elements.

A *duality pair* is a pair $(\mathcal{A}, \mathcal{D})$ of families of graphs satisfying that for every graph G we have either $A \to G$ for some $A \in \mathcal{A}$ or $G \to D$ for some $D \in \mathcal{D}$ but not both. If $(\mathcal{A}, \mathcal{D})$ is a duality pair we call $\mathcal{D}$ a *dual* of $\mathcal{A}$. Note, however, that this relation is not symmetric. Duality pairs were first introduced in [6] and investigated in detail in [7] for the special case when $|\mathcal{A}| = 1$.

Clearly, each graph in $\mathcal{A}$ and $\mathcal{D}$ in a duality pair $(\mathcal{A}, \mathcal{D})$ can be replaced with its core to obtain another duality pair $(\mathcal{A}', \mathcal{D}')$ so we can (and often will) assume that both sides of a duality pair consist of cores. Further if $A \to A'$ with $A \neq A'$ and $A, A' \in \mathcal{A}$ we can remove A' from $\mathcal{A}$ without ruining the duality pair property. This way, if $\mathcal{A}$ is *finite* we can replace it with the antichain $\mathcal{A}'$ of its minimal elements and the resulting pair $(\mathcal{A}', \mathcal{D})$ is still a duality pair. Similarly, if $\mathcal{D}$ is finite we can replace it with the set $\mathcal{D}'$ of its maximal elements to obtain a duality pair $(\mathcal{A}, \mathcal{D}')$ with $\mathcal{D}'$ being an antichain. Note, however, that such transformation is not possible in general for infinite families.

It is a trivial observation that any family $\mathcal{A}$ has a dual set $\mathcal{D}$, simply take $\mathcal{D} = \{G \mid \nexists A \in \mathcal{A} : A \to G\}$. For any family $\mathcal{D}$ of graphs one can similarly set $\mathcal{A} = \{G \mid \nexists D \in \mathcal{D} : G \to D\}$ making $(\mathcal{A}, \mathcal{D})$ a duality pair. Because of this abundance it is not reasonable to hope for a meaningful characterization of all

duality pairs. But characterization of restricted classes of duality pairs have been done successfully already.

Theorem 1.1 ([7])**.** *Each tree T has a well-defined, unique (up to equivalence) graph $D(T)$ making $(\{T\},\{D(T)\})$ a duality pair. In all singleton duality pairs $(\{A\},\{D\})$ the graph A is equivalent to a tree.*

Theorem 1.2 ([5])**.** *For any given finite family $\mathcal{A}$ of forests there exists a unique (up to the equivalence) antichain dual $\mathcal{D}(\mathcal{A})$ and it is finite. For any duality pair $(\mathcal{A},\mathcal{D})$ with both $\mathcal{A}$ and $\mathcal{D}$ finite antichains all graphs $A \in \mathcal{A}$ are equivalent to forests.*

We saw above that having antichains as the members of duality pairs can be considered as a relaxation of the finiteness condition. Having characterized the duality pairs with both sides finite it is natural to consider this relaxation. We start with quoting a result showing that there are probably too many infinite-infinite antichain duality pairs for a meaningful characterization.

Theorem 1.3 ([1])**.** *Each finite antichain $\mathcal{A}$ of graphs that is not maximal can be extended*

(i) *to a duality pair $(\mathcal{B},\mathcal{C})$ such that $\mathcal{A} \subset \mathcal{B}$ and both $\mathcal{B}$ and $\mathcal{C}$ are infinite antichains;*

(ii) *to a maximal infinite antichain, which is not a union of the sides of any duality pair.*

This naturally leads to the question of finding or characterizing antichain duality pairs with one side finite while the other is an infinite antichain. Erdős and Soukup [2] proved that no finite-infinite antichain duality pair exists and asked if infinite-finite ones do.

Theorem 1.4 ([2])**.** *There exists no duality pair $(\mathcal{A},\mathcal{D})$ with $\mathcal{A}$ finite and $\mathcal{D}$ an infinite antichain.*

In this paper we answer the question of Erdős and Soukup by giving several examples of infinite-finite antichain duality pairs and also study what families can appear in the left side of such a duality pair. The final answer (a characterization of such families) will follow from the upcoming paper [4] that studies the problem in the more general context of relational structures.

In Section 2 we limit the complexity of any graph appearing in an antichain with a finite dual: it must be equivalent to a forest. For finite antichains this is implied by Theorem 1.2. We also show that such a family has to have bounded maximum degree and bounded number of components.

When the forests in a duality pair have only one component and maximum degree two we deal with families of paths. In Section 3 we exhibit specific infinite antichains of paths, some with, and some without a finite dual.

In Section 4 we give a simple transformation turning the duality pairs in Section 3 into ones with non-path trees on the left side. However, these trees are still close to paths. We also give examples of infinite-singleton duality pairs where the left side consists of more complex trees constructed from arbitrary binary trees. One of these examples has an antichain on the left side. In another example of an infinite-finite antichain duality pair the left side consists of forests with several components.

2. Why forests?

In this section we prove that all graphs in the left side of an infinite-finite antichain duality pair must be equivalent to forests. This is an extension of the corresponding result for finite antichains in Theorem 1.2.

Theorem 2.1. *Let $(\mathcal{A}, \mathcal{D})$ be a duality pair, where $\mathcal{D}$ is finite and $\mathcal{A}$ consists of cores. Then for each graph $A \in \mathcal{A}$ that is not a forest there exists another graph $B \in \mathcal{A}$ with $B \to A$ but $A \not\to B$.*

This result can be proved from the Directed Sparse Incomparability Lemma, see [1, 2]. We present a self contained proof instead.

Proof. Let $A \in \mathcal{A}$ be a graph that is not a forest. Let (x, y) be an edge of A contained in a cycle C. Let A' be the graph obtained from A by removing this edge, adding a new vertex x' and the edge (x', y). Notice that the map moving x' to x and fixing all other vertices is an $A' \to A$ homomorphism.

Let X be an arbitrary tournament with more vertices than any of the (finitely many) graphs in $\mathcal{D}$. Let us consider the vertex set $V(X)$ (no edges yet) and disjoint copies A'_{uv} of A' for every edge $(u, v) \in E(X)$. We obtain the graph Y by identifying the copy of x in A'_{uv} with u and the copy of x' in A'_{uv} with v for all $(u, v) \in E(X)$.

Note that the natural $A' \to A$ homomorphism can be applied to each copy A'_{uv} of A' as all the identified vertices are mapped to x. This gives us a natural homomorphism $g : Y \to A$.

As $(\mathcal{A}, \mathcal{D})$ is a duality pair we either have a graph $B \in \mathcal{A}$ with $B \to Y$ or a graph $D \in \mathcal{D}$ with $Y \to D$. In the latter case we have $|V(X)| > |V(D)|$, so by the pigeonhole principle we must have $f(u) = f(v)$ for an edge $(u, v) \in E(X)$. But this means that f restricted to A'_{uv} is an $A \to D$ homomorphism, a clear contradiction. This leaves the former possibility only. We show that $B \in \mathcal{A}$ with $B \to Y$ satisfies the statement of the theorem.

Indeed we have $B \to Y \to A$. We will show $A \not\to Y$ and this implies $A \not\to B$. In the degenerate case when A consists of a single loop edge $A \not\to Y$ holds, since Y is a tournament in this case. So we may assume A is not a loop and as it is a core it does not even contain a loop. In particular $x \neq y$ and C has length at least 2. Assume for a contradiction that a homomorphism $f : A \to Y$ exists. As A is a core the homomorphism $f \circ g : A \to A$ must be an automorphism. Modifying f

appropriately, one can assume without loss of generality that $f \circ g$ is the identity, so $f(z) \in g^{-1}(z)$ for each vertex $z \in V(A)$. We must have $f(x) \in g^{-1}(x) = V(X)$ and $f(z) \notin V(X)$ for any other vertex z of A. The vertices of the cycle C except x itself must be mapped in a single connected component of $Y \setminus V(X)$, in particular, in a single copy A'_{uv} of A'. The image $f(y)$ of y must be the copy of y in A'_{uv}, so to have $(f(x), f(y)) \in E(Y)$ we must have $f(x) = v$. This forces the image of the other edge incident to x in the cycle C outside $E(Y)$. The contradiction finishes the proof of the theorem. □

Note that this theorem implies that if we have antichains in an infinite-finite duality pair of cores, then the left side contains forests only. Furthermore, something can be said without restricting attention to antichains. Let $(\mathcal{A}, \mathcal{D})$ be an infinite-finite duality pair of cores. We can remove from $\mathcal{A}$ all the graphs which are not forests but are "dominated" by one: the graphs $A \in \mathcal{A}$ for which a forest $B \in \mathcal{A}$ exists with $B \to A$, but A itself is not an forest. Clearly, the set of graphs to which a homomorphism exists from a member of the remaining family $\mathcal{A}'$ did not change, so $(\mathcal{A}', \mathcal{D})$ is still a duality pair. This duality pair may still contain a graph $A \in \mathcal{A}'$ that is not a forest, but such a graph must have *infinitely many* distinct graphs $B \in \mathcal{A}'$ with $B \to A$. Indeed, if A has only finitely many such dominating B, then any minimal graph in this finite set would violate the preceding theorem. From the Directed Sparse Incomparability Lemma one can also show that if a graph $A \in \mathcal{A}'$ is *imbalanced* (containing a cycle with an unequal number of forward and reverse oriented edges), then the underlying undirected graphs of the graphs $B \in \mathcal{A}'$ have unbounded girth.

In the following lemma we state the connection between having connected graphs on the left side of a duality pair and having a single graph on the right side. Recall that we call a graph *connected* if the underlying undirected graph is connected and use the term *connected component* in a similar way.

Lemma 2.2. *Let $(\mathcal{A}, \mathcal{D})$ be an antichain duality pair with $\mathcal{A}$ consisting of cores. If a graph $A \in \mathcal{A}$ has k connected components we have $|\mathcal{D}| \geq k$. But if all graphs in $\mathcal{A}$ are connected, then $|\mathcal{D}| = 1$.*

Proof. Take a graph $A \in \mathcal{A}$. Let $A_1, \ldots, A_k$ be the graphs obtained from A by removing a single one of its k components. For any $1 \leq i \leq k$ we have $A \not\to A_i$ (since A is a core) furthermore we have $A_i \to A$, so as $\mathcal{A}$ is an antichain it contains no graph that has a homomorphism to A_i. As $(\mathcal{A}, \mathcal{D})$ is a duality pair each A_i has a graph $D_i \in \mathcal{D}$ with $A_i \to D_i$. If we have $D_i = D_j$ for some $1 \leq i < j \leq k$ we can construct an $A \to D_i$ homomorphism by extending the $A_i \to D_i$ homomorphism to the missing component using the corresponding restriction of the $A_j \to D_i$ homomorphism. Since we must have $A \not\to D$ for $D \in \mathcal{D}$ all graphs D_i are distinct and thus $|\mathcal{D}| \geq k$ as claimed.

Now assume that every $A \in \mathcal{A}$ is connected but still we have two graphs $D_1 \neq D_2$ in $\mathcal{D}$. As $\mathcal{D}$ is an antichain the disjoint union D of D_1 and D_2 does not have a homomorphism to any member of $\mathcal{D}$. By the duality pair property we must

have a graph $A \in \mathcal{A}$ such that $A \to D$. As A is connected this homomorphism maps A either to D_1 or to D_2, giving $A \to D_1$ or $A \to D_2$, a contradiction. □

An immediate corollary of this lemma is that if an antichain has a finite dual its members have a bounded number of components. For this we do not need the full strength of the antichain condition, it is enough to assume that we do not have a homomorphism between two graphs of $\mathcal{A}$ that avoids an entire connected component of the target graph. While replacing the left hand side of a duality pair with an equivalent antichain is not always possible, it is easy to see that replacing the left hand side of a duality pair with an equivalent family satisfying this constraint is always possible. If the right side is finite, then after this transformation the graphs in the left side have a bounded number of components.

We end this section by showing that the maximum degree is also bounded in an antichain of core graphs that has a finite dual.

Lemma 2.3. *Let $(\mathcal{A}, \mathcal{D})$ be a duality pair with $\mathcal{A}$ an antichain consisting of cores and $\mathcal{D}$ finite. Any vertex of any graph $A \in \mathcal{A}$ has total degree (this is the sum of the in-degree and out-degree) at most $d_0 = \sum_{D \in \mathcal{D}} |V(D)|$.*

Proof. Let $A \in \mathcal{A}$ and $v \in V(A)$ and suppose the total degree d of v is larger than d_0. By Theorem 2.1 A is a forest, so v cuts its component of A into d parts. Let us form the subgraphs $A_1, \ldots, A_d$ of A by removing a single one of these parts from A. That is, each A_i is obtained from A by removing an edge e from A that connects v to another vertex w (with either orientation) and also removing the connected component of w from the resulting graph. As $\mathcal{A}$ is an antichain of cores no member of $\mathcal{A}$ has a homomorphism to any of these subgraphs A_i, so by the duality pair property, there must be homomorphisms $f_i : A_i \to D_i$ from A_i to certain graphs $D_i \in D$. From $d > d_0$ we must have $1 \leq i < j \leq d$ with $D_i = D_j$ and $f_i(v) = f_j(v)$. We construct an $A \to D_i$ homomorphism by extending f_i with the restriction of f_j to the part of A missing from A_i. The contradiction finishes the proof. □

3. Antichains of paths

In this section we give concrete examples of infinite antichains of paths with or without a finite dual. As paths are connected, Lemma 2.2 tells us that when looking for a finite dual it is always enough to consider duals consisting of a single graph.

To speak of (oriented) paths we use the natural correspondence between them and words over the binary alphabet $\{+, -\}$. We use standard notation with respect to these words, namely a word is a member of $\{+, -\}^* = \cup_{k \geq 0} \{+, -\}^k$, where $\{+, -\}^k$ is the set of length k sequences from the alphabet. For $x, y \in \{+, -\}^*$ and $k \geq 0$ we write xy for the concatenation of x and y and x^k for the word obtained by concatenating k copies of x.

The correspondence is given by the map p mapping $\{+, -\}^*$ to paths. For a word $x = x_1 \ldots x_k \in \{+, -\}^k$ let $p(x)$ stand for the path consisting of k edges

with the i'th edge oriented forward if $x_i = +$ and backward otherwise. A bit informally we will refer to the first and last vertices of $p(x)$ in their obvious meaning, although formally the end vertices of the path $p(x)$ cannot be distinguished without knowing x. We say that a homomorphism $f : p(x) \to G$ *maps* $p(x)$ *from* u *to* v if the image of the first vertex of $p(x)$ is $u \in V(G)$ and the image of the last vertex is $v \in V(G)$. Note that although all (isomorphism classes of) paths will be obtained as images in this map the correspondence is not one-one: p maps up to two distinct words to isomorphic paths (with the role of the first and last vertices reversed), for example $p(++-)$ and $p(+--)$ are isomorphic.

3.1. Antichains without a finite dual

The following observation is trivial: Take any infinite antichain of paths (as we will see such antichains are easy to find). The cardinality of the set of its subsets is continuum, and no two can have the same set for dual. Thus many have no finite dual, as the set of finite families of graphs is countable. This cardinality argument gives no explicit family without a finite dual. Here we set out to construct such a set.

Lemma 3.1. *Let G be a graph with $|V(G)| \leq k$ and assume that for some $x \in \{+,-\}^*$ we have $p(x^k) \to G$. Then for each $\ell \geq 0$ we also have $p(x^\ell) \to G$.*

Proof. There is nothing to prove if x is the empty string. Otherwise, by considering the homomorphism $p(x^k) \to G$, one can find vertices $v_0, \ldots, v_k$ in G such that for each $1 \leq i \leq k$ a suitable restriction of the homomorphism maps $p(x)$ from v_{i-1} to v_i. By the pigeonhole principle we find $v_i = v_j$ for some $0 \leq i < j \leq k$. Thus, we can map the $p(x^{j-i})$ to G with both endpoints mapping to the same vertex. This closed walk can take the homomorphic image of $p(x^\ell)$ for any ℓ. □

Example 3.2. *Let $Q_k = p((+(+-)^k)^k + +)$ and consider any infinite family $\mathcal{A} \subseteq \{Q_k \mid k \geq 1\}$. Then $\mathcal{A}$ is an antichain of paths and has no finite dual.*

Proof. To see that $\mathcal{A}$ is an antichain observe that the *height* of a path defined as the maximal difference between the number of forward edges and the number of backward edges in a sub-path cannot be decreased by a homomorphism. As the height of Q_k is $k+2$ we have $Q_k \not\to Q_\ell$ for $\ell < k$. But $p(+(+-)^k++)$ is a sub-path of Q_k and even this sub-path does not map to Q_ℓ for $\ell > k$. A similar argument also shows that all Q_k are cores: as deleting either the first or the last edge of Q_k decreases its height any homomorphism $Q_k \to Q_k$ must be onto and thus an isomorphism.

Assume $(\mathcal{A}, \mathcal{D})$ is a duality pair. Let $Q_k \in \mathcal{A}$ and consider $Q'_k = p((+(+-)^k)^k)$. Clearly, $Q'_k \to Q_k$, so we have $Q_\ell \not\to Q'_k$ for $\ell \neq k$ by the antichain property and $Q_k \not\to Q'_k$ since Q_k is a core. So we must have $D \in \mathcal{D}$ with $Q'_k \to D$. We claim that $|V(D)| > k$. Indeed, otherwise by Lemma 3.1 we have $Q_k \to p((+(+-)^k)^{k+1}) \to D$, a contradiction. As $\mathcal{A}$ is infinite k could be chosen arbitrarily large, so $\mathcal{D}$ must have arbitrarily large graphs and, thus, cannot be finite. □

3.2. Infinite antichains of paths with a finite dual

Our first infinite-finite antichain duality pair, the $s = 3$ case of the next example, is the smallest possible such example in the sense that the dual is a single graph on four vertices, while no graph or family of graphs on fewer vertices is a dual of an infinite antichain.

Example 3.3. *Let $P^s_k = p(+^s(-+^{s-1})^k+)$ for $s \geq 1$, $k \geq 0$ and let D_s be the graph obtained from the transitive tournament on $s+1$ vertices by deleting the edge connecting the source and the sink. Then $(\{P^s_k \mid k \geq 0\}, \{D_s\})$ is an antichain duality pair of cores for $s \geq 3$.*

Proof. To see that the infinite side is an antichain of cores we partition P^s_k into $k+2$ parts, the first being the directed path $p(+^s)$, the next k parts being $p(-+^{s-1})$, the last part being a single edge. In any $P^s_k \to P^s_\ell$ homomorphism the first part of P^s_k must not map to last s edges of P^s_ℓ because that would make the mapping of the next part impossible. So it must be mapped identically to the first part of P^s_ℓ and then the next k parts of P^s_k must also map identically to the next k parts of P^s_ℓ. This only works if $\ell \geq k$. But if $\ell > k$ the last edge of P^s_k cannot be mapped anywhere. So we must have $k = \ell$ and the homomorphism must be the identity.

To see that D_s is also a core it is enough to note that it is acyclic and has a directed Hamiltonian path. Let us denote the vertices along this path by $v_0, \ldots, v_s$.

We show $P^s_k \not\to D_s$ similarly to the antichain property. Indeed, the first part of P^s_k (forming a directed path) has a single homomorphism to D_s ending at v_s. Each of the next k parts must map to the path $v_s v_1 v_2 \cdots v_s$. But as v_s is a sink, this homomorphism cannot be extended to the last edge of P^s_k.

Let G be an arbitrary graph. By the statement in the last paragraph we cannot have $P^s_k \to G \to D_s$ for any $k \geq 0$. So it remains to prove that either $P^s_k \to G$ for some $k \geq 0$ or we have $G \to D_s$.

We call a vertex $v \in V(G)$ *type* i for $0 \leq i \leq s$ if it is the image under a homomorphism of the last vertex of a path $p(+(+^{s-1}-)^k+^i)$ for some $k \geq 0$. Note that for $i \geq 1$ a type i vertex is the image of the last vertex of the path $p(+^i)$ so it is also a type $i-1$ vertex.

If there is a type s vertex in G we clearly have $P^s_k \to G$ for some $k \geq 0$ and we are done.

If there is no type s vertex in G we define $\phi : V(G) \to V(D_s)$ by setting $\phi(v) = v_0$ if v is not type 0 and for $1 \leq i \leq s$ setting $\phi(v) = v_i$ if v is not type i but v is type $i-1$.

We claim that ϕ is a $G \to D_s$ homomorphism.

Let (u, v) be an edge of G. This makes v the endpoint of an edge, so it is type 0. Moreover, if u is type i, then the path $p(+(+^{s-1}-)^k+^i)$ mapping to G and ending at u can be extended by the (u, v) edge, making v a type $i+1$ vertex. Thus if $\phi(u) = v_j$ and $\phi(v) = v_{j'}$ we must have $j < j'$. It remains to prove that $\phi(u) = v_0$ and $\phi(v) = v_s$ is impossible. Indeed, $\phi(v) = v_s$ implies v is type $s-1$, so it is the image of the last vertex of a path $p(+(+^{s-1}-)^k+^{s-1})$. Extending

this with the (u, v) edge we get that u is the image of the last vertex of the path $p(+(+^{s-1}-)^{k+1})$ making u a type 0 vertex. This finishes the proof. □

3.3. Regularity

From the two examples considered so far one can notice the relevance of regular languages. Indeed, while the family of words $\{+^s(-+^{s-1})^k+ \mid k \geq 0\}$ is a regular language for any s, the family $\{+(+-)^k)^k + + \mid k \geq 1\}$ or any of its infinite subfamilies are not regular. This connection was the basis of our upcoming paper [3] that establishes regularity as a necessary and sufficient condition for having a finite dual in this case. We state the following easy observation regarding regularity to further motivate this connection.

Lemma 3.4. *Let G be an arbitrary graph. The set $\{x \in \{+,-\}^* \mid p(x) \to G\}$ is a regular language.*

Proof. We turn the graph G into a nondeterministic finite automaton. The states of the automaton are the vertices of G and each state is an initial and also a terminal state. For each edge (u, v) of G we make the transition from u to v possible for the letter $+$ and the transition from v to u possible for the letter $-$. It is straightforward to see that this automaton accepts the language in the lemma. □

4. Antichains of trees

In this section we give infinite-finite antichain duality pairs where the infinite side has trees that are not paths. The following lemma is instructive for this.

Lemma 4.1. *Let $(\mathcal{A}, \mathcal{D})$ be a duality pair. Let us modify each $A \in \mathcal{A}$ by enriching it with new vertices and edges: from each sink of A we start a new edge to a separate new vertex . Let $\mathcal{A}'$ be the family of these modified graphs. Let us modify each graph $D \in \mathcal{D}$ by adding a single new vertex and edges to this vertex from every vertex of D. Let $\mathcal{D}'$ be the family of these modified graphs. Then $(\mathcal{A}', \mathcal{D}')$ is a duality pair. If $\mathcal{A}$ is an antichain so is $\mathcal{A}'$, if $\mathcal{D}$ is an antichain, so is $\mathcal{D}'$, furthermore if $\mathcal{A}$ and $\mathcal{D}$ consist of cores so do $\mathcal{A}'$ and $\mathcal{D}'$.*

Proof. Let $A' \in \mathcal{A}'$ be the modification of $A \in \mathcal{A}$ and $D' \in \mathcal{D}'$ be the modification of $D \in \mathcal{D}$. If we have a homomorphism $f : A' \to D'$, then its restriction to A must map to D as the single vertex of $D' \setminus D$ is a sink in D', but no vertex of A is a sink in A'. But the existence of an $A \to D$ homomorphism contradicts the fact that $(\mathcal{A}, \mathcal{D})$ is a duality pair.

Let G' be an arbitrary graph. We cannot have $A' \to G' \to D'$ for some $A' \in \mathcal{A}'$ and $D' \in \mathcal{D}'$ by the previous paragraph. It remains to show that $A' \to G'$ for some $A' \in \mathcal{A}'$ or $G' \to D'$ for some $D' \in \mathcal{D}'$.

Let G be the subgraph of G' induced by the non-sink vertices. As $(\mathcal{A}, \mathcal{D})$ is a duality pair we either have $A \to G$ for some $A \in \mathcal{A}$ or $G \to D$ for some $D \in \mathcal{D}$. In the former case we can extend the homomorphism $A \to G$ to a homomorphism

$A' \to G'$, where $A' \in \mathcal{A}'$ is the modified version of A. In the latter case we can extend the homomorphism $G \to D$ to a homomorphism $G' \to D'$, where $D' \in \mathcal{D}'$ is the modified version of D, by sending all vertices of $G' \setminus G$ to the single vertex in $D' \setminus D$.

To see that the antichain and core properties are inherited to the modified sets consider two graphs X and Y from the same family $\mathcal{A}$ or $\mathcal{D}$ and their modifications X' and Y'. Restricting a homomorphism $X' \to Y'$ to X we get a homomorphism $X \to Y$. Indeed, all vertices in $Y' \setminus Y$ are sinks and no vertex in X is sink in X'. So if the family was antichain, then $X = Y$ and so the modified family is also an antichain. If $X = Y$ is a core, then the $X \to Y$ homomorphism must be an isomorphism and it is easy to see that the original $X' \to Y'$ homomorphism must also be an isomorphism. □

Applying this lemma (possibly several times) to our earlier examples of infinite-finite antichain duality pairs we get several new such examples. Although the graphs on the left side of these pairs are no longer paths, they are still very similar to paths in structure.

The examples in the next lemma show better the complexity that families with a finite dual can exhibit.

Let us consider the family T_0 of all finite rooted (undirected) binary trees satisfying that each vertex is either a leaf (it has no children) or it has two children: a left child and a right child. Although these are "trees" of some kind, they are not in the category of finite directed graphs we study here and we will not apply homomorphisms to these binary trees. To emphasize the difference we denote these binary trees by lower case letters as opposed to using capitals for directed graphs. Note that the smallest member of T_0 has a single vertex.

Let $x, y, s, z \in \{+, -\}^*$ be words. We define the family of oriented trees $T(x, y, s, z) = \{t(x, y, s, z) \mid t \in T_0\}$, where $t(x, y, s, z)$ is an oriented tree obtained from t by

(A) replacing each edge connecting a vertex u to its left child v by a copy of $p(x)$ from u to v,

(B) replacing each edge connecting a vertex u to its right child w by a copy of $p(y)$ from u to w,

(C) adding a path $p(s)$ from each leaf vertex of t and

(D) adding a path $p(z)$ from the root of t.

Let G_1 and G_2 be the graphs depicted on Figures 1 and 2, furthermore let

$$\mathcal{T}_1 = T(+-, -+, --, ++) \quad \text{and} \quad \mathcal{T}_2 = T(+--, -+-, --, +++).$$

Theorem 4.2. (i) *$(\mathcal{T}_1, \{G_1\})$ is a duality pair of core graphs.*

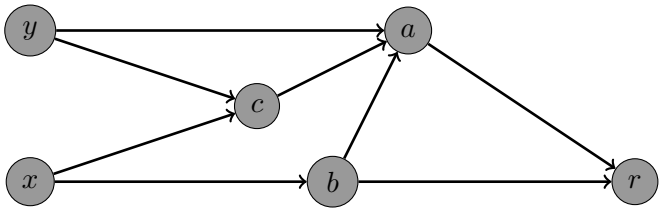

Figure 1: The graph G_1.

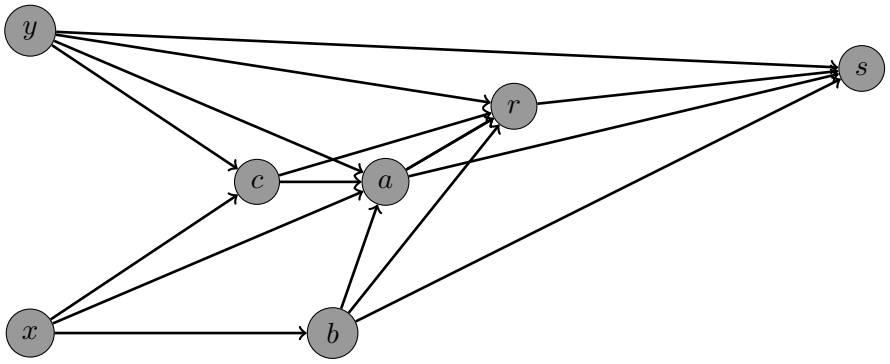

Figure 2: The graph G_2.

(ii) $(\mathcal{T}_2, \{G_2\})$ *is an antichain duality pair of core graphs.*

Proof. We leave the simple proofs that all involved graphs are cores and that $\mathcal{T}_2$ is an antichain to the diligent reader. Note that $\mathcal{T}_1$ is not an antichain: if $t \in T_0$ and t' is subtree of t containing the root, then $t(+-, -+, --, ++) \to t'(+-, -+, --, ++)$. As a result (once (i) is proved) one also has that $(\{t_k(+-, -+, --, ++) \mid k \geq 100\}, \{G_1\})$ is a duality pair if $t_k \in T_0$ is the depth k full binary tree with 2^k leaves.

Assume for a contradiction that $f : t(+-, -+, --, ++) \to G_1$. We claim that all vertices of t must map to the vertices a or r. This is certainly true for the leaves because of the attached paths $p(--)$ could not map to G_1 otherwise. Working with a bottom up induction assume that both the left and the right children of the vertex u map to a or r. In this case the paths from u to its children must be mapped to G_1 from $f(u)$ to either a or r. Then we must have $f(u) = a$ as from no other vertex of G_1 is there both a path $p(+-)$ and a path $p(-+)$ to either a or r. So the root vertex must also be mapped to a or r and the contradiction comes from there being no place in G_1 for the path $p(++)$ attached to the root.

For (i) it is left to prove that for any graph X we either have $A \to X$ for an $A \in \mathcal{T}_1$ or we have $X \to G_1$. For this we define the "level" $L_i \subseteq V(X)$ to consist of the vertices of X with a homomorphism $P_i \to X$ ending at u but no homomorphism $P_{i+1} \to X$ ending at u. Here $P_i = p(+^i)$ is the directed path of i edges and the levels L_0, L_1, L_2 and L_3 partition $V(x)$ or we have $P_4 \to X$ and we are done since $P_4 \in \mathcal{T}_1$.

We construct the map $\phi : V(X) \to V(G_1)$ as follows:

(1) Set $\phi(u) = a$ for any vertex $u \in L_1$ that has a $t \in T_0$ and a homomorphism $f : t(+-, -+, --, \epsilon) \to X$ mapping the root of t to a. Here ϵ stands for the empty word.

(2) Set $\phi(u) = b$ for any vertex $u \in L_1$ not yet mapped to a that either has an edge (u, v) to a vertex $v \in L_3$ or two edges (u, v) and (w, v) with $w \in L_1$ already mapped to a.

(3) Set $\phi(u) = c$ for the remaining vertices $u \in L_1$.

(4) Set $\phi(u) = x$ if $u \in L_0$ and there exists an edge (u, v) with $\phi(v) = b$.

(5) Set $\phi(u) = y$ for the remaining vertices $u \in L_0$.

(6) Set $\phi(u) = a$ if $u \in L_2$ and there is no $v \in L_1$ with $\phi(u) = a$ and (v, u) an edge.

(7) Set $\phi(u) = r$ for all remaining vertices in $V(X)$.

If ϕ is a homomorphism $X \to G_1$ we are done. Otherwise one of the steps above made an edge in X map outside X.

Steps 1–3 map the independent set L_1 so they caused no problem. Step 4 can create a problem if a vertex $u \in L_0$ has edges (u, v) and (u, w) with $v, w \in L_1$, $\phi(v) = b$ and $\phi(w) = a$. But in this case the homomorphisms triggering $\phi(v) = b$ and $\phi(w) = a$ in steps 2 and 1 can be combined (together with the wuv path) with a homomorphism triggering $\phi(v) = a$ in the first step, a contradiction.

Step 5 cannot cause trouble as both (y, a) and (y, c) are edges in G_1.

Steps 6 or 7 cause trouble if there is a vertex $u \in L_2 \cup L_3$ with (v, u) an edge from a vertex $v \in L_0$ with $\phi(v) = x$. But then there is a vertex $w \in L_1$ with (v, w) an edge and $\phi(w) = b$. Here again, the homomorphism of $p(++-+)$ to X ending in the vertices uvw can be combined to the homomorphism triggering $\phi(w) = b$ to obtain a homomorphism triggering $\phi(w) = a$, a contradiction.

Finally in step 7 we can map both ends of an (u, v) edge to r. This happens if there exists an edge (w, u) from a vertex $w \in L_1$ with $\phi(w) = a$. This may indeed happen, but then the homomorphism triggering $\phi(w) = a$ can be combined to the directed path wuv to get $A \to X$ for a tree $A \in \mathcal{T}_1$. This finishes the proof of part (i).

(ii) The proof of this part is only slightly more complicated.

Assume for a contradiction that $f : t(+--, -+-, --, +++) \to G_2$. We claim that all vertices of t must map to the vertices a, r or s. This can be shown exactly like the corresponding statement in part (i). So the root vertex must also be mapped to a or u and the contradiction comes from there being no place in G_2 for the path $p(+++)$ attached to the root.

Finally we assume X is graph with no $A \to X$ homomorphism for any $A \in \mathcal{T}_2$. We construct the homomorphism $\phi : X \to G_2$ similarly to part (i). We partition $V(X)$ into levels L_i as we did above. As $P_5 \in \mathcal{T}_2$ does not map to X, $V(X)$ is partitioned into the sets L_0, L_1, L_2, L_3 and L_4.

(1) Set $\phi(u) = a$ for any vertex $u \in L_1$ that has a $t \in T_0$ and a homomorphism $f : t(+--, -+-, --, \epsilon) \to X$ mapping the root of t to a.

(2) Set $\phi(u) = b$ for any vertex $u \in L_1$ not yet mapped to a that has either a homomorphism mapping $p(+----)$ to G_2 from u or a homomorphism of $p(+--)$ to G_2 from u to a vertex $v \in L_1$ with $\phi(v) = a$.

(3) Set $\phi(u) = c$ for the remaining vertices $u \in L_1$.

(4) Set $\phi(u) = x$ if $u \in L_0$ and there exists an edge (u, v) with $\phi(u) = b$.

(5) Set $\phi(u) = y$ for the remaining vertices $u \in L_0$.

(6) Set $\phi(u) = a$ if $u \in L_2$ and there is no $v \in L_1$ with $\phi(u) = a$ and (v, u) an edge.

(7) Set $\phi(u) = r$ for all remaining vertices $u \in L_2$.

(8) Also set $\phi(u) = r$ for vertices $u \in L_3$ with no edge (v, u) from a vertex $v \in L_2$ with $\phi(v) = r$.

(9) Set $\phi(u) = s$ for all remaining vertices $u \in V(X)$.

The proof that ϕ is indeed a homomorphism is almost identical to the corresponding argument in part (i). □

We finish the paper with a simple observation that shows how to combine duality pairs to obtain new pairs with several graphs on the right side. For simplicity we restrict attention to combining two duality pairs with single graphs on the right hand side that are incomparable.

In the following proposition and example $A_1 \cup A_2$ denotes the disjoint union of the graphs A_1 and A_2.

Proposition 4.3. *Let $(\mathcal{A}_1, \mathcal{D}_1)$ and $(\mathcal{A}_2, \mathcal{D}_2)$ be duality pairs and let us partition $\mathcal{A}_i$ into $\mathcal{A}'_i = \{A \in \mathcal{A}_i \mid \exists B \in \mathcal{A}_{3-i} : B \to A\}$ and $\mathcal{A}''_i = \mathcal{A}_i \setminus \mathcal{A}'_i$ for $i = 1, 2$.*

(i) *$(\mathcal{A}, \mathcal{D})$ is a duality pair, where $\mathcal{D} = \mathcal{D}_1 \bigcup \mathcal{D}_2$ and $\mathcal{A} = \{A_1 \cup A_2 \mid A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\}$.*

(ii) *$(\mathcal{A}', \mathcal{D})$ is also a duality pair, where $\mathcal{A}' = \mathcal{A}'_1 \bigcup \mathcal{A}'_2 \bigcup \{A_1 \cup A_2 \mid A_1 \in \mathcal{A}''_1, A_2 \in \mathcal{A}''_2\}$.*

(iii) *If each $\mathcal{A}_i$ is an antichain and $|\mathcal{D}_1| = |\mathcal{D}_2| = 1$, then $\mathcal{A}'$ can be made an antichain by removing possible duplicates: leaving one member only from each equivalent pair of graphs from $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Proof. For (i) it is enough to note that $A_1 \cup A_2 \to B$ if and only if $A_1 \to B$ and $A_2 \to B$.

For (ii) take $A_2 \in \mathcal{A}'_2$ and a graph $A_1 \in \mathcal{A}_1$ with $A_1 \to A_2$. As $A_1 \cup A_2$ is equivalent to A_2 we can put A_2 into the left side of the duality pair $(\mathcal{A}, \mathcal{D})$. But then all graphs $A'_1 \cup A_2$ can be removed from there as A_2 maps to these graphs. Doing this for all $A_2 \in \mathcal{A}'_2$ and similar changes for the graphs in $\mathcal{A}'_1$ one obtains $\mathcal{A}'$ and (ii) is proved.

To prove (iii) take $A \in \mathcal{A}'$ and consider the sets $S_i(A) = \{B \in \mathcal{A}_i \mid B \to A\}$ for $i = 1, 2$. For a graph $A = A_1 \cup A_2$ with $A_1 \in \mathcal{A}''_1$, $A_2 \in \mathcal{A}''_2$ we have $S_1(A) = \{A_1\}$ and $S_2(A) = \{A_2\}$ since the graphs in $\mathcal{A}_1 \bigcup \mathcal{A}_2$ are connected (Lemma 2.2), thus they map to A if and only if they map to A_1 or A_2. For $A \in \mathcal{A}'_i$ and $i = 1$ or 2 we have $S_i(A) = \{A\}$. Since $A \to A'$ implies $S_i(A) \subseteq S_i(A')$ for $A, A' \in \mathcal{A}'$ and $i = 1, 2$ the only possibility of such a map with $A \neq A'$ is $A \in \mathcal{A}'_i$ and $A' \in \mathcal{A}'_{3-i}$. From $A \in \mathcal{A}'_i$ we have $A'' \in \mathcal{A}_{3-i}$ with $A'' \to A \to A'$. As $\mathcal{A}_{3-i}$ is an antichain we must have $A'' = A$ and thus A and A' are equivalent. □

We can apply this lemma to combine any two of the several examples of infinite-finite duality pairs in this paper or even one such example with a simple duality with a single tree on the left hand side. We chose the duality pairs $(\{P^4_k \mid k \geq 0\}, \{D_4\})$ from Example 3.3 and $(\mathcal{T}_2, \{G_2\})$ from Theorem 4.2. Note that P^4_0 is the directed path with five edges and it appears in $\mathcal{T}_2$ but no homomorphism exist from a member of $\mathcal{T}_2$ to some P^4_k with $k \geq 1$ or vice versa. Thus from Lemma 4.3 we get the following

Example 4.4. *The following is an antichain duality pair of core graphs:*

$$(\{P^4_0\} \cup \{P^4_k \cup A \mid k \geq 1, A \in \mathcal{T}_2 \setminus \{P^4_0\}\}, \{D_4, G_2\}).$$

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