# MAXIMAL CHAINS OF ISOMORPHIC SUBORDERS OF COUNTABLE ULTRAHOMOGENEOUS PARTIAL ORDERS 

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#### Abstract

We investigate the poset $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$, where $\mathbb{P}(\mathbb{X})$ is the set of isomorphic suborders of a countable ultrahomogeneous partial order $\mathbb{X}$. For $\mathbb{X}$ different from (resp. equal to) a countable antichain the order types of maximal chains in $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$ are characterized as the order types of compact (resp. compact and nowhere dense) sets of reals having the minimum non-isolated. 2000 Mathematics Subject Classification: 06A06, 06A05, 03C15, 03 C 50. Keywords: ultrahomogeneous partial order, isomorphic substructure, maximal chain, compact set.


## 1 Introduction

The general concept - to explore the relationship between the properties of a relational structure $\mathbb{X}$ and the properties of the poset $\mathbb{P}(\mathbb{X})$ of its isomorphic substructures - can be developed in several ways. For example, regarding the forcing theoretic aspect, the poset of copies of each countable non-scattered linear order is forcing equivalent to the two-step iteration of the Sacks forcing and a $\sigma$-closed forcing [9], while the posets of copies of countable scattered linear orders have $\sigma$-closed forcing equivalents (separative quotients) [10].

Regarding the order-theoretic aspect, one of extensively investigated order invariants of a poset is the class of order types of its maximal chains [2, 5, 6, 11] and, for the poset of isomorphic suborders of the rational line, $\langle\mathbb{Q},<\mathbb{Q}\rangle$, this class is characterized in [8]. The main result of the present paper is the following generalization of that result.

Theorem 1.1 If $\mathbb{X}$ is a countable ultrahomogeneous partial order different from a countable antichain, then for each linear order $L$ the following conditions are equivalent:
(a) $L$ is isomorphic to a maximal chain in the poset $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$;
(b) $L$ is an $\mathbb{R}$-embeddable complete linear order with $0_{L}$ non-isolated;
(c) $L$ is isomorphic to a compact set $K \subset \mathbb{R}$ such that $0_{K} \in K^{\prime}$.

If $\mathbb{X}$ is a countable antichain, then the corresponding characterization is obtained if we replace "complete" by "Boolean" in (b) and "compact" by "compact and nowhere dense" in (c).

[^0]So, for example, there are maximal chains of copies of the random poset isomorphic to $(0,1]$, to the Cantor set without 0 , and to $\alpha^{*}$, for each countable limit ordinal $\alpha$. Although it is not a usual practice, we start with a proof in the introduction. The equivalence of (b) and (c) is a known fact (see, for example, Theorem 6 of [8]) and the implication (a) $\Rightarrow$ (b) follows from the general result on ultrahomogeneous structures given in Theorem [2.2] of the present paper. Thus, only the implication (b) $\Rightarrow$ (a) remains to be proved. Naturally, we will use the following, well known classification of countable ultrahomogeneous partial orders - the Schmerl list [13]:

Theorem 1.2 (Schmerl) A countable strict partial order is ultrahomogeneous iff it is isomorphic to one of the following partial orders:
$\mathbb{A}_{\omega}$, a countable antichain (that is, the empty relation on $\omega$ );
$\mathbb{B}_{n}=n \times \mathbb{Q}$, for $1 \leq n \leq \omega$, where $\left\langle i_{1}, q_{1}\right\rangle<\left\langle i_{2}, q_{2}\right\rangle \Leftrightarrow i_{1}=i_{2} \wedge q_{1}<_{\mathbb{Q}} q_{2} ;$
$\mathbb{C}_{n}=n \times \mathbb{Q}$, for $1 \leq n \leq \omega$, where $\left\langle i_{1}, q_{1}\right\rangle<\left\langle i_{2}, q_{2}\right\rangle \Leftrightarrow q_{1}<_{\mathbb{Q}} q_{2}$;
$\mathbb{D}$, the unique countable homogeneous universal poset (the random poset).
For the antichain $\mathbb{A}_{\omega}$ the implication (b) $\Rightarrow$ (a) follows from Theorem 1.4 and the fact that $\mathbb{P}\left(\mathbb{A}_{\omega}\right)=[\omega]^{\omega}$ is a positive family. The most difficult part of the proof of (b) $\Rightarrow$ (a) - for the random poset $\mathbb{D}$ - is given in Section 4 In Sections 5 and 6 , using the constructions from [8], we prove (b) $\Rightarrow$ (a) for the posets $\mathbb{B}_{n}$ and $\mathbb{C}_{n}$.

The rest of this section contains two facts which will be used in the sequel. We remind the reader that a linear order $\langle L,<\rangle$ is called Boolean iff it is complete (has 0,1 and has no gaps) and has dense jumps, which means that for each $x, y \in L$ satisfying $x<y$ there are $a, b \in L$ such that $x \leq a<b \leq y$ and $(a, b)_{L}=\emptyset$.

Fact 1.3 Each countable complete linear order is Boolean.
We recall that a family $\mathcal{P} \subset P(\omega)$ is called a positive family iff:
(P1) $\emptyset \notin \mathcal{P}$;
(P2) $\mathcal{P} \ni A \subset B \subset \omega \Rightarrow B \in \mathcal{P}$;
(P3) $A \in \mathcal{P} \wedge|F|<\omega \Rightarrow A \backslash F \in \mathcal{P}$;
(P4) $\exists A \in \mathcal{P} \quad|\omega \backslash A|=\omega$.
Theorem 1.4 ([7]) If $\mathcal{P} \subset P(\omega)$ is a positive family, then for each linear order $L$ the following conditions are equivalent:
(a) $L$ is isomorphic to a maximal chain in the poset $\langle\mathcal{P} \cup\{\emptyset\}, \subset\rangle$;
(b) $L$ is an $\mathbb{R}$-embeddable Boolean linear order with $0_{L}$ non-isolated;
(c) $L$ is isomorphic to a compact nowhere dense set $K \subset \mathbb{R}$ such that $0_{K} \in K^{\prime}$. In addition, (b) implies that there is a maximal chain $\mathcal{L}$ in $\langle\mathcal{P} \cup\{\emptyset\}, \subset\rangle$ satisfying $\bigcap(\mathcal{L} \backslash\{\emptyset\})=\emptyset$ and isomorphic to $L$.

## 2 Copies of countable ultrahomogeneous structures

Let $L=\left\{R_{i}: i \in I\right\}$ be a relational language, where $\operatorname{ar}\left(R_{i}\right)=n_{i}, i \in I$. An $L$-structure $\mathbb{X}=\left\langle X,\left\{\rho_{i}: i \in I\right\}\right\rangle$ is called countable iff $|X|=\omega$. If $A \subset X$, then $\left\langle A,\left\{\left(\rho_{i}\right)_{A}: i \in I\right\}\right\rangle$ (shortly denoted by $\left\langle A,\left\{\rho_{i}: i \in I\right\}\right\rangle$, whenever this abuse of notation does not produce a confusion) is a substructure of $\mathbb{X}$, where $\left(\rho_{i}\right)_{A}=$ $\rho_{i} \cap A^{n_{i}}, i \in I$. If $\mathbb{Y}=\left\langle Y,\left\{\sigma_{i}: i \in I\right\}\right\rangle$ is an $L$-structure too, a mapping $f: X \rightarrow Y$ is an embedding (we write $\mathbb{X} \hookrightarrow_{f} \mathbb{Y}$ ) iff it is an injection and

$$
\forall i \in I \quad \forall\left\langle x_{1}, \ldots x_{n_{i}}\right\rangle \in X^{n_{i}} \quad\left(\left\langle x_{1}, \ldots, x_{n_{i}}\right\rangle \in \rho_{i} \Leftrightarrow\left\langle f\left(x_{1}\right), \ldots, f\left(x_{n_{i}}\right)\right\rangle \in \sigma_{i}\right)
$$

If $\mathbb{X}$ embeds in $\mathbb{Y}$ we write $\mathbb{X} \hookrightarrow \mathbb{Y}$. Let $\operatorname{Emb}(\mathbb{X}, \mathbb{Y})=\left\{f: \mathbb{X} \hookrightarrow_{f} \mathbb{Y}\right\}$ and $\operatorname{Emb}(\mathbb{X})=\left\{f: \mathbb{X} \hookrightarrow_{f} \mathbb{X}\right\}$. If, in addition, $f$ is a surjection, it is an isomorphism (we write $\mathbb{X} \cong_{f} \mathbb{Y}$ ) and the structures $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic, in notation $\mathbb{X} \cong \mathbb{Y}$.

A finite isomorphism of $\mathbb{X}$ is each isomorphism between finite substructures of $\mathbb{X}$. A structure $\mathbb{X}$ is ultrahomogeneous iff each finite isomorphism on $\mathbb{X}$ can be extended to an automorphism of $\mathbb{X}$. The age of $\mathbb{X}$, Age $\mathbb{X}$, is the class of all finite $L$-structures embeddable in $\mathbb{X}$. We will use the following well known facts from the Fraïssé theory.

Theorem 2.1 (Fraïssé) Let $L$ be an at most countable relational language. Then
(a) A countable $L$-structure $\mathbb{X}$ is ultrahomogeneous iff for each finite isomorphism $\varphi$ of $\mathbb{X}$ and each $x \in X \backslash \operatorname{dom} \varphi$ there is a finite isomorphism $\psi$ of $\mathbb{X}$ extending $\varphi$ to $x$ (see [3] p. 389 or [4] p. 326).
(b) If $\mathbb{X}$ and $\mathbb{Y}$ are countable ultrahomogeneous $L$-structures and Age $\mathbb{X}=$ Age $\mathbb{Y}$, then $\mathbb{X} \cong \mathbb{Y}$ (see [3] p. 333 or [4] p. 326).

Concerning the order types of maximal chains in the posets of the form $\langle\mathbb{P}(\mathbb{X}), \subset\rangle$, where $\mathbb{X}=\left\langle X,\left\{\rho_{i}: i \in I\right\}\right\rangle$ is a relational structure and $\mathbb{P}(\mathbb{X})$ the set of the domains of its isomorphic substructures, that is

$$
\mathbb{P}(\mathbb{X})=\left\{A \subset X:\left\langle A,\left\{\left(\rho_{i}\right)_{A}: i \in I\right\}\right\rangle \cong \mathbb{X}\right\}=\{f[X]: f \in \operatorname{Emb}(\mathbb{X})\}
$$

we have the following general statement.
Theorem 2.2 Let $\mathbb{X}$ be a countable ultrahomogeneous structure of an most countable relational language and $\mathbb{P}(\mathbb{X}) \neq\{X\}$. If $\mathcal{L}$ is a maximal chain in the poset $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$, then
(a) $\mathcal{L}$ is an $\mathbb{R}$-embeddable complete linear order with $0_{\mathcal{L}}(=\emptyset)$ non-isolated;
(b) If there is a positive family $\mathcal{P} \subset \mathbb{P}(\mathbb{X})$, then for each countable linear order $L$ satisfying (a), there is a maximal chain in $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$ isomorphic to $L$.

Proof. (a) First we prove that

$$
\begin{equation*}
\bigcup \mathcal{A} \in \mathbb{P}(\mathbb{X}), \text { for each chain } \mathcal{A} \text { in the poset }\langle\mathbb{P}(\mathbb{X}), \subset\rangle \tag{1}
\end{equation*}
$$

Let $\varphi$ be a finite isomorphism of $\bigcup \mathcal{A}$ and $x \in \bigcup \mathcal{A}$. Since $\mathcal{A}$ is a chain there is $A \in \mathcal{A}$ such that $\operatorname{dom} \varphi \cup \operatorname{ran} \varphi \cup\{x\} \subset A$. Since $A \cong \mathbb{X}$, by Theorem 2.1(a) there is $y \in A$ such that $\psi=\varphi \cup\{\langle x, y\rangle\}$ is an isomorphism so $\psi$ is a finite isomorphism of $\cup \mathcal{A}$. Thus, by Theorem 2.1 a), the structure $\cup \mathcal{A}$ is ultrahomogeneous. Since $\mathbb{X} \cong A \subset \bigcup \mathcal{A} \subset X$ we have Age $\mathbb{X}=$ Age $A \subset$ Age $\bigcup \mathcal{A} \subset$ Age $\mathbb{X}$, which, by Theorem 2.11(b), implies $\cup \mathcal{A} \cong \mathbb{X}$, that is $\cup \mathcal{A} \in \mathbb{P}(\mathbb{X})$.

Let $X=\left\{x_{n}: n \in \omega\right\}$ be an enumeration. Since $\mathcal{L} \subset[X]^{\omega} \cup\{\emptyset\}$, the function $f: \mathcal{L} \rightarrow \mathbb{R}$ defined by $f(A)=\sum_{n \in \omega} 2^{-n} \cdot \chi_{A}\left(x_{n}\right)$ (where $\chi_{A}: X \rightarrow\{0,1\}$ is the characteristic function of the set $A \subset X)$ is an embedding of $\langle\mathcal{L}, \subset\rangle$ into $\langle\mathbb{R},<\mathbb{R}\rangle$.

Clearly, $\min \mathcal{L}=\emptyset$ and $\max \mathcal{L}=X$. Let $\langle\mathcal{A}, \mathcal{B}\rangle$ be a cut in $\mathcal{L}$. If $\mathcal{A}=\{\emptyset\}$ then $\max \mathcal{A}=\emptyset$. If $\mathcal{A} \neq\{\emptyset\}$, by (1) we have $\cup \mathcal{A} \in \mathbb{P}(\mathbb{X})$ and, since $A \subset \cup \mathcal{A} \subset B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, the maximality of $\mathcal{L}$ implies $\cup \mathcal{A} \in \mathcal{L}$. So, if $\cup \mathcal{A} \in \mathcal{A}$ then $\max \mathcal{A}=\bigcup \mathcal{A}$. Otherwise $\bigcup \mathcal{A} \in \mathcal{B}$ and $\min \mathcal{B}=\bigcup \mathcal{A}$. Thus $\langle\mathcal{L}, \subset\rangle$ is complete.

Suppose that $A$ is the successor of $\emptyset$ in $\mathcal{L}$. Since $\mathbb{P}(\mathbb{X}) \neq\{X\}$ there is $B \in$ $\mathbb{P}(\mathbb{X}) \backslash\{X\}$ and, if $f: \mathbb{X} \hookrightarrow A$, then $f[B] \in \mathbb{P}(\mathbb{X}), f[B] \varsubsetneqq A$ and, hence, $\mathcal{L} \cup\{f[B]\}$ is a chain in $\mathbb{P}(\mathbb{X})$. A contradiction to the maximality of $\mathcal{L}$.
(b) By Fact $1.3, L$ is a Boolean order and, by Theorem 1.4 in the poset $\langle\mathcal{P} \cup$ $\{\emptyset\}, \subset\rangle$ there is a maximal chain $\mathcal{L}$ isomorphic to $L$ and such that $\bigcap(\mathcal{L} \backslash\{\emptyset\})=\emptyset$. Now, $\mathcal{L}$ is a chain in $\langle\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}, \subset\rangle$ and we check its maximality. Suppose that $\mathcal{L} \cup\{A\}$ is a chain, where $A \in \mathbb{P}(\mathbb{X}) \backslash \mathcal{L}$. Then $A \varsubsetneqq S$ or $S \varsubsetneqq A$, for each $S \in \mathcal{L} \backslash\{\emptyset\}$ and, since $\bigcap(\mathcal{L} \backslash\{\emptyset\})=\emptyset$, there is $S \in \mathcal{L} \backslash\{\emptyset\}$ such that $S \subset A$, which implies $A \in \mathcal{P}$. But $\mathcal{L} \backslash\{\emptyset\}$ is a maximal chain in $\mathcal{P}$. A contradiction.

Remark 2.3 Concerning the assumption $\mathbb{P}(\mathbb{X}) \neq\{X\}$ we note that there are countable ultrahomogeneous structures satisfying $\mathbb{P}(\mathbb{X})=\{X\}$ (see [3], p. 399).

For $1<n<\omega$ the set $\mathbb{P}\left(\mathbb{C}_{n}\right)$ does not contain a positive family, since (P3) is not satisfied. Namely, if $A \in \mathbb{P}\left(\mathbb{C}_{n}\right)$ and $x \in A$, then $A \backslash\{x\}$ is not a copy of $\mathbb{C}_{n}$ (one class of incompatible elements is of size $n-1$ ).

For some $\omega$-saturated, $\omega$-homogeneous-universal relational structures the implication (b) $\Rightarrow$ (a) of Theorem 1.1 is not true. Let $L$ be the language with one binary relational symbol $\rho$ and $\mathcal{T}$ the $L$-theory of empty relations $(\forall x, y \neg x \rho y)$. Then $\mathbb{X}=\langle\omega, \emptyset\rangle$ is the $\omega$-saturated model of $\mathcal{T}$. But $\mathbb{P}(\mathbb{X})=[\omega]^{\omega}$ is a positive family and, by Theorem 1.4 maximal chains in $\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}$ are Boolean. Thus, for example, $\mathbb{P}(\mathbb{X}) \cup\{\emptyset\}$ does not contain a maximal chain isomorphic to $[0,1]_{\mathbb{R}}$.

## 3 Copies of the countable random poset

Let $\mathbb{P}=\langle P,<\rangle$ be a partial order. By $C(\mathbb{P})$ we denote the set of all triples $\langle L, G, U\rangle$ of pairwise disjoint finite subsets of $P$ such that:
(C1) $\forall l \in L \forall g \in G \quad l<g$,
(C2) $\forall u \in U \forall l \in L \neg u<l$ and
(C3) $\forall u \in U \forall g \in G \neg g<u$.

For $\langle L, G, U\rangle \in C(\mathbb{P})$, let $P_{\langle L, G, U\rangle}$ be the set of all $p \in P \backslash(L \cup G \cup U)$ satisfying:
(S1) $\forall l \in L p>l$,
(S2) $\forall g \in G p<g$ and
(S3) $\forall u \in U p \| u$ (where $p \| q$ denotes that $p \neq q \wedge \neg p<q \wedge \neg q<p$ ).
Fact 3.1 Let $\mathbb{P}=\langle P,<\rangle$ be a partial order and $\emptyset \neq A \subset P$. Then
(a) $C(A,<)=\{\langle L, G, U\rangle \in C(\mathbb{P}): L, G, U \subset A\}$;
(b) $A_{\langle L, G, U\rangle}=P_{\langle L, G, U\rangle} \cap A$, for each $\langle L, G, U\rangle \in C(A,<)$.
(c) $\langle\emptyset, \emptyset, \emptyset\rangle \in C(\mathbb{P})$ and $P_{\langle\emptyset, \emptyset, \emptyset\rangle}=P$.

Proof. For pairwise disjoint sets $L, G, U \subset A$ we have: $L \times G \subset<\operatorname{iff} L \times G \subset<_{A}$ and $((U \times L) \cup(G \times U)) \cap<=\emptyset$ iff $((U \times L) \cup(G \times U)) \cap<_{A}=\emptyset$.

Fact 3.2 A countable strict partial order $\mathbb{D}=\langle D,\langle \rangle$ is a countable random poset iff $D_{\langle L, G, U\rangle} \neq \emptyset$, for each $\langle L, G, U\rangle \in C(\mathbb{D})$ (see [1]).

Lemma 3.3 Let $\mathbb{D}=\langle D,<\rangle$ be a countable random poset. Then
(a) $D_{\langle L, G, U\rangle} \in \mathbb{P}(\mathbb{D})$ and, hence, $\left|D_{\langle L, G, U\rangle}\right|=\omega$, for each $\langle L, G, U\rangle \in C(\mathbb{D})$;
(b) $D \backslash F \in \mathbb{P}(\mathbb{D})$, for each finite $F \subset D$;
(c) If $D=A \dot{\cup} B$, then either $A$ or $B$ contains an element of $\mathbb{P}(\mathbb{D})$;
(d) If $\mathcal{L} \subset \mathbb{P}(\mathbb{D})$ is a chain, then $\cup \mathcal{L} \in \mathbb{P}(\mathbb{D})$;
(e) If $C \subset D$ and $A \not \subset C$ for each $A \in \mathbb{P}(\mathbb{D})$, then $D \backslash C \in \mathbb{P}(\mathbb{D})$.

Proof. (a) Let $\langle L, G, U\rangle \in C(\mathbb{D})$. Then $L, G$ and $U$ are disjoint subsets of $D$,

$$
\begin{equation*}
\forall l \in L \quad \forall g \in G \quad \forall u \in U \quad(u \nless l<g \nless u), \tag{2}
\end{equation*}
$$

and $D_{\langle L, G, U\rangle} \cap(L \cup G \cup U)=\emptyset$. Let $\left\langle L_{1}, G_{1}, U_{1}\right\rangle \in C\left(D_{\langle L, G, U\rangle}\right)$. Then $L_{1}$, $G_{1}$ and $U_{1}$ are disjoint subsets of $D_{\langle L, G, U\rangle}$ and, by Fact $3.1\left\langle L_{1}, G_{1}, U_{1}\right\rangle \in C(\mathbb{D})$ which implies

$$
\begin{equation*}
\forall l_{1} \in L_{1} \forall g_{1} \in G_{1} \forall u_{1} \in U_{1} \quad\left(u_{1} \nless l_{1}<g_{1} \nless u_{1}\right) . \tag{3}
\end{equation*}
$$

Since $L_{1} \cup G_{1} \cup U_{1} \subset D_{\langle L, G, U\rangle}$, by (S1)-(S3) we have

$$
\begin{equation*}
\forall x \in L_{1} \cup G_{1} \cup U_{1} \quad \forall l \in L \quad \forall g \in G \quad \forall u \in U \quad(l<x<g \wedge x \nless u \wedge u \nless x) . \tag{4}
\end{equation*}
$$

First we show that $\left\langle L \cup L_{1}, G \cup G_{1}, U \cup U_{1}\right\rangle \in C(\mathbb{D})$. (C1) Let $l^{\prime} \in L \cup L_{1}$ and $g^{\prime} \in G \cup G_{1}$. Then $l^{\prime}<g^{\prime}$ follows from: (2), if $l^{\prime} \in L$ and $g^{\prime} \in G$; (3), if $l^{\prime} \in L_{1}$ and $g^{\prime} \in G_{1}$; (4), if $l^{\prime} \in L$ and $g^{\prime}=x \in G_{1}$ or $l^{\prime}=x \in L_{1}$ and $g^{\prime} \in G$. (C2) Let $l^{\prime} \in L \cup L_{1}$ and $u^{\prime} \in U \cup U_{1}$. Then $u^{\prime} \nless l^{\prime}$ follows from: (2), if $l^{\prime} \in L$ and $u^{\prime} \in U$; (3), if $l^{\prime} \in L_{1}$ and $u^{\prime} \in U_{1}$; (4), if $l^{\prime} \in L$ and $u^{\prime}=x \in U_{1}$ (since $l^{\prime}<u^{\prime}$ ) or $l^{\prime}=x \in L_{1}$ and $u^{\prime} \in U$. In the same way we prove (C3).

So there is $x \in D_{\left\langle L \cup L_{1}, G \cup G_{1}, U \cup U_{1}\right\rangle}$, which implies $x \in D_{\langle L, G, U\rangle} \cap D_{\left\langle L_{1}, G_{1}, U_{1}\right\rangle}$ $=\left(D_{\langle L, G, U\rangle}\right)_{\left\langle L_{1}, G_{1}, U_{1}\right\rangle}\left(\right.$ Fact 3.1). Thus $D_{\langle L, G, U\rangle}$ is a random poset and, hence a copy of $\mathbb{D}$.
(b) Let $\langle L, G, U\rangle \in C(D \backslash F)$. By Fact 3.1 we have $\langle L, G, U\rangle \in C(\mathbb{D})$ and, by (a), $\emptyset \neq(D \backslash F) \cap D_{\langle L, G, U\rangle}=(D \backslash F)_{\langle L, G, U\rangle}$. Thus $D \backslash F$ is a copy of $\mathbb{D}$.
(c) Suppose that $P(A) \cap \mathbb{P}(\mathbb{D})=\emptyset$. Then $A \notin \mathbb{P}(\mathbb{D})$ and, hence, there is $\langle L, G, U\rangle \in C(A)$ such that $A_{\langle L, G, U\rangle}=D_{\langle L, G, U\rangle} \cap A=\emptyset$. By Fact 3.1 we have $\langle L, G, U\rangle \in C(\mathbb{D})$ and, by $(\mathrm{a}), \mathbb{P}(\mathbb{D}) \ni D_{\langle L, G, U\rangle} \subset B$.
(d) See (1) in the proof of Theorem 2.2
(e) Let $\langle L, G, U\rangle \in C(D \backslash C)$. Then, by Fact $3.1\langle L, G, U\rangle \in C(\mathbb{D})$ and, by (a), $D_{\langle L, G, U\rangle} \in \mathbb{P}(\mathbb{D})$. By the assumption we have $D_{\langle L, G, U\rangle} \cap(D \backslash C) \neq \emptyset$ and, by Fact 3.1 $(D \backslash C)_{\langle L, G, U\rangle} \neq \emptyset$ and $D \backslash C$ is a random poset.

Lemma 3.4 Let $\mathbb{D}=\langle D,<\rangle$ be a countable random poset, $C \in[D]^{\omega}$ and $A \not \subset C$ for each $A \in \mathbb{P}(\mathbb{D})$ (for example, $C$ can be an infinite antichain). Then
(a) $\mathcal{P}=\left\{B \subset D: D \backslash C \subset^{*} B\right\} \subset \mathbb{P}(\mathbb{D})$;
(b) $\mathcal{P}$ is a positive family on $D$.

Proof. (a) Suppose that $A \subset D \backslash B$, for some $A \in \mathbb{P}(\mathbb{D})$. Since $D \backslash C \subset^{*} B$ we have $D \backslash B \subset^{*} C$ and, hence, $A \subset^{*} C$, that is $|A \backslash C|<\omega$. By Lemma 3.3(b), $A \cap C=A \backslash(A \backslash C) \in \mathbb{P}(\mathbb{D})$, which is not true. So $D \backslash B$ does not contain copies of $\mathbb{D}$ and, by Lemma 3.3(e), $B \in \mathbb{P}(\mathbb{D})$.
(b) Conditions (P1) and (P2) are evident. If $D \backslash C \subset^{*} B$ and $|F|<\omega$, then, clearly, $D \backslash C \subset^{*} B \backslash F$ and (P3) is true. Since the set $D \backslash C$ is co-infinite (P4) is true as well.

Lemma 3.5 Let $A \subset B \subset \omega$ and let $L$ be a complete linear ordering, such that $|B \backslash A|=|L|-1$. Then there is a chain $\mathcal{L}$ in $[A, B]_{P(B)}$ satisfying $A, B \in \mathcal{L} \cong L$ and such that $\bigcup \mathcal{A}, \cap \mathcal{B} \in \mathcal{L}$ and $|\bigcap \mathcal{B} \backslash \bigcup \mathcal{A}| \leq 1$, for each cut $\langle\mathcal{A}, \mathcal{B}\rangle$ in $\mathcal{L}$.

Proof. If $|B \backslash A|$ is a finite set, say $B=A \cup\left\{a_{1}, \ldots a_{n}\right\}$, then $|L|=n+1$ and $\mathcal{L}=\left\{A, A \cup\left\{a_{1}\right\}, A \cup\left\{a_{1}, a_{2}\right\}, \ldots, B\right\}$ is a chain with the desired properties.

If $|B \backslash A|=\omega$, then $L$ is a countable and, hence, $\mathbb{R}$-embeddable complete linear order. It is known that an infinite linear order is isomorphic to a maximal chain in $P(\omega)$ iff it is $\mathbb{R}$-embeddable and Boolean (see, for example, [7]). By Fact $1.4 L$ is a Boolean order and, thus, there is a maximal chain $\mathcal{L}_{1}$ in $P(B \backslash A)$ isomorphic to $L$. Let $\mathcal{L}=\left\{A \cup C: C \in \mathcal{L}_{1}\right\}$. Since $\emptyset, B \backslash A \in \mathcal{L}_{1}$ we have $A, B \in \mathcal{L}$ and the function $f: \mathcal{L}_{1} \rightarrow \mathcal{L}$, defined by $f(C)=A \cup C$, witnesses that $\left\langle\mathcal{L}_{1}, \nsubseteq\right\rangle \cong\langle\mathcal{L}, \nsubseteq\rangle$ so $\mathcal{L}$ is isomorphic to $L$. For each cut $\langle\mathcal{A}, \mathcal{B}\rangle$ in $\mathcal{L}_{1}$ we have $\cup \mathcal{A} \subset \cap \mathcal{B}$ and, by the maximality of $\mathcal{L}_{1}, \bigcup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_{1}$ and $|\cap \mathcal{B} \backslash \cup \mathcal{A}| \leq 1$. Clearly, the same is true for each cut in $\mathcal{L}$.

## 4 Maximal chains of copies of the random poset

Theorem 4.1 For each $\mathbb{R}$-embeddable complete linear order $L$ with $0_{L}$ non-isolated there is a maximal chain in $\langle\mathbb{P}(\mathbb{D}) \cup\{\emptyset\}, \subset\rangle$ isomorphic to $L$.

Proof. By Lemma 3.4 and Theorem 2.2 it remains to prove the statement for uncountable $L$ 's. So let $L$ be an uncountable linear order with the given properties.

Claim 4.2 $L \cong \sum_{x \in[-\infty, \infty]} L_{x}$, where
(L1) $L_{x}, x \in[-\infty, \infty]$, are at most countable complete linear orders,
(L2) The set $M=\left\{x \in[-\infty, \infty]:\left|L_{x}\right|>1\right\}$ is at most countable,
(L3) $\left|L_{-\infty}\right|=1$ or $0_{L_{-\infty}}$ is non-isolated.
Proof. $L=\sum_{i \in I} L_{i}$, where $L_{i}$ are the equivalence classes corresponding to the condensation relation $\sim$ on $L$ given by: $x \sim y \Leftrightarrow|[\min \{x, y\}, \max \{x, y\}]| \leq \omega$ (see [12]). Since $L$ is complete and $\mathbb{R}$-embeddable $I$ is too and, since the cofinalities and coinitialities of $L_{i}$ 's are countable, $I$ is a dense linear order; so $I \cong$ $[0,1] \cong[-\infty, \infty]$. Hence $L_{i}$ 's are complete and, since $\min L_{i} \sim \max L_{i}$, countable. If $\left|L_{i}\right|>1, L_{i}$ has a jump (Fact 1.3) so, $L \hookrightarrow \mathbb{R}$ gives $|M| \leq \omega$.

Case I: $-\infty \notin M \ni \infty$. First we take the rational line $\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle$ and construct a set $\triangleleft \subset \mathbb{Q}^{2}$ such that $\langle\mathbb{Q}, \triangleleft\rangle$ is a random poset with additional, convenient properties. Let $\mathbb{P}$ be the set of pairs $p=\left\langle P_{p}, \triangleleft_{p}\right\rangle$ satisfying
(i) $P_{p} \in[\mathbb{Q}]^{<\omega}$,
(ii) $\triangleleft_{p} \subset P_{p} \times P_{p}$ is a strict partial order on $P_{p}$,
(iii) $<_{\mathbb{Q}}$ extends $\triangleleft_{p}$, that is $\forall q_{1}, q_{2} \in P_{p}\left(q_{1} \triangleleft_{p} q_{2} \Rightarrow q_{1}<_{\mathbb{Q}} q_{2}\right)$, and let the relation $\leq$ on $\mathbb{P}$ be defined by:

$$
\begin{equation*}
p \leq q \Leftrightarrow P_{p} \supset P_{q} \wedge \triangleleft_{p} \cap\left(P_{q} \times P_{q}\right)=\triangleleft_{q} \tag{5}
\end{equation*}
$$

Claim $4.3\langle\mathbb{P}, \leq\rangle$ is a partial order.
Proof. The reflexivity of $\leq$ is obvious. If $p \leq q \leq p$, then $P_{p}=P_{q}$ and, hence, $\triangleleft_{p}=\triangleleft_{p} \cap\left(P_{p} \times P_{p}\right)=\triangleleft_{p} \cap\left(P_{q} \times P_{q}\right)=\triangleleft_{q}$ so $p=q$ and $\leq$ is antisymmetric.

If $p \leq q \leq r$, then $P_{p} \supset P_{q} \supset P_{r}$ and, consequently, $\triangleleft_{p} \cap\left(P_{r} \times P_{r}\right)=$ $\triangleleft_{p} \cap\left(P_{q} \times P_{q}\right) \cap\left(P_{r} \times P_{r}\right)=\triangleleft_{q} \cap\left(P_{r} \times P_{r}\right)=\triangleleft_{r}$. Thus $p \leq r$.

Claim 4.4 The sets $\mathcal{D}_{q}=\left\{p \in \mathbb{P}: q \in P_{p}\right\}, q \in \mathbb{Q}$, are dense in $\mathbb{P}$.
Proof. If $p \in \mathbb{P} \backslash \mathcal{D}_{q}$, that is $q \notin P_{p}$, then $\triangleleft_{p}$ is an irreflexive and transitive relation on the set $P_{p}$ and on the set $P_{p} \cup\{q\}$ as well. Also $\triangleleft_{p} \subset<\mathbb{Q}$ thus $p_{1}=$ $\left\langle P_{p} \cup\{q\}, \triangleleft_{p}\right\rangle \in \mathbb{P}$. Thus $p_{1} \in \mathcal{D}_{q}$ and, clearly, $p_{1} \leq p$.
Let $\mathbb{Q}=J \cup \bigcup_{y \in M} J_{y}$ be a partition of $\mathbb{Q}$ into $|M|+1$ dense subsets of $\mathbb{Q}$. For $\langle L, G, U\rangle \in\left([\mathbb{Q}]^{<\omega}\right)^{3} \backslash\{\langle\emptyset, \emptyset, \emptyset\rangle\}$, let $m_{\langle L, G, U\rangle}=\max _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle}(L \cup G \cup U)$.

Claim 4.5 For each $\langle L, G, U\rangle \in\left([\mathbb{Q}]^{<\omega}\right)^{3} \backslash\{\langle\emptyset, \emptyset, \emptyset\rangle\}$ and each $m \in \mathbb{N}$ the set $\mathcal{D}_{\langle L, G, U\rangle, m}$ is dense in $\mathbb{P}$, where

$$
\begin{aligned}
\mathcal{D}_{\langle L, G, U\rangle, m}= & \left\{p \in \mathbb{P}: L \cup G \cup U \subset P_{p} \wedge(\langle L, G, U\rangle \notin C(p)\right. \\
& \vee\left(G \neq \emptyset \wedge p_{\langle L, G, U\rangle} \cap J \neq \emptyset\right) \\
& \left.\left.\vee\left(G=\emptyset \wedge p_{\langle L, G, U\rangle} \cap\left(m_{\langle L, G, U\rangle}, m_{\langle L, G, U\rangle}+\frac{1}{m}\right) \cap J \neq \emptyset\right)\right)\right\} .
\end{aligned}
$$

Proof. Let $p^{\prime} \in \mathbb{P} \backslash \mathcal{D}_{\langle L, G, U\rangle, m}$. By Claim 4.4 there is $p \in \mathbb{P}$ such that $p \leq p^{\prime}$ and $L \cup G \cup P \subset P_{p}$. If $\langle L, G, U\rangle \notin C(p)$ then $p \in \mathcal{D}_{\langle L, G, U\rangle, m}$ and we are done. If

$$
\begin{equation*}
\langle L, G, U\rangle \in C(p), \tag{6}
\end{equation*}
$$

then we continue the proof distinguishing the following two cases.
Case 1: $G \neq \emptyset$. Let us define $\max _{\left\langle\mathbb{Q},<_{Q}\right\rangle} \emptyset=-\infty$. By (6) and (C1) for $p$, if $L \neq \emptyset$, then $\max _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle} L \triangleleft_{p} \min _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle} G$ and, by (iii), $\max _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle} L<_{\mathbb{Q}}$ $\min _{\left\langle\mathbb{Q},<_{Q}\right\rangle} G$. Now, since $J$ is a dense set in $\langle\mathbb{Q},\langle\mathbb{Q}\rangle$ we choose

$$
\begin{equation*}
q \in\left(\max _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle} L, \min _{\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle} G\right) \cap J \backslash P_{p} \tag{7}
\end{equation*}
$$

and define $p_{1}=\left\langle P_{p} \cup\{q\}, \triangleleft_{p_{1}}\right\rangle$ where

$$
\begin{equation*}
\triangleleft_{p_{1}}=\triangleleft_{p} \cup\left\{\langle x, q\rangle: \exists l \in L \quad x \unlhd_{p} l\right\} \cup\left\{\langle q, y\rangle: \exists g \in G g \unlhd_{p} y\right\} . \tag{8}
\end{equation*}
$$

First we prove that $p_{1} \in \mathbb{P}$. Clearly, $p_{1}$ satisfies condition (i).
(ii) Since $\triangleleft_{p}$ is an irreflexive relation and, by (7), $q \notin P_{p}$, by (8) the relation $\triangleleft_{p_{1}}$ is irreflexive as well.

Suppose that $\triangleleft_{p_{1}}$ is not asymmetric. Then, since $\triangleleft_{p}$ is asymmetric, there is $t \in P_{p}$ such that $\langle t, q\rangle,\langle q, t\rangle \in \unlhd_{p_{1}}$ and by (8), $g \unlhd_{p} t \unlhd_{p} l$, for some $l \in L$ and $g \in G$ which, by the transitivity of $\unlhd_{p}$ implies $g \unlhd_{p} l$. But, by (6) and (C1) we have $l \triangleleft_{p} g$. A contradiction.

Let $\langle a, b\rangle,\langle b, c\rangle \in \triangleleft_{p_{1}}$. Then, since the relation $\triangleleft_{p_{1}}$ is irreflexive and asymmetric, we have $a \neq b \neq c \neq a$. If $q \notin\{a, b, c\}$, then $\langle a, c\rangle \in \triangleleft_{p_{1}}$ by the transitivity of $\triangleleft_{p}$. Otherwise we have three possibilities:
$a=q$. Then $\langle b, c\rangle \in \triangleleft_{p}$ and there is $g \in G$ such that $g \unlhd_{p} b$. Hence $g \triangleleft_{p} c$ which, by (8), implies $\langle q, c\rangle \in \triangleleft_{p_{1}}$, that is $\langle a, c\rangle \in \triangleleft_{p_{1}}$.
$b=q$. Then there are $l \in L$ and $g \in G$ such that $a \unlhd_{p} l$ and $g \unlhd_{p} c$. By (C1) we have $l \triangleleft_{p} g$ and, by the transitivity of $\triangleleft_{p}, a \triangleleft_{p} c$ and, hence, $\langle a, c\rangle \in \triangleleft_{p_{1}}$.
$c=q$. Then $\langle a, b\rangle \in \triangleleft_{p}$ and there is $l \in L$ such that $b \unlhd_{p} l$. Hence $a \unlhd_{p} l$ which, by (8), implies $\langle a, q\rangle \in \triangleleft_{p_{1}}$, that is $\langle a, c\rangle \in \triangleleft_{p_{1}}$.
(iii) Since $p \in \mathbb{P}$, we have $\triangleleft_{p} \subset<_{\mathbb{Q}}$. If $\langle x, q\rangle \in \triangleleft_{p_{1}}$ and $l \in L$, where $x \unlhd_{p} l$, then, since $\triangleleft_{p}$ satisfies (iii), we have $x \leq_{\mathbb{Q}} l$. By (7) we have $l<_{\mathbb{Q}} q$ and, thus, $x<_{\mathbb{Q}} q$. In a similar way we show that $\langle q, y\rangle \in \triangleleft_{p_{1}}$ implies $q<_{\mathbb{Q}} y$.

Thus $p_{1} \in \mathbb{P}, P_{p_{1}} \supset P_{p} \supset L \cup G \cup U$ and, by (8), $\triangleleft_{p_{1}} \cap\left(P_{p} \times P_{p}\right)=\triangleleft_{p}$, which implies that $p_{1} \leq p\left(\leq p^{\prime}\right)$. So $p$ is a suborder of $p_{1}$ and, by (6) and Fact
3.1. $\langle L, G, U\rangle \in C\left(p_{1}\right)$. Since $G \neq \emptyset$ and $q \in J$, for a proof that $p_{1} \in \mathcal{D}_{\langle L, G, U\rangle, m}$ it remains to be shown that $q \in\left(p_{1}\right)_{\langle L, G, U\rangle}$. By (8) $l \triangleleft_{p_{1}} q \triangleleft_{p_{1}} g$, for each $l \in L$ and $g \in G$, so (S1) and (S2) are true. For $u \in U,\langle u, q\rangle \in \triangleleft_{p_{1}}$ would give $l \in L$ satisfying $u \unlhd_{p} l$ and, since $U \cap L=\emptyset, u \unlhd_{p} l$, which is impossible by (6) and (C2). Similarly, $\langle q, u\rangle \in \triangleleft_{p_{1}}$ is not possible and, thus, $q \|_{p_{1}} u$ and (S3) is satisfied.

Case 2: $G=\emptyset$. Again, since $J$ is a dense set in the linear order $\left\langle\mathbb{Q},<_{\mathbb{Q}}\right\rangle$ we choose

$$
\begin{equation*}
q \in\left(m_{\langle L, G, U\rangle}, m_{\langle L, G, U\rangle}+\frac{1}{m}\right) \cap J \backslash P_{p} \tag{9}
\end{equation*}
$$

and define $p_{1}=\left\langle P_{p} \cup\{q\}, \triangleleft_{p_{1}}\right\rangle$, where

$$
\begin{equation*}
\triangleleft_{p_{1}}=\triangleleft_{p} \cup\left\{\langle x, q\rangle: \exists l \in L \quad x \unlhd_{p} l\right\} \tag{10}
\end{equation*}
$$

First we prove that $p_{1} \in \mathbb{P}$. Clearly, $p_{1}$ satisfies condition (i).
(ii) By (9) we have $q \notin P_{p}$ so, by (10) the relation $\triangleleft_{p_{1}}$ is irreflexive.

Let $\langle a, b\rangle,\langle b, c\rangle \in \triangleleft_{p_{1}}$. If $q \notin\{a, b, c\}$, then $\langle a, c\rangle \in \triangleleft_{p_{1}}$ by (10) and the transitivity of $\triangleleft_{p}$. Otherwise, by (10) again, $a, b \neq q$ and, thus, $c=q$. Hence there is $l \in L$ such that $b \unlhd_{p} l$. Since $a, b \neq q$, by (10) we have $a \triangleleft_{p} b$ and, hence $a \triangleleft_{p} l$, which implies $\langle a, q\rangle \in \triangleleft_{p_{1}}$, that is $\langle a, c\rangle \in \triangleleft_{p_{1}}$.
(iii) Since $p \in \mathbb{P}$, we have $\triangleleft_{p} \subset<_{\mathbb{Q}}$. If $\langle x, q\rangle \in \triangleleft_{p_{1}}$ and $l \in L$, where $x \unlhd_{p} l$, then, since $\triangleleft_{p}$ satisfies (iii), we have $x \leq_{\mathbb{Q}} l$. By (9) we have $l \leq_{\mathbb{Q}} m_{\langle L, G, U\rangle}<_{\mathbb{Q}} q$ and, thus, $x<_{\mathbb{Q}} q$.

Thus $p_{1} \in \mathbb{P}$. As in Case 1 we show that $L \cup G \cup U \subset P_{p_{1}}, p_{1} \leq p\left(\leq p^{\prime}\right)$ and $\langle L, G, U\rangle \in C\left(p_{1}\right)$. By (9) and since $G=\emptyset$, for a proof that $p_{1} \in \mathcal{D}_{\langle L, G, U\rangle, m}$ it remains to be shown that $q \in\left(p_{1}\right)_{\langle L, G, U\rangle}$. (S2) is trivial and, by (10), for $l \in L$ we have $\langle l, q\rangle \in \triangleleft_{p_{1}}$ thus (S1) holds as well. Suppose that $\neg q \|_{p_{1}} u$, for some $u \in U$. Then, by (9) and (10), $\langle u, q\rangle \in \triangleleft_{p_{1}}$ and, hence, there is $l \in L$ satisfying $u \triangleleft_{p} l$, which is impossible by (6) and (C2) for $p$. So (S3) is true.

By the Rasiowa Sikorski theorem there is a filter $\mathcal{G}$ in $\langle\mathbb{P}, \leq\rangle$ intersecting the sets $\mathcal{D}_{q}, q \in \mathbb{Q}$, and $\mathcal{D}_{\langle L, G, U\rangle, m},\langle L, G, U\rangle \in\left([\mathbb{Q}]^{<\omega}\right)^{3}, m \in \mathbb{N}$.

Claim 4.6 (a) $\bigcup_{p \in \mathcal{G}} P_{p}=\mathbb{Q}$;
(b) $\triangleleft=\bigcup_{p \in \mathcal{G}} \triangleleft_{p}$ is a strict partial order on $\mathbb{Q}$;
(c) $\triangleleft \cap\left(P_{p} \times P_{p}\right)=\triangleleft_{p}$, for each $p \in \mathcal{G}$;
(d) $<_{\mathbb{Q}}$ extends $\triangleleft$, that is $\forall q_{1}, q_{2} \in \mathbb{Q}\left(q_{1} \triangleleft q_{2} \Rightarrow q_{1}<_{\mathbb{Q}} q_{2}\right)$.

Proof. (a) For $q \in \mathbb{Q}$ let $p_{0} \in \mathcal{G} \cap \mathcal{D}_{q}$. Then $q \in P_{p_{0}} \subset \bigcup_{p \in \mathcal{G}} P_{p}$.
(b) The relation $\triangleleft$ is irreflexive since all the relations $\triangleleft_{p}$ are irreflexive.

Let $\langle a, b\rangle,\langle b, c\rangle \in \triangleleft,\langle a, b\rangle \in \triangleleft_{p_{1}}$ and $\langle b, c\rangle \in \triangleleft_{p_{2}}$, where $p_{1}, p_{2} \in \mathcal{G}$. Since $\mathcal{G}$ is a filter there is $p \in \mathcal{G}$ such that $p \leq p_{1}, p_{2}$, which by (5) implies $\triangleleft_{p_{1}}, \triangleleft_{p_{2}} \subset \triangleleft_{p}$. Thus $\langle a, b\rangle,\langle b, c\rangle \in \triangleleft_{p}$ and, by the transitivity of $\triangleleft_{p},\langle a, c\rangle \in \triangleleft_{p} \subset \triangleleft$.
(c) The inclusion " $\supset$ " follows from (ii) and the definition of $\triangleleft$. If $\langle a, b\rangle \in$ $\triangleleft \cap\left(P_{p} \times P_{p}\right)$, then there is $p_{1} \in \mathcal{G}$ such that $\langle a, b\rangle \in \triangleleft_{p_{1}}$ and, since $\mathcal{G}$ is a filter, there is $p_{2} \in \mathcal{G}$ such that $p_{2} \leq p, p_{1}$. By (5) we have $\triangleleft_{p_{1}} \subset \triangleleft_{p_{2}}$, which implies $\langle a, b\rangle \in \triangleleft_{p_{2}}$ and, by (5) again, $\langle a, b\rangle \in \triangleleft_{p_{2}} \cap\left(P_{p} \times P_{p}\right)=\triangleleft_{p}$.
(d) If $\left\langle q_{1}, q_{2}\right\rangle \in \triangleleft$ and $p \in \mathcal{G}$ where $\left\langle q_{1}, q_{2}\right\rangle \in \triangleleft_{p}$, then by (iii), $q_{1}<_{\mathbb{Q}} q_{2}$.

Claim 4.7 (a) $\langle A, \triangleleft\rangle$ is a random poset, for each $x \in(-\infty, \infty]$ and each set $A$ satisfying

$$
\begin{equation*}
(-\infty, x) \cap J \subset A \subset(-\infty, x) \cap \mathbb{Q} \tag{11}
\end{equation*}
$$

(b) If $J \subset A \subset \mathbb{Q}$ then $\langle A, \triangleleft\rangle$ (in particular, $\langle\mathbb{Q}, \triangleleft\rangle$ ) is a random poset.
(c) If $C \subset \mathbb{Q}$ and $\max _{\langle\mathbb{Q},<\mathbb{Q}\rangle} C$ exists, then $\langle C, \triangleleft\rangle$ is not a random poset.

Proof. (a) By Claim 4.6(b), $\langle A, \triangleleft\rangle$ is a strict partial order. Let $\langle L, G, U\rangle \in$ $C(A, \triangleleft)$. Then

$$
\begin{gather*}
L \cup G \cup U \subset A \wedge L \cap G=G \cap U=U \cap A=\emptyset  \tag{12}\\
\forall l \in L \quad \forall g \in G \quad \forall u \in U \quad(\langle l, g\rangle \in \triangleleft \wedge\langle u, l\rangle \notin \triangleleft \wedge\langle g, u\rangle \notin \triangleleft) . \tag{13}
\end{gather*}
$$

We show that $\langle A, \triangleleft\rangle_{\langle L, G, U\rangle} \neq \emptyset$. For $\langle L, G, U\rangle \neq\langle\emptyset, \emptyset, \emptyset\rangle$ we have two cases.
Case 1: $G \neq \emptyset$. Let $p \in \mathcal{G} \cap \mathcal{D}_{\langle L, G, U\rangle, 1}$. Then

$$
\begin{equation*}
L \cup G \cup U \subset P_{p} . \tag{14}
\end{equation*}
$$

First we show that $\langle L, G, U\rangle \in C(p)$. Let $l \in L, g \in G$ and $u \in U$. By (13), (14) and Claim4.6(c) we have $\langle l, g\rangle \in \triangleleft_{p}$ and (C1) is true. Since $\triangleleft_{p} \subset \triangleleft$ by (13) we have $\langle u, l\rangle \notin \triangleleft_{p}$ and $\langle g, u\rangle \notin \triangleleft_{p}$ and (C2) and (C3) are true as well.

Since $p \in \mathcal{D}_{\langle L, G, U\rangle, 1}$ there is $q \in p_{\langle L, G, U\rangle} \cap J$. We prove that $q \in\langle A, \triangleleft\rangle_{\langle L, G, U\rangle}$. For a $g \in G$ we have $q \triangleleft_{p} g$ and, by (iii), $q<_{\mathbb{Q}} g$. By (11) and (12) we have $g \in G \subset A \subset(-\infty, x)$ and, hence $q<_{\mathbb{Q}} g<_{\mathbb{R}} x$, thus $q \in(-\infty, x) \cap J \subset A$. Let $l \in L, g \in G$ and $u \in U$. Since $q \in p_{\langle L, G, U\rangle}$ we have $l \triangleleft_{p} q \triangleleft_{p} g$ and $\triangleleft_{p} \subset \triangleleft$ implies $l \triangleleft q \triangleleft g$. Thus (S1) and (S2) are true. Suppose that $\neg q \|_{\langle A, \triangleleft\rangle} u$. Since $q \notin U$ we have $q \neq u$ and, hence, $q \triangleleft u$ or $u \triangleleft q$. But then, since $u, q \in P_{p}$, by Claim 4.6(c) we would have $q \triangleleft_{p} u$ or $u \triangleleft_{p} q$, which is impossible because $q \in p_{\langle L, G, U\rangle}$. So (S3) is true as well.

Case 2: $G=\emptyset$. By (11) and (12) we have $L \cup G \cup U \subset(-\infty, x)$, which implies $m_{\langle L, G, U\rangle}<x$ and, hence, there is $m \in \mathbb{N}$ such that

$$
\begin{equation*}
m_{\langle L, G, U\rangle}+\frac{1}{m}<x . \tag{15}
\end{equation*}
$$

Let $p \in \mathcal{G} \cap \mathcal{D}_{\langle L, G, U\rangle, m}$. Then (14) holds again and exactly like in Case 1 we show that $\langle L, G, U\rangle \in C(p)$. Thus, since $p \in \mathcal{D}_{\langle L, G, U\rangle, m}$ there is $q \in p_{\langle L, G, U\rangle} \cap$ $\left(m_{\langle L, G, U\rangle}, m_{\langle L, G, U\rangle}+\frac{1}{m}\right) \cap J$ and, by (15), $q \in J \cap(-\infty, x)$. Thus, by (11), $q \in A$ and exactly like in Case 1 we prove that $q \in\langle A, \triangleleft\rangle_{\langle L, G, U\rangle}$.
(b) Follows from (a) for $x=\infty$.
(c) Suppose that $\max _{\left\langle\mathbb{Q},<_{Q}\right\rangle} C=q$ and that $\langle C, \triangleleft\rangle$ is a random poset. Then $C_{\langle\{q\}, \emptyset, \emptyset\rangle} \neq \emptyset$ and, by (S1), there is $q_{1} \in C$ such that $q \triangleleft q_{1}$, which, by Claim 4.6(d) implies $q<\mathbb{Q} q_{1}$. A contradiction with the maximality of $q$.

For $y \in M$ let us take $I_{y} \in\left[J_{y} \cap(-\infty, y)\right]^{\left|L_{y}\right|-1}$ and define $A_{-\infty}=\emptyset$ and

$$
A_{x}=(J \cap(-\infty, x)) \cup \bigcup_{y \in M \cap(-\infty, x)} I_{y}, \quad \text { for } x \in(-\infty, \infty] ;
$$

$$
A_{x}^{+}=A_{x} \cup I_{x}, \quad \text { for } x \in M
$$

Since $J \subset A_{\infty}^{+} \subset \mathbb{Q}$, by Claim4.7 (b) $\left\langle A_{\infty}^{+}, \triangleleft\right\rangle$ is a random poset and we construct a maximal chain $\mathcal{L}$ in $\left\langle\mathbb{P}\left(A_{\infty}^{+}, \triangleleft\right), \subset\right\rangle$, such that $\mathcal{L} \cong L$.

Claim 4.8 The sets $A_{x}, x \in[-\infty, \infty]$ and $A_{x}^{+}, x \in M$ are subsets of the set $A_{\infty}^{+}$ and of $\mathbb{Q}$. In addition, for each $x, x_{1}, x_{2} \in[-\infty, \infty]$ we have
(a) $A_{x} \subset(-\infty, x)$;
(b) $A_{x}^{+} \subset(-\infty, x)$, if $x \in M$;
(c) $x_{1}<x_{2} \Rightarrow A_{x_{1}} \nsubseteq A_{x_{2}}$;
(d) $M \ni x_{1}<x_{2} \Rightarrow A_{x_{1}}^{+} \varsubsetneqq A_{x_{2}}$;
(e) $\left|A_{x}^{+} \backslash A_{x}\right|=\left|L_{x}\right|-1$, if $x \in M$;
(f) $A_{x} \in \mathbb{P}\left(A_{\infty}^{+}\right)$, for each $x \in(-\infty, \infty]$.
(g) $A_{x}^{+} \in \mathbb{P}\left(A_{\infty}^{+}\right)$and $\left[A_{x}, A_{x}^{+}\right]_{\mathbb{P}\left(A_{\infty}^{+}\right)}=\left[A_{x}, A_{x}^{+}\right]_{P\left(A_{x}^{+}\right)}$, for each $x \in M$.

Proof. Statements (c) and (d) are true since $J$ is a dense subset of $\mathbb{Q}$; (a), (b) and (e) follow from the definitions of $A_{x}$ and $A_{x}^{+}$and the choice of the sets $I_{y}$. Since $J \cap(-\infty, x) \subset A_{x} \subset A_{x}^{+} \subset(-\infty, x) \cap \mathbb{Q}$, (f) and (g) follow from Claim4.7(a).

Now, for $x \in[-\infty, \infty]$ we define chains $\mathcal{L}_{x} \subset \mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}$ in the following way.
For $x \notin M$ we define $\mathcal{L}_{x}=\left\{A_{x}\right\}$. In particular, $\mathcal{L}_{-\infty}=\{\emptyset\}$.
For $x \in M$, using Claim4.8 and Lemma 3.5we obtain a set $\mathcal{L}_{x} \subset\left[A_{x}, A_{x}^{+}\right]_{P\left(A_{x}^{+}\right)}$ such that $\left\langle\mathcal{L}_{x}, \mp\right\rangle \cong\left\langle L_{x},\left\langle_{x}\right\rangle\right.$ and

$$
\begin{equation*}
A_{x}, A_{x}^{+} \in \mathcal{L}_{x} \subset\left[A_{x}, A_{x}^{+}\right]_{\mathbb{P}\left(A_{\infty}^{+}\right)} \tag{16}
\end{equation*}
$$

$\cup \mathcal{A}, \cap \mathcal{B} \in \mathcal{L}_{x}$ and $|\cap \mathcal{B} \backslash \cup \mathcal{A}| \leq 1$, for each cut $\langle\mathcal{A}, \mathcal{B}\rangle$ in $\mathcal{L}_{x}$.
For $\mathcal{A}, \mathcal{B} \subset \mathbb{P}\left(A_{\infty}^{+}\right)$we will write $\mathcal{A} \prec \mathcal{B}$ iff $A \varsubsetneqq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
Claim 4.9 Let $\mathcal{L}=\bigcup_{x \in[-\infty, \infty]} \mathcal{L}_{x}$. Then
(a) If $-\infty \leq x_{1}<x_{2} \leq \infty$, then $\mathcal{L}_{x_{1}} \prec \mathcal{L}_{x_{2}}$ and $\cup \mathcal{L}_{x_{1}} \subset A_{x_{2}} \subset \bigcup \mathcal{L}_{x_{2}}$.
(b) $\mathcal{L}$ is a chain in $\left\langle\mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L=\sum_{x \in[-\infty, \infty]} L_{x}$.
(c) $\mathcal{L}$ is a maximal chain in $\left\langle\mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}, \subset\right\rangle$.

Proof. (a) Let $A \in \mathcal{L}_{x_{1}}$ and $B \in \mathcal{L}_{x_{2}}$. If $x_{1} \in(-\infty, \infty] \backslash M$, then, by (16) and Claim4.8(c) we have $A=A_{x_{1}} \nsubseteq A_{x_{2}} \subset B$. If $x_{1} \in M$, then, by (16) and Claim 4.8(d), $A \subset A_{x_{1}}^{+} \nsubseteq A_{x_{2}} \subset B$. The second statement follows from $A_{x_{2}} \in \mathcal{L}_{x_{2}}$.
(b) By (a), $\langle[-\infty, \infty],<\rangle \cong\left\langle\left\{\mathcal{L}_{x}: x \in[-\infty, \infty]\right\}, \prec\right\rangle$. Since $\mathcal{L}_{x} \cong L_{x}$, for $x \in[-\infty, \infty]$, we have $\langle\mathcal{L}, \mp\rangle \cong \sum_{x \in[-\infty, \infty]}\left\langle\mathcal{L}_{x}, \mp\right\rangle \cong \sum_{x \in[-\infty, \infty]} L_{x}=L$.
(c) Suppose that $C \in \mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Clearly $\mathcal{L}=\mathcal{A} \dot{\mathcal{B}}$ and $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A}=\{A \in \mathcal{L}: A \nsubseteq C\}$ and $\mathcal{B}=\{B \in \mathcal{L}:$ $C \varsubsetneqq B\}$. Now $\emptyset \in \mathcal{L}_{-\infty}$ and, since $\infty \in M$, by (16) we have $A_{\infty}^{+} \in \mathcal{L}_{\infty}$. Thus $\emptyset, A_{\infty}^{+} \in \mathcal{L}$, which implies $\mathcal{A}, \mathcal{B} \neq \emptyset$ and, hence, $\langle\mathcal{A}, \mathcal{B}\rangle$ is a cut in $\langle\mathcal{L}, \mp\rangle$. By (16)
we have $\left\{A_{x}: x \in(-\infty, \infty]\right\} \subset \mathcal{L} \backslash\{\emptyset\}$ and, by Claim4.8(a), $\bigcap(\mathcal{L} \backslash\{\emptyset\}) \subset$ $\bigcap_{x \in(-\infty, \infty]} A_{x} \subset \bigcap_{x \in(-\infty, \infty]}(-\infty, x)=\emptyset$, which implies $\mathcal{A} \neq\{\emptyset\}$. Clearly,

$$
\begin{equation*}
\cup \mathcal{A} \subset C \subset \bigcap \mathcal{B} \tag{18}
\end{equation*}
$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_{0}} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_{0}} \neq \emptyset$, for some $x_{0} \in(-\infty, \infty]$. Then $\left|\mathcal{L}_{x_{0}}\right|>1$, $x_{0} \in M$ and $\left\langle\mathcal{A} \cap \mathcal{L}_{x_{0}}, \mathcal{B} \cap \mathcal{L}_{x_{0}}\right\rangle$ is a cut in $\mathcal{L}_{x_{0}}$ satisfying (17). By (a), $\mathcal{A}=$ $\bigcup_{x<x_{0}} \mathcal{L}_{x} \cup\left(\mathcal{A} \cap \mathcal{L}_{x_{0}}\right)$ and, consequently, $\bigcup \mathcal{A}=\bigcup\left(\mathcal{A} \cap \mathcal{L}_{x_{0}}\right) \in \mathcal{L}$. Similarly, $\bigcap \mathcal{B}=\bigcap\left(\mathcal{B} \cap \mathcal{L}_{x_{0}}\right) \in \mathcal{L}$ and, since $|\bigcap \mathcal{B} \backslash \cup \mathcal{A}| \leq 1$, by (18) we have $C \in \mathcal{L}$. A contradiction.

Case 2: $\neg$ Case 1. Then for each $x \in(-\infty, \infty]$ we have $\mathcal{L}_{x} \subset \mathcal{A}$ or $\mathcal{L}_{x} \subset \mathcal{B}$. Since $\mathcal{L}=\mathcal{A} \dot{\mathcal{B}}, \mathcal{A} \neq\{\emptyset\}$ and $\mathcal{A}, \mathcal{B} \neq \emptyset$, the sets $\mathcal{A}^{\prime}=\left\{x \in(-\infty, \infty]: \mathcal{L}_{x} \subset \mathcal{A}\right\}$ and $\mathcal{B}^{\prime}=\left\{x \in(-\infty, \infty]: \mathcal{L}_{x} \subset \mathcal{B}\right\}$ are non-empty and $(-\infty, \infty]=\mathcal{A}^{\prime} \dot{\cup} \mathcal{B}^{\prime}$. Since $\mathcal{A} \prec \mathcal{B}$, for $x_{1} \in \mathcal{A}^{\prime}$ and $x_{2} \in \mathcal{B}^{\prime}$ we have $\mathcal{L}_{x_{1}} \prec \mathcal{L}_{x_{2}}$ so, by (a), $x_{1}<x_{2}$. Thus $\left\langle\mathcal{A}^{\prime}, \mathcal{B}^{\prime}\right\rangle$ is a cut in $(-\infty, \infty]$ and, consequently, there is $x_{0} \in(-\infty, \infty]$ such that $x_{0}=\max \mathcal{A}^{\prime}$ or $x_{0}=\min \mathcal{B}^{\prime}$.

Subcase 2.1: $x_{0}=\max \mathcal{A}^{\prime}$. Then $x_{0}<\infty$ because $\mathcal{B} \neq \emptyset$ and $\mathcal{A}=\bigcup_{x \leq x_{0}} \mathcal{L}_{x}$ so, by (a), $\cup \mathcal{A}=\bigcup_{x \leq x_{0}} \cup \mathcal{L}_{x}=\bigcup_{x<x_{0}} \cup \mathcal{L}_{x} \cup \bigcup \mathcal{L}_{x_{0}}=\bigcup \mathcal{L}_{x_{0}}$ which, together with (16) implies

$$
\cup \mathcal{A}= \begin{cases}A_{x_{0}} & \text { if } x_{0} \notin M  \tag{19}\\ A_{x_{0}}^{+} & \text {if } x_{0} \in M .\end{cases}
$$

Since $\mathcal{B}=\bigcup_{x \in\left(x_{0}, \infty\right]} \mathcal{L}_{x}$, we have $\bigcap \mathcal{B}=\bigcap_{x \in\left(x_{0}, \infty\right]} \cap \mathcal{L}_{x}$. By (16) $\cap \mathcal{L}_{x}=A_{x}$, so we have $\bigcap \mathcal{B}=\left(\bigcap_{x \in\left(x_{0}, \infty\right]}(-\infty, x) \cap J\right) \cup\left(\bigcap_{x \in\left(x_{0}, \infty\right]} \bigcup_{y \in M \cap(-\infty, x)} I_{y}\right)=$ $\left(\left(-\infty, x_{0}\right] \cap J\right) \cup \bigcup_{y \in M \cap\left(-\infty, x_{0}\right]} I_{y}=A_{x_{0}} \cup\left(\left\{x_{0}\right\} \cap J\right) \cup \bigcup_{y \in M \cap\left\{x_{0}\right\}} I_{y}$, so

$$
\cap \mathcal{B}=\left\{\begin{array}{lllll}
A_{x_{0}} & \text { if } & x_{0} \notin J & \wedge & x_{0} \notin M,  \tag{20}\\
A_{x_{0}} \cup\left\{x_{0}\right\} & \text { if } & x_{0} \in J & \wedge & x_{0} \notin M, \\
A_{x_{0}}^{+} & \text {if } & x_{0} \notin J & \wedge & x_{0} \in M, \\
A_{x_{0}}^{+} \cup\left\{x_{0}\right\} & \text { if } & x_{0} \in J & \wedge & x_{0} \in M
\end{array}\right.
$$

If $x_{0} \notin J$, then, by (18), (19) and (20), we have $\bigcup \mathcal{A}=\bigcap \mathcal{B}=C \in \mathcal{L}$. A contradiction.

If $x_{0} \in J$ and $x_{0} \notin M$, then $\bigcup \mathcal{A}=A_{x_{0}}$ and $\bigcap \mathcal{B}=A_{x_{0}} \cup\left\{x_{0}\right\}$. So, by (18) and since $C \notin \mathcal{L}$ we have $C=\bigcap \mathcal{B}$. But, by Claim4.8(a), $x_{0}=\max \bigcap \mathcal{B}$ so, by Claim4.7(c), $C \notin \mathbb{P}\left(A_{\infty}^{+}\right)$. A contradiction.

If $x_{0} \in J$ and $x_{0} \in M$, then $\cup \mathcal{A}=A_{x_{0}}^{+}$and $\bigcap \mathcal{B}=A_{x_{0}}^{+} \cup\left\{x_{0}\right\}$. Again, by (18) and since $C \notin \mathcal{L}$ we have $C=\bigcap \mathcal{B}$. By Claim4.8(b), $x_{0}=\max \cap \mathcal{B}$ so, by Claim 4.7(c), $C \notin \mathbb{P}\left(A_{\infty}^{+}\right)$. A contradiction.

Subcase 2.2: $x_{0}=\min \mathcal{B}^{\prime}$. Then, by (16), $A_{x_{0}} \in \mathcal{L}_{x_{0}} \subset \mathcal{B}$ which, by (a), implies $\bigcap \mathcal{B}=A_{x_{0}}$. Since $A_{x} \in \mathcal{L}_{x}$, for $x \in(-\infty, \infty]$ and $\mathcal{A}=\bigcup_{x<x_{0}} \mathcal{L}_{x}$ we have $\bigcup \mathcal{A}=\bigcup_{x<x_{0}} \cup \mathcal{L}_{x} \supset \bigcup_{x<x_{0}} A_{x}=\bigcup_{x<x_{0}}((-\infty, x) \cap J) \cup \bigcup_{x<x_{0}} \bigcup_{y \in M \cap(-\infty, x)} I_{y}$
$=\left(\left(-\infty, x_{0}\right) \cap J\right) \cup \bigcup_{y \in M \cap\left(-\infty, x_{0}\right)} I_{y}=A_{x_{0}}$ so $A_{x_{0}} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B}=A_{x_{0}}$, which implies $C=A_{x_{0}} \in \mathcal{L}$. A contradiction.

Case II: $-\infty \notin M \not \ngtr \infty$. Then $L_{\infty}=\{\max L\}$ and the sum $L+1$ belongs to Case I. So, there are a maximal chain $\mathcal{L}$ in $\langle\mathbb{P}(\mathbb{D}) \cup\{\emptyset\}, \subset\rangle$ and an isomorphism $f:\langle L+1,<\rangle \rightarrow\langle\mathcal{L}, \subset\rangle$. Then $A=f(\max L) \in \mathbb{P}(\mathbb{D})$ and $\mathcal{L}^{\prime}=f[L] \cong L$. By the maximality of $\mathcal{L}, \mathcal{L}^{\prime}$ is a maximal chain in $\langle\mathbb{P}(A) \cup\{\emptyset\}, \subset\rangle$.
Case III: $-\infty \in M$. Then $L=\sum_{x \in[-\infty, \infty]} L_{x}$, (L1) and (L2) of Claim4.2 hold and
(L3') $L_{-\infty}$ is a countable complete linear order with $0_{L_{-\infty}}$ non-isolated.
Clearly $L=L_{-\infty}+L^{+}$, where $L^{+}=\sum_{x \in(-\infty, \infty]} L_{x}=\sum_{y \in(0, \infty]} L_{\ln y}$ (here $\ln \infty=\infty)$. Let $L_{y}^{\prime}, y \in[-\infty, \infty]$, be disjoint linear orders such that $L_{y}^{\prime} \cong 1$, for $y \in[-\infty, 0]$, and $L_{y}^{\prime} \cong L_{\ln y}$, for $y \in(0, \infty]$. Now $\sum_{y \in[-\infty, \infty]} L_{y}^{\prime} \cong[-\infty, 0]+$ $L^{+}$belongs to Case I or Case II and we obtain a maximal chain $\mathcal{L}$ in $\mathbb{P}(\mathbb{D}) \cup\{\emptyset\}$ and an isomorphism $f:\left\langle[-\infty, 0]+L^{+},\langle \rangle \rightarrow\langle\mathcal{L}, \subset\rangle\right.$. Clearly, for $A_{0}=f(0)$ and $\mathcal{L}^{+}=f\left[L^{+}\right]$we have $A_{0} \in \mathcal{L}$ and $\mathcal{L}^{+} \cong L^{+}$.

By ( $\mathrm{L}^{\prime}$ ) and the fact that $(\mathrm{b}) \Rightarrow(\mathrm{a})$ for countable $L$ 's, $\mathbb{P}\left(A_{0}\right) \cup\{\emptyset\}$ contains a maximal chain $\mathcal{L}_{-\infty} \cong L_{-\infty}$. Clearly $A_{0} \in \mathcal{L}_{-\infty}$ and $\mathcal{L}_{-\infty} \cup \mathcal{L}^{+} \cong L_{-\infty}+L^{+}=$ $L$. Suppose that $B$ witnesses that $\mathcal{L}_{-\infty} \cup \mathcal{L}^{+}$is not a maximal chain in $\mathbb{P}(\mathbb{D}) \cup\{\emptyset\}$. Then either $A_{0} \nsubseteq B$, which is impossible since $\mathcal{L}$ is maximal in $\mathbb{P}(\mathbb{D}) \cup\{\emptyset\}$, or $B \nsubseteq A_{0}$, which is impossible since $\mathcal{L}_{-\infty}$ is maximal in $\mathbb{P}\left(A_{0}\right) \cup\{\emptyset\}$.

## 5 Maximal chains in $\mathbb{P}\left(\mathbb{B}_{n}\right)$

Theorem 5.1 For $n \in \mathbb{N}$ and each $\mathbb{R}$-embeddable complete linear order $L$ with $0_{L}$ non-isolated there is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{B}_{n}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L$.

Proof. Let the order on $\mathbb{B}_{n}=\bigcup_{i<n} \mathbb{Q}_{i}=\bigcup_{i<n}\{i\} \times \mathbb{Q}$ be given by

$$
\left\langle i_{1}, q_{1}\right\rangle<\left\langle i_{2}, q_{2}\right\rangle \Leftrightarrow i_{1}=i_{2} \wedge q_{1}<_{\mathbb{Q}} q_{2} .
$$

Clearly, $\langle\mathbb{Q},<\mathbb{Q}\rangle \cong_{f_{i}}\left\langle\mathbb{Q}_{i},<\right\rangle$, where $f_{i}(q)=\langle i, q\rangle$, for all $q \in \mathbb{Q}$ and, hence, $\mathbb{P}\left(\mathbb{Q}_{i}\right)=\{\{i\} \times C: C \in \mathbb{P}(\mathbb{Q})\}$. If $f: \mathbb{B}_{n} \hookrightarrow \mathbb{B}_{n}$, then for each $i<n$ the restriction $f \mid \mathbb{Q}_{i}$ is an isomorphism, thus there is $j_{i}<n$ such that $f\left[\mathbb{Q}_{i}\right] \subset \mathbb{Q}_{j_{i}}$ and, moreover, $f\left[\mathbb{Q}_{i}\right] \in \mathbb{P}\left(\mathbb{Q}_{j_{i}}\right)$. Clearly, $i_{1} \neq i_{2}$ implies $j_{i_{1}} \neq j_{i_{2}}$ and, thus, we have

$$
\begin{equation*}
\mathbb{P}\left(\mathbb{B}_{n}\right)=\left\{\bigcup_{i<n}\{i\} \times C_{i}: \forall i<n \quad C_{i} \in \mathbb{P}(\mathbb{Q})\right\} . \tag{21}
\end{equation*}
$$

Now, by Theorem 6 of [8], there is a maximal chain $\mathcal{L}$ in $\langle\mathbb{P}(\mathbb{Q}) \cup\{\emptyset\}, \subset\rangle$ isomorphic to $L$. For $A \in \mathcal{L} \backslash\{\emptyset\}$ let

$$
\begin{equation*}
A^{*}=(\{0\} \times A) \cup \bigcup_{0<i<n}\{i\} \times \mathbb{Q} . \tag{22}
\end{equation*}
$$

By (21) we have $\mathcal{L}^{*}=\left\{A^{*}: A \in \mathcal{L} \backslash\{\emptyset\}\right\} \cup\{\emptyset\} \subset \mathbb{P}\left(\mathbb{B}_{n}\right) \cup\{\emptyset\}$ and, clearly, $\left\langle\mathcal{L}^{*}, \subset\right\rangle$ is a chain in $\left\langle\mathbb{P}\left(\mathbb{B}_{n}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $\langle\mathcal{L}, \subset\rangle$ and, hence, to $L$.

Suppose that some $C=\bigcup_{i<n}\{i\} \times C_{i} \in \mathbb{P}\left(\mathbb{B}_{n}\right)$ witnesses that $\mathcal{L}^{*}$ is not a maximal chain. By (21) and (22) $C \subset \bigcap_{A \in \mathcal{L} \backslash\{\emptyset\}} A^{*}$ would imply $\mathbb{P}(\mathbb{Q}) \ni C_{0} \subset \bigcap(\mathcal{L} \backslash\{\emptyset\})$, which is impossible $\left(\mathcal{L}\right.$ is a maximal chain in $\mathbb{P}(\mathbb{Q}) \cup\{\emptyset\}$ and $C_{0} \backslash F \in \mathbb{P}(\mathbb{Q})$ for each finite $F \subset C_{0}$ ). Thus there is $A \in \mathcal{L} \backslash\{\emptyset\}$ such that $A^{*} \subset C$ and, by (22),

$$
\begin{equation*}
C=\{0\} \times C_{0} \cup \bigcup_{0<i<n}\{i\} \times \mathbb{Q} . \tag{23}
\end{equation*}
$$

Since $\mathcal{L}^{*} \cup\{C\}$ is a chain, for each $A \in \mathcal{L} \backslash\{\emptyset\}$ we have $A^{*} \subsetneq C \vee C \subsetneq A^{*}$ which together with (22) and (23) implies $A \subsetneq C_{0}$ or $C_{0} \subsetneq A$. A contradiction to the maximality of $\mathcal{L}$.

Theorem 5.2 For each $\mathbb{R}$-embeddable complete linear order $L$ with $0_{L}$ non-isolated there is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L$.

Proof. Let $x_{0}=\infty$, let $\left\langle x_{n}: n \in \mathbb{N}\right\rangle$ be a descending sequence in $\mathbb{R} \backslash \mathbb{Q}$ without a lower bound and let $\mathbb{B}_{\omega}=\left\langle\mathbb{Q},<_{\omega}\right\rangle=\bigcup_{i \in \omega}\left\langle\left(x_{i+1}, x_{i}\right) \cap \mathbb{Q},<_{i}\right\rangle$ where

$$
q_{1}<_{\omega} q_{2} \Leftrightarrow \exists i \in \omega\left(q_{1}, q_{2} \in\left(x_{i+1}, x_{i}\right) \wedge q_{1}<_{\mathbb{Q}} q_{2}\right)
$$

Then for the sets $\mathbb{Q}_{i}=\left(x_{i+1}, x_{i}\right) \cap \mathbb{Q}, i \in \omega$, we have $\left\langle\mathbb{Q}_{i},<_{i}\right\rangle \cong\langle\mathbb{Q},<\mathbb{Q}\rangle$, which implies $\mathbb{P}\left(\mathbb{Q}_{i},<_{i}\right) \cong \mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right)$. As in the proof of Theorem5.1]we obtain

$$
\begin{equation*}
\mathbb{P}\left(\mathbb{B}_{\omega}\right)=\left\{\bigcup_{i \in S} C_{i}: S \in[\omega]^{\omega} \wedge \forall i \in S \quad C_{i} \in \mathbb{P}\left(\mathbb{Q}_{i}\right)\right\} . \tag{24}
\end{equation*}
$$

Let $L$ be a linear order with the given properties and, first, let $|L|=\omega$. Clearly the family Dense $\left(\mathbb{Q}_{i}\right)$ of dense subsets of $\mathbb{Q}_{i}$ is a subset of $\mathbb{P}\left(\mathbb{Q}_{i}\right)$ and by (24) we have $\mathcal{P}=\left\{\bigcup_{i \in \omega} C_{i}: \forall i \in \omega \quad C_{i} \in \operatorname{Dense}\left(\mathbb{Q}_{i}\right)\right\} \subset \mathbb{P}\left(\mathbb{B}_{\omega}\right)$. It is easy to check that $\mathcal{P}$ is a positive family on $\mathbb{Q}$ so, by Theorem 2.2 (b), there is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L$.

Now, let $|L|>\omega$. Then, by Claim4.2] we can assume that $L=\sum_{x \in[-\infty, \infty]} L_{x}$, where conditions (L1-L3) from Claim 4.2 are satisfied. We distinguish two cases.

Case 1: $-\infty \notin M$. Then, by the construction from [8] (if $(0,1]$ is replaced by $(-\infty, \infty]$ and $A_{1}^{+}$by $\left.\mathbb{Q}\right)$, there is a maximal chain $\mathcal{L}$ in $\langle\mathbb{P}(\mathbb{Q}) \cup\{\emptyset\}, \subset\rangle$ such that

$$
\begin{equation*}
\forall A \in \mathcal{L} \backslash\{\emptyset\} \exists x \in(-\infty, \infty](A \subset(-\infty, x) \wedge A \text { is dense in }(-\infty, x)) \tag{25}
\end{equation*}
$$

and $\mathcal{L} \cong L$. Now we prove

$$
\begin{equation*}
\mathcal{L} \backslash\{\emptyset\} \subset \mathbb{P}\left(\mathbb{B}_{\omega}\right) \subset \mathbb{P}(\mathbb{Q},<\mathbb{Q}) . \tag{26}
\end{equation*}
$$

Let $A \in \mathcal{L} \backslash\{\emptyset\}$, let $x$ be the real corresponding to $A$ in the sense of (25) and let $i_{0}=\min \left\{i \in \omega:(-\infty, x) \cap\left(x_{i+1}, x_{i}\right) \neq \emptyset\right\}$. Then $x_{i_{0}+1}<x \leq x_{i_{0}}$ and, by (25) the set $C_{i_{0}}=A \cap\left(x_{i_{0}+1}, x\right)$ is dense in $\left(x_{i_{0}+1}, x\right)$ and, hence, $C_{i_{0}} \in \mathbb{P}\left(\mathbb{Q}_{i_{0}}\right)$. Similarly, $C_{i}=A \cap\left(x_{i+1}, x_{i}\right) \in \mathbb{P}\left(\mathbb{Q}_{i}\right)$, for all $i>i_{0}$. Since $A \subset \mathbb{Q}$, we have $A=\bigcup_{i \geq i_{0}} C_{i}$ and, by (24), $A \in \mathbb{P}\left(\mathbb{B}_{\omega}\right)$. So the first inclusion of (26) is proved.

Let $C=\bigcup_{i \in S} C_{i} \in \mathbb{P}\left(\mathbb{B}_{\omega}\right)$. By (24) for each $i \in S$ we have $C_{i} \cong \mathbb{Q}_{i} \cong \mathbb{Q}$ and, hence, $C \cong \sum_{\omega^{*}} \mathbb{Q} \cong \mathbb{Q}$. The second inclusion of (26) is proved as well.

By (26) we have $\mathcal{L} \subset \mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\} \subset \mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right) \cup\{\emptyset\}$ and, clearly, $\mathcal{L}$ is a chain in $\mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\}$. Suppose that $\mathcal{L} \cup\{C\}$ is a chain, for some $C \in\left(\mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\}\right) \backslash \mathcal{L}$. Then, by [26), $C \in \mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right)$ and $\mathcal{L}$ would not be a maximal chain in the poset $\langle\mathbb{P}(\mathbb{Q},<\mathbb{Q}) \cup\{\emptyset\}, \subset\rangle$. So $\mathcal{L}$ is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{B}_{\omega}\right) \cup\{\emptyset\}, \subset\right\rangle$ and $\mathcal{L} \cong L$.

Case 2: $-\infty \in M$. Then we proceed as in (III) of the proof of Theorem 4.1

## 6 Maximal chains in $\mathbb{P}\left(\mathbb{C}_{n}\right)$

Theorem 6.1 For all $n \in \mathbb{N}$ and each $\mathbb{R}$-embeddable complete linear order $L$ with $0_{L}$ non-isolated there is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{C}_{n}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L$.

Proof. Let the order $<$ on $\mathbb{C}_{n}=\mathbb{Q} \times n$ be given by $\left\langle q_{1}, i_{1}\right\rangle<\left\langle q_{2}, i_{2}\right\rangle \Leftrightarrow q_{1}<\mathbb{Q}$ $q_{2}$. Clearly, the incomparability relation $a \| b \Leftrightarrow a \nless b \wedge b \nless a$ on $\mathbb{C}_{n}$ is an equivalence relation with the equivalence classes $\{q\} \times n, q \in \mathbb{Q}$, of size $n$ and the corresponding quotient, $\mathbb{C}_{n} / \|$, is isomorphic to $\langle\mathbb{Q},<\mathbb{Q}\rangle$. Since each element of $\mathbb{P}\left(\mathbb{C}_{n}\right)$ has such classes we have $\mathbb{P}\left(\mathbb{C}_{n}\right)=\{A \times n: A \in \mathbb{P}(\mathbb{Q},<\mathbb{Q})\}$. It is easy to see that the mapping $f: \mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right) \cup\{\emptyset\} \rightarrow \mathbb{P}\left(\mathbb{C}_{n}\right) \cup\{\emptyset\}$, given by $f(A)=A \times n$, is an isomorphism of partial orders $\left\langle\mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right) \cup\{\emptyset\}, \subset\right\rangle$ and $\left\langle\mathbb{P}\left(\mathbb{C}_{n}\right) \cup\{\emptyset\}, \subset\right\rangle$. Hence the statement follows from Theorem 6 of [8].

Theorem 6.2 For each $\mathbb{R}$-embeddable complete linear order $L$ with $0_{L}$ non-isolated there is a maximal chain in $\left\langle\mathbb{P}\left(\mathbb{C}_{\omega}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L$.

Proof. Let the strict order $<$ on $\mathbb{C}_{\omega}=\mathbb{Q} \times \omega=\bigcup_{q \in \mathbb{Q}}\{q\} \times \omega=\bigcup_{q \in \mathbb{Q}} \omega_{q}$ be given by $\left\langle q_{1}, i_{1}\right\rangle<\left\langle q_{2}, i_{2}\right\rangle \Leftrightarrow q_{1}<\mathbb{Q} q_{2}$. For a set $X \subset \mathbb{C}_{\omega}$ let us define $\operatorname{supp} X=\left\{q \in \mathbb{Q}: X \cap \omega_{q} \neq \emptyset\right\}$. Now the incomparability classes $\omega_{q}$ are infinite and, again, the corresponding quotient, $\mathbb{C}_{\omega} / \|$, is isomorphic to the rational line $\left\langle\mathbb{Q},\left\langle_{\mathbb{Q}}\right\rangle\right.$. Since the same holds for the copies of $\mathbb{C}_{\omega}$ it is easy to check that

$$
\begin{align*}
\mathbb{P}\left(\mathbb{C}_{\omega}\right)= & \left\{\bigcup_{q \in A}\{q\} \times C_{q}: A \in \mathbb{P}\left(\mathbb{Q},<_{\mathbb{Q}}\right) \wedge \forall q \in A C_{q} \in[\omega]^{\omega}\right\} .  \tag{27}\\
& X \subset \mathbb{C}_{\omega} \wedge \text { there is max supp } X \Rightarrow X \notin \mathbb{P}\left(\mathbb{C}_{\omega}\right) . \tag{28}
\end{align*}
$$

By (27), $\mathcal{P}=\left\{\bigcup_{q \in \mathbb{Q}}\{q\} \times C_{q}: \forall q \in \mathbb{Q} C_{q} \in[\omega]^{\omega}\right\} \subset \mathbb{P}\left(\mathbb{C}_{\omega}\right)$ and, clearly, $\mathcal{P}$ is a positive family so for a countable $L$ the statement follows from Theorem 2.2(b).

Now, let $L$ be an uncountable linear order. Then, by Claim4.2, we can assume that $L=\sum_{x \in[-\infty, \infty]} L_{x}$, where conditions (L1-L3) from Claim4.2 are satisfied.
Case I: $-\infty \notin M \ni \infty$. Let $\mathbb{Q}=\bigcup_{y \in M} J_{y}$ be a partition of $\mathbb{Q}$ into $|M|$ disjoint dense sets and, for $y \in M$, let $I_{y} \in\left[J_{y} \cap(-\infty, y)\right]^{\left|L_{y}\right|-1}$. Let $(-\infty, x)_{\mathbb{Q}}=$ $(-\infty, x) \cap \mathbb{Q}$ and $\omega^{+}=\omega \backslash\{0\}$. Let us define $A_{-\infty}=\emptyset$ and, for $x \in(-\infty, \infty]$,

$$
A_{x}=\left((-\infty, x)_{\mathbb{Q}} \times \omega^{+}\right) \cup \bigcup_{y \in M \cap(-\infty, x)} I_{y} \times\{0\},
$$

$$
A_{x}^{+}=A_{x} \cup\left(I_{x} \times\{0\}\right), \quad \text { for } x \in M
$$

By (27), $A_{\infty}^{+} \cong \mathbb{C}_{\omega}$ and we will construct a maximal chain $\mathcal{L} \cong L$ in the poset $\left\langle\mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}, \subset\right\rangle$. By (27), for each $x \in(-\infty, \infty]$ and each set $A \subset \mathbb{C}_{\omega}$ we have

$$
\begin{equation*}
(-\infty, x)_{\mathbb{Q}} \times \omega^{+} \subset A \subset(-\infty, x)_{\mathbb{Q}} \times \omega \Rightarrow A \in \mathbb{P}\left(\mathbb{C}_{\omega}\right) . \tag{29}
\end{equation*}
$$

Claim 6.3 The sets $A_{x}, x \in[-\infty, \infty]$ and $A_{x}^{+}, x \in M$ are subsets of the set $A_{\infty}^{+}$. In addition, for each $x, x_{1}, x_{2} \in[-\infty, \infty]$ we have
(a) $A_{x} \subset(-\infty, x)_{\mathbb{Q}} \times \omega$;
(b) $A_{x}^{+} \subset(-\infty, x)_{\mathbb{Q}} \times \omega$, if $x \in M$;
(c) $x_{1}<x_{2} \Rightarrow A_{x_{1}} \varsubsetneqq A_{x_{2}}$;
(d) $M \ni x_{1}<x_{2} \Rightarrow A_{x_{1}}^{+} \varsubsetneqq A_{x_{2}}$;
(e) $\left|A_{x}^{+} \backslash A_{x}\right|=\left|L_{x}\right|-1$, if $x \in M$;
(f) $A_{x} \in \mathbb{P}\left(A_{\infty}^{+}\right)$, for each $x \in(-\infty, \infty]$.
(g) $A_{x}^{+} \in \mathbb{P}\left(A_{\infty}^{+}\right)$and $\left[A_{x}, A_{x}^{+}\right]_{\mathbb{P}\left(A_{\infty}^{+}\right)}=\left[A_{x}, A_{x}^{+}\right]_{P\left(A_{x}^{+}\right)}$, for each $x \in M$.

Proof. Statements (c) and (d) are true since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$; (a), (b) and (e) follow from the definitions of $A_{x}$ and $A_{x}^{+}$and the choice of the sets $I_{y}$. Since $(-\infty, x)_{\mathbb{Q}} \times \omega^{+} \subset A_{x} \subset A_{x}^{+} \subset(-\infty, x)_{\mathbb{Q}} \times \omega$, (f) and (g) follow from (29).
Now, for $x \in[-\infty, \infty]$ we define chains $\mathcal{L}_{x} \subset \mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}$ in the following way.
For $x \notin M$ we define $\mathcal{L}_{x}=\left\{A_{x}\right\}$. In particular, $\mathcal{L}_{-\infty}=\{\emptyset\}$.
For $x \in M$, by Claim 6.3 and Lemma 3.5 there is a set $\mathcal{L}_{x} \subset\left[A_{x}, A_{x}^{+}\right]_{P\left(A_{x}^{+}\right)}$ such that $\left\langle\mathcal{L}_{x}, \mp\right\rangle \cong\left\langle L_{x},<_{x}\right\rangle$ and

$$
\begin{gather*}
A_{x}, A_{x}^{+} \in \mathcal{L}_{x} \subset\left[A_{x}, A_{x}^{+}\right]_{\mathbb{P}\left(A_{\infty}^{+}\right)}  \tag{30}\\
\cup \mathcal{A}, \bigcap \mathcal{B} \in \mathcal{L}_{x} \quad \text { and }|\bigcap \mathcal{B} \backslash \bigcup \mathcal{A}| \leq 1, \text { for each cut }\langle\mathcal{A}, \mathcal{B}\rangle \text { in } \mathcal{L}_{x} . \tag{31}
\end{gather*}
$$

For $\mathcal{A}, \mathcal{B} \subset \mathbb{P}\left(A_{\infty}^{+}\right)$we will write $\mathcal{A} \prec \mathcal{B}$ iff $A \varsubsetneqq B$, for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
Claim 6.4 Let $\mathcal{L}=\bigcup_{x \in[-\infty, \infty]} \mathcal{L}_{x}$. Then
(a) If $-\infty \leq x_{1}<x_{2} \leq \infty$, then $\mathcal{L}_{x_{1}} \prec \mathcal{L}_{x_{2}}$ and $\bigcup \mathcal{L}_{x_{1}} \subset A_{x_{2}} \subset \bigcup \mathcal{L}_{x_{2}}$.
(b) $\mathcal{L}$ is a chain in $\left\langle\mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}, \subset\right\rangle$ isomorphic to $L=\sum_{x \in[-\infty, \infty]} L_{x}$.
(c) $\mathcal{L}$ is a maximal chain in $\left\langle\mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}, \subset\right\rangle$.

Proof. The proof of (a) and (b) is a copy of the proof of (a) and (b) of Claim4.9, if we replace (16) and Claim 4.8 by (30) and Claim 6.3
(c) Suppose that $C \in \mathbb{P}\left(A_{\infty}^{+}\right) \cup\{\emptyset\}$ witnesses that $\mathcal{L}$ is not maximal. Using (30) and Claim6.3, as in the proof of Claim4.9(c) for $\mathcal{A}=\{A \in \mathcal{L}: A \nsubseteq C\}$ and $\mathcal{B}=\{B \in \mathcal{L}: C \nsubseteq B\}$ we show that $\langle\mathcal{A}, \mathcal{B}\rangle$ is a cut in $\langle\mathcal{L}, \mp\rangle, \mathcal{A} \neq\{\emptyset\}$ and

$$
\begin{equation*}
\cup \mathcal{A} \subset C \subset \bigcap \mathcal{B} . \tag{32}
\end{equation*}
$$

Case 1: $\mathcal{A} \cap \mathcal{L}_{x_{0}} \neq \emptyset$ and $\mathcal{B} \cap \mathcal{L}_{x_{0}} \neq \emptyset$, for some $x_{0} \in(-\infty, \infty]$. Then we obtain a contradiction exactly like in Claim4.9

Case 2: $\neg$ Case 1. Then like in Claim4.9 for $\mathcal{A}^{\prime}=\left\{x \in(-\infty, \infty]: \mathcal{L}_{x} \subset \mathcal{A}\right\}$ and $\mathcal{B}^{\prime}=\left\{x \in(-\infty, \infty]: \mathcal{L}_{x} \subset \mathcal{B}\right\}$ we show that $\left\langle\mathcal{A}^{\prime}, \mathcal{B}^{\prime}\right\rangle$ is a cut in $(-\infty, \infty]$. Thus, there is $x_{0} \in(-\infty, \infty]$ such that $x_{0}=\max \mathcal{A}^{\prime}$ or $x_{0}=\min \mathcal{B}^{\prime}$.
Subcase 2.1: $x_{0}=\max \mathcal{A}^{\prime}$. Then like in Claim4.9we prove

$$
\cup \mathcal{A}= \begin{cases}A_{x_{0}} & \text { if } x_{0} \notin M  \tag{33}\\ A_{x_{0}}^{+} & \text {if } x_{0} \in M\end{cases}
$$

Since $\mathcal{B}=\bigcup_{x \in\left(x_{0}, \infty\right]} \mathcal{L}_{x}$, we have $\bigcap \mathcal{B}=\bigcap_{x \in\left(x_{0}, \infty\right]} \cap \mathcal{L}_{x}$. By (30) $\cap \mathcal{L}_{x}=A_{x}$, so $\bigcap \mathcal{B}=\left(\bigcap_{x \in\left(x_{0}, \infty\right]}(-\infty, x)_{\mathbb{Q}} \times \omega^{+}\right) \cup\left(\bigcap_{x \in\left(x_{0}, \infty\right]} \bigcup_{y \in M \cap(-\infty, x)} I_{y} \times\{0\}\right)=$ $\left(\left(-\infty, x_{0}\right]_{\mathbb{Q}} \times \omega^{+}\right) \cup \bigcup_{y \in M \cap\left(-\infty, x_{0}\right]} I_{y} \times\{0\}=A_{x_{0}} \cup\left(\left(\left\{x_{0}\right\} \cap \mathbb{Q}\right) \times \omega^{+}\right) \cup$ $\bigcup_{y \in M \cap\left\{x_{0}\right\}} I_{y} \times\{0\}$, so

$$
\bigcap \mathcal{B}=\left\{\begin{array}{llll}
A_{x_{0}} & \text { if } x_{0} \notin \mathbb{Q} & \wedge & x_{0} \notin M,  \tag{34}\\
A_{x_{0}} \cup\left(\left\{x_{0}\right\} \times \omega^{+}\right) & \text {if } x_{0} \in \mathbb{Q} & \wedge & x_{0} \notin M, \\
A_{x_{0}}^{+} & \text {if } x_{0} \notin \mathbb{Q} & \wedge & x_{0} \in M, \\
A_{x_{0}}^{+} \cup\left(\left\{x_{0}\right\} \times \omega^{+}\right) & \text {if } x_{0} \in \mathbb{Q} & \wedge & x_{0} \in M,
\end{array}\right.
$$

If $x_{0} \notin \mathbb{Q}$, then, by $\sqrt{32,34}$, we have $\cup \mathcal{A}=\bigcap \mathcal{B}=C \in \mathcal{L}$. A contradiction.
If $x_{0} \in \mathbb{Q}$ and $x_{0} \notin M$, then $\cup \mathcal{A}=A_{x_{0}}$ and $\cap \mathcal{B}=A_{x_{0}} \cup\left(\left\{x_{0}\right\} \times \omega^{+}\right)$. So, by (32) and since $C \notin \mathcal{L}$ we have $C=A_{x_{0}} \cup S$, where $\emptyset \neq S \subset\left\{x_{0}\right\} \times \omega^{+}$. By Claim6.3 (a), $x_{0}=\max \operatorname{supp} C$ so, by (28), $C \notin \mathbb{P}\left(A_{\infty}^{+}\right)$. A contradiction.

If $x_{0} \in \mathbb{Q}$ and $x_{0} \in M$, then $\bigcup \mathcal{A}=A_{x_{0}}^{+}$and $\bigcap \mathcal{B}=A_{x_{0}}^{+} \cup\left(\left\{x_{0}\right\} \times \omega^{+}\right)$. Again, by (32) and since $C \notin \mathcal{L}$ we have $C=A_{x_{0}} \cup S$, where $\emptyset \neq S \subset\left\{x_{0}\right\} \times \omega^{+}$. By Claim6.3(b), $x_{0}=\max \operatorname{supp} C$ so, by [28), $C \notin \mathbb{P}\left(A_{\infty}^{+}\right)$. A contradiction.
Subcase 2.2: $x_{0}=\min \mathcal{B}^{\prime}$. Then, by (30), $A_{x_{0}} \in \mathcal{L}_{x_{0}} \subset \mathcal{B}$ which, by (a), implies $\bigcap \mathcal{B}=A_{x_{0}}$. Since $A_{x} \in \mathcal{L}_{x}$, for all $x \in(-\infty, \infty]$ and $\mathcal{A}=\bigcup_{x<x_{0}} \mathcal{L}_{x}$ we have $\cup \mathcal{A} \supset \bigcup_{x<x_{0}} A_{x}=\bigcup_{x<x_{0}}\left((-\infty, x)_{\mathbb{Q}} \times \omega^{+}\right) \cup \bigcup_{x<x_{0}} \bigcup_{y \in M \cap(-\infty, x)} I_{y} \times\{0\}=$ $\left(\left(-\infty, x_{0}\right)_{\mathbb{Q}} \times \omega^{+}\right) \cup \bigcup_{y \in M \cap\left(-\infty, x_{0}\right)} I_{y} \times\{0\}=A_{x_{0}}$ so $A_{x_{0}} \subset \bigcup \mathcal{A} \subset \bigcap \mathcal{B}=A_{x_{0}}$, which implies $C=A_{x_{0}} \in \mathcal{L}$. A contradiction.
Case II: $-\infty \notin M \nexists \infty$ or $-\infty \in M$. Then we proceed like in Claim4.9

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