THE HOMOMORPHISM LATTICE INDUCED BY A FINITE ALGEBRA

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ABSTRACT. Each finite algebra **A** induces a lattice $\mathbf{L}_{\mathbf{A}}$ via the quasi-order \rightarrow on the finite members of the variety generated by **A**, where $\mathbf{B} \rightarrow \mathbf{C}$ if there exists a homomorphism from **B** to **C**. In this paper, we introduce the question: 'Which lattices arise as the homomorphism lattice $\mathbf{L}_{\mathbf{A}}$ induced by a finite algebra **A**?' Our main result is that each finite distributive lattice arises as $\mathbf{L}_{\mathbf{Q}}$, for some quasi-primal algebra **Q**. We also obtain representations of some other classes of lattices as homomorphism lattices, including all finite partition lattices, all finite subspace lattices and all lattices of the form $\mathbf{L} \oplus \mathbf{1}$, where **L** is an interval in the subgroup lattice of a finite group.

For any category \mathcal{C} , there is a natural quasi-order \rightarrow on the objects of \mathcal{C} , given by $\mathbf{A} \rightarrow \mathbf{B}$ if there exists a morphism from \mathbf{A} to \mathbf{B} . The associated equivalence relation on \mathcal{C} is given by

$$\mathbf{A} \equiv \mathbf{B} \iff \mathbf{A} \to \mathbf{B} \text{ and } \mathbf{B} \to \mathbf{A}.$$

Thus \rightarrow induces an order on the class C/\equiv . One way to ensure that C/\equiv is a set is to take the category C to be a class of finite structures with all homomorphisms between them. In this case, we can define the ordered set

$$\mathbf{P}_{\mathfrak{C}} := \langle \mathfrak{C} / \equiv; \rightarrow \rangle_{\mathfrak{C}}$$

which we refer to as the homomorphism order on \mathcal{C} . If \mathcal{C} has pairwise products and coproducts, then $\mathbf{P}_{\mathcal{C}}$ is a lattice: the meet and join of \mathbf{A}/\equiv and \mathbf{B}/\equiv are $(\mathbf{A}\times\mathbf{B})/\equiv$ and $(\mathbf{A}\sqcup\mathbf{B})/\equiv$, respectively; see Figure 1.

The homomorphism order has been studied extensively for the category \mathcal{G} of finite directed graphs; see Hell and Nešetřil [18]. The ordered set $\mathbf{P}_{\mathcal{G}}$, which forms a bounded distributive lattice, is very complicated: every countable ordered set embeds into $\mathbf{P}_{\mathcal{G}}$ [16, 32, 20]. More generally, the homomorphism order has been studied for various categories of finite relational structures [25, 24, 10, 23].

In this paper, we introduce the study of the homomorphism order for categories of the form $\mathcal{V}_{\mathrm{fin}}$, consisting of the finite members of a variety \mathcal{V} of algebras. To ensure that the homomorphism order forms a lattice, we shall restrict our attention to locally finite varieties and, more particularly, to finitely generated varieties.

Given a finite algebra **A**, we can define the lattice

$$\mathbf{L}_{\mathbf{A}} := \langle \operatorname{Var}(\mathbf{A})_{\operatorname{fin}} / \equiv; \rightarrow \rangle,$$

which we refer to as the homomorphism lattice induced by \mathbf{A} . We shall see that such a lattice $\mathbf{L}_{\mathbf{A}}$ may be just as complicated as the homomorphism order $\mathbf{P}_{\mathcal{G}}$ for

²⁰¹⁰ Mathematics Subject Classification. Primary: 08B25; Secondary: 06B15, 08A40.

Key words and phrases. homomorphism order, finitely generated variety, quasi-primal algebra, distributive lattice, covering forest.

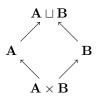


FIGURE 1. Meet and join in $\mathbf{P}_{\mathcal{C}}$ when \mathcal{C} has pairwise products and coproducts.

finite directed graphs. For example, there is a five-element unary algebra U such that $\mathbf{P}_{\mathcal{G}}$ order-embeds into $\mathbf{L}_{\mathbf{U}}$; see Example 1.3.

We are interested in the question:

Which lattices arise as L_A , for some finite algebra A?

Our main result (proved over Sections 2–4) is that each finite distributive lattice arises as the homomorphism lattice $\mathbf{L}_{\mathbf{Q}}$, for some quasi-primal algebra \mathbf{Q} . In the proof, we use Behncke and Leptin's construction [3] of the *covering forest* of a finite ordered set, which is analogous to the universal covering tree from graph theory.

In Section 5, we obtain representations for some other classes of finite lattices. We consider finite algebras **A** such that each element is the value of a nullary term function, and prove a simple result (Lemma 5.2) characterising when such an algebra satisfies $\mathbf{L}_{\mathbf{A}} \cong \operatorname{Con}(\mathbf{A})$. This allows us to represent a range of finite lattices as the homomorphism lattice induced by a finite algebra:

- every finite partition lattice (Example 5.4);
- the lattice of subspaces of a finite vector space (Example 5.4);
- the lattice [H, G] ⊕ 1, where [H, G] denotes an interval in the subgroup lattice of a finite group G (Example 5.6);
- the five-element non-modular lattice N_5 (Example 5.7).

We use this representation for finite partition lattices to see that the only universal first-order sentences true in all homomorphism lattices are those true in all lattices.

There are many unanswered questions concerning the homomorphism lattices induced by finite algebras. For example:

- Does every countable bounded lattice arise as the homomorphism lattice L_A induced by a finite algebra A? In particular, does every finite lattice arise in this way?
- For which finite algebras \mathbf{A} is the lattice $\mathbf{L}_{\mathbf{A}}$ finite? Is this decidable?

1. The homomorphism order

In this introductory section, we motivate the *homomorphism lattice* induced by a finite algebra and give some examples.

The homomorphism order on finite directed graphs. The motivation for this paper came from Hell and Nešetřil's text *Graphs and Homomorphisms* [18].

Recall that \mathcal{G} denotes the category of finite directed graphs. Since \mathcal{G} has pairwise products, given by direct product, and coproducts, given by disjoint union, the ordered set $\mathbf{P}_{\mathcal{G}} = \langle \mathcal{G} / \equiv; \rightarrow \rangle$ forms a lattice. Since product distributes over disjoint union, the lattice $\mathbf{P}_{\mathcal{G}}$ is distributive. In fact, the lattice $\mathbf{P}_{\mathcal{G}}$ is relatively pseudocomplemented, via the exponential construction (see [18, Section 2.4]).

Both the lattice $\mathbf{P}_{\mathcal{G}}$ and its sublattice $\mathbf{P}_{\mathcal{S}}$ have been studied extensively, where \mathcal{S} is the category of finite symmetric directed graphs (i.e., finite graphs). For example, it is known that every countable ordered set embeds into $\mathbf{P}_{\mathcal{S}}$ [32, 20], and that $\mathbf{P}_{\mathcal{S}}$ is dense above the complete graph \mathbf{K}_2 [34] (see [18, Section 3.7]).

The homomorphism order on categories of algebras. Within many natural categories of finite algebras, all the algebras are homomorphically equivalent, and so the homomorphism order is trivial: groups, semigroups, rings and lattices, for example. However, there are also many natural categories of finite algebras for which the homomorphism order is extremely complicated.

Example 1.1. Consider the category \mathcal{L}_{01} of finite bounded lattices. A simple observation is that there is an infinite ascending chain $\mathbf{M}_3 \to \mathbf{M}_4 \to \mathbf{M}_5 \to \cdots$ in the homomorphism order $\mathbf{P}_{\mathcal{L}_{01}}$, where \mathbf{M}_n is the bounded lattice of height 2 with n atoms. In fact, we can say much more.

A variety \mathcal{V} of algebras is *finite-to-finite universal* if the category of directed graphs has a finiteness-preserving full embedding into \mathcal{V} . (This is equivalent to requiring that every variety of algebras has a finiteness-preserving full embedding into \mathcal{V} ; see [31, 17, 32]). Since the variety of bounded lattices is finite-to-finite universal (Adams and Sichler [2]), it follows that there is an order-embedding of $\mathbf{P}_{\mathcal{G}}$ into $\mathbf{P}_{\mathcal{L}_{01}}$, and therefore every countable ordered set embeds into $\mathbf{P}_{\mathcal{L}_{01}}$.

The homomorphism lattice induced by a finite algebra. In general, the coproduct of two finite algebras in a variety does not have to be finite. Consequently, it is not clear whether the homomorphism order $\mathbf{P}_{\mathcal{L}_{01}}$ from Example 1.1 is a lattice. We can avoid this problem if we restrict our attention to locally finite varieties.

Lemma 1.2. Let \mathcal{V} be a locally finite variety. Then the homomorphism order $\mathbf{P}_{\mathcal{V}_{\text{fin}}} = \langle \mathcal{V}_{\text{fin}} / \equiv; \rightarrow \rangle$ is a countable bounded lattice.

Proof. Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}_{\text{fin}}$. To see that $\mathbf{P}_{\mathcal{V}_{\text{fin}}}$ is a lattice, it suffices to observe that the product $\mathbf{A} \times \mathbf{B}$ is finite and the coproduct $\mathbf{A} \sqcup \mathbf{B}$ in \mathcal{V} is finite (since \mathcal{V} is locally finite); see Figure 1. The top element of $\mathbf{P}_{\mathcal{V}_{\text{fin}}}$ contains all the trivial algebras in \mathcal{V} (and consists of all the finite algebras in \mathcal{V} with a trivial subalgebra). The bottom element of $\mathbf{P}_{\mathcal{V}_{\text{fin}}}$ contains all the finitely generated free algebras in \mathcal{V} . Note that $\mathbf{P}_{\mathcal{V}_{\text{fin}}}$ is countable as every finite algebra in \mathcal{V} is a homomorphic image of a finitely generated free algebra.

In this paper, we focus on finitely generated varieties. Given a finite algebra \mathbf{A} , we can define the *homomorphism lattice* induced by \mathbf{A} to be

$$\mathbf{L}_{\mathbf{A}} = \langle \operatorname{Var}(\mathbf{A})_{\operatorname{fin}} / \equiv; \rightarrow \rangle.$$

If the variety $Var(\mathbf{A})$ is finite-to-finite universal, then the lattice $\mathbf{L}_{\mathbf{A}}$ is just as complicated as the homomorphism order $\mathbf{P}_{\mathcal{G}}$ for finite directed graphs, as $\mathbf{P}_{\mathcal{G}}$ order-embeds into $\mathbf{L}_{\mathbf{A}}$. Examples of finite algebras that generate a finite-to-finite universal variety include:

- the bounded lattice \mathbf{M}_3 [13];
- the algebra $\mathbf{A} = \langle A; \lor, \land, 0, 1, a_1, a_2 \rangle$, where $\langle A; \lor, \land, 0, 1 \rangle$ is the bounded distributive lattice $\mathbf{1} \oplus \mathbf{2}^2 \oplus \mathbf{1}$ freely generated by $\{a_1, a_2\}$ [1].

Even a small unary algebra can generate a finite-to-finite universal variety, as in the following example.

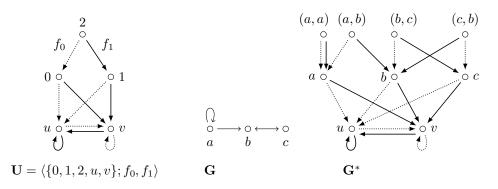


FIGURE 2. A finite unary algebra \mathbf{U} with infinite lattice $\mathbf{L}_{\mathbf{U}}$.

Example 1.3. Let $\mathbf{U} = \langle \{0, 1, 2, u, v\}; f_0, f_1 \rangle$ be the five-element unary algebra shown in Figure 2. Then $\mathbf{P}_{\mathcal{G}}$ order-embeds into $\mathbf{L}_{\mathbf{U}}$, and therefore every countable ordered set embeds into $\mathbf{L}_{\mathbf{U}}$.

Proof. The values of the constant term functions of \mathbf{U} are u and v. To simplify the proof, we will add u and v to the signature of \mathbf{U} as nullary operations; this has no effect on Var(\mathbf{U}), up to term equivalence.

We will use a construction of Hedrlín and Pultr [17]. Given a directed graph $\mathbf{G} = \langle G; r \rangle$, define the algebra $\mathbf{G}^* = \langle G \cup r \cup \{u, v\}; f_0, f_1, u, v \rangle$, where

$$\begin{aligned} f_0(g) &= u, \qquad f_0((g_0,g_1)) = g_0, \qquad f_0(u) = v, \qquad f_0(v) = v, \\ f_1(g) &= v, \qquad f_1((g_0,g_1)) = g_1, \qquad f_1(u) = u, \qquad f_1(v) = u, \end{aligned}$$

for all $g \in G$ and $(g_0, g_1) \in r$. (We assume that $u, v \notin G \cup r$.) See Figure 2 for an example of the algebra \mathbf{G}^* constructed from a directed graph \mathbf{G} .

Each one-generated subalgebra of \mathbf{G}^* is a homomorphic image of a subalgebra of \mathbf{U} , and therefore belongs to $\operatorname{Var}(\mathbf{U})$. Thus \mathbf{G}^* satisfies all one-variable equations that are true in \mathbf{U} . Since each constant term function of \mathbf{U} has value u or v, the two-variable equations $t_1(x) \approx t_2(y)$ that are true in \mathbf{U} follow from one-variable equations of the form $t(x) \approx u$ and $t(x) \approx v$. Hence $\mathbf{G}^* \in \operatorname{Var}(\mathbf{U})$.

Hedrlín and Pultr [17] showed that there is a bijection between hom(\mathbf{G}, \mathbf{H}) and hom($\mathbf{G}^*, \mathbf{H}^*$). It follows immediately that $\mathbf{P}_{\mathcal{G}}$ order-embeds into $\mathbf{L}_{\mathbf{U}}$. \Box

To contrast with the previous example, we finish this section by describing the lattice $\mathbf{L}_{\mathbf{A}}$, for each finite monounary algebra $\mathbf{A} = \langle A; f \rangle$. We say that a non-empty subset $\{a_0, a_1, \ldots, a_{n-1}\}$ of A is a *cycle* if $f(a_i) = a_{i+1 \pmod{n}}$. For each $k \in \mathbb{N}$, we use \mathbf{k} to denote the k-element chain.

Example 1.4. Let $\mathbf{A} = \langle A; f \rangle$ be a finite monounary algebra and let n be the least common multiple of the sizes of the cycles of \mathbf{A} . If n = 1, then $\mathbf{L}_{\mathbf{A}} \cong \mathbf{1}$. Otherwise, let $p_1^{k_1} \cdots p_{\ell}^{k_{\ell}}$ be the prime decomposition of n. Then

$$\mathbf{L}_{\mathbf{A}} \cong (\mathbf{k}_1 \sqcup \cdots \sqcup \mathbf{k}_\ell) \oplus \mathbf{1},$$

where the coproduct is taken in the variety \mathcal{D} of distributive lattices.

Proof. If n = 1, then every algebra in Var(**A**)_{fin} has a trivial subalgebra, and so $|L_{\mathbf{A}}| = 1$. Now assume that $n \ge 2$. Let D_n be the set of positive divisors of n, and

let $\mathbf{D}_n = \langle D_n; \text{lcm}, \text{gcd} \rangle$ be the divisor lattice of n, where $a \leq b$ in \mathbf{D}_n if and only if a divides b. Then we have $\mathbf{D}_n \cong (\mathbf{k}_1 \oplus \mathbf{1}) \times \cdots \times (\mathbf{k}_\ell \oplus \mathbf{1})$.

Let $Up^+(\mathbf{D}_n)$ denote the lattice of all non-empty up-sets of \mathbf{D}_n , ordered by inclusion. We start by showing that $\mathbf{L}_{\mathbf{A}}$ is isomorphic to $Up^+(\mathbf{D}_n)$.

For each finite monounary algebra \mathbf{B} , define

$$\operatorname{Cyc}(\mathbf{B}) := \{ |C| : C \text{ is a cycle of } \mathbf{B} \}$$

For all $\mathbf{B} \in \operatorname{Var}(\mathbf{A})_{\operatorname{fin}}$, we have $\operatorname{Cyc}(\mathbf{B}) \subseteq D_n$, as there exists $m \in \mathbb{N}$ such that \mathbf{A} satisfies the equation $f^m(x) \approx f^{m+n}(x)$. For all $\mathbf{B}_1, \mathbf{B}_2 \in \operatorname{Var}(\mathbf{A})_{\operatorname{fin}}$, we have $\mathbf{B}_1 \to \mathbf{B}_2$ if and only if $\operatorname{Cyc}(\mathbf{B}_1) \subseteq \uparrow \operatorname{Cyc}(\mathbf{B}_2)$ in \mathbf{D}_n . It follows that we can define an order-embedding $\psi \colon \mathbf{L}_{\mathbf{A}} \to \operatorname{Up}^+(\mathbf{D}_n)$ by

$$\psi(\mathbf{B}/\equiv) := \uparrow \operatorname{Cyc}(\mathbf{B})$$

To see that the map ψ is surjective, let $\mathcal{U} \in \mathrm{Up}^+(\mathbf{D}_n)$. Choose a finite monounary algebra **B** that is a disjoint union of cycles and satisfies $\mathrm{Cyc}(\mathbf{B}) = \mathcal{U}$. Since each equation true in **A** is of the form $f^j(x) \approx f^{j+kn}(x)$, for some $j, k \in \mathbb{N} \cup \{0\}$, it follows that $\mathbf{B} \in \mathrm{Var}(\mathbf{A})_{\mathrm{fin}}$. Clearly, we have $\psi(\mathbf{B}/\equiv) = \mathcal{U}$. Hence $\mathbf{L}_{\mathbf{A}}$ is isomorphic to the lattice $\mathrm{Up}^+(\mathbf{D}_n)$.

The lattice \mathbf{D}_n is self-dual, and therefore $\mathbf{L}_{\mathbf{A}}$ is also isomorphic to the lattice $\mathcal{O}^+(\mathbf{D}_n)$ of all non-empty down-sets of \mathbf{D}_n , ordered by inclusion. Using Priestley duality for the variety \mathcal{D}_{01} of bounded distributive lattices (see [5]), we first describe the lattice $\mathcal{O}(\mathbf{D}_n)$ of all down-sets of \mathbf{D}_n :

$$egin{aligned} \mathcal{O}(\mathbf{D}_n) &\cong \mathcal{O}((\mathbf{k}_1 \oplus \mathbf{1}) imes \cdots imes (\mathbf{k}_\ell \oplus \mathbf{1})) \ &\cong \mathcal{O}(\mathbf{k}_1 \oplus \mathbf{1}) \sqcup_{01} \cdots \sqcup_{01} \mathcal{O}(\mathbf{k}_\ell \oplus \mathbf{1}) \ &\cong (\mathbf{1} \oplus \mathbf{k}_1 \oplus \mathbf{1}) \sqcup_{01} \cdots \sqcup_{01} (\mathbf{1} \oplus \mathbf{k}_\ell \oplus \mathbf{1}) \ &\cong \mathbf{1} \oplus (\mathbf{k}_1 \sqcup \cdots \sqcup \mathbf{k}_\ell) \oplus \mathbf{1}, \end{aligned}$$

where \sqcup_{01} denotes coproduct in \mathcal{D}_{01} and \sqcup denotes coproduct in \mathcal{D} . Hence

 $\mathbf{L}_{\mathbf{A}} \cong \mathcal{O}^+(\mathbf{D}_n) \cong (\mathbf{k}_1 \sqcup \cdots \sqcup \mathbf{k}_\ell) \oplus \mathbf{1},$

as claimed.

Note that, if a finite unary algebra \mathbf{A} has no constant term functions, then coproduct in Var(\mathbf{A}) is disjoint union; so product distributes over coproduct, and therefore $\mathbf{L}_{\mathbf{A}}$ is distributive. (In fact, such a variety has a natural exponentiation, and therefore $\mathbf{L}_{\mathbf{A}}$ is relatively pseudocomplemented; see [24].) However, the homomorphism lattice induced by a finite unary algebra is not necessarily distributive. Using Corollary 5.3 and Example 5.4(i), each finite partition lattice arises as $\mathbf{L}_{\mathbf{A}}$, for some finite unary algebra \mathbf{A} .

Remark 1.5. While we will not be making use of cores in this paper, they serve as natural representatives for the elements of the homomorphism lattice $\mathbf{L}_{\mathbf{A}}$, for a finite algebra \mathbf{A} . So the lattice $\mathbf{L}_{\mathbf{A}}$ is finite if and only if there is a finite bound on the sizes of the cores in Var $(\mathbf{A})_{\text{fin}}$.

A finite algebra \mathbf{C} is a *core* if every endomorphism of \mathbf{C} is an automorphism. For each finite algebra \mathbf{B} , there is a retraction $\varphi \colon \mathbf{B} \to \mathbf{C}$ such that \mathbf{C} is a core (unique up to isomorphism). If we let \mathcal{C} consist of one copy (up to isomorphism) of each core in $\operatorname{Var}(\mathbf{A})_{\operatorname{fin}}$, then \mathcal{C} is a transversal of the equivalence classes of $\mathbf{L}_{\mathbf{A}}$. Moreover, the cores are precisely the algebras that are minimal-sized within their equivalence classes.

2. The homomorphism lattice induced by a quasi-primal algebra

This section focuses on the homomorphism lattice $\mathbf{L}_{\mathbf{Q}}$ in the case that \mathbf{Q} is a quasi-primal algebra. We give a simple description of the lattice $\mathbf{L}_{\mathbf{Q}}$ that could be converted into an algorithm for computing this lattice. The description will play a pivotal role in Section 4, where we show that each finite distributive lattice can be obtained as $\mathbf{L}_{\mathbf{Q}}$, for some quasi-primal algebra \mathbf{Q} .

A finite algebra \mathbf{Q} is *quasi-primal* if the *ternary discriminator* operation τ is a term function, where

$$\tau(x, y, z) := \begin{cases} x & \text{if } x \neq y, \\ z & \text{if } x = y. \end{cases}$$

(This implies that every finitary operation on Q that preserves the partial automorphisms of \mathbf{Q} is a term function; see Pixley [28] and Werner [35]). We will use the following two general results about quasi-primal algebras.

Theorem 2.1 (Pixley [27, Theorem 5.1]). A finite algebra \mathbf{Q} is quasi-primal if and only if every non-trivial subalgebra of \mathbf{Q} is simple and the variety $\operatorname{Var}(\mathbf{Q})$ is both congruence permutable and congruence distributive.

Theorem 2.2 (Pixley [27, Theorem 4.1]). Let \mathbf{Q} be a quasi-primal algebra. Then every finite algebra in Var(\mathbf{Q}) is isomorphic to a product of subalgebras of \mathbf{Q} .

We can now give our description of the lattice $\mathbf{L}_{\mathbf{Q}}$. For an ordered set \mathbf{P} , we again use $\mathcal{O}(\mathbf{P})$ to denote the lattice of all down-sets of \mathbf{P} , ordered by inclusion. We use $\mathrm{Sub}(\mathbf{A})$ to denote the set of all subalgebras of an algebra \mathbf{A} .

Theorem 2.3. Let \mathbf{Q} be a quasi-primal algebra. Define the ordered set

 $\mathbf{P}:=\langle \mathrm{Sub}(\mathbf{Q})/{\equiv}; \rightarrow \rangle$

and let $\overline{\mathbf{P}}$ denote \mathbf{P} without its top.

- (i) If **Q** has no trivial subalgebras, then $\mathbf{L}_{\mathbf{Q}} \cong \mathcal{O}(\mathbf{P})$.
- (ii) If **Q** has a trivial subalgebra, then $\mathbf{L}_{\mathbf{Q}} \cong \mathcal{O}(\overline{\mathbf{P}})$.

Proof. To simplify the notation, let \mathcal{P} be a transversal of $\operatorname{Sub}(\mathbf{Q})/\equiv$ and define the ordered set $\mathbf{P} := \langle \mathcal{P}; \rightarrow \rangle$. Without loss of generality, we can assume that $\mathbf{Q} \in \mathcal{P}$, whence \mathbf{Q} is the top element of \mathbf{P} .

We prove the theorem via a sequence of four claims.

Claim 1. For each non-trivial finite algebra $\mathbf{A} \in \operatorname{Var}(\mathbf{Q})$, there exists a non-empty up-set \mathcal{U} of \mathbf{P} such that $\mathbf{A} \equiv \prod \mathcal{U}$.

Proof of Claim 1. Let \mathbf{A} be a non-trivial finite algebra in Var(\mathbf{Q}). By Theorem 2.2, we know that \mathbf{A} is isomorphic to a product $\prod_{i \in I} \mathbf{B}_i$ of subalgebras of \mathbf{Q} , for some non-empty set I. For each $i \in I$, there exists $\mathbf{Q}_i \in \mathcal{P}$ such that $\mathbf{Q}_i \equiv \mathbf{B}_i$. Now define the up-set \mathcal{U} of \mathbf{P} by

$$\mathcal{U} := \uparrow \{ \mathbf{Q}_i \mid i \in I \}.$$

For each $i \in I$, we have $\mathbf{Q}_i \in \mathcal{U}$ with $\mathbf{Q}_i \to \mathbf{B}_i$. Thus $\prod \mathcal{U} \to \prod_{i \in I} \mathbf{B}_i$. For each $\mathbf{C} \in \mathcal{U}$, there exists $i \in I$ such that $\mathbf{Q}_i \to \mathbf{C}$ and so $\mathbf{B}_i \to \mathbf{C}$. Thus $\prod_{i \in I} \mathbf{B}_i \to \prod \mathcal{U}$. We have shown that $\mathbf{A} \cong \prod_{i \in I} \mathbf{B}_i \equiv \prod \mathcal{U}$, as required.

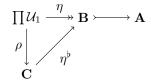
Claim 2. Let $Up^+(\mathbf{P})$ denote the set of all non-empty up-sets of \mathbf{P} . Then there is an order-embedding $\sigma: \langle Up^+(\mathbf{P}); \supseteq \rangle \to \mathbf{L}_{\mathbf{Q}}$ given by $\sigma(\mathcal{U}) = (\prod \mathcal{U}) / \equiv$.

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Proof of Claim 2. Note that σ is well defined, as each up-set \mathcal{U} of \mathbf{P} is a finite set of subalgebras of \mathbf{Q} . To see that σ is order-preserving, let $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Up}^+(\mathbf{P})$ with $\mathcal{U}_1 \supseteq \mathcal{U}_2$. Then $\prod \mathcal{U}_1 \to \prod \mathcal{U}_2$, via a projection, which gives $\sigma(\mathcal{U}_1) \to \sigma(\mathcal{U}_2)$.

To prove that σ is an order-embedding, let $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Up}^+(\mathbf{P})$ with $\sigma(\mathcal{U}_1) \to \sigma(\mathcal{U}_2)$. Then there is a homomorphism $\alpha \colon \prod \mathcal{U}_1 \to \prod \mathcal{U}_2$. We want to show that $\mathcal{U}_1 \supseteq \mathcal{U}_2$, so let $\mathbf{A} \in \mathcal{U}_2$. Then there is a projection $\pi \colon \prod \mathcal{U}_2 \to \mathbf{A}$. Thus, we obtain a homomorphism $\eta \colon \prod \mathcal{U}_1 \to \mathbf{A}$, where $\eta = \pi \circ \alpha$. Define $\mathbf{B} := \mathrm{Im}(\eta) \leq \mathbf{A}$. To prove that $\mathbf{A} \in \mathcal{U}_1$, we consider two cases.

Case (a): **B** is trivial. Then **A** has a trivial subalgebra, so $\mathbf{Q} \to \mathbf{A}$, via a constant map. Since $\mathbf{A} \in \mathcal{P}$ and \mathbf{Q} is the top element of \mathbf{P} , we have $\mathbf{A} = \mathbf{Q} \in \mathcal{U}_1$, as required. Case (b): **B** is non-trivial. Since the subalgebra **B** of **Q** is simple and Var(**Q**) is congruence distributive (see Theorem 2.1), we can apply the finite-product version of Jónsson's Lemma to the surjection $\eta \colon \prod \mathcal{U}_1 \to \mathbf{B}$. For some $\mathbf{C} \in \mathcal{U}_1$, there is a projection $\rho \colon \prod \mathcal{U}_1 \to \mathbf{C}$ and a homomorphism $\eta^{\flat} \colon \mathbf{C} \to \mathbf{B}$ such that the following diagram commutes.



We have found $\mathbf{C} \in \mathcal{U}_1$ such that $\mathbf{C} \to \mathbf{A}$. Since $\mathbf{A} \in \mathcal{P}$ and \mathcal{U}_1 is an up-set of \mathbf{P} , we have shown that $\mathbf{A} \in \mathcal{U}_1$, as required.

Claim 3. If Q has no trivial subalgebras, then $L_Q \cong \mathcal{O}(P)$.

Proof of Claim 3. Assume that **Q** has no trivial subalgebras. Let $Up(\mathbf{P})$ denote the set of all up-sets of **P**. We want to extend the order-embedding σ from Claim 2 to an order-isomorphism $\sigma^{\sharp}: \langle Up(\mathbf{P}); \supseteq \rangle \to \mathbf{L}_{\mathbf{Q}}$ given by $\sigma^{\sharp}(\mathcal{U}) = (\prod \mathcal{U}) / \equiv$.

For all $\mathcal{U} \in \mathrm{Up}(\mathbf{P})$, we have $\prod \mathcal{U} \to \prod \emptyset$, via the constant map. Thus it follows that σ^{\sharp} is order-preserving. To see that σ^{\sharp} is an order-embedding, it remains to show that $\sigma^{\sharp}(\emptyset) \not\to \sigma^{\sharp}(\mathcal{U})$, for all $\mathcal{U} \in \mathrm{Up}^+(\mathbf{P})$.

Let $\mathcal{U} \in \mathrm{Up}^+(\mathbf{P})$ and suppose that $\sigma^{\sharp}(\emptyset) \to \sigma^{\sharp}(\mathcal{U})$. Then $\prod \emptyset \to \prod \mathcal{U}$. Since the up-set \mathcal{U} is non-empty, it contains the top element \mathbf{Q} of \mathbf{P} . So $\prod \mathcal{U} \to \mathbf{Q}$, via a projection, and thus $\prod \emptyset \to \mathbf{Q}$. But this implies that \mathbf{Q} has a trivial subalgebra, which is a contradiction. Hence σ^{\sharp} is an order-embedding.

The map σ^{\sharp} is surjective, by Claim 1, since $\sigma^{\sharp}(\emptyset) = 1/\equiv$, for any trivial algebra $1 \in \operatorname{Var}(\mathbf{Q})$. Hence $\mathbf{L}_{\mathbf{Q}} \cong \langle \operatorname{Up}(\mathbf{P}); \supseteq \rangle \cong \mathcal{O}(\mathbf{P})$.

Claim 4. If **Q** has a trivial subalgebra, then $\mathbf{L}_{\mathbf{Q}} \cong \mathcal{O}(\overline{\mathbf{P}})$, where $\overline{\mathbf{P}}$ denotes the ordered set **P** without its top.

Proof of Claim 4. Let **1** be a trivial subalgebra of **Q**. Note that $\mathbf{1} \equiv \mathbf{Q}$. By Claim 2, there is an order-embedding $\sigma: \langle \mathrm{Up}^+(\mathbf{P}); \supseteq \rangle \to \mathbf{L}_{\mathbf{Q}}$ given by $\sigma(\mathcal{U}) = (\prod \mathcal{U}) / \equiv$. The map σ is surjective, by Claim 1, since $\{\mathbf{Q}\}$ is an up-set of **P** with $\prod\{\mathbf{Q}\} \cong \mathbf{Q} \equiv \mathbf{1}$. Hence we have $\mathbf{L}_{\mathbf{Q}} \cong \langle \mathrm{Up}^+(\mathbf{P}); \supseteq \rangle \cong \mathcal{O}(\overline{\mathbf{P}})$.

Claims 3 and 4 complete the proof of the theorem, as $\mathbf{P} = \langle \mathcal{P}; \rightarrow \rangle$ is isomorphic to $\langle \operatorname{Sub}(\mathbf{Q}) / \equiv; \rightarrow \rangle$.

It follows immediately from Theorem 2.3 that $\mathbf{L}_{\mathbf{Q}}$ is a finite distributive lattice, for each quasi-primal algebra \mathbf{Q} . Moreover, since the ordered set \mathbf{P} always has a top, it also follows from the theorem that, if \mathbf{Q} has no trivial subalgebras, then the lattice $\mathbf{L}_{\mathbf{Q}}$ has a join-irreducible top.

Remark 2.4. For a quasi-primal algebra \mathbf{Q} with no trivial subalgebras, the variety $\operatorname{Var}(\mathbf{Q})$ satisfies a stronger property that implies the lattice $\mathbf{L}_{\mathbf{Q}}$ is distributive. We can see this using the full duality for $\operatorname{Var}(\mathbf{Q})$ described by Davey and Werner [7, Section 2.7] (based on the duality given by Keimel and Werner [22]).

If \mathbf{Q} has no trivial subalgebras, then there is a dual equivalence between $\operatorname{Var}(\mathbf{Q})$ and a category \mathcal{X} of topological partial unary algebras, where the product in \mathcal{X} is direct product and the pairwise coproduct in \mathcal{X} is disjoint union. Therefore product distributes over coproduct in \mathcal{X} , and so coproduct distributes over product in $\operatorname{Var}(\mathbf{Q})$, which implies that $\mathbf{L}_{\mathbf{Q}}$ is distributive.

Varieties of algebras in which coproduct distributes over product are, in a sense, more common than those in which product distributes over coproduct; see [6].

3. The covering forest of a finite ordered set

In the next section, we will prove that every finite distributive lattice arises as a homomorphism lattice $\mathbf{L}_{\mathbf{Q}}$, for some quasi-primal algebra \mathbf{Q} . We use a construction, introduced by Behncke and Leptin [3], for converting a finite ordered set into a forest. In this section, we establish some useful properties of this construction, and show the relationship with the universal covering tree from graph theory.

A forest is an ordered set \mathbf{F} such that, for all $a \in F$ and all $b, c \in \uparrow a$, we have $b \leq c$ or $c \leq b$. A tree is a connected forest. Given a finite ordered set \mathbf{P} , we shall construct a finite forest \mathbf{F} with a surjective order-preserving map $\varphi \colon \mathbf{F} \to \mathbf{P}$.

Recall that, given an alphabet P, the set of all finite words in P is denoted by P^* . For elements a and b of an ordered set \mathbf{P} , we write $a \prec b$ to indicate that a is covered by b in \mathbf{P} . We use $Max(\mathbf{P})$ for the set of all maximal elements of \mathbf{P} and $Min(\mathbf{P})$ for the set of all minimal elements of \mathbf{P} .

Definition 3.1 (Behncke and Leptin [3]). Let **P** be a finite ordered set.

(1) Consider the set of all covering chains in \mathbf{P} that reach a maximal element, viewed as words in the alphabet P:

 $F := \{ a_1 a_2 \dots a_n \in P^* \mid n \in \mathbb{N}, a_1 \prec a_2 \prec \dots \prec a_n \text{ and } a_n \in \operatorname{Max}(\mathbf{P}) \}.$

We define the *covering forest* of **P** to be the ordered set $\mathbf{F} = \langle F; \leq \rangle$, where

$$w \leq v \iff (\exists u \in P^*) \ w = uv.$$

(That is, we have $w \leq v$ if and only if v is a final segment of w.) It is easy to check that the ordered set **F** is indeed a forest.

(2) We can define the order-preserving surjection $\varphi \colon \mathbf{F} \to \mathbf{P}$ by

$$\varphi(a_1a_2\ldots a_n):=a_1,$$

for each $a_1 a_2 \ldots a_n \in F$.

(3) If the ordered set **P** has a top element, then its covering forest $\mathbf{F} = \langle F; \leq \rangle$ is a tree, and we refer to **F** as the *covering tree* of **P**.

In Figure 3, we see an example of the covering forest **F** of an ordered set **P** and the map $\varphi \colon F \to P$.

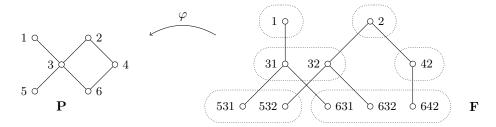


FIGURE 3. The covering forest \mathbf{F} of an ordered set \mathbf{P} .

We now begin to consider the relationship between the covering forest \mathbf{F} of a finite ordered set \mathbf{P} and the universal covering tree from graph theory.

Definition 3.2. First define the *lower-neighbourhood* of an element a of a finite ordered set \mathbf{A} to be the set

$$N_a := \{ c \in A \mid c \prec a \}.$$

For finite ordered sets **A** and **B**, we will say that $\gamma: A \to B$ is a *covering map* if γ restricts to a bijection on the maximals and on lower-neighbourhoods. That is:

- (1) $\gamma \upharpoonright_{\operatorname{Max}(\mathbf{A})} \colon \operatorname{Max}(\mathbf{A}) \to \operatorname{Max}(\mathbf{B})$ is a bijection, and
- (2) $\gamma \upharpoonright_{N_a} : N_a \to N_{\gamma(a)}$ is a bijection, for all $a \in A$.

Remark 3.3. This is analogous to the definitions of covering map for connected graphs and for connected directed graphs; see [33, 9]. Our condition (1) replaces the condition that γ is surjective. In our condition (2), we use 'lower-neighbourhood' instead of 'neighbourhood' in the case of graphs, and instead of 'in-neighbourhood' and 'out-neighbourhood' in the case of directed graphs.

Covering maps for ordered sets have been considered by Hoffman [19] in a setting that is both more special than ours (ranked ordered sets) and more general (covers in the ordered sets are assigned positive integer weights).

Before stating some useful properties of covering maps, we require a definition.

Definition 3.4. Let **A** and **B** be ordered sets. We shall say that an orderpreserving map $\alpha : \mathbf{A} \to \mathbf{B}$ is a *quotient map* if, for all $b_1 \leq b_2$ in **B**, there exists $a_1 \leq a_2$ in **A** such that $\alpha(a_1) = b_1$ and $\alpha(a_2) = b_2$. Note that, because \leq is reflexive, a quotient map is necessarily surjective.

Lemma 3.5. Let $\gamma: A \to B$ be a covering map for finite ordered sets **A** and **B**.

- (i) The map γ is cover-preserving (and is therefore order-preserving).
- (ii) For each covering chain $b_1 \prec b_2 \prec \cdots \prec b_n$ in **B** with $b_n \in \text{Max}(\mathbf{B})$, there exists a covering chain $a_1 \prec a_2 \prec \cdots \prec a_n$ in **A** with $a_n \in \text{Max}(\mathbf{A})$ such that $\gamma(a_i) = b_i$, for all $i \in \{1, 2, \dots, n\}$.
- (iii) The map γ is a quotient map (and is therefore surjective).

Proof. Part (i) is trivial, and part (ii) is an easy induction: condition (1) in the definition of a covering map gets the induction started at n = 1, and condition (2) yields the inductive step. Part (iii) follows directly from part (ii): let $x \leq y$ in **B** and apply (ii) to a covering chain in **B** that starts at x, passes through y and ends at a maximal element of **B**.

The next result follows almost immediately from the construction of the covering forest, together with part (iii) of the previous lemma.

Lemma 3.6. Let \mathbf{F} be the covering forest of a finite ordered set \mathbf{P} . Then $\varphi \colon \mathbf{F} \to \mathbf{P}$ is a covering map (and is therefore a quotient map).

Remark 3.7. It can be shown that the covering forest **F** from Definition 3.1 is the unique forest (up to isomorphism) with a covering map to **P**. Moreover, the covering map $\varphi : \mathbf{F} \to \mathbf{P}$ is the *universal cover* of **P**: for every covering map $\gamma : \mathbf{A} \to \mathbf{P}$, there is a (necessarily unique) order-preserving map $\alpha : \mathbf{F} \to \mathbf{A}$ with $\varphi = \gamma \circ \alpha$.

We now establish some properties of the covering forest \mathbf{F} that will be used in the next section. In particular, we will be using the fact that, if two elements of Fare identified by the covering map φ , then the corresponding principal down-sets of \mathbf{F} are order-isomorphic.

Definition 3.8. Let **F** be the covering forest of a finite ordered set **P**. Then, since $\varphi \colon \mathbf{F} \to \mathbf{P}$ is a covering map, for each $x \in F$, we can define the bijection $\psi_x \colon N_{\varphi(x)} \to N_x$ to be the inverse of the bijection $\varphi \upharpoonright_{N_x} \colon N_x \to N_{\varphi(x)}$.

Lemma 3.9. Let **F** be the covering forest of a finite ordered set **P**, and let $u, v \in F$ with $\varphi(u) = \varphi(v)$. Then there is an order-isomorphism $\mu: \downarrow u \to \downarrow v$ such that

- (i) $\varphi \circ \mu = \varphi \restriction_{\downarrow u}$, and
- (ii) $\mu \circ \psi_x = \psi_{\mu(x)}$, for all $x \in \downarrow u$.

Proof. Recall that the elements of F are covering chains in **P** that reach a maximal, and that each element of $\downarrow u$ is of the form su, for some $s \in P^*$. We want to define the map $\mu : \downarrow u \to \downarrow v$ by

$$\mu(su) := s\iota$$

for all $su \in \downarrow u$. Since $\varphi(u) = \varphi(v)$, the covering chains u and v start at the same element of P, so it follows that $sv \in F$, as required.

To see that μ is order-preserving, let $y \leq z$ in $\downarrow u$. Then z = su, for some $s \in P^*$, and y = tsu, for some $t \in P^*$. Thus $\mu(y) = tsv \leq sv = \mu(z)$, whence μ is order-preserving. By symmetry, there is an order-preserving map $\nu : \downarrow v \to \downarrow u$ such that $\nu \circ \mu = id_{\downarrow u}$ and $\mu \circ \nu = id_{\downarrow v}$. Therefore μ and ν are mutually inverse order-isomorphisms.

For (i), let $x \in \downarrow u$. Then x = su and $\mu(x) = sv$, for some $s \in P^*$. If s is not the empty word, then clearly $\varphi(\mu(x)) = \varphi(x)$. If s is the empty word, then we still have $\varphi(\mu(x)) = \varphi(x)$, since $\varphi(u) = \varphi(v)$.

For (ii), let $x \in \downarrow u$ and define $y := \mu(x)$. Since μ is an order-isomorphism, we have $\mu(N_x) = N_y$. Using (i), we obtain $\varphi \upharpoonright_{N_y} \circ \mu \upharpoonright_{N_x} = \varphi \upharpoonright_{N_x}$. Therefore

$$\psi_x = (\varphi \upharpoonright_{N_x})^{-1} = (\varphi \upharpoonright_{N_y} \circ \mu \upharpoonright_{N_x})^{-1} = (\mu \upharpoonright_{N_x})^{-1} \circ (\varphi \upharpoonright_{N_y})^{-1} = (\mu \upharpoonright_{N_x})^{-1} \circ \psi_y,$$

from which (ii) follows easily. \Box

4. Obtaining each finite distributive lattice

In Section 2 we saw that, for each quasi-primal algebra \mathbf{Q} , the homomorphism lattice $\mathbf{L}_{\mathbf{Q}}$ is a finite distributive lattice. In this section, we prove that all finite distributive lattices arise in this way.

Theorem 4.1. For each finite distributive lattice \mathbf{L} , there exists a quasi-primal algebra \mathbf{Q} such that $\mathbf{L}_{\mathbf{Q}}$ is isomorphic to \mathbf{L} .

We start by proving a special case of a result of Birkhoff and Frink [4], since it serves as motivation for the approach used in the proof of Theorem 4.1. We see that, given a finite join-semilattice \mathbf{S} , we can construct an algebra \mathbf{A} on the same universe as \mathbf{S} such that the subalgebras of \mathbf{A} correspond to the principal ideals of \mathbf{S} . Recall that we use Sub(\mathbf{A}) to denote the set of all subalgebras of \mathbf{A} .

Lemma 4.2 (Birkhoff and Frink [4]). For each finite join-semilattice \mathbf{S} , there exists a finite algebra \mathbf{A} such that $\mathrm{Sub}(\mathbf{A})$ is order-isomorphic to \mathbf{S} , where $\mathrm{Sub}(\mathbf{A})$ is ordered by inclusion.

Proof. Let $\mathbf{S} = \langle S; \lor \rangle$ be a finite semilattice. For each covering pair $s \prec t$ in \mathbf{S} , define the unary operation $f_{ts} \colon S \to S$ by

$$f_{ts}(x) := \begin{cases} s & \text{if } x = t, \\ x & \text{otherwise.} \end{cases}$$

Now define $F := \{ f_{ts} \mid s \prec t \text{ in } \mathbf{S} \}$ and $\mathbf{A} := \langle S; \lor, F \rangle$.

Since **S** is finite, a subset *B* of *A* is closed under all $f_{ts} \in F$ if and only if *B* is a down-set. It follows that the subalgebras of **A** correspond precisely to the ideals of **S**. For each $x \in S$, let \mathbf{A}_x denote the subalgebra of **A** with universe $\downarrow x$. Then, again since **S** is finite, we have $\operatorname{Sub}(\mathbf{A}) = \{\mathbf{A}_x \mid x \in S\}$. We can now define the order-isomorphism $\alpha \colon \mathbf{S} \to \operatorname{Sub}(\mathbf{A})$ by $\alpha(x) \coloneqq \mathbf{A}_x$.

For our result, we are starting with a finite distributive lattice **L**. By Birkhoff's Representation Theorem (see [5]), there is a finite ordered set **P** such that **L** is isomorphic to the lattice $\mathcal{O}(\mathbf{P})$ of all down-sets of **P**. Let \mathbf{P}^{\top} denote the ordered set obtained from **P** by adding a new top element \top .

We want to construct a quasi-primal algebra \mathbf{Q} , with a trivial subalgebra, such that $(\operatorname{Sub}(\mathbf{Q})/\equiv; \rightarrow)$ is isomorphic to \mathbf{P}^{\top} . By Theorem 2.3(ii), we will then have

$$\mathbf{L}_{\mathbf{Q}} \cong \mathcal{O}(\mathbf{P}^{\top}) = \mathcal{O}(\mathbf{P}) \cong \mathbf{L},$$

as required.

To mimic the construction from the proof of the previous lemma, the general idea is to build the algebra \mathbf{Q} on a semilattice \mathbf{S} that will correspond to $\langle \operatorname{Sub}(\mathbf{Q}); \subseteq \rangle$. There is an order-preserving map from $\langle \operatorname{Sub}(\mathbf{Q}); \subseteq \rangle$ onto $\langle \operatorname{Sub}(\mathbf{Q})/\equiv; \rightarrow \rangle$. Therefore, we need an order-preserving map from the semilattice \mathbf{S} onto \mathbf{P}^{\top} . We will take \mathbf{S} to be the covering tree of \mathbf{P}^{\top} , as given in Definition 3.1.

Note that this general idea will need tweaking to avoid problems with trivial subalgebras.

Assumptions 4.3. Throughout the rest of this section, we fix a non-trivial finite distributive lattice \mathbf{L} , with $\mathbf{L} \cong \mathcal{O}(\mathbf{P})$. Let \mathbf{P}^{\top} be the ordered set obtained from \mathbf{P} by adding a new top element \top , and let \mathbf{S} be the covering tree of \mathbf{P}^{\top} . By Lemma 3.6, we have a quotient map $\varphi \colon \mathbf{S} \to \mathbf{P}^{\top}$. Note that \top is also the top element of \mathbf{S} .

We will construct a quasi-primal algebra \mathbf{Q} such that there is a quotient map $\eta: \mathbf{S} \to \langle \operatorname{Sub}(\mathbf{Q}) / \equiv; \rightarrow \rangle$ with $\operatorname{ker}(\eta) = \operatorname{ker}(\varphi)$. It will then follow that the two ordered sets \mathbf{P}^{\top} and $\langle \operatorname{Sub}(\mathbf{Q}) / \equiv; \rightarrow \rangle$ are isomorphic, as required.

To avoid problems with trivial subalgebras, we will not use the universe of \mathbf{S} as the universe of \mathbf{Q} , but will make a slight modification.

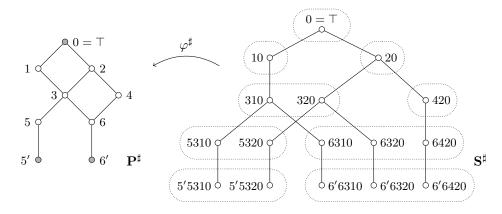


FIGURE 4. The covering tree \mathbf{S}^{\sharp} of the ordered set \mathbf{P}^{\sharp} .

Definition 4.4. Let \mathbf{P}^{\sharp} be the ordered set obtained from \mathbf{P}^{\top} by adding a new minimal element m' below each minimal element m of \mathbf{P}^{\top} , so that m is the unique upper cover of m'.

Now let \mathbf{S}^{\sharp} be the covering tree of \mathbf{P}^{\sharp} . By Lemma 3.6, we have a covering map $\varphi^{\sharp} : \mathbf{S}^{\sharp} \to \mathbf{P}^{\sharp}$. See Figure 4 for an example of this construction, where \mathbf{P} is the six-element ordered set from Figure 3.

We can easily see that **S** is a subordered set of S^{\sharp} , with

$$S = S^{\sharp} \setminus \operatorname{Min}(\mathbf{S}^{\sharp}) \quad \text{and} \quad \varphi = \varphi^{\sharp} \upharpoonright_{S}$$

Since \mathbf{S}^{\sharp} is a finite tree, each minimal element x of \mathbf{S}^{\sharp} has a unique upper cover, which we denote by x^{\uparrow} .

We will base our quasi-primal algebra \mathbf{Q} on the semilattice \mathbf{S}^{\sharp} and adapt the approach used in the proof of Lemma 4.2. We want to ensure that the non-trivial subalgebras of \mathbf{Q} correspond to the non-trivial principal ideals of \mathbf{S}^{\sharp} . However, we also need to ensure that two such subalgebras are homomorphically equivalent if their maximum elements are identified by φ .

Definition 4.5. To define the quasi-primal algebra \mathbf{Q} on the universe S^{\sharp} , we first define some families of operations on S^{\sharp} . Recall from Definition 3.8 that, since $\varphi^{\sharp} \colon \mathbf{S}^{\sharp} \to \mathbf{P}^{\sharp}$ is a covering map, for each $x \in S^{\sharp}$, we let $\psi_x \colon N_{\varphi^{\sharp}(x)} \to N_x$ denote the inverse of the bijection $\varphi^{\sharp}|_{N_x} \colon N_x \to N_{\varphi^{\sharp}(x)}$.

F: These operations will ensure that the subuniverses of \mathbf{Q} not containing \top are down-sets. For all $a \prec b$ in \mathbf{P}^{\sharp} with $b \neq \top$, define $f_{ba} \colon S^{\sharp} \to S^{\sharp}$ by

$$f_{ba}(x) := \begin{cases} \psi_x(a) & \text{if } \varphi^{\sharp}(x) = b, \\ x & \text{otherwise.} \end{cases}$$

Let $F := \{ f_{ba} \mid a \prec b \text{ in } P^{\sharp} \setminus \{\top\} \}.$

G: These next operations will ensure that the minimal elements of \mathbf{S}^{\sharp} do not form trivial subalgebras of \mathbf{Q} . For $m \in \operatorname{Min}(\mathbf{P})$, define $g_m \colon S^{\sharp} \to S^{\sharp}$ by

$$g_m(x) := \begin{cases} x^{\uparrow} & \text{if } \varphi^{\sharp}(x) = m', \\ x & \text{otherwise.} \end{cases}$$

Let $G := \{ g_m \mid m \in \operatorname{Min}(\mathbf{P}) \}.$

h: Finally, this operation will ensure that the only non-trivial subuniverse of **Q** containing \top is Q itself. Fix a cyclic permutation $\lambda: S^{\sharp} \setminus \{\top\} \to S^{\sharp} \setminus \{\top\}$. Define the binary operation $h: S^{\sharp} \times S^{\sharp} \to S^{\sharp}$ by

$$h(x,y) := \begin{cases} \lambda(x) & \text{if } x \neq \top \text{ and } y = \top, \\ x & \text{otherwise.} \end{cases}$$

Let \vee be the join-semilattice operation on the tree \mathbf{S}^{\sharp} , and let τ denote the ternary discriminator operation on S^{\sharp} . We can now define the quasi-primal algebra

$$\mathbf{Q} := \langle S^{\sharp}; \vee, F, G, h, \tau \rangle.$$

Note 4.6. We will use the fact that, for each covering pair $s \prec t$ in \mathbf{S}^{\sharp} with $t \neq \top$, we have $f_{\varphi^{\sharp}(t)\varphi^{\sharp}(s)}(t) = s$ in \mathbf{Q} . To see this, first note that, since φ^{\sharp} is a covering map, we must have $\varphi^{\sharp}(s) \prec \varphi^{\sharp}(t)$ in \mathbf{P}^{\sharp} with $\varphi^{\sharp}(t) \neq \top$. We now calculate

$$f_{\varphi^{\sharp}(t)\varphi^{\sharp}(s)}(t) = \psi_t \left(\varphi^{\sharp}(s) \right) = \psi_t \circ \varphi^{\sharp} \upharpoonright_{N_t}(s) = s,$$

as required.

Throughout this section, we use $\downarrow u$ to denote the down-set of u in the tree \mathbf{S}^{\sharp} .

Lemma 4.7. The non-empty subuniverses of \mathbf{Q} are $\{\top\}$ and $\downarrow u$, for all $u \in S$.

Proof. Let A be a non-empty subuniverse of \mathbf{Q} with $A \neq \{\top\}$. We will show that $A = \downarrow u$, for some $u \in S$. Since A is closed under \lor , there is a maximum element u of A in \mathbf{S}^{\sharp} . Suppose that $u \in \operatorname{Min}(\mathbf{S}^{\sharp})$. Then $\varphi^{\sharp}(u) = m'$, for some $m \in \operatorname{Min}(\mathbf{P})$. This gives $u^{\uparrow} = g_m(u) \in A$, which is a contradiction. Thus $u \in S$.

First assume that $u = \top$. Since A is closed under h, it follows that A is closed under the cyclic permutation λ of $S^{\sharp} \setminus \{\top\}$. As $A \setminus \{\top\} \neq \emptyset$, we have $A = Q = \downarrow u$.

Now we can assume that $u \neq \top$, and so $\top \notin A$. We want to show that A is a down-set of \mathbf{S}^{\sharp} . Consider a covering pair $s \prec t$ in \mathbf{S}^{\sharp} with $t \neq \top$. By Note 4.6, we have $f_{\varphi^{\sharp}(t)\varphi^{\sharp}(s)}(t) = s$. As A is closed under each $f_{ba} \in F$ and \mathbf{S}^{\sharp} is finite, it follows that A is a down-set. Thus $A = \downarrow u$.

It remains to check that the sets $\{\top\}$ and $\downarrow u$, for $u \in S$, are indeed subuniverses of **Q**. First consider $\{\top\}$. This set is clearly closed under \lor , τ and h. Since $\varphi^{\sharp}(\top) = \top$, the set $\{\top\}$ is also closed under each $f_{ba} \in F$ and each $g_m \in G$. Thus $\{\top\}$ is a subuniverse of **Q**.

Now consider $\downarrow u$, for some $u \in S \setminus \{\top\}$. The set $\downarrow u$ is closed under each $f_{ba} \in F$, since $f_{ba}(x) \in N_x \cup \{x\}$, for all $x \in S^{\sharp}$. For each $m \in \operatorname{Min}(\mathbf{P})$, if $\varphi^{\sharp}(x) = m'$, then $x \in \operatorname{Min}(\mathbf{S}^{\sharp})$ and so $x^{\uparrow} \in \downarrow u$, as $u \notin \operatorname{Min}(\mathbf{S}^{\sharp})$. Thus $\downarrow u$ is closed under each $g_m \in G$. As $\top \notin \downarrow u$, the set $\downarrow u$ is closed under h. Since $\downarrow u$ is also closed under \lor and τ , it is a subuniverse of \mathbf{Q} .

Definition 4.8. Using the previous lemma, for each $u \in S$, we can define \mathbf{Q}_u to be the subalgebra of \mathbf{Q} with universe $\downarrow u$.

Lemma 4.9. Each subalgebra of \mathbf{Q} is homomorphically equivalent to \mathbf{Q}_u , for some $u \in S$.

Proof. This follows immediately from Lemma 4.7, since the trivial subalgebra of \mathbf{Q} is homomorphically equivalent to $\mathbf{Q} = \mathbf{Q}_{\top}$.

Lemma 4.10. Let $u, v \in S$. Then $\varphi(u) = \varphi(v)$ if and only if $\mathbf{Q}_u \cong \mathbf{Q}_v$.

Proof. First assume that there is an isomorphism $\alpha \colon \mathbf{Q}_u \to \mathbf{Q}_v$. We can assume that $u \neq \top$, since otherwise $u = v = \top$. As α preserves \lor , we have an orderisomorphism $\alpha \colon \downarrow u \to \downarrow v$ and so $\alpha(u) = v$. Since $u \notin \operatorname{Min}(\mathbf{S}^{\sharp})$, we can choose $s \in S^{\sharp}$ with $s \prec u$. Using Note 4.6, we have $f_{\varphi^{\sharp}(u)\varphi^{\sharp}(s)}(u) = s \neq u$. Since α is an isomorphism with $\alpha(u) = v$, this implies that $f_{\varphi^{\sharp}(u)\varphi^{\sharp}(s)}(v) \neq v$. Hence $\varphi^{\sharp}(v) = \varphi^{\sharp}(u)$ and so $\varphi(u) = \varphi(v)$, as $u, v \in S$.

Now assume that $\varphi(u) = \varphi(v)$. We want to show that $\mathbf{Q}_u \cong \mathbf{Q}_v$. Since we have $\varphi^{-1}(\top) = \{\top\}$, we can assume that $u, v \neq \top$. By applying Lemma 3.9 to the covering map $\varphi^{\sharp} : \mathbf{S}^{\sharp} \to \mathbf{P}^{\sharp}$, we obtain an order-isomorphism $\mu : \downarrow u \to \downarrow v$ such that

- (i) $\varphi^{\sharp} \circ \mu = \varphi^{\sharp} \upharpoonright_{\downarrow u}$, and
- (ii) $\mu \circ \psi_x = \psi_{\mu(x)}$, for all $x \in \downarrow u$.

As μ is an order-isomorphism, it must preserve \vee and τ . To prove that $\mu: \mathbf{Q}_u \to \mathbf{Q}_v$ is an isomorphism, it remains to check that μ preserves the operations in $F \cup G$ and the operation h.

F: Let $a \prec b$ in $P^{\sharp} \setminus \{\top\}$. We want to show that μ preserves f_{ba} . Let $x \in \downarrow u$. By (i), we have $\varphi^{\sharp}(x) = \varphi^{\sharp}(\mu(x))$. Since f_{ba} acts as the identity map on $S^{\sharp} \setminus (\varphi^{\sharp})^{-1}(b)$, we can assume that $\varphi^{\sharp}(x) = \varphi^{\sharp}(\mu(x)) = b$. Using (ii), we have

$$\mu(f_{ba}(x)) = \mu(\psi_x(a)) = \psi_{\mu(x)}(a) = f_{ba}(\mu(x)).$$

Thus μ preserves f_{ba} .

G: Let $m \in \text{Min}(\mathbf{P})$. We want to show that μ preserves g_m . Let $x \in \downarrow u$. By (i), we have $\varphi^{\sharp}(x) = \varphi^{\sharp}(\mu(x))$. Since g_m acts as the identity map on $S^{\sharp} \setminus (\varphi^{\sharp})^{-1}(m')$, we can assume that $\varphi^{\sharp}(x) = \varphi^{\sharp}(\mu(x)) = m'$. Since $\mu : \downarrow u \to \downarrow v$ is an order-isomorphism, we obtain

$$\mu(g_m(x)) = \mu(x^{\uparrow}) = \mu(x)^{\uparrow} = g_m(\mu(x)).$$

Thus μ preserves g_m .

h: We are assuming that $u, v \neq \top$. Thus the binary operation h acts as the first projection on both $\downarrow u$ and $\downarrow v$. Hence $\mu : \downarrow u \rightarrow \downarrow v$ preserves h.

We have shown that $\mu: \mathbf{Q}_u \to \mathbf{Q}_v$ is an isomorphism.

We can now construct the required quotient map from the covering tree \mathbf{S} of \mathbf{P}^{\top} to the homomorphism order on $\mathrm{Sub}(\mathbf{Q})$.

Lemma 4.11. There exists a quotient map $\eta: \mathbf{S} \to \langle \operatorname{Sub}(\mathbf{Q}) / \equiv; \to \rangle$ such that $\ker(\eta) = \ker(\varphi)$.

Proof. We can define the map $\eta: S \to \operatorname{Sub}(\mathbf{Q}) / \equiv$ by

$$\eta(x) := \mathbf{Q}_x / \equiv,$$

for each $x \in S$, using Definition 4.8.

To show that $\ker(\varphi) \subseteq \ker(\eta)$, let $u, v \in S$ with $\varphi(u) = \varphi(v)$. Then $\mathbf{Q}_u \cong \mathbf{Q}_v$, by Lemma 4.10. Thus $\mathbf{Q}_u \equiv \mathbf{Q}_v$, giving $\eta(u) = \eta(v)$.

To show that $\ker(\eta) \subseteq \ker(\varphi)$, let $u, v \in S$ with $\eta(u) = \eta(v)$. Then $\mathbf{Q}_u \equiv \mathbf{Q}_v$, so $\mathbf{Q}_u \to \mathbf{Q}_v$ and $\mathbf{Q}_v \to \mathbf{Q}_u$. Since \mathbf{Q} is quasi-primal, each of these homomorphisms is either constant or an embedding (see Theorem 2.1). If both are embeddings, then $\mathbf{Q}_u \cong \mathbf{Q}_v$ and therefore $\varphi(u) = \varphi(v)$, by Lemma 4.10. Without loss of generality, we can now consider the case that there is a constant homomorphism $\mathbf{Q}_u \to \mathbf{Q}_v$,

whence \mathbf{Q}_v has a trivial subalgebra. Since $\mathbf{Q}_v \to \mathbf{Q}_u$, it follows that \mathbf{Q}_u also has a trivial subalgebra. By Lemma 4.7, the only trivial subuniverse of \mathbf{Q} is $\{\top\}$. So we can conclude that $u = v = \top$. We have shown that $\ker(\eta) = \ker(\varphi)$.

To see that η is order-preserving, let $u \leq v$ in **S**. Then $\downarrow u \subseteq \downarrow v$ in **S**^{\sharp} and therefore $\mathbf{Q}_u \to \mathbf{Q}_v$, via the inclusion. Hence $\eta(u) \to \eta(v)$.

To prove that η is a quotient map, let $\mathbf{A}, \mathbf{B} \in \mathrm{Sub}(\mathbf{Q})$ with $\mathbf{A} \to \mathbf{B}$. We want to find $x \leq y$ in \mathbf{S} such that $\mathbf{Q}_x \equiv \mathbf{A}$ and $\mathbf{Q}_y \equiv \mathbf{B}$. By Lemma 4.9, there exist $u, v \in S$ with $\mathbf{Q}_u \equiv \mathbf{A}$ and $\mathbf{Q}_v \equiv \mathbf{B}$. Since $u \leq \top$ in \mathbf{S} , we can assume that $v \neq \top$. As $\mathbf{A} \to \mathbf{B}$, we have a homomorphism $\mathbf{Q}_u \to \mathbf{Q}_v$. Since $v \neq \top$, the algebra \mathbf{Q}_v has no trivial subalgebras, so this homomorphism is an embedding. Therefore, using Lemma 4.7, we must have

$$\mathbf{A} \equiv \mathbf{Q}_u \cong \mathbf{Q}_w \leqslant \mathbf{Q}_v \equiv \mathbf{B}_v$$

for some $w \in S$ with $w \leq v$, as required. Hence η is a quotient map.

We are now able to prove Theorem 4.1 by showing that $\mathbf{L}_{\mathbf{Q}} \cong \mathbf{L}$. Recall that $\mathbf{L} \cong \mathcal{O}(\mathbf{P})$, from Assumptions 4.3. Since we have quotient maps

$$\varphi \colon \mathbf{S} \to \mathbf{P}^{\top} \quad \text{and} \quad \eta \colon \mathbf{S} \to \langle \operatorname{Sub}(\mathbf{Q}) / \equiv; \to \rangle$$

with $\ker(\varphi) = \ker(\eta)$, it follows that $\mathbf{P}^{\top} \cong (\operatorname{Sub}(\mathbf{Q})/\equiv; \rightarrow)$. Since \mathbf{Q} has a trivial subuniverse $\{\top\}$, we can use Theorem 2.3(ii) to obtain

$$\mathbf{L}_{\mathbf{Q}} \cong \mathcal{O}(\overline{\mathbf{P}^{\top}}) = \mathcal{O}(\mathbf{P}) \cong \mathbf{L},$$

which completes the proof of Theorem 4.1.

5. The class of homomorphism lattices

In this section, we consider the class \mathcal{L}_{hom} consisting of all lattices **L** such that $\mathbf{L} \cong \mathbf{L}_{\mathbf{A}}$, for some finite algebra **A**. We pose the problem:

Does every finite lattice belong to \mathcal{L}_{hom} ?

By Theorem 4.1, we know that every finite distributive lattice belongs to \mathcal{L}_{hom} .

Lemma 5.1. The class \mathcal{L}_{hom} is closed under finite products.

Proof. Consider finite algebras $\mathbf{A}_1 = \langle A_1; F_1 \rangle$ and $\mathbf{A}_2 = \langle A_2; F_2 \rangle$. Up to term equivalence, we can assume that F_1 and F_2 are disjoint and do not contain nullary operations. So we can define algebras \mathbf{B}_1 and \mathbf{B}_2 of signature $F_1 \cup F_2 \cup \{*\}$ such that \mathbf{B}_i is term equivalent to \mathbf{A}_i , where * is a binary operation that acts as the first projection on \mathbf{B}_1 and the second projection on \mathbf{B}_2 .

Now the algebra $\mathbf{C} := \mathbf{B}_1 \times \mathbf{B}_2$ is the independent product of \mathbf{B}_1 and \mathbf{B}_2 , and it follows that $\mathbf{L}_{\mathbf{C}} \cong \mathbf{L}_{\mathbf{B}_1} \times \mathbf{L}_{\mathbf{B}_2}$; see [15].

It is not clear whether \mathcal{L}_{hom} is closed under forming homomorphic images or taking sublattices. Indeed, if we could show that \mathcal{L}_{hom} were closed under taking sublattices, then it would follow from Example 5.4(i) below that \mathcal{L}_{hom} contained all finite lattices.

Congruence lattices. One of the most famous unsolved problems in universal algebra, the *Finite Lattice Representation Problem*, asks whether every finite lattice arises as the congruence lattice of a finite algebra; see Problem 13 of Grätzer [14, p. 116]. It is therefore natural to ask whether the congruence lattice of each finite algebra belongs to \mathcal{L}_{hom} . That is, given a finite algebra \mathbf{A} , does there exist a finite algebra \mathbf{B} such that $\mathbf{L}_{\mathbf{B}} \cong \text{Con}(\mathbf{A})$?

We can obtain some interesting applications by focussing on the special case where $\mathbf{L}_{\mathbf{A}} \cong \operatorname{Con}(\mathbf{A})$. We can assume, without affecting the lattice $\operatorname{Con}(\mathbf{A})$, that every element of \mathbf{A} is the value of a nullary term function. Then, for all $\mathbf{B} \in \operatorname{Var}(\mathbf{A})$, the values of the nullary term functions of \mathbf{B} form a subalgebra that we will denote by \mathbf{B}_{0} .

Lemma 5.2. Let \mathbf{A} be a finite algebra such that each element is the value of a nullary term function. Then the following are equivalent:

- (i) $\mathbf{L}_{\mathbf{A}} \cong \operatorname{Con}(\mathbf{A});$
- (ii) for every finite algebra $\mathbf{B} \in \operatorname{Var}(\mathbf{A})$, we have $\mathbf{B} \to \mathbf{B}_0$;
- (iii) (a) for every finite subdirectly irreducible algebra $\mathbf{B} \in \text{Var}(\mathbf{A})$, we have $\mathbf{B} \to \mathbf{B}_0$, and
 - (b) for all $\theta_1, \theta_2 \in \text{Con}(\mathbf{A})$, we have $(\mathbf{A}/\theta_1) \times (\mathbf{A}/\theta_2) \to \mathbf{A}/(\theta_1 \cap \theta_2)$.

Proof. We will be using the following consequence of our assumption that each element of \mathbf{A} is named via a nullary term function:

(\natural) For all $\theta \in \text{Con}(\mathbf{A})$, if $\mathbf{C} \equiv \mathbf{A}/\theta$ and \mathbf{B} embeds into \mathbf{C} , then $\mathbf{B} \equiv \mathbf{A}/\theta$. First we define the map ψ : Con $(\mathbf{A}) \to \mathbf{L}_{\mathbf{A}}$ by

$$\psi(\theta) := (\mathbf{A}/\theta) / \equiv,$$

for all $\theta \in \text{Con}(\mathbf{A})$. Note that ψ is order-preserving, and that it follows that ψ is an order-embedding as every element of \mathbf{A} is named. Hence, since $\text{Con}(\mathbf{A})$ is finite, we have $\mathbf{L}_{\mathbf{A}} \cong \text{Con}(\mathbf{A})$ if and only if the map ψ is surjective.

(i) \Rightarrow (ii): Assume that (i) holds and let $\mathbf{B} \in \operatorname{Var}(\mathbf{A})_{\operatorname{fin}}$. The map ψ is surjective, so $\mathbf{B} \equiv \mathbf{A}/\theta$, for some $\theta \in \operatorname{Con}(\mathbf{A})$. Since \mathbf{B}_0 embeds into \mathbf{B} , it follows by (\natural) that $\mathbf{B}_0 \equiv \mathbf{A}/\theta$. Thus $\mathbf{B} \equiv \mathbf{B}_0$, whence (ii) holds.

(ii) \Rightarrow (iii): Now assume that (ii) holds. Clearly (iii)(a) holds. To prove (iii)(b), let $\theta_1, \theta_2 \in \text{Con}(\mathbf{A})$ and define $\mathbf{B} := (\mathbf{A}/\theta_1) \times (\mathbf{A}/\theta_2)$. Since $\mathbf{A}/(\theta_1 \cap \theta_2)$ embeds into \mathbf{B} , there must be an embedding $\alpha : \mathbf{B}_0 \to \mathbf{A}/(\theta_1 \cap \theta_2)$. By (ii), there exists a homomorphism $\beta : \mathbf{B} \to \mathbf{B}_0$. Thus $\alpha \circ \beta : (\mathbf{A}/\theta_1) \times (\mathbf{A}/\theta_2) \to \mathbf{A}/(\theta_1 \cap \theta_2)$, whence condition (iii)(b) holds.

(iii) \Rightarrow (i): Finally, assume that (iii) holds. We must show that ψ is surjective. Let $\mathbf{B} \in \operatorname{Var}(\mathbf{A})_{\operatorname{fin}}$. Then \mathbf{B} embeds into a finite product $\prod_{i \in I} \mathbf{C}_i$, where each \mathbf{C}_i is a finite subdirectly irreducible algebra in $\operatorname{Var}(\mathbf{A})$.

Let $i \in I$. Then $\mathbf{C}_i \to (\mathbf{C}_i)_0$, by (iii)(a), and therefore $\mathbf{C}_i \equiv (\mathbf{C}_i)_0$. As \mathbf{A} is the zero-generated free algebra in Var(\mathbf{A}), we have $(\mathbf{C}_i)_0 \cong \mathbf{A}/\theta_i$ and so $\mathbf{C}_i \equiv \mathbf{A}/\theta_i$, for some $\theta_i \in \text{Con}(\mathbf{A})$. It follows that $\prod_{i \in I} \mathbf{C}_i \equiv \prod_{i \in I} \mathbf{A}/\theta_i$.

for some $\theta_i \in \operatorname{Con}(\mathbf{A})$. It follows that $\prod_{i \in I} \mathbf{C}_i \equiv \prod_{i \in I} \mathbf{A}/\theta_i$. Define $\theta := \bigcap_{i \in I} \theta_i$ in $\operatorname{Con}(\mathbf{A})$. Since $\mathbf{A}/\theta \to \prod_{i \in I} \mathbf{A}/\theta_i$ always holds, it follows by induction from (iii)(b) that $\prod_{i \in I} \mathbf{A}/\theta_i \equiv \mathbf{A}/\theta$. We now have $\prod_{i \in I} \mathbf{C}_i \equiv \mathbf{A}/\theta$, and so $\mathbf{B} \equiv \mathbf{A}/\theta$, using (\natural). Hence the map ψ is surjective, as required. \Box

Given an algebra \mathbf{A} , let \mathbf{A}^+ denote the algebra obtained from \mathbf{A} by naming each element via a nullary operation. Since $\operatorname{Con}(\mathbf{A}^+) = \operatorname{Con}(\mathbf{A})$, we obtain the following result by applying Lemma 5.2 (ii) \Rightarrow (i) to the algebra \mathbf{A}^+ .

Corollary 5.3. Let **A** be a finite algebra, and assume that every finite algebra in $Var(\mathbf{A})$ has a retraction onto each of its subalgebras. Then $\mathbf{L}_{\mathbf{A}^+} \cong Con(\mathbf{A})$.

We can now easily show that \mathcal{L}_{hom} contains all finite partition lattices and all finite subspace lattices. Note that every finite lattice embeds into a finite partition lattice (Pudlák and Tůma [30]).

Example 5.4.

- (i) For every non-empty finite set A, the lattice Equiv(A) of all equivalence relations on A belongs to L_{hom}.
- (ii) For every finite vector space V, the lattice Sub(V) of all subspaces of V belongs to L_{hom}.

Proof. Both parts follow from Corollary 5.3. For (i), note that every non-empty subset of a set is a retract. For (ii), note that every subspace of a vector space is a retract and that congruences correspond to subspaces. \Box

We can use this example to say something about the first-order theory of \mathcal{L}_{hom} .

Lemma 5.5. The only universal first-order sentences true in the class \mathcal{L}_{hom} are those true in all lattices.

Proof. Consider a universal first-order sentence $\sigma = \forall x_1 \dots \forall x_n \ \Phi(x_1, \dots, x_n)$ that fails in a lattice **L**. We want to show that σ fails in a finite lattice. By Pudlák and Tůma [30], it will then follow that σ fails in a finite partition lattice, whence σ fails in \mathcal{L}_{hom} by Example 5.4(i).

The proof of Dean [8, Theorem 1] for equations can be extended to universal sentences. Choose witnesses $a_1, \ldots, a_n \in L$ of the failure of σ in **L**. Let A be the subset of L consisting of the corresponding evaluations of all subterms of terms appearing in Φ , and let $\mathbf{A} = \langle A; \leqslant \rangle$ inherit the order from **L**. Then **A** is a finite ordered set, and the Dedekind–MacNeille completion DM(**A**) is a finite lattice that preserves all existing joins and meets from **A**. Hence σ fails in DM(**A**).

Intervals in subgroup lattices. Pálfy and Pudlák [26] have proved the equivalence of the following two statements:

- Every finite lattice arises as the congruence lattice of a finite algebra.
- Every finite lattice arises as an interval in the subgroup lattice of a finite group.

So, given the Finite Lattice Representation Problem, it is also natural to investigate which intervals in subgroup lattices belong to \mathcal{L}_{hom} .

The next example was provided by Keith Kearnes, and shows that \mathcal{L}_{hom} contains every lattice that can be obtained by adding a new top element to an interval in the subgroup lattice of a finite group.

Example 5.6. Let G be a finite group.

- (i) Let $\mathbf{A} = \langle A; \{ \lambda_g \mid g \in G \} \rangle$ be a finite \mathbf{G} -set regarded as a unary algebra, and assume that \mathbf{A} has a trivial subalgebra. Then $\mathbf{L}_{\mathbf{A}^+} \cong \operatorname{Con}(\mathbf{A})$.
- (ii) Now let **H** be a subgroup of **G**, and define **A** to be the **G**-set with universe $A = \{aH \mid a \in G\} \cup \{\infty\}$ such that $\lambda_g(aH) = gaH$ and $\lambda_g(\infty) = \infty$, for all $g, a \in G$. Then $\mathbf{L}_{\mathbf{A}^+} \cong [\mathbf{H}, \mathbf{G}] \oplus \mathbf{1}$.

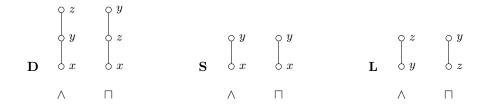


FIGURE 5. The subdirectly irreducible distributive bisemilattices.

Proof. (i): We will apply Lemma 5.2 (ii) \Rightarrow (i) to the algebra \mathbf{A}^+ . By assumption, there exists $a \in A$ such that $\{a\}$ is a subuniverse of \mathbf{A} . Let \mathbf{B} be a finite algebra in Var(\mathbf{A}^+). Since \mathbf{B} is a model of the equational theory of \mathbf{A}^+ , both of the subsets $B \setminus B_0$ and $\{a^{\mathbf{B}}\}$ of B are closed under each λ_g . Hence, we may define a homomorphism $\varphi \colon \mathbf{B} \to \mathbf{B}_0$ by $\varphi(b) = b$, for all $b \in B_0$, and $\varphi(b) = a^{\mathbf{B}}$, for all $b \in B \setminus B_0$. It follows from Lemma 5.2 that $\mathbf{L}_{\mathbf{A}^+} \cong \operatorname{Con}(\mathbf{A})$.

(ii): By part (i), it suffices to show that $\operatorname{Con}(\mathbf{A}) \cong [\mathbf{H}, \mathbf{G}] \oplus \mathbf{1}$. Note that $\infty/\theta = \{\infty\}$, for all $\theta \in \operatorname{Con}(\mathbf{A}) \setminus \{\mathbf{1}_{\mathbf{A}}\}$, as the action of \mathbf{G} on $A \setminus \{\infty\}$ is transitive. There are mutually inverse order-isomorphisms between $\operatorname{Con}(\mathbf{A}) \setminus \{\mathbf{1}_{\mathbf{A}}\}$ and $[\mathbf{H}, \mathbf{G}]$, given by

$$\theta \mapsto \mathbf{K}_{\theta}, \text{ where } K_{\theta} := \{ a \in G \mid aH \equiv H \pmod{\theta} \}, \text{ and}$$
$$\mathbf{K} \mapsto \theta_{\mathbf{K}}, \text{ where } aH \equiv bH \pmod{\theta_{\mathbf{K}}} \iff a^{-1}b \in K.$$

Five-element lattices. The lattice \mathbf{M}_3 belongs to \mathcal{L}_{hom} by Example 5.4, since it can be represented as Equiv($\{0, 1, 2\}$) or as $\text{Sub}(\mathbb{Z}_2^2)$.

With the exception of the pentagon lattice N_5 , we now know that all lattices of size up to 5 belong to \mathcal{L}_{hom} . Our aim for the remainder of this section is to apply Lemma 5.2 to help us find an example of a finite algebra \mathbf{A} such that $\mathbf{L}_{\mathbf{A}} \cong \mathbf{N}_5$. The algebra \mathbf{A} will be a four-element distributive bisemilattice with all elements named via nullary operations.

An algebra $\mathbf{A} = \langle A; \wedge, \sqcap \rangle$ is said to be a *distributive bisemilattice* (alternatively, a distributive quasilattice) if both \wedge and \sqcap are semilattice operations on A and each distributes over the other; see [29, 21, 12]. Kalman [21] has proved that, up to isomorphism, the only subdirectly irreducible distributive bisemilattices are the algebra \mathbf{D} and its subalgebras \mathbf{S} and \mathbf{L} , shown in Figure 5. (Note that we depict both \wedge and \sqcap as meet operations.) The algebra \mathbf{S} is term equivalent to the two-element semilattice, and the algebra \mathbf{L} is the two-element lattice.

Example 5.7. Let $\mathbf{A} = \langle \{0, a, b, 1\}; \land, \sqcap, 0, a, b, 1 \rangle$ be the distributive bisemilattice $\mathbf{S} \times \mathbf{L}$, labelled as in Figure 6, with all four elements named via nullary operations. Then $\mathbf{L}_{\mathbf{A}} \cong \mathbf{N}_{5}$.

Proof. We will apply Lemma 5.2 (iii) \Rightarrow (i). The congruences α , β and γ on **A** are shown in Figure 7. Note that the underlying bisemilattices of \mathbf{A}/α , \mathbf{A}/β and \mathbf{A}/γ are isomorphic to **S**, **L** and **D**, respectively; see Figure 8. We shall see that these are the only subdirectly irreducible algebras in Var(**A**).

As nullary operations play no role in determining the congruences on an algebra, the underlying bisemilattice of a subdirectly irreducible algebra in $Var(\mathbf{A})$ must be a subdirectly irreducible distributive bisemilattice and hence must be isomorphic

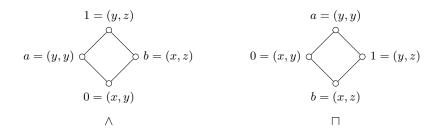


FIGURE 6. The distributive bisemilattice $\mathbf{S} \times \mathbf{L}$.

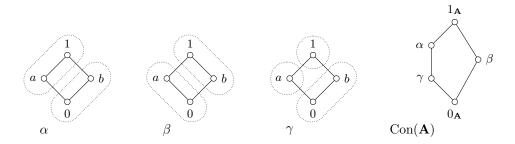


FIGURE 7. The congruences on **A** (shown relative to \wedge).

							01	$\circ a$
$\mathbf{A}/lpha$	$\circ a1 \\ \circ 0b$	$\circ a1 \\ \circ 0b$	\mathbf{A}/eta	$\circ b1 \\ \circ 0a$	$\circ 0a$ $\circ b1$	\mathbf{A}/γ	$\circ 1 \\ \circ a \\ \circ 0b$	$ \bigcirc 1 $ $ \bigcirc 0b $
	\wedge	Π		\wedge	Π		\wedge	П

FIGURE 8. The subdirectly irreducible algebras in $Var(\mathbf{A})$.

to \mathbf{S} , \mathbf{L} or \mathbf{D} , by Kalman's result [21]. But the values of the four nullary operations on an algebra in Var(\mathbf{A}) are determined by the two semilattice operations:

- 1 and 0 are the top and bottom for \wedge , respectively;
- a and b are the top and bottom for \Box , respectively.

Hence, the values of the nullary operations 0, a, b, 1 shown in Figure 8 are the only possible labellings of the bisemilattices **S**, **L** and **D** that produce an algebra in Var(**A**). So these are the only subdirectly irreducible algebras in Var(**A**). Each of these algebras is zero-generated, so it follows that condition 5.2(iii)(a) holds.

Since $\gamma \subseteq \alpha$ and $\alpha \cap \beta = 0_{\mathbf{A}}$, we can establish condition 5.2(iii)(b) by showing $(\mathbf{A}/\alpha) \times (\mathbf{A}/\beta) \to \mathbf{A}$. But $\alpha \cap \beta = 0_{\mathbf{A}}$ and $\alpha \cdot \beta = 1_{\mathbf{A}}$ imply $\mathbf{A} \cong (\mathbf{A}/\alpha) \times (\mathbf{A}/\beta)$. Hence, by Lemma 5.2, we have $\mathbf{L}_{\mathbf{A}} \cong \operatorname{Con}(\mathbf{A}) \cong \mathbf{N}_5$.

We have established that all lattices of size up to 5 arise as the homomorphism lattice induced by a finite algebra.

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Acknowledgement. We are indebted to Keith Kearnes for suggestions that led to improvements in Section 5. In particular, Example 5.6 is due to Keith.

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