# A PROBABILISTIC CHARACTERIZATION OF THE DOMINANCE ORDER ON PARTITIONS

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#### Abstract

A probabilistic characterization of the dominance partial order on the set of partitions is presented. This extends work in "Symmetric polynomials and symmetric mean inequalities". Electron. J. Combin., 20(3): Paper 34, 2013.

Let *n* be a positive integer and let  $\nu$  be a partition of *n*. Let *F* be the Ferrers diagram of  $\nu$ , a table of rows of cells, the *i*th row containing  $\nu(i)$  cells. Let *m* be a positive integer and let  $p \in (0, 1)$ . Fill each cell of *F* with balls, the number of which is independently drawn from the random variable X = Bin(m, p). Given non-negative integers *j* and *t*, let  $P(\nu, j, t)$  be the probability that the total number of balls in *F* is *j* and that no row of *F* contains more that *t* balls. We show that if  $\nu$  and  $\mu$  are partitions of *n*, then  $\nu$  dominates  $\mu$ , i.e.  $\sum_{i=1}^{k} \nu(i) \geq \sum_{i=1}^{k} \mu(i)$  for all positive integers *k*, if and only if  $P(\nu, j, t) \leq$  $P(\mu, j, t)$  for all non-negative integers *j* and *t*. It is also shown that this same result holds when *X* is replaced by any one member of a large class of random variables.

Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. Let  $\mathbb{N}$  be the set of nonnegative integers together with the usual order. Let  $\mathcal{T}_p$  be the  $\mathbb{N}$  by  $\mathbb{N}$  matrix with  $(\mathcal{T}_p)_{i,j} = p_{j-i}$  for all  $i, j \in \mathbb{N}$ . Here we take  $p_n = 0$  for all negative integers n. For all  $i, j \in \mathbb{N}$ , let  $(p^i)_j$  be the coefficient of  $x^j$  in  $(p(x))^i$  where  $p(x) = \sum_{n=0}^{\infty} p_n x^n$ . Here we take  $(p(x))^0 = 1$ . Let  $\mathcal{S}_p$  be the  $\mathbb{N}$  by  $\mathbb{N}$  matrix with  $(\mathcal{S}_p)_{i,j} = (p^i)_j$  for all  $i, j \in \mathbb{N}$ . We say that a matrix M is totally non-negative of order k if all of the minors of M of order k or less are non-negative. We

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show that if  $\mathcal{T}_p$  is totally non-negative of order k then so is  $\mathcal{S}_p$ . The case k = 2 of this result is a key step in the proof of the result on domination. We also show that the case k = 2 would follow from a combinatorial conjecture that might be of independent interest.

### 1 Introduction

Let  $\mathbb{N}$  denote the set of non-negative integers. A *partition* is a function  $\lambda : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N}$  that is *non-increasing* and has *finite support*, i.e. such that  $\lambda(s) \geq \lambda(t)$  for all  $s, t \in \mathbb{N}$  with s < t and  $\text{supp}(\lambda) = \{i \in \mathbb{N} \setminus \{0\} : \lambda(i) \neq 0\}$  is finite. The *weight* of  $\lambda$  is  $|\lambda| = \sum_{i=0}^{\infty} \lambda(i)$ . If  $n \in \mathbb{N}$  and  $\lambda$  is a partition we say that  $\lambda$  *is a partition of* n if  $|\lambda| = n$ . Let  $\mathcal{P}$  be the set of all partitions and for all  $n \in \mathbb{N}$ , let  $\mathcal{P}_n$  be the set of partitions of weight n.

We define the *dominance partial order*,  $\trianglelefteq$ , on  $\mathcal{P}$  as follows. If  $\lambda, \mu \in \mathcal{P}$ , we say  $\lambda$  is *dominated by*  $\mu$  (or  $\mu$  *dominates*  $\lambda$ ) if and only if  $|\lambda| = |\mu|$  and  $\sum_{i=1}^{j} \lambda(i) \leq \sum_{i=1}^{j} \mu(i)$  for all positive integers j. We denote this by  $\lambda \trianglelefteq \mu$  (or  $\mu \trianglerighteq \lambda$ ).

The dominance order is a special case of the more general majorization order. If m is a positive integer, the *majorization order* on  $\mathbb{R}^m$  is defined as follows. If  $x \in \mathbb{R}^m$  let  $x_p \in \mathbb{R}^m$  be the *non-increasing rearrangement of* x. I.e.  $(x_p)(i) = (x_p)_{\pi(i)}$  for all  $i \in [m]$  where  $\pi \in S_m$  is chosen so that  $(x_p)_1 \ge (x_p)_2 \ge \cdots \ge (x_p)_m$ . If  $x, y \in \mathbb{R}^m$  we say x is majorized by y if and only if  $\sum_{i=1}^m x_i = \sum_{i=1}^m y_i$  and  $\sum_{i=1}^j (x_p)_i \le \sum_{i=1}^j (y_p)_i$  for all integers j with  $1 \le j \le m$ . Evidently for all  $\lambda, \mu \in \mathcal{P}^m$  or  $\mathcal{U}^m$ , we have  $\lambda \le \mu$  if and only if  $\lambda$  is majorized by  $\mu$ .

The dominance and majorization orders frequently come up as definitions of key importance in many disparate fields in mathematics from the social sciences to representation theory, see [2,4].

Continuing work begun in [3], the author presents a probabilistic characterization of the dominance partial order,  $(\mathcal{P}, \trianglelefteq)$ .

Let *M* be a finite or infinite matrix and let  $k \in \mathbb{N}$ . We say *M* is *totally non-negative* (respectively, *totally positive*) of order *k* if and only if every minor of *M* of size *k* or less is non-negative (respectively, positive). We will abbreviate this by writing  $M \in TN_k$  (respectively,  $M \in TP_k$ ). See [1,5].

Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. We define the  $\mathbb{N}$  by  $\mathbb{N}$  matrix  $\mathcal{T}_p$  with  $(\mathcal{T}_p)_{i,j} = p_{j-i}$  for all  $i, j \in \mathbb{N}$ . Here we take  $p_n = 0$  for negative integers n.

We say that p is *totally non-negative* (respectively, *totally positive*) of order k if and only if  $\mathcal{T}_p \in TN_k$  (respectively,  $\mathcal{T}_p \in TP_k$ ) and denote this by  $p \in TN_k$  (respectively,  $p \in TP_k$ ).

Let  $\mathbb{R}[[x]]$  be the ring of formal power series over  $\mathbb{R}$ . If  $p = \{p_n\}_{n=0}^{\infty}$ , then the *ordinary generating function of* p is the element  $p(x) = \sum_{n=0}^{\infty} p_n x^n$  of  $\mathbb{R}[[x]]$ . We say p(x) is  $TN_k$  (respectively,  $TP_k$ ) if and only if p has the same property.

Let X be an  $\mathbb{N}$ -valued random variable. We say  $p = p_X = \{P(X = n)\}_{n=0}^{\infty}$  is the sequence of probabilities of X and  $p_X(x) = \mathbb{E}x^X = \sum_{n=0}^{\infty} P(X = n)x^n$  is the probability generating function of X. We say X is  $TN_k$  (respectively,  $TP_k$ ) if  $p_X$  (or, equivalently,  $p_X(x)$ ) has the same property. We define the range of X to be range $(X) = \{n \in \mathbb{N} : P(X = n) \neq 0\}$ .

If  $n \in \mathbb{N}$  and  $\nu$  is a partition of n, let  $\mathcal{Y}(\nu) = (Y_i : i \in \mathbb{N} \setminus \{0\})$  be a sequence of independent random variables where, for each  $i \in \mathbb{N}$ ,  $Y_i$  is distributed as the sum of  $\nu(i)$  independent copies of X. If  $\nu(i) = 0$ , we define  $Y_i$  to be identically 0. If  $j, t \in \mathbb{N}$ , let  $E(\nu, X, j, t)$  be the event that  $Y_i \leq t$  for all  $i \in \mathbb{N} \setminus \{0\}$  and  $\sum_{i=1}^{\infty} Y_i = j$ . If  $\lambda$  and  $\mu$  are partitions, let  $C(\lambda, \mu, X)$  be the condition that

$$P(E(\lambda, X, j, t)) \le P(E(\mu, X, j, t)), \text{ for all } j, t \in \mathbb{N}.$$
(1)

Our main theorems are Theorems 1.1, 1.2 and 1.4, listed below.

**Theorem 1.1.** Let X be an  $\mathbb{N}$ -valued random variable. Suppose  $X \in TN_2$  and range $(X) = \{0, 1, ..., r\}$  for some positive integer r. Then, for all  $n \in \mathbb{N}$  and all partitions  $\lambda$  and  $\mu$  of  $n, \lambda \succeq \mu$  if and only if  $C(\lambda, \mu, X)$ .

**Theorem 1.2.** For all  $n \in \mathbb{N}$  and all partitions  $\lambda$  and  $\mu$  of n,  $\lambda \supseteq \mu$  if and only if we have  $C(\lambda, \mu, X)$  for all  $\mathbb{N}$ -valued random variables X with  $X \in TN_2$ .

The line of investigation that led to Theorems 1.1 and 1.2 began in [3] where it was proved that for any  $p \in (0, 1)$  if X = Bin(1, p) then  $\lambda \ge \mu$  implies  $C(\lambda, \mu, X)$ .

Corollary 1.3, listed below, is a pictorial description of two special cases of Theorem 1.1. Let  $F(\lambda) = \{(i, j) : i \in \mathbb{Z}_{>0}, 1 \leq j \leq \lambda(i)\}$  be the *Ferrers diagram* of  $\lambda$ . Customarily, the (i, j)th *cell* of  $F(\lambda)$  is represented as the box  $[-(i-1), -i] \times [j-1, j]$ in  $\mathbb{R}^2$  so that  $F(\lambda)$  represents the parts of  $\lambda$  as a left-aligned stack of rows of cells in  $\mathbb{R}^2$ , the *i*th topmost row corresponding to  $\lambda(i)$  in that it consists of  $\lambda(i)$  cells. Let *r* be a positive integer and let  $p \in (0, 1)$ . Let  $U_r$  be the random variable that is distributed uniformly on  $\{0, 1, \ldots r\}$ . Let Bin(r, p) be the random variable *X* with  $P(X = k) = {n \choose k} p^k (1 - p)^{n-k}$  for  $0 \leq k \leq r$  and P(X = k) = 0 for k > r.

**Corollary 1.3.** Independently fill each cell of the Ferrers diagrams of  $\lambda$  and  $\mu$  with balls, the number put in each cell drawn from the distribution  $U_r$ . Then  $\lambda \geq \mu$  if and only if for all integers  $j, t \geq 0$  the probability that the Ferrers diagram for  $\lambda$  contains j balls with at most t balls in each row is less than or equal to the corresponding probability for  $\mu$ . The same remains true if  $U_r$  is replaced by Bin(r, p).

We prove Theorems 1.1 and 1.2 via the case k = 2 of Theorem 1.4, listed below. Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. For all  $i, j \in \mathbb{N}$ , let  $(p^i)_j$  be the coefficient of  $x^j$  in  $(p(x))^i$  where  $p(x) = \sum_{n=0}^{\infty} p_n x^n$ . Here we take  $(p(x))^0 = 1$ . Let  $S_p$  be the  $\mathbb{N}$  by  $\mathbb{N}$  matrix with  $(S_p)_{i,j} = (p^i)_j$  for all  $i, j \in \mathbb{N}$ .

**Theorem 1.4.** Let  $k \in \mathbb{N}$  and let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. If  $\mathcal{T}_p \in TN_k$ , then  $\mathcal{S}_p \in TN_k$ . Also, if  $\mathcal{T}_p \in TP_k$ , then  $\mathcal{S}_p \in TP_k$ .

The case k = 2 of Theorem 1.4 is implied by Conjecture 1.5, listed below, a combinatorial conjecture that might be of independent interest.

If *m* is a positive integer, let  $[m] = \{1, 2, ..., m\}$ . Let  $[0] = \emptyset$ . If  $\lambda : [m] \to \mathbb{N}$ , we say  $\lambda$  is a *composition with m non-negative parts*. If  $\lambda$  is also *non-increasing*, i.e.  $\lambda(i) \ge \lambda(j)$  for all integers *i* and *j* with  $1 \le i < j \le m$ , then we say  $\lambda$  is a *partition with m non-negative parts*. Let  $\mathcal{U}^m$  (respectively,  $\mathcal{P}^m$ ) be the sets of compositions (respectively, partitions) with *k* non-negative parts. If  $\lambda \in \mathcal{U}^m$ , let  $|\lambda| = \sum_{i \in [m]} \lambda(i)$ be the *weight* of  $\lambda$ . If  $\lambda \in \mathcal{U}^m$  and  $|\lambda| = n$ , we say  $\lambda$  is a *composition* of *n*. If  $\lambda \in \mathcal{U}^m$ and  $|\lambda| = n$ , we say  $\lambda$  is a *partition* of *n*. If  $m, n \in \mathbb{N}$ , let  $\mathcal{U}_n^m = \{\lambda \in \mathcal{U}^m : |\lambda| = n\}$ and  $\mathcal{P}_n^m = \{\lambda \in \mathcal{P}^m : |\lambda| = n\}$ .

If  $\lambda, \mu \in \mathcal{U}^m$ , let  $\lambda_p, \mu_p$  be the non-increasing rearrangements of  $\lambda$  and  $\mu$  into partitions. We define the *dominance* partial order on  $\mathcal{U}^m$  (and  $\mathcal{P}^m$ ),  $\trianglelefteq$ , by setting  $\lambda \trianglelefteq \mu$  if and only if  $\lambda_p \trianglelefteq \mu_p$ .

Let  $A, a, B, b \in \mathbb{N}$ . If  $\lambda \in \mathcal{U}_a^A$  and  $\mu \in \mathcal{U}_b^B$ , let  $\lambda \mu \in \mathcal{U}_{a+b}^{A+B}$  be the *concatenation of*  $\lambda$  and  $\mu$ , defined by setting  $(\lambda \mu)(i) = \lambda(i)$  for  $i \in [A]$  and  $(\lambda \mu)(i) = \mu(i - A)$  for  $i \in [A + B] \setminus [A]$ .

**Conjecture 1.5.** For all integer A, a, B, b with  $A \ge B \ge 0$  and  $a \ge b \ge 0$  there is an injection  $\gamma : \mathcal{U}_b^A \times \mathcal{U}_a^B \hookrightarrow \mathcal{U}_a^A \times \mathcal{U}_b^B$  such that for all  $(\lambda_1, \mu_1) \in \mathcal{U}_b^A \times \mathcal{U}_a^B$ ,  $(\lambda_2, \mu_2) = \gamma((\lambda_1, \mu_1)) \in \mathcal{U}_a^A \times \mathcal{U}_b^B$  satisfies  $\lambda_1 \mu_1 \ge \lambda_2 \mu_2$ .

In Section 2, we give two useful characterizations of  $TN_2$  in Lemma 2.1 and prove Lemma 2.3 which states a number of basic results on how the properties  $TN_2$ , nonnegativity, positivity, unimodality and log-concavity of a sequence p relate to one another. In Section 3, we will prove Theorem 1.4 and discuss how Conjecture 1.5 implies the case k = 2 of this theorem. In Section 4, we will use this case to prove Theorems 1.1 and 1.2. In Section 5, we record some observations on the roles of the assumptions in Theorem 1.1.

### **2 Basic Results on** $TN_2$

Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. We say p is *non-negative* (respectively, *positive*) if  $p_n \ge 0$  (respectively,  $p_n > 0$ ) for all  $n \in \mathbb{N}$ . If  $\lambda \in \mathcal{P}^m$ , let  $p_{\lambda} = \prod_{i=1}^{m} p_{\lambda(i)}$ .

**Lemma 2.1.** Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. Then the following statements are equivalent.

- (i)  $p \in TN_2$
- (ii) p is non-negative and  $p_a p_d \le p_b p_c$  for all integers a, b, c, d with  $a \ge b \ge c \ge d \ge 0$ and a + d = b + c.
- (iii) p is non-negative and for all positive integers m, if  $\lambda, \mu \in \mathcal{P}^m$  and  $\lambda \succeq \mu$ , then  $p_{\lambda} \leq p_{\mu}$ .

In order to prove this lemma, we need some basic results on the cover relation in the dominance order.

Let  $(P, \leq)$  be a partially ordered set. If  $a, b \in P$  we say a is covered by b or, equivalently, b covers a if  $a \leq b$  and  $\{x \in P : a \leq x \leq b\} = \{a, b\}$ . The following lemma, stated without proof, is a standard characterization of the cover relation  $\triangleleft$  corresponding to the dominance order  $\trianglelefteq$  on  $\mathcal{P}$ .

**Lemma 2.2.** (See [2], 1.4.21, p.28) Let  $\lambda$ ,  $\mu$  be partitions. Then  $\lambda \cdot \triangleright \mu$  if and only if there exist integers i, j with  $j > i \ge 1$  such that (a),  $\lambda(j) = \mu(j) - 1$  and  $\lambda(i) = \mu(i) + 1$ , while for  $\nu \notin \{i, j\}$  we have  $\lambda(\nu) = \mu(\nu)$ , and (b), i = j - 1 or  $\mu(j) = \mu(i)$ .

Let *A* be an  $\mathbb{N}$  by  $\mathbb{N}$  matrix. If  $x, y, z, w \in \mathbb{N}$  with x < y and z < w, let  $A_{\{x,y\} \times \{z,w\}}$  be the 2 by 2 submatrix of *A* whose rows are indexed by *x* and *y* and whose columns are indexed by *z* and *w*. We define the following 2 by 2 minor of  $\mathcal{T}_p$ ,  $M_{\{x,y\} \times \{z,w\}} = \det((\mathcal{T}_p)_{\{x,y\} \times \{z,w\}})$ .

**Proof of Lemma 2.1.** We will first show that (i) and (ii) are equivalent.

Suppose we have (i). Suppose  $a, b, c, d \in \mathbb{N}$ ,  $a \ge b \ge c \ge d \ge 0$  and a + d = b + c. If a = b then c = d and  $p_b p_c - p_a p_d = 0$ . Suppose a > b. Let x = 0, y = a - b, z = c, w = a. Then  $0 \le x < y$  and  $0 \le z < w$  and  $p_b p_c - p_a p_d = M_{\{x,y\} \times \{z,w\}} \ge 0$ . Thus we have (ii).

Now we assume (ii). If  $x, y, z, w \in \mathbb{N}$  with x < y and z < w,  $M_{\{x,y\}\times\{z,w\}} = p_b p_c - p_a p_d$  where a = w - x, b = z - x, c = w - y, d = z - y, Note a > b, c > d, and a + d = b + c. If d < 0 then  $p_d = 0$  and  $M_{\{x,y\}\times\{z,w\}} = p_b p_c \ge 0$ . If  $d \ge 0$  then

 $M_{\{x,y\}\times\{z,w\}} = p_b p_c - p_a p_d \ge 0$ . Thus we have (i). Note that the non-negativity of p is necessary: if  $p_n = (-2)^n$  then the second condition of (ii) holds but  $p \notin TN_2$ .

The condition in (iii) for m = 2 is the condition in (ii), thus (iii) implies (ii). Now we assume (ii). This implies the cases m = 1 and m = 2 of (iii) are true. We now assume  $m \ge 3$ . Since  $\lambda \trianglerighteq \mu$  implies  $|\lambda| = |\mu|$  and since  $\mathcal{P}_n^m$  is finite, we need only show  $p_\lambda \le p_\mu$  if  $\lambda \cdot \bowtie \mu$ .

Let  $j > i \ge 1$  be the indices witnessing  $\lambda \cdot \triangleright \mu$ , i.e. those satisfying (a) and (b) of Lemma 2.2. Since (a) implies  $\lambda(i) > \mu(i) \ge \mu(j) > \lambda(j) \ge 0$  and  $\mu(i) + \mu(j) = \lambda(i) + \lambda(j)$ , (ii) implies  $p_{\lambda(i)}p_{\lambda(j)} \le p_{\mu(i)}p_{\mu(j)}$ . Thus  $p_{\lambda} = (\prod_{\nu \notin \{i,j\}} p_{\lambda(\nu)})p_{\lambda(i)}p_{\lambda(j)} = (\prod_{\nu \notin \{i,j\}} p_{\mu(\nu)})p_{\lambda(i)}p_{\lambda(j)} \le (\prod_{\nu \notin \{i,j\}} p_{\mu(\nu)})p_{\mu(i)}p_{\mu(j)} = p_{\mu}$ . The second equality holds by (a) of Lemma 2.2 while the inequality holds by the non-negativity of p. Thus we have (iii).

We say that p is unimodal if there is a  $k \in \mathbb{N}$  such that  $p_i \leq p_j \leq p_k \geq p_l \geq p_m$ for all  $i, j, l, m \in \mathbb{N}$  with  $i \leq j \leq k \leq l \leq m$ . Alternatively, p is unimodal if and only if there are no  $i, j, k \geq 0$  such that i < j < k and  $p_i > p_j < p_k$ . We say say that p is *log-concave* (respectively, *strictly log-concave*) if and only if  $p_k^2 \geq p_{k+1}p_{k-1}$ (respectively,  $p_k^2 > p_{k+1}p_{k-1}$ ) for all positive integers k. We say that p is *k-nonnegative* (respectively, *k-positive*) if and only if  $\mathcal{T}_p$  has all minors of order k nonnegative (respectively, positive).

**Lemma 2.3.** Let  $p = \{p_n\}_{n=0}^{\infty}$  be a sequence of real numbers. Then the following statements hold.

- *(i)* If p is 2-non-negative, then p is log-concave. If p is 2-positive, then p is strictly log-concave.
- (ii) If  $p \in TN_2$  then p is unimodal.
- (iii) Suppose p is non-negative. If p is unimodal or log-concave then p is not necessarily  $TN_2$ .
- (iv) Suppose p is positive. If p is log-concave, then p is  $TN_2$ . If p is strictly log-concave, then p is  $TP_2$ .

#### Proof of Lemma.

We prove (i). Suppose p is 2-non-negative. Let k be a positive integer. Then  $p_k^2 - p_{k+1}p_{k-1} = M_{\{0,1\}\times\{k,k+1\}} \ge 0$  and thus p is log-concave. If p is 2-positive  $p_k^2 - p_{k+1}p_{k-1} = M_{\{0,1\}\times\{k,k+1\}} > 0$  and p is strictly log-concave.

We now prove (ii). Suppose that  $p \in TN_2$ . Suppose, for the sake of deriving a contradiction, that p is not unimodal. Then there must be  $i, j, k \in \mathbb{N}$  with i < j < k

such that  $p_i > p_j < p_k$ . Let  $d = \max\{x \in \mathbb{N} : (i \le x < j) \text{ and } (p_x \ge p_i)\}$ . Let  $a = \min\{y : (i < y \le k) \text{ and } (p_x \ge p_k)\}$ . Let b = a - 1 and c = d + 1. Since  $d < j < a, a > b \ge c > d \ge 0$ . Also a + d = b + c. But  $0 \le p_c < p_d$  and  $0 \le p_b < p_a$  so  $p_a p_d > p_b p_c$ , a contradiction to Lemma 2.1 (ii).

We prove (iii) by noting that the sequence  $p_n = 2^n + 1$  is unimodal but not logconcave and the sequence (1, 0, 0, 1, 1, ...) is log-concave but not 2-non-negative. We now prove (iv). Let  $a \ge b \ge c \ge d \ge 0$  with a + d = b + c. Let t = a - c = b - d. If t = 0, a = b = c = d and we are done. Now suppose  $t \ge 1$ . Since p is positive and log-concave,  $p_k/p_{k-1} > 0$  and  $p_k/p_{k-1} \ge p_{k+1}/p_k$  for all  $k \ge 1$ . Since c > d,  $p_{d+i}/p_{d+i-1} \ge p_{c+i}/p_{c+i-1}$  for all integers i with  $1 \le i \le t$ . Thus  $p_b/p_d = \prod_{i=1}^t (p_{d+i}/p_{d+i-1}) \ge \prod_{i=1}^t (p_{c+i}/p_{c+i-1}) = p_a/p_c$ , hence  $p_b p_c - p_a p_d \ge 0$ . Thus  $p \in TN_2$  by Lemma 2.1 (ii). The proof that  $p \in TP_2$  when p is strictly logconcave is analogous.

#### **3 Proof of Theorem 1.4**

#### **Proof of Theorem 1.4.**

If  $p(x) = \sum_{n \ge 0} p_n x^n \in \mathbb{R}[[x]]$ , we define  $\mathcal{T}(x) = \mathcal{T}_{p(x)}(x)$ , an  $\mathbb{N}$  by  $\mathbb{N}$  matrix with entries in  $\mathbb{R}[[x]]$ , by setting

$$(\mathcal{T}(x))_{i,j} = \frac{1}{(j-i)!} \left(\frac{d}{dx}\right)^{j-i} p(x)$$

if  $j \ge i$  and  $(\mathcal{T}(x))_{i,j} = 0$  otherwise. Here, we define  $(d/dx)^0 p(x) = p(x)$ . Let  $\mathcal{S}(x) = \mathcal{S}_{p(x)}(x)$  be an  $\mathbb{N}$  by  $\mathbb{N}$  matrix with entries in  $\mathbb{R}[[x]]$  defined by

$$(\mathcal{S}(x))_{i,j} = \frac{1}{j!} \left(\frac{d}{dx}\right)^j p^i(x),$$

where  $p^0(x) = 1$ . The derivatives that occur in the definition of  $\mathcal{T}$  and  $\mathcal{S}$  are iterations of the purely formal operation  $d/dx : \mathbb{R}[[x]] \to \mathbb{R}[[x]]$  defined by

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}p_nx^n\right) = \sum_{n=0}^{\infty}(n+1)p_{n+1}x^n.$$

Note that  $\mathcal{T}(0) = \mathcal{T}_p$  and  $\mathcal{S}(0) = \mathcal{S}_p$ .

Suppose *M* is an  $\mathbb{N}$  by  $\mathbb{N}$  matrix with entries in a set  $\mathcal{E}$ . Let  $\ell \geq 1$  and let  $A, a \in \mathbb{N}^{\ell}$ . We define  $M_{A \times a}$  to be the  $\ell$  by  $\ell$  matrix with  $(M_{A \times a})_{i,j} = M_{A(i),a(j)}$  for all  $1 \leq i, j \leq j$   $\ell$ . If *A* and *a* are strictly increasing (i.e.  $A_1 < \cdots < A_\ell$  and  $a_1 < \cdots < a_\ell$ ) then  $M_{A \times a}(x)$  is just the size  $\ell$  square sub-matrix of *M* restricted to the rows in *A* and the columns in *a*.

By assumption, we have  $k \in \mathbb{N}$  and  $\det(\mathcal{T}(0)_{A \times a}) \ge 0$  for all strictly increasing  $A, a \in \mathbb{N}^{\ell}$  for all integers  $\ell$  with  $0 \le \ell \le k$ . We wish to show  $\det(\mathcal{S}(0)_{A \times a}) \ge 0$  for all strictly increasing  $A, a \in \mathbb{N}^{\ell}$  for all integers  $\ell$  with  $0 \le \ell \le k$ . We will prove this by induction successively on  $k, \ell$ , and  $A_1$ .

Since there is nothing to show when  $\ell = 0$  we may assume  $k, \ell \ge 1$ . If  $\mathcal{T}_p$  is  $TN_1$  then p is non-negative and thus  $\mathcal{S}_p$  is  $TN_1$ . Thus we may assume that  $k, \ell \ge 2$ . Suppose  $A_1 = 0$ . If  $a_1 = 0$  as well, the first row of  $\mathcal{S}(0)_{A \times a}$  has a 1 as its first entry and every other entry 0. Thus,  $\det(\mathcal{S}(0)_{A \times a}) = \det(\mathcal{S}(0)_{(A_2,...,A_l) \times (a_2,...,a_l)})$  and we have the result by induction on  $\ell$ . If  $a_1 > 0$  then the first row of  $\mathcal{S}(0)_{A \times a}$  is the zero row and  $\det(\mathcal{S}(0)_{A \times a}) = 0$ . Thus we may now assume that  $A_1 \ge 1$ .

Let  $B, b \in \mathbb{N}^{\ell}$ . Let  $\sigma \in S_{\ell}$  such that  $B_{\sigma 1} \leq B_{\sigma 2} \leq \cdots \leq B_{\sigma \ell}$ . Let  $B' = (B_{\sigma 1}, B_{\sigma 2}, \dots, B_{\sigma \ell})$ . We define sgn(B) to be 0 if B has a repeated entry and, otherwise, sgn(B) = sgn $(\sigma)$  where sgn $(\sigma) = 1$  if  $\sigma$  is an even permutation and -1 if  $\sigma$  is an odd permutation. We define b' and sgn(b) analogously. Note that det $(\mathcal{S}(x)_{B \times b}) =$  sgn(B)sgn(b) det $(\mathcal{S}(x)_{B' \times b'}(x))$ .

Since the formal derivative d/dx on  $\mathbb{R}[[x]]$  satisfies the product rule, we also have the generalized product rule on  $\mathbb{R}[[x]]$ , namely

$$\left(\frac{d}{dx}\right)^n \left(p(x)q(x)\right) = \sum_{k=0}^n \binom{n}{k} \left(\left(\frac{d}{dx}\right)^{n-k} p(x)\right) \left(\left(\frac{d}{dx}\right)^k q(x)\right) \text{ for all } p(x), q(x) \in \mathbb{R}[[x]]$$

where  $(d/dx)^0 p(x) = p(x)$ . For each  $i, j \in [\ell]$ , we use this rule to write

$$(\mathcal{S}(x)_{A \times a})_{i,j} = \frac{1}{a_j!} \sum_{b_j=0}^{a_j} {a_j \choose b_j} \left( \left(\frac{d}{dx}\right)^{a_j-b_j} p(x) \right) \left( \left(\frac{d}{dx}\right)^{b_j} p^{A_i-1}(x) \right)$$
$$= \sum_{b_j \ge 0} 1_{b_j \le a_j} \left( \frac{1}{(a_j-b_j)!} \left(\frac{d}{dx}\right)^{a_j-b_j} p(x) \right) \left( \frac{1}{b_j!} \left(\frac{d}{dx}\right)^{b_j} p^{A_i-1}(x) \right)$$

where  $1_{b_j \leq a_j} = 1$  when  $b_j \leq a_j$  and 0 otherwise. Let  $B = (A_1 - 1, \dots, A_\ell - 1)$ . Since  $det(S(x)_{A \times a})$  is multilinear in its columns, we get

$$\det(\mathcal{S}(x)_{A\times a}) = \sum_{b\in\mathbb{N}^{\ell}} \prod_{j=1}^{\ell} \left( \frac{1_{b_j \le a_j}}{(a_j - b_j)!} \left( \frac{d}{dx} \right)^{a_j - b_j} p(x) \right) \det(\mathcal{S}(x)_{B\times b})$$

$$= \sum_{b \in \mathbb{N}^{\ell}} \det(\mathcal{S}(x)_{B \times b'}) \prod_{j=1}^{\ell} \operatorname{sgn}(b) \left( \frac{1_{b_j \le a_j}}{(a_j - b_j)!} \left( \frac{d}{dx} \right)^{a_j - b_j} p(x) \right)$$
$$= \sum_{0 \le b_1 < \dots < b_{\ell}} \det(\mathcal{S}(x)_{B \times b}) \sum_{\sigma \in S(\{b_1, \dots, b_{\ell}\})} \operatorname{sgn}(\sigma) \prod_{j=1}^{\ell} \left( \frac{1_{\sigma_j \le a_j}}{(a_j - \sigma_j)!} \left( \frac{d}{dx} \right)^{a_j - \sigma_j} p(x) \right)$$

Thus

$$\det(\mathcal{S}(x)_{A \times a}) = \sum_{0 \le b_1 < \dots < b_\ell} \det(\mathcal{S}(x)_{B,b}) \det(\mathcal{T}(x)_{b \times a}).$$

It is important to realize that this is a finite sum: when  $b_{\ell} > a_{\ell}$  all entries in the  $b_{\ell}$  row of  $\mathcal{T}(x)_{b \times a}$  are 0 and thus  $\det(\mathcal{T}(x)_{b \times a}) = 0$ . Setting x = 0 gives

$$\det(\mathcal{S}(0)_{A \times a}) = \sum_{0 \le b_1 < \dots < b_\ell} \det(\mathcal{S}(0)_{B,b}) \det(\mathcal{T}(0)_{b \times a}).$$

By induction on  $A_1$ , all of the minors appearing in this last sum are non-negative. It is easily seen that with a little modification this argument will also furnish a proof of the fact that  $\mathcal{T}_p \in TP_k$  implies  $\mathcal{S}_p \in TP_k$ .

**Theorem 3.1.** Conjecture 1.5 implies the case k = 2 of Theorem 1.4.

*Proof.* By Lemma 2.1 (ii),  $S_p \in TN_2$  if and only if  $(p^A)_b(p^B)_a \leq (p^A)_a(p^B)_b$  for all  $A, B, a, b \in \mathbb{N}$  with  $A \geq B$  and  $a \geq b$ . We have

$$(p^A)_b(p^B)_a = \left(\sum_{\lambda_1 \in \mathcal{U}_b^A} p_{\lambda_1}\right) \left(\sum_{\mu_1 \in \mathcal{U}_a^B} p_{\mu_1}\right) = \sum_{(\lambda_1, \mu_1) \in \mathcal{U}_b^A \times \mathcal{U}_a^B} p_{\lambda_1 \mu_1}$$

and

$$(p^A)_a(p^B)_b = \sum_{(\lambda_2,\mu_2)\in\mathcal{U}_a^A\times\mathcal{U}_b^B} p_{\lambda_2\mu_2}.$$

Conjecture 1.5 would imply there is an injection  $\gamma : \mathcal{U}_b^A \times \mathcal{U}_a^B \hookrightarrow \mathcal{U}_a^A \times \mathcal{U}_b^B$  such that if  $(\lambda_2, \mu_2) = \gamma((\lambda_1, \mu_1))$  then  $\lambda_1 \mu_1 \supseteq \lambda_2 \mu_2$ . By Lemma 2.1 (iii) this means that  $p_{\lambda_1 \mu_1} \ge p_{\lambda_2 \mu_2}$ . But this means  $(p^A)_b (p^B)_a \le (p^A)_a (p^B)_b$  as every term  $p_{\lambda_1 \mu_1}$  in  $(p^A)_b (p^B)_a$  is matched by  $\gamma$  to its own distinct term  $p_{\lambda_2 \mu_2}$  of  $(p^A)_a (p^B)_b$  with  $p_{\lambda_1 \mu_1} \le p_{\lambda_2 \mu_2}$ .  $\Box$ 

### 4 **Proofs of Theorems 1.1 and 1.2**

Theorems 1.1 and 1.2 are immediate corollaries of Theorems 4.1 and 4.2 below.

**Theorem 4.1.** Let X be an  $\mathbb{N}$ -valued random variable with  $X \in TN_2$ . Then, for all partitions  $\lambda$  and  $\mu$ ,  $\lambda \succeq \mu$  implies  $C(\lambda, \mu, X)$ .

**Theorem 4.2.** Let X be a  $\mathbb{N}$ -valued random variable with range $(X) = \{0, 1, ..., r\}$  for some positive integer r. Then, for all partitions  $\lambda$  and  $\mu$  with  $|\lambda| = |\mu|$ ,  $C(\lambda, \mu, X)$  implies  $\lambda \ge \mu$ .

In order to prove Theorems 4.1 and 4.2 we will rephrase the condition  $C(\lambda, \mu, X)$  as a condition on  $p_X(x)$ , its probability generating function.

Let  $p(x) = \sum_{n=0}^{\infty} p_n x^n \in \mathbb{R}[[x]]$ . For any  $t \in \mathbb{N}$ , we define the *truncation of*  $p(x) = \sum_{n=0}^{\infty} p_n x^n \in \mathbb{R}[[x]]$  to degree t to be  $p(x)|_t = \sum_{n=0}^t p_n x^n$ . Given a positive integer m,  $\lambda \in \mathcal{P}^m$  and  $t \in \mathbb{N}$ , let

$$f(\lambda, p(x), t, x) = \prod_{i=1}^{m} (p^{\lambda(i)}(x)|_t),$$

where  $p^0(x) = 1$  by definition.

Let  $\mathbb{R}_{\geq 0}[[x]] = \{\sum_{n=0}^{\infty} p_n x^n : p_n \ge 0 \text{ for all } n \in \mathbb{N}\}$ . Given  $p(x) \in \mathbb{R}_{\geq 0}[[x]]$  let  $p(1) = \sum_{n=0}^{\infty} p_n$ . If  $p(1) \in (0, +\infty)$ , we say X is distributed according to p(x) if and only if  $p_X(x) = p(x)/p(1)$ . We denote this by  $X \sim p(x)$ .

If we also have  $q(x) = \sum_{n=0}^{\infty} q_n x^n \in \mathbb{R}[[x]]$ , we say p(x) coefficient-wise dominates q(x) if and only if  $p_n \ge q_n$  for all  $n \in \mathbb{N}$ . We denote this by  $q(x) \sqsubseteq p(x)$  or, equivalently,  $p(x) \sqsupseteq q(x)$ .

Let  $C(\lambda, \mu, p(x))$  be the condition that

$$\forall t \in \mathbb{Z}_{\geq 0}, \ f(\lambda, p(x), t, x) \sqsubseteq f(\mu, p(x), t, x).$$

**Lemma 4.3.** If  $p(x) \in \mathbb{R}_{\geq 0}[[x]]$  with  $p(1) \in (0, \infty)$  and X is an  $\mathbb{N}$ -valued random variable with  $X \sim p(x)$  then  $C(\lambda, \mu, p(x))$  is equivalent to  $C(\lambda, \mu, X)$ .

*Proof.* It is easy enough to see that  $f(\lambda, p(x), t, x) = \sum_{j=0}^{\infty} (p(1))^{|\lambda|} P(E(\lambda, X, j, t)) x^j$  for all  $t \in \mathbb{N}$ .

**Proof of Theorem 4.1.** Let X be an N-valued random with  $X \in TN_2$ . For all  $n \in \mathbb{N}$ , let  $p_n = P(X = n)$ . Then  $p(x) = p_X(x) = \sum_{n=0}^{\infty} p_n x^n$  and  $p = p_X = \{p_n\}_{n=0}^{\infty}$ .

Since  $X \in TN_2$ ,  $\mathcal{T}_p \in TN_2$  by definition. By Theorem 1.4  $\mathcal{S}_p$  is  $TN_2$ . (This would also follow from Conjecture 1.5 if it were true.) This means

$$(p^A)_b(p^B)_a \le (p^A)_a(p^B)_b$$
 for all  $A \ge B \ge 0$  and  $a \ge b \ge 0$ . (2)

We now mimic the proof of Lemma 2.1 (ii). To show that  $\lambda \ge \mu$  implies  $C(\lambda, \mu, X)$  for all  $\lambda, \mu \in \mathcal{P}_n$  we may assume  $\lambda \cdot \triangleright \mu$ . Let j and i with  $j > i \ge 1$  witness this fact as in Lemma 2.2. Let  $A = \mu(i)$  and  $B = \lambda(j)$ . Then A > B and  $\lambda(i) = A + 1$  and  $\mu(j) = B + 1$ .

We now show that that for all  $A > B \ge 0$  and for all  $t \in \mathbb{N}$ ,

$$(p^{A+1}(x)|_t)(p^B(x)|_t) \sqsubseteq (p^A(x)|_t)(p^{B+1}(x)|_t).$$
(3)

This will be enough to prove  $C(\lambda, \mu, p(x))$  and hence, by Lemma 4.3,  $C(\lambda, \mu, X)$ . It is easy enough to verify that if  $f(x) \in \mathbb{R}_{\geq 0}[[x]]$  and  $g(x), h(x) \in \mathbb{R}[[x]]$  with  $g(x) \sqsubseteq h(x)$  then  $f(x)g(x) \sqsubseteq f(x)h(x)$ . Since (3) implies  $p^{\lambda(i)}(x)p^{\lambda(j)}(x) \sqsubseteq p^{\mu(i)}(x)p^{\mu(j)}(x)$ ,

$$f(\lambda, p(x), t, x) = \left(\prod_{\nu \notin \{i, j\}} (p^{\lambda(\nu)}(x)|_t)\right) (p^{\lambda(i)}(x)|_t) (p^{\lambda(j)}(x)|_t)$$
$$= \left(\prod_{\nu \notin \{i, j\}} (p^{\mu(\nu)}(x)|_t)\right) (p^{\lambda(i)}(x)|_t) (p^{\lambda(j)}(x)|_t)$$
$$\sqsubseteq \left(\prod_{\nu \notin \{i, j\}} (p^{\mu(\nu)}(x)|_t\right) (p^{\mu(i)}(x)|_t) (p^{\mu(j)}(x)|_t) = f(\mu, p(x), t, x).$$

The first equality holds by (a) of Lemma 2.2.

It remains to show (3). Fixing  $i \in \mathbb{N}$ , we must show that the corresponding coefficients of  $x^i$  in the two polynomials in (3) satisfy

$$\sum_{b+c+a=i,b+c\leq t,a\leq t} (p^A)_b p_c(p^B)_a \leq \sum_{b+c+a=i,b\leq t,c+a\leq t} (p^A)_b p_c(p^B)_a.$$

In this last inequality and in the ones that follow, the indices a, b, c range over  $\mathbb{N}$ . Canceling the terms that appear in both summations, we get

$$\sum_{b+c+a=i, b+c \le t, a \le t, c+a > t} (p^A)_b p_c(p^B)_a \le \sum_{b+c+a=i, b \le t, c+a \le t, b+c > t} (p^A)_b p_c(p^B)_a.$$

Exchanging the role of the variables a, b in the summation on on the right of this last inequality, we get

$$\sum_{b+c+a=i,b+c \le t, a \le t, c+a>t} (p^A)_b p_c(p^B)_a \le \sum_{b+c+a=i,b+c \le t, a \le t, c+a>t} (p^A)_a p_c(p^B)_b p_c(p^B)_b$$

Fix *c*. For the tuples (b, c, a) in the summations, we have  $b \le t - c < a$  and (2) implies  $(p^A)_b p_c(p^B)_a \le (p^A)_a p_c(p^B)_b$  as  $p_c \ge 0$ .  $\Box$ We complete the paper by giving a proof of Theorem 4.2. Given  $\lambda \in \mathcal{P}$ , let  $\lambda' \in \mathcal{P}$  be the *dual* partition given by

 $\lambda'(i) = |\{j : (j \ge 1) \text{ and } (\lambda(j) \ge i)\}|, \text{ for all positive integers } i.$ 

It is a standard result that for all  $\lambda, \mu \in \mathcal{P}, \lambda \supseteq \mu$  if and only if  $\lambda' \leq \mu'$ , see [2], 1.4.11, p.26.

**Proof of Theorem 4.2.** Let *X* and *r* be as in the statement of the theorem. Let  $p(x) = p_X(x) = \sum_{n=0}^r p_n x^n$ . By Lemma 4.3, we need to prove  $f(\lambda, p(x), t, x) \sqsubseteq f(\mu, p(x), t, x)$  implies  $\lambda \supseteq \mu$ . Since range $(X) = \{0, 1, \ldots, r\}$ , for any  $a, t \in \mathbb{N}$  and  $\lambda \in \mathcal{P}$ , p(x),  $p^a(x)$ ,  $p^a(x)|_t$  and  $f(\lambda, p(x), t, x)$  will all have their coefficients supported on initial segments of  $\mathbb{N}$ . Let  $1 \le t \le \lambda(1)$ . Note that

$$\deg\left(\prod(p^{\lambda(i)}(x)|_{rt})\right) = \sum_{\lambda(i) \ge t} rt + \sum_{\lambda(i) < t} r\lambda(i) = r\sum_{k=1}^{t} \lambda'(k).$$

Since  $\prod (p^{\lambda(i)}(x)|_{rt}) \subseteq \prod (p^{\mu(i)}(x)|_{rt})$ ,

$$\deg(\prod(p^{\lambda(i)}(x)|_{rt}) \le \deg(\prod(p^{\mu(i)}(x)|_{rt}))$$

or

$$r\sum_{k=1}^{t}\lambda'(k) \le r\sum_{k=1}^{t}\mu'(k).$$

Thus  $\lambda' \trianglelefteq \mu'$ , and hence,  $\lambda \trianglerighteq \mu$ .

#### 5 Notes

We record the following observations on the assumptions of Theorem 1.1. Let X be an  $\mathbb{N}$ -valued random variable. If range $(X) = \{0, 1\}$ , then X is automatically 2-non-negative and  $C(\lambda, \mu, X)$ , i.e. (1), always holds. If range(X) =

 $\{0, 1, 2\}$ , some evidence suggests that  $C(\lambda, \mu, X)$  holds without the restriction that X be 2-non-negative. If range $(X) = \{0, 1, \ldots, r\}$  for an integer r with  $r \ge 3$ , some additional constraint on X is necessary for  $C(\lambda, \mu, X)$  to hold. For example, if Y is uniformly distributed on  $\{0, 1, 3\}$ , then even though  $(4, 2) \ge (3, 3)$ , P(E((4, 2), Y, 12, 6)) = 10/729 > 9/729 = P(E((3, 3), Y, 12, 6)). For  $q \in [0, 1)$ , define the random variable X with range $(X) = \{0, 1, 2, 3\}$  by setting P(X = k) = q/3 if  $k \in \{0, 1, 3\}$  and P(X = 2) = 1 - q. For q sufficiently close to 1, P(E((4, 2), X, 12, 6)) > P(E((3, 3), X, 12, 6)).

If  $|\operatorname{range}(X)| \in \{1, 2\}$  then  $X \in TN_2$ . However some additional restriction on X is still needed  $C(\lambda, \mu, X)$  to hold. For example, if P(X = 0) = 1 then for every  $\lambda \in \mathcal{P}$ ,  $P(E(\lambda, X, j, t)) = 1$  if j = 0 and is 0 otherwise. Thus in this case  $C(\lambda, \mu, X)$  holds for *any* partitions  $\lambda$  and  $\mu$ . If P(X = 1) = 1, then  $C(\lambda, \mu, X)$  holds if and only if  $|\lambda| = |\mu|$  and  $\mu(1) \leq \lambda(1)$ .

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