TAYLOR'S MODULARITY CONJECTURE AND RELATED PROBLEMS FOR IDEMPOTENT VARIETIES

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ABSTRACT. We provide a partial result on Taylor's modularity conjecture, and several related problems. Namely, we show that the interpretability join of two idempotent varieties that are not congruence modular is not congruence modular either, and we prove an analogue for idempotent varieties with a cube term. Also, similar results are proved for linear varieties and the properties of congruence modularity, having a cube term, congruence n-permutability for a fixed n, and satisfying a non-trivial congruence identity.

1. INTRODUCTION

An *interpretation* ι of a variety $\mathcal{V}$ in a variety $\mathcal{W}$ is a mapping that maps basic operations of $\mathcal{V}$ to terms of $\mathcal{W}$ of the same arity such that for every algebra $\mathbf{A} \in \mathcal{W}$, the algebra $(A, (\iota(f)^{\mathbf{A}})_{f\in\sigma})$ (where σ is the signature of $\mathcal{V}$) is an algebra in $\mathcal{V}$. We say that a variety $\mathcal{V}$ is interpretable in a variety $\mathcal{W}$ if there exist an interpretation of $\mathcal{V}$ in $\mathcal{W}$. The *lattice of interpretability types of varieties* (see [Neu74, GT84]) is then constructed by quasi-ordering all varieties by interpretability, and factoring out varieties that are interpretable in each other. This gives a partially ordered class such that every set has a join and a meet. The lattice of interpretability types of varieties is a suitable object for expressing properties of Mal'cev conditions (for a formal definition see [Tay73]): The varieties that satisfy a given Mal'cev condition form a filter in this lattice, thus, e.g. implications among Mal'cev conditions translate into inclusions among the corresponding filters.

In this paper, we contribute to the line of research whose aim is to understand which of the important Mal'cev conditions are indecomposable in the following strong sense: if two sets of identities in disjoint languages together imply the Mal'cev condition, then one of the sets already do. An equivalent formulation using the interpretability lattice is especially simple: which of the important Mal'cev conditions determine a prime filter? Some of the Mal'cev conditions with this property have been described in the monograph by Garcia and Taylor [GT84], e.g. having a cyclic term of given prime arity. Garcia and Taylor conjectured that the filter of congruence permutable varieties and the filter of congruence modular varieties are prime.

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For congruence permutability, this was confirmed by Tschantz [Tsc96]. Unfortunately, this proof has never been published. The congruence modular case is still open:

Conjecture 1.1 (Taylor's modularity conjecture)**.** *The filter of congruence modular varieties is prime, that is, if $\mathcal{V}$ and $\mathcal{W}$ are two varieties such that $\mathcal{V} \vee \mathcal{W}$ is congruence modular, then either $\mathcal{V}$ or $\mathcal{W}$ is congruence modular.*

In [BS14], Bentz and Sequeira proved that this is true if $\mathcal{V}$ and $\mathcal{W}$ are idempotent varieties that can be defined by linear identities (such varieties are called linear idempotent varieties), and later in [BOP15], Barto, Pinsker, and the author generalized their result to linear varieties that do not need to be idempotent. In this paper we generalize Bentz and Sequeira's result in a different direction.

Theorem 1.2. *If $\mathcal{V}$, $\mathcal{W}$ are two idempotent varieties such that $\mathcal{V} \vee \mathcal{W}$ is congruence modular then either $\mathcal{V}$ or $\mathcal{W}$ is congruence modular.*

Several similar partial results on primeness of some Mal'cev filters have been obtained before. Bentz and Sequeira in [BS14] also proved for two linear idempotent varieties: if the join of the two varieties is congruence n-permutable for some n, then so is one of the two varieties; and similarly if their join satisfies a non-trivial congruence identities, then so does one of the two varieties. Stronger versions of these results also follow from the work of Valeriote and Willard [VW14], who proved that every idempotent variety that is not n-permutable for any n is interpretable in the variety of distributive lattices, and the work of Kearnes and Kiss [KK13], who proved that any idempotent variety which does not satisfy a non-trivial congruence identity is interpretable in the variety of semilattices. Recently, a similar result has been obtained by Kearnes and Szendrei [KS16] for the filter of varieties having a cube term.

Theorem 1.3. *Suppose that $n \geq 2$, and let $\mathcal{V}$ and $\mathcal{W}$ be two idempotent varieties such that $\mathcal{V} \vee \mathcal{W}$ has an n-cube term. Then so does either $\mathcal{V}$ or $\mathcal{W}$.*

We provide an alternative proof of this result using the fact (obtained in [KS16] and recently also by [MM17]) that idempotent varieties that do not have a cube term contain an algebra with a cube term blocker.

All of the filters mentioned so far share the following property: the varieties from their complements have a so-called strong coloring of their terms by a finite relational structure that depends on the particular filter. The precise definition is given in Section 3.2. The notion is a reformulation of colorings described in [BOP15], and a generalization of compatibility with projections introduced in [Seq01]. In the present manuscript, we describe these coloring structures. These characterizations by the means of colorings allow us to give analogous results for linear varieties. Moreover, we are also able to connect these results with their analogues for idempotent varieties (when those are available).

Theorem 1.4. *Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties such that each of them is either linear or idempotent.*

(i) *If $\mathcal{V} \vee \mathcal{W}$ is congruence modular, then so is either $\mathcal{V}$ or $\mathcal{W}$;*
(ii) *if $\mathcal{V} \vee \mathcal{W}$ is congruence k-permutable for some k, then so is either $\mathcal{V}$ or $\mathcal{W}$;*
(iii) *if $\mathcal{V} \vee \mathcal{W}$ satisfies a non-trivial congruence identity, then so does either $\mathcal{V}$ or $\mathcal{W}$;*

(iv) *for all $n \geq 2$: if $\mathcal{V} \in \mathcal{W}$ has an n-cube term, then so does either $\mathcal{V}$ or $\mathcal{W}$.*

Theorem 1.5. *Let $n \geq 2$. If $\mathcal{V}$ and $\mathcal{W}$ are two linear varieties such that $\mathcal{V} \vee \mathcal{W}$ is congruence n-permutable, then so is either $\mathcal{V}$ or $\mathcal{W}$.*

However, we are not able to answer the following. (Note that the case $n = 2$ has been resolved in [Tsc96] as well as Theorem 1.4(iv).)

Problem 1.6. *Given $n > 2$ and two idempotent varieties $\mathcal{V}$ and $\mathcal{W}$ such that $\mathcal{V} \vee \mathcal{W}$ is n-permutable. Is it always true that either $\mathcal{V}$ or $\mathcal{W}$ is n-permutable?*

2. Varieties, clones, and relational structures

Before we get to prove the results of this paper, we would like to recall another constructions of a class-size lattices that is equivalent to the lattice of interpretability types of varieties. That is the lattice of (homomorphism classes of) clones ordered by an existence of a clone homomorphism: We start with a preorder on clones defined by the existence of a clone homomorphism between the two clones, and follow by factoring out the homomorphically equivalent clones. The equivalence of the two construction is implicitly given by Birkhoff's theorem. There is also a third construction that lacks being completely equivalent to the two: taking relational structures and pp-interpretability between them. Those are connected to clones by the Galois correspondence between relations and function clones on a fixed (finite) set. Even though this construction of the lattice of interpretability types lacks to be equivalent in general, we will often describe some function clones as clones of polymorphisms of a relational structure.

2.1. **Function clones.** A function clone (or just a clone) is a set of operations $\mathscr{A}$ on a fixed set A that contains projections and is closed under composition. We will always use the same letter in italic font to denote the underlying set of the clone. By a *clone homomorphism* from a function clone $\mathscr{A}$ to a function clone $\mathscr{B}$ we mean a mapping $\xi\colon \mathscr{A} \to \mathscr{B}$ that preserves composition and projections.

The correspondence between clones and varieties is given by the following: Any clone $\mathscr{A}$ can be naturally understood as an algebra $\mathbf{A} = (A, (f)_{f \in \mathscr{A}})$ having the signature $\mathscr{A}$; the corresponding variety is then the variety generated by $\mathbf{A}$. The same variety can be also understood as the variety of all actions of $\mathscr{A}$ on sets (each action of $\mathscr{A}$ gives an algebra in $\mathsf{HSP}\,\mathbf{A}$). We will call this variety *the variety of actions of $\mathscr{A}$*. For the other way, to obtain a clone corresponding to a variety $\mathcal{V}$ we take the clone of term functions of its countably-generated free algebra $\mathscr{F}$, or in fact a clone of term functions of any generator of $\mathcal{V}$ (any two such clones are isomorphic). When we refer to a *clone of a variety* we mean the function clone $\mathscr{F}$ above.

2.2. **Relational structures and pp-interpretations.** We say that a relation R is *pp-definable* in the structure $\mathbb{B}$ if there is a primitive positive formula (using only conjunction and existential quantifiers) ψ such that $(b_1, \dots, b_k) \in R$ if and only if $\mathbb{B} \models \psi(b_1, \dots, b_n)$, and we say that a structure $\mathbb{A}$ is *pp-interpretable* in $\mathbb{B}$ if there is a pp-definable relation R and a surjective mapping $f\colon R \to A$ such that the kernel of f and the relations $f^{-1}(S^{\mathbb{A}})$, where S is a relation S of $\mathbb{A}$, are pp-definable in $\mathbb{B}$.

The following relation between pp-interpretations and clone homomorphisms for finite underlying sets is a consequence of the Galois correspondence between function clones and pp-closed systems of relations (see e.g. [Gei68]) and Birkhoff's HSP theorem.

Theorem 2.1. *Let $\mathbb{A}$, $\mathbb{B}$ be finite relational structures and $\mathscr{A}$, $\mathscr{B}$, respective, their polymorphism clones. Then the following are equivalent.*

(i) $\mathbb{B}$ *is pp-interpretable in* $\mathbb{A}$.
(ii) *There exists a clone homomorphism from* $\mathscr{A}$ *into* $\mathscr{B}$. □

For infinite structures, there are a few issues that can be resolved by restricting to countable ω-categorical structures and considering uniformly continuous clone homomorphisms, or relaxing a definition of relational structure to allow infinitary relations and allow infinite 'pp-definitions' (see [BP15] and [Rom77]). Nevertheless, the implication (i) $\to$ (ii) is valid in general, i.e., if the structures $\mathbb{A}$ and $\mathbb{B}$ are infinite and not necessarily ω-categorical.

2.3. **Notation.** We use letters in italic for underlying sets of clones, algebras, and relational structures that are denoted by the same letters. That means the symbol A is used to denote an underlying set of: an algebra $\mathbf{A}$, a clone $\mathscr{A}$, and a relational structure $\mathbb{A}$. We will keep this consistent, i.e., if we denote two structures (algebraic and relational) by the same letter, they have the same underlying set. Moreover, $\mathscr{A}$ denotes the clone of polymorphisms of the relational structure $\mathbb{A}$. We will also keep some consistence between algebras, relational structures, and clones: all operations of an algebra $\mathbf{A}$ will be compatible with relations of $\mathbb{A}$ and belong to the clone $\mathscr{A}$.

We also say that an algebra $\mathbf{A}$ is *compatible* with a relational structure $\mathbb{B}$, if the above is the case, that is, they share a universe ($A = B$) and each operation of $\mathbf{A}$ is compatible with every relation of $\mathbb{B}$.

3. Overview of the method

Each of the Mal'cev filters given by one of the conditions mentioned in Theorems 1.4 and 1.5 can be described as the class of varieties that do not contain an algebra that has some compatible relations with a special property (e.g. congruence modular are those varieties that do not contain an algebra having three congruences that do not satisfy the modular law, finitely generated varieties with cube terms are those that do not contain an algebra with 'too many subpowers', etc.). In order to prove that these filters are prime, we have to prove that for any two varieties $\mathcal{V}$, $\mathcal{W}$ that contain such 'ugly' algebras, also $\mathcal{V} \vee \mathcal{W}$ contains such an algebra. More precisely, Taylor's modularity conjecture in fact states that for any two varieties $\mathcal{V}$, $\mathcal{W}$ that are not congruence modular, there exists an algebra $\mathbf{A}$ in $\mathcal{V} \vee \mathcal{W}$ that has congruences α, β, and γ that do not satisfy the modularity law. Such an algebra has two natural reducts $\mathbf{A}_1 \in \mathcal{V}$, $\mathbf{A}_2 \in \mathcal{W}$ which are obtained by taking only those basic operations of $\mathbf{A}$ that belong to the signature of the respective variety. Both these algebras share a universe and the three congruences α, β, and γ. Therefore, Taylor's conjecture states that for any two congruence non-modular varieties, there exist a set A and equivalence relations α, β, γ on A such that both varieties contain an algebra with the universe A and congruences α, β, and γ. In other words (see Lemma 3.1), both varieties are interpretable in the variety of actions of the polymorphism clone $\mathscr{A}$ of the relational structure $(A; \alpha, \beta, \gamma)$. Although it would be convenient, we cannot find a single relational structure $\mathbb{B}$ such that a variety is not congruence modular if and only if it is interpretable in the variety of actions of $\mathscr{B}$. This is due to the fact, that congruence non-modular varieties can omit algebras of size smaller then any fixed cardinal. Instead, we will find a chain of clones $\mathscr{P}_\kappa$ indexed by cardinals, that are polymorphism clones of relational structures of increasing sizes, such that every non-modular idempotent variety is interpretable in the variety of actions of $\mathscr{P}_\kappa$ for every big enough cardinal κ. The same method is also applied for the other filters in the interpretability lattice with the only difference being use of different relational structures.

Contrary to the general case, in the case that a variety $\mathcal{V}$ is linear and not congruence modular, we are able to prove that it is interpretable in the variety of actions of the clone $\mathscr{P}_0$, that is defined as the clone of polymorphisms of $\mathbb{P}_0 = (\{0,1,2,3\}, \alpha, \beta, \gamma)$ where α, β, and γ are equivalences defined by partitions $01|23$, $03|12$, and $0|12|3$, respectively. This is due to the fact that non-modular varieties can be described as those varieties whose terms are colorable by $\mathbb{P}_0$ (see Proposition 4.5). This turns out to be, in the case of linear varieties, equivalent to any of the conditions in Lemma 3.1 for the relational structure $\mathbb{P}_0$ (see Lemma 3.5). Moreover, we will also show that there is a clone homomorphism from $\mathscr{P}_0$ to $\mathscr{P}_\kappa$ for any infinite cardinal κ, connecting the results for linear varieties and idempotent varieties.

The rest of this section is dedicated to describing several general results to be used in the following sections.

3.1. **Varieties of actions of polymorphism clones.** Our method is based on the following lemma that describes the class of varieties that are interpretable in the variety of actions of the polymorphism clone of a relational structure. Here, we denote the signature of a variety $\mathcal{V}$ by $\sigma(\mathcal{V})$.

Lemma 3.1. *Let $\mathcal{V}$ be a variety, $\mathscr{F}$ the clone of its countably generated free algebra, $\mathbb{B}$ be a relational structure. The following are equivalent:*

1. *There is a clone homomorphism from $\mathscr{F}$ to $\mathscr{B}$;*
2. *$\mathcal{V}$ is interpretable in the variety of actions of $\mathscr{B}$;*
3. *$\mathcal{V}$ contains an algebra compatible with $\mathbb{B}$.*

Proof. (1) $\to$ (2): Let $\mathbf{F}$ denote the countable generated free algebra, and let $\xi\colon \mathscr{F} \to \mathscr{B}$ be a clone homomorphism. We will use it to define an interpretation ι from $\mathcal{V}$ to the variety of action of $\mathscr{B}$. Note that the signature of the variety of actions of $\mathscr{B}$ is the set $\mathscr{B}$. For a k-ary symbol f in the signature of $\mathcal{V}$, we define $\iota(f) = \xi(f^{\mathbf{F}})$. We get that $(B; \iota(f)_{f\in\sigma(\mathcal{V})}) \in \mathcal{V}$ since ξ preserves identities, and consequently, since $(B; (f)_{f\in\mathscr{B}})$ generates the variety of actions of $\mathscr{B}$, ι is an interpretation.

(2) $\to$ (3): Suppose that ι is an interpretation of $\mathcal{V}$ in the variety $\mathcal{W}$ of actions of $\mathscr{B}$. Note that, since $\mathscr{B}$ is closed under compositions, we can without loss of generality suppose that ι maps basic operations of $\mathcal{V}$ to basic operations of $\mathcal{W}$ (i.e., the elements of $\mathscr{B}$). From the definition of an interpretation, we have that for any action of $\mathscr{B}$ on a set C, $(C, (\iota(f))_{f\in\sigma(\mathcal{V})})$ is an algebra in $\mathcal{V}$ (here $\iota(f)$ is understood as the action of the element $\iota(f) \in \mathscr{B}$). Considering in particular the natural action of $\mathscr{B}$ on B, we claim that $\mathbf{A} = (B, (\iota(f))_{f\in\sigma(\mathcal{V})})$ has the required properties. Indeed, it has the right universe, and moreover for any $f \in \sigma(\mathcal{V})$, the function $\iota(f)$ is a polymorphism of $\mathbb{B}$.

(3) $\to$ (1): Let $\mathbf{A}$ be the algebra satisfying the condition (3), and let $\mathbf{F}$ be the countably generated free algebra in $\mathcal{V}$. Define $\xi\colon \mathscr{F} \to \mathscr{B}$ by putting $\xi(f^{\mathbf{F}}) = f^{\mathbf{A}}$

for every term f of $\mathcal{V}$. This mapping is well-defined since $\mathbf{A}$ satisfies all identities that are satisfied in $\mathbf{F}$ and every term operation of $\mathbf{A}$ is a polymorphism of $\mathbb{B}$. It is also a clone homomorphism since it preserves projections and composition. $\square$

Also from the above, we get the following property of the interpretability join of two varieties.

Lemma 3.2. *Let $\mathbb{B}$ be a relational structure, and let $\mathcal{V}$ and $\mathcal{W}$ be two varieties such that both contain an algebra compatible with $\mathbb{B}$. Then also the interpretability join $\mathcal{V} \vee \mathcal{W}$ contains an algebra compatible with $\mathbb{B}$.*

Proof. From the previous lemma, we get that both $\mathcal{V}$ and $\mathcal{W}$ are interpretable in the variety of actions of $\mathscr{B}$, therefore also $\mathcal{V} \vee \mathcal{W}$ is by the definition of interpretability join. Again by the previous lemma, this implies that $\mathcal{V} \vee \mathcal{W}$ contain an algebra compatible with $\mathbb{B}$.

Alternatively, we can construct such an algebra directly: Suppose that $\mathbf{C} \in \mathcal{V}$ and $\mathbf{D} \in \mathcal{W}$ are compatible with $\mathbb{B}$ (in particular $C = D = B$). The algebra $\mathbf{B}$ in $\mathcal{V} \vee \mathcal{W}$ compatible with $\mathbb{B}$ is obtained by putting the structures of these two algebras on top of each other, i.e., as $\mathbf{B} = (B; (f^{\mathbf{C}})_{f \in \sigma(\mathcal{V})}, (g^{\mathbf{D}})_{g \in \sigma(\mathcal{W})})$. $\square$

3.2. **Colorings of terms by relational structures.** Usually we are unable to find a single relational structure $\mathbb{B}$ such that variety falls in the complement of a filter if and only if it is interpretable in the variety of actions of $\mathscr{B}$. Nevertheless, we can relax from taking clone homomorphism, corresponding to interpretation of varieties, to h1 clone homomorphisms. An h1 clone homomorphism is a mapping between two clones that preserves identities of height 1 in a similar way as clone homomorphism preserves all identities [BOP15, Definition 5.1]. Existence of an h1 clone homomorphism from the clone of a variety $\mathcal{V}$ to the polymorphism clone of a relational structure $\mathbb{B}$ is described by an existence of a coloring described below. As noted in [BOP15], h1 clone homomorphisms encapsulate not only the usual algebraical (primitive positive) construction, but also homomorphic equivalence of relational structures. This said, coloring is defined as a homomorphism from a certain 'freely generated' relational structure given by $\mathcal{V}$ and $\mathbb{B}$ to the relational structure $\mathbb{B}$. There will always be a natural homomorphism from $\mathbb{B}$ to this freely generated structure.

Definition 3.3. Given a variety $\mathcal{V}$ and $\mathbb{B}$ a relational structure. We define a *free structure generated by* $\mathbb{B}$ to be a relational structure $\mathbb{F}$ of the same signature as $\mathbb{B}$ whose underlying set is the universe of the free B-generated algebra $\mathbf{F}$, and a relation $R^{\mathbb{F}}$ is the smallest compatible relation on F containing $R^{\mathbb{B}}$.

As for a free algebra, the elements of $\mathbb{F}$ are represented by terms of $\mathcal{V}$ over the set B as their set of variables. Therefore, it is sensible to denote the elements of the copy of B in F as variables x_b, $b \in B$. Another way is to define $\mathbf{F}$ as being generated by the set $\{x_b : b \in B\}$ rather then B itself, and $R^{\mathbb{F}}$ being the relations generated by $\{(x_{b_1}, \dots, x_{b_k}) : (b_1, \dots, b_k) \in R^{\mathbb{B}}\}$. It is clear that both relational structures are isomorphic, moreover they are also isomorphic as (mixed) structures with both relations $R^{\mathbb{F}}$ and operations $f^{\mathbf{F}}$.

Definition 3.4. A *coloring of terms* of a variety $\mathcal{V}$ by $\mathbb{B}$ (or $\mathbb{B}$-*coloring of terms*) is any homomorphism from the free structure generated by $\mathbb{B}$ to $\mathbb{B}$. A coloring is *strong* if its restriction to B is an indentity mapping (maps x_b to b).

The following is a consequence of the theory developed in [BOP15]; it follows from Propositions 7.4 and 7.5 of the mentioned paper. Nevertheless, we present a variant of the proof of equivalence of (3) and (5) for completeness.

Lemma 3.5. *Let $\mathcal{V}$ be a linear variety, $\mathscr{F}$ its clone, and $\mathbb{B}$ a relational structure. The following are equivalent:*

1. *There is a clone homomorphism from $\mathscr{F}$ to $\mathscr{B}$;*
2. *there is a strong h1 clone homomorphism from $\mathscr{F}$ to $\mathscr{B}$;*
3. *the terms of $\mathcal{V}$ are strongly $\mathbb{B}$-colorable;*
4. *$\mathcal{V}$ is interpretable in the variety of actions of $\mathscr{B}$;*
5. *$\mathcal{V}$ contains an algebra compatible with $\mathbb{B}$.*

Proof. The equivalence of (1), (4), and (5) follows from Lemma 3.1. The implication (1) → (2) is trivial, and its converse is given by [BOP15, Proposition 7.5]. To finish the proof, we show that (3) and (5) are equivalent. We denote $\mathbf{F}_B$, the free algebra in $\mathcal{V}$ generated by B, and $\mathbb{F}_B$, the free structure generated by $\mathbb{B}$.

(3) → (5): A strong coloring is a mapping $c\colon F_B \to B$. We define a structure of a $\mathcal{V}$-algebra on the set $\mathbb{B}$ as a retraction (see [BOP15, Definition 4.1]) of $\mathbf{F}_B$ by i and c where $i\colon B \to F_B$ is an inclusion ($i(b) = x_b$, resp.), i.e.,

$$f^{\mathbf{B}}(b_1, \dots, b_n) = c(f^{\mathbf{F}_B}(x_{b_1}, \dots, x_{b_n})).$$

One can observe that operations defined this way satisfy all linear identities satisfied by $\mathbf{F}_B$ (see also [BOP15, Corollary 5.4]), therefore $\mathbf{B} \in \mathcal{V}$. To prove that $\mathbf{B}$ is compatible with $\mathbb{B}$, we consider a relation $R^{\mathbb{B}}$ of $\mathbb{B}$ and $\mathbf{b}_1, \dots, \mathbf{b}_n \in R^{\mathbb{B}}$. Then by the definition of $\mathbb{F}_B$, $\mathbf{b}_1, \dots, \mathbf{b}_n \in R^{\mathbb{F}_B}$, therefore $f^{\mathbf{F}_B}(\mathbf{b}_1, \dots, \mathbf{b}_n) \in R^{\mathbb{F}_B}$, and $c(f^{\mathbf{F}_B}(\mathbf{b}_1, \dots, \mathbf{b}_n)) \in R^{\mathbb{B}}$ from the definition of coloring. This shows that $f^{\mathbf{B}}$ defined as above is a polymorphism of $\mathbb{B}$, and hence $\mathbf{B}$ is compatible with $\mathbb{B}$.

(5) → (3): Let $\mathbf{B}$ be an algebra in $\mathcal{V}$ compatible with $\mathbb{B}$, and let $c\colon \mathbf{F}_B \to \mathbf{B}$ be the natural homomorphism from $\mathbf{F}_B$ that extends the identity mapping (the mapping $x_b \mapsto b$, resp.). We claim that this mapping is a coloring. Indeed, if $\mathbf{f} \in R^{\mathbb{F}_B}$ then there is a term f and $\mathbf{b}_1, \dots, \mathbf{b}_n \in R^{\mathbb{B}}$ such that $\mathbf{f} = f^{\mathbf{F}_B}(\mathbf{b}_1, \dots, \mathbf{b}_n)$, therefore

$$c(\mathbf{f}) = c(f^{\mathbf{F}_B}(\mathbf{b}_1, \dots, \mathbf{b}_n)) = f^{\mathbf{B}}(\mathbf{b}_1, \dots, \mathbf{b}_n) \in R^{\mathbb{B}},$$

since $\mathbb{B}$ is compatible with $\mathbf{B}$. The coloring c is strong by definition. □

3.3. **Coloring by transitive relations.** Some of the relational structures we use for coloring have binary transitive relations, or even equivalence relations. In these cases, we can refine the free structure $\mathbb{F}$ generated by a relational structure $\mathbb{B}$ in the following way.

Lemma 3.6. *Let $\mathbb{B}$ be a relational structure, $\mathcal{V}$ a variety, $\mathbb{F}$ the free structure in $\mathcal{V}$ generated by $\mathbb{B}$, and let $\mathbb{F}'$ be a structure obtained from $\mathbb{F}$ by replacing every relation $R^{\mathbb{F}}$, for which $R^{\mathbb{B}}$ is transitive, by its transitive closure. Then a mapping $c\colon F \to B$ is a coloring if and only if it is a homomorphism $c\colon \mathbb{F}' \to \mathbb{B}$.*

Proof. This directly follows from the corresponding result about relational structures in general. In detail, given that $\mathbb{A}$ and $\mathbb{B}$ are relational structures sharing a signature such that $R^{\mathbb{B}}$ is transitive, then every homomorphism from $\mathbb{A}$ to $\mathbb{B}$ maps transitive closure of $R^{\mathbb{A}}$ to $R^{\mathbb{B}}$. Conversely, if $\mathbb{A}'$ is obtained from $\mathbb{A}$ as described in the proposition, then any homomorphism from $\mathbb{A}'$ to $\mathbb{B}$ is automatically a homomorphism from $\mathbb{A}$ to $\mathbb{B}$. □

Following the notation of the lemma, note that, if the relation $R^{\mathbb{B}}$ is symmetric binary relation, then also $R^{\mathbb{F}}$ is. This follows from the fact that if $(f,g) \in R^{\mathbb{F}}$ then there is a term t of $\mathcal{V}$ and tuples $(a_1,b_1),\dots,(a_n,b_n) \in R^{\mathbb{B}}$ such that $t(a_1,\dots,a_n) = f$ and $t(b_1,\dots,b_n) = g$. Applying the same term t to pairs $(b_1,a_1),\dots,(b_n,a_n) \in R^{\mathbb{B}}$, we get that $(g,f) \in R^{\mathbb{F}}$. From this observation and the lemma above, we get that if $R^{\mathbb{B}}$ is an equivalence relation, we can take $R^{\mathbb{F}}$ to be the congruence generated by $R^{\mathbb{B}}$. This case is further described in the following lemma.

Lemma 3.7. *Let $\mathbb{B}$ be a finite relational structure with $B = \{1,\dots,n\}$, $\alpha^{\mathbb{B}}$ an equivalence relation, $\mathbf{F}$ the free B-generated algebra, and $\mathbb{F}$ the free structure in $\mathcal{V}$ generated by $\mathbb{B}$. Then the the following three definitions give the same relation:*

1. *the transitive closure of $\alpha^{\mathbb{F}}$,*
2. *the congruence of $\mathbf{F}$ generated by $\alpha^{\mathbb{B}}$,*
3. *the set of all pairs $(f^{\mathbf{F}}(1,\dots,n), g^{\mathbf{F}}(1,\dots,n))$ where f,g are terms of $\mathcal{V}$ such that $f(x_1,\dots,x_n) \approx g(x_1,\dots,x_n)$ is satisfied in $\mathcal{V}$ whenever $x_i \approx x_j$ for all $(i,j) \in \alpha^{\mathbb{B}}$.*

Proof. Let us denote the relation defined in the item (i) of the above list by α_i. Clearly, $\alpha_1 \subseteq \alpha_2$ since α_2 is transitive and contains $\alpha^{\mathbb{F}}$. We will prove $\alpha_2 \subseteq \alpha_3 \subseteq \alpha_1$.

$\alpha_2 \subseteq \alpha_3$: Clearly, α_3 is symmetric, transitive, and reflexive. It is straightforward to check that it is compatible with operations of $\mathbf{F}$. Finally, we claim that $\alpha^{\mathbb{B}} \subseteq \alpha_3$. Indeed, pick $(i,j) \in \alpha^{\mathbb{B}}$ and choose $f(x_1,\dots,x_n) = x_i$ and $g(x_1,\dots,x_n) = x_j$. Clearly, f and g satisfies the condition in (3), therefore $(i,j) = (f^{\mathbf{F}}(1,\dots,n), g^{\mathbf{F}}(1,\dots,n)) \in \alpha_3$. We proved that α_3 is a congruence of $\mathbf{F}$ containing $\alpha^{\mathbb{B}}$, which concludes $\alpha_2 \subseteq \alpha_3$.

$\alpha_3 \subseteq \alpha_1$: Let $(f^{\mathbf{F}}(1,\dots,n), g^{\mathbf{F}}(1,\dots,n))$ be a typical pair in α_3. Fix a representative for each $\alpha^{\mathbb{B}}$ class, and let i_k denote the representative of the class containing k. From the definition of $\alpha^{\mathbb{F}}$, we obtain $(f^{\mathbf{F}}(1,\dots,n), f^{\mathbf{F}}(i_1,\dots,i_n)) \in \alpha_{\mathbb{F}}$ and $(g^{\mathbf{F}}(i_1,\dots,i_n), g^{\mathbf{F}}(1,\dots,n)) \in \alpha_{\mathbb{F}}$. Since $f(x_{i_1},\dots,x_{i_n}) \approx g(x_{i_1},\dots,x_{i_n})$ is true in $\mathcal{V}$, we get that $f^{\mathbf{F}}(i_1,\dots,i_n) = g^{\mathbf{F}}(i_1,\dots,i_n)$, and consequently

$$\big(f^{\mathbf{F}}(1,\dots,n), g^{\mathbf{F}}(1,\dots,n)\big) \in \alpha^{\mathbb{F}} \circ \alpha^{\mathbb{F}} \subseteq \alpha_1$$

which concludes the proof of $\alpha_3 \subseteq \alpha_1$. □

As a consequence of the previous lemma, we also obtain that coloring by a relational structure $\mathbb{B}$ whose all relations are equivalence relation is equivalent to having terms compatible with projections as defined in [Seq06] (compare item (3) with Definition 5.2 of [Seq06]). The corresponding set $(\alpha_1,\dots,\alpha_k)$ of projections can be obtained from the relations of $\mathbb{B}$.

3.4. **Tarski's construction.** In this section we describe a transfinite construction that appeared in [Bur72] and is attributed to Tarski. This construction is an algebraical version of Löwenheim-Skolem theorem, and we will use it to show that the chains of clones $\mathscr{P}_\kappa$, $\mathscr{B}_\kappa$, and $\mathscr{C}_\kappa$ defined in the following sections are indeed chains as ordered by an existence of a clone homomorphism.

Given an algebra $\mathbf{A}$, we define a sequence $\mathbf{A}_\lambda$ indexed by ordinals by the following transfinite construction: We start with $\mathbf{A}_0 = \mathbf{A}$. For an ordinal successor $\lambda+1$ define $\mathbf{A}_{\lambda+1}$ as an algebra isomorphic to $\mathbf{A}_\lambda^2$ while identifying the diagonal with $\mathbf{A}_\lambda$, i.e., take $A_{\lambda+1} \subseteq A_\lambda$ with a bijection $f_\lambda\colon A_\lambda^2 \to A_{\lambda+1}$ such that $f_\lambda(a,a) = a$, and define the structure of $\mathbf{A}_{\lambda+1}$ in such a way that f_λ is an isomorphism. For a limit ordinal λ, we set $\mathbf{A}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{A}_\alpha$. This construction produces algebras in the variety generated by $\mathbf{A}$ of all infinite cardinalities larger than $|A|$. Indeed, one can observe that $|A_\lambda| = \max\{|\lambda|, |A|\}$ for all infinite λ. We can also follow this construction to get from a fixed proper subset $U \subset A$, $U \neq \emptyset$ a set $U_\lambda \subseteq A_\lambda$ such that both its cardinality and cardinality of its complement is $|\lambda|$ (given that $|\lambda| \geq |A|$): We put $U_0 = U$; for ordinal successor $\lambda+1$, we take $U_{\lambda+1} = f_\lambda(A_\lambda \times U_\lambda)$ (observe that $U_{\lambda+1} \cap A_\lambda = U_\lambda$); and for limit λ, we take $U_\lambda = \bigcup_{\alpha<\lambda} U_\alpha$.

4. Taylor's conjecture

In this section, we prove Theorem 1.2 about Taylor's conjecture on congruence modular varieties. Congruence modular varieties have been thoroughly investigated, and they have many nice properties, nevertheless we need only the definition and the following Mal'cev characterization of these varieties by A. Day [Day69]:

Theorem 4.1. *The following are equivalent for any variety $\mathcal{V}$.*

(1) *Every algebra in $\mathcal{V}$ has modular congruence lattice;*
(2) *there exists n, and quaternary terms $d_0, \dots, d_n$ such that the following identities are satisfied in $\mathcal{V}$:*

$$d_0(x,y,z,w) \approx x \text{ and } d_n(x,y,z,w) \approx w,$$
$$d_i(x,y,y,z) \approx d_{i+1}(x,y,y,z) \text{ for even } i,$$
$$d_i(x,x,y,y) \approx d_{i+1}(x,x,y,y) \text{ and } d_i(x,y,y,x) \approx d_{i+1}(x,y,y,x) \text{ for odd } i.$$

A sequence of terms satisfying the identities in (2) is referred to as *Day terms*; we will also say that some functions (or polymorphisms) are *Day functions* if they satisfy these identities.

4.1. **Pentagons.** For the description of a cofinal chain of clones in the complement of the filter of clones containing Day terms, we will use relational structures of the form $\mathbb{P} = (P; \alpha, \beta, \gamma)$ where α, β, and γ are equivalence relations on P that do not satisfy the modularity law. A very similar structures have been used in [BCV13] to prove that the problem of comparison of pp-formulae is coNP-hard for algebras that do not generate congruence modular varieties; it was also used in [McG09]. The following definition of a pentagon is almost identical to the one in [BCV13] (we focus on those pentagon which are by themselves 'interesting'). For our purpose we need pentagons of even more special shape; we call them special and very special pentagons.

Definition 4.2. A *pentagon* is a relational structure $\mathbb{P}$ in the signature $\{\alpha, \beta, \gamma\}$ with three binary relations that are all equivalence relations on P satisfying:

- $\alpha^{\mathbb{P}} \wedge \beta^{\mathbb{P}} = 0_P$,
- $\alpha^{\mathbb{P}} \circ \beta^{\mathbb{P}} = 1_P$,
- $\gamma^{\mathbb{P}} \vee \beta^{\mathbb{P}} = 1_P$, and
- $\gamma^{\mathbb{P}} < \alpha^{\mathbb{P}}$.

The first two items in this definition ensure that every pentagon naturally factors as a direct product $P = A \times B$ in such a way that α and β are kernels of projections on the first and the second coordinate, respectively. In this setting, γ is an equivalence relation that relates some pairs of the form $((a,b),(a,c))$. For an equivalence γ on a product $A \times B$, and $a \in A$, we define γ^a to be the following equivalence on B:

$$\gamma^a := \{(b,c) : ((a,b),(a,c)) \in \gamma\}.$$

A pentagon on the set $A \times B$ (with α and β being the two kernels of projections) is said to be *special* if γ^a for $a \in A$ gives exactly two distinct congruences with one of them being the full congruence on B, i.e., there exists $\eta < 1_B$ such that

$$\{\gamma^a : a \in A\} = \{1_B, \eta\}.$$

Such pentagon is *very special* if the above is true for $\eta = 0_B$. When defining a special, or a very special pentagon $\mathbb{P}$ on a product $A \times B$, we will usually speak only about the relation $\gamma^{\mathbb{P}}$ since the other two relations are implicitly given as the kernels of projections.

In this section we will use pentagons $\mathbb{P}_0$ and $\mathbb{P}_\kappa$ where κ is an infinite cardinal. All of these pentagons are either themselves very special, or isomorphic to a very special pentagon.

Definition 4.3. We define $\mathbb{P}_0$ as the smallest possible pentagon. In detail: $P_0 = \{0, 1, 2, 3\}$ and $\alpha^{\mathbb{P}_0}$, $\beta^{\mathbb{P}_0}$, and $\gamma^{\mathbb{P}_0}$ are equivalences defined by partitions $12|03$, $01|23$, and $12|0|3$, respectively.

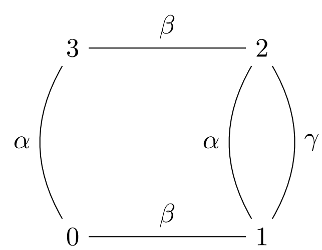

FIGURE 1. The pentagon $\mathbb{P}_0$

The pentagon $\mathbb{P}_0$ is isomorphic to a very special pentagon on the set $P = \{0, 1\} \times \{0, 1\}$ with $\gamma^0 = 0_{\{0,1\}}$ and $\gamma^1 = 1_{\{0,1\}}$; an isomorphism is given by the map: $0 \mapsto (0, 0)$, $1 \mapsto (1, 0)$, $2 \mapsto (1, 1)$, and $3 \mapsto (0, 1)$.

Definition 4.4. For an infinite cardinal κ, we define a pentagon $\mathbb{P}_\kappa$: Fix $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$, and define $P_\kappa = \kappa \times \kappa$, $\alpha^{\mathbb{P}_\kappa}$ and $\beta^{\mathbb{P}_\kappa}$ to be the kernels of the first and the second projection, respectively, and

$$\gamma^{\mathbb{P}_\kappa} = \{((a, b), (a, c)) : a, b, c \in \kappa \text{ such that } a \in U_\kappa \text{ or } b = c\}.$$

4.2. **Coloring and linear varieties.** Using Day terms, we obtain a characterization of congruence non-modular varieties by the means of coloring by the pentagon $\mathbb{P}_0$ defined above.

Proposition 4.5. *A variety $\mathcal{V}$ does not have Day terms if and only if it has strongly $\mathbb{P}_0$-colorable terms.*

Proof. Let $\mathbb{F}$ denote the structure obtained from the free structure in $\mathcal{V}$ generated by $\mathbb{P}_0$ by replacing every its relation by its transitive closure, and note that all three relations of $\mathbb{F}$ are equivalences (see Lemma 3.7), and also that $c\colon F \to P_0$ is a coloring if and only if it is a homomorphism from $\mathbb{F}$ to $\mathbb{P}_0$ (see Lemma 3.6). To simplify the notation we identify a 4-ary term d with the element $d(x_0, x_1, x_2, x_3) \in F$.

First, we prove the implication from left to right. For a contradiction, suppose that $\mathcal{V}$ has Day terms $d_0, \dots, d_n$, and that there is a strong coloring $c\colon \mathbb{F} \to$

$\mathbb{P}_0$. Any strong coloring c maps x_0, and therefore also d_0, to 0, and it satisfies $(c(d_i), c(d_{i+1})) \in \gamma$ for even i, and $(c(d_i), c(d_{i+1})) \in \alpha \wedge \beta$ for odd i; this follows from Day identities that can be reformulated as: $(d_i, d_{i+1}) \in \gamma^{\mathbb{F}}$ for even i and $(d_i, d_{i+1}) \in \alpha^{\mathbb{F}} \wedge \beta^{\mathbb{F}}$ for odd i. Using these observations, we obtain by induction on i that c maps d_i to 0 for all i, in particular $c(x_3) = c(d_n) = 0$. This gives us a contradiction with $c(x_3) = 3$.

For the converse, we want to prove that Day terms are the only obstruction for having strongly $\mathbb{P}_0$-colorable terms. We will do that by defining a valid strong coloring of terms of any $\mathcal{V}$ that does not have Day terms. Observe that even in this case we can repeat the argumentation from the above paragraph to get that $c(f) = 0$ for any f that is connected to x_0 by a Day-like chain, i.e., terms $d_0, \dots, d_n$ satisfying Day identities where $d_n \approx x_3$ is replaced by $d_n \approx f$. Therefore, we put $c(f) = 0$ if this is the case. Furthermore, for any f with $(f, g) \in \beta^{\mathbb{F}}$ for some g having the property above, we have $c(f) \in \{0, 1\}$. We put $c(f) = 1$ if this is the case and we have not defined the value $c(f)$ yet. Similarly, if $(f, g) \in \alpha^{\mathbb{F}}$ for some g with $c(g) = 0$ and $c(g)$ undefined, we put $c(f) = 3$ following the rule that $c(f) \in \{0, 3\}$. Note that in this step we have defined $c(x_3) = 3$ since $(x_0, x_3) \in \alpha^{\mathbb{F}}$. Finally, for the remaining $f \in F$ we put $c(f) = 2$. This definition is summarized as follows: $c(f) = 0$ if $(f, x_0) \in (\alpha^{\mathbb{F}} \wedge \beta^{\mathbb{F}}) \vee \gamma^{\mathbb{F}}$, otherwise:

$$c(f) = \begin{cases} 1 & \text{if } (f,g) \in \beta^{\mathbb{F}} \text{ for some } g \text{ with } c(g) = 0, \\ 3 & \text{if } (f,g) \in \alpha^{\mathbb{F}} \text{ for some } g \text{ with } c(g) = 0, \text{ and} \\ 2 & \text{in all remaining cases.} \end{cases}$$

Note that if f satisfies both the first and the second row of the above, say there are g_1, g_2 with $c(g_i) = 0$, $(f, g_1) \in \beta^{\mathbb{F}}$, and $(f, g_2) \in \alpha^{\mathbb{F}}$, then also $(f, g_1) \in \alpha^{\mathbb{F}}$ since $(\alpha^{\mathbb{F}} \wedge \beta^{\mathbb{F}}) \vee \gamma^{\mathbb{F}} \leq \alpha^{\mathbb{F}}$, and $c(f)$ is assigned value 0 in the first step. Therefore, c is well-defined.

We claim that c defined this way is a homomorphism from $\mathbb{F}$ to $\mathbb{P}_0$, and therefore a coloring. To prove that we need to show that c preserves all α, β, and γ. First, let $(f, g) \in \alpha^{\mathbb{F}}$. Then either there is $h \in F$ with $c(h) = 0$ in the $\alpha^{\mathbb{F}}$-class of f and g, and both $c(f)$ and $c(g)$ are assigned values in $\{0, 3\}$, or there is no such h and both $c(f)$ and $c(g)$ are assigned values in $\{1, 2\}$. Either way $(c(f), c(g)) \in \alpha^{\mathbb{P}_0}$. The same argument can be used for showing $(c(f), c(g)) \in \beta^{\mathbb{P}_0}$ given that $(f, g) \in \beta^{\mathbb{F}}$. Finally, suppose that $(f, g) \in \gamma^{\mathbb{F}}$. If $c(f) \in \{1, 2\}$, we immediately get that $c(g) \in \{1, 2\}$ since c preserves α. If this is not the case, and f and g are assigned values that are not in the same class of $\gamma^{\mathbb{P}_0}$, then $c(f) = 0$ and $c(g) = 3$, or vice-versa. But this implies that $(f, g) \notin \gamma^{\mathbb{F}}$ as otherwise both would be assigned the value 0. This concludes that c is a coloring.

We are left to prove that c is strong, i.e., $c(x_i) = i$ for $i = 0, 1, 2, 3$. We know (from the defintion of c) that $c(x_0) = 0$ and $c(x_3) = 3$. Now, x_1 and x_2 are assigned values that are in the same class of $\gamma^{\mathbb{P}_0}$, and moreover $(c(x_1), c(x_0)) \in \beta^{\mathbb{P}_0}$ and $(c(x_3), c(x_2)) \in \beta^{\mathbb{P}_0}$. This leaves us with the only option: $c(x_1) = 1$ and $c(x_2) = 2$. □

Note that in the previous proof we use the same three congruences ($\alpha^{\mathbb{F}}$, $\beta^{\mathbb{F}}$, and $\gamma^{\mathbb{F}}$) of the free algebra generated by a four element set that also appeared in the original proof of Day's results. This is not a pure coincidence, we will use a very similar method of deriving a suitable structure for coloring in Section 5.

4.3. **Idempotent varieties.** In this subsection, we prove a generalization of the following theorem of McGarry [McG09] for idempotent varieties that do not need to be locally finite.

Theorem 4.6 ([McG09])**.** *A locally finite idempotent variety is not congruence modular if and only if it contains and algebra compatible with a special pentagon.* $\square$

Throughout this proof we will work with several variants of special and very special pentagons $\mathbb{P}$ defined on a product $A \times B$ for two algebras $\mathbf{A}$, $\mathbf{B}$ in our idempotent variety, and in particular with the corresponding relations $\gamma^{\mathbb{P}}$. We say that γ is a *modularity blocker* if $\mathbb{P} = (A \times B; \alpha, \beta, \gamma)$ is a special pentagon given that α and β are kernels of the two projections, and γ is a *special modularity blocker* if $\mathbb{P}$ is a very special pentagon.

The first step of the proof coincides with McGarry's proof. Nevertheless, we present an alternative proof for completeness.

Lemma 4.7. *Let $\mathcal{V}$ be an idempotent variety, and $\mathbf{F}$ the free algebra in $\mathcal{V}$ generated by $\{x, y\}$. Then $\mathcal{V}$ is congruence modular if and only if*

$$\big((y,x),(y,y)\big) \in \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{\big((x,x),(x,y)\big)\}. \tag{$*$}$$

Proof. Given that $\mathcal{V}$ is congruence modular, one obtains $(*)$ from the modularity law applied on the two kernels of projections and the congruence on the right hand side of $(*)$.

To show that $(*)$ also implies congruence modularity, we will prove that $(*)$ is not true in any $\mathcal{V}$ that is idempotent and not congruence modular. First, we consider the free algebra $\mathbf{F}_4$ in $\mathcal{V}$ generated by the four-element set $\{x_0, x_1, x_2, x_3\}$ and its congruences $\alpha = \mathrm{Cg}\{(x_0,x_3),(x_1,x_2)\}$, $\beta = \mathrm{Cg}\{(x_0,x_1),(x_2,x_3)\}$, and $\gamma = \mathrm{Cg}\{(x_1,x_2)\}$. Note that this is the structure $\mathbb{F}$ that has been used in the proof of Proposition 4.5. Either from this proposition, or by a standard argument from the proof of Day's result, we obtain that α, β, and γ don't satisfy the modularity law, and in particular, $(x_0,x_3) \notin \gamma \vee (\alpha \wedge \beta)$.

Next, we shift this property to the second power of the two-generated free algebra; consider the homomorphism $h\colon \mathbf{F}_4 \to \mathbf{F} \times \mathbf{F}$ defined on the generators by $x_0 \mapsto (y,x)$, $x_1 \mapsto (x,x)$, $x_2 \mapsto (x,y)$, and $x_3 \mapsto (y,y)$. The homomorphism h is surjective, since for every two binary idempotent terms t, s we have $h\big(t(s(x_1,x_2), s(x_0,x_3))\big) = \big(t(x,y), s(x,y)\big)$. Finally, since the kernel of h is $\alpha \wedge \beta$, we get that

$$h^{-1}\big(\,\mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{((x,x),(x,y))\}\big) = \gamma \vee (\alpha \wedge \beta) \not\ni (x_0,x_3),$$

and consequently $((y,x),(y,y)) = (h(x_0),h(x_3)) \notin \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{((x,x),(x,y))\}$. $\square$

Lemma 4.8. *Let $\mathcal{V}$ be an idempotent variety which is not congruence modular, and $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(x,y)$. Then there is a modularity blocker γ in $\mathbf{F} \times \mathbf{F}$ such that $((x,x),(x,y)) \in \gamma$ and $((y,x),(y,y)) \notin \gamma$,*

Proof. Let $\gamma_0 = \mathrm{Cg}_{\mathbf{F}\times\mathbf{F}}\{\big((x,x),(x,y)\big)\}$. From the previous lemma, we know that $((y,x),(y,y)) \notin \gamma_0$. Let η be a maximal equivalence relation on F such that

$$\big((y,x),(y,y)\big) \notin \gamma_0 \vee \mathrm{Cg}\{\big((y,a),(y,b)\big) : (a,b) \in \eta\}$$

(such equivalence exists from Zorn's lemma). We claim that the equivalence on the right-hand side, let us call it γ, is a modularity blocker. This is proven in two steps:

Claim 1. $\gamma^p \geq \eta$ for all $p \in F$.

Let f be binary term such that $f(x,y) = p$. Observe that $\gamma^x = 1_F \geq \eta$ and $\gamma^y = \eta$, therefore we obtain that $\big((p,a),(p,b)\big) = \big(f\big((x,a),(y,a)\big), f\big((x,b),(y,b)\big)\big) \in \eta$ for all $(a,b) \in \eta$. Which shows that $\gamma^p \geq \eta$. $\triangle$

Claim 2. If $\gamma^p > \eta$ for some $p \in F$ then $\gamma^p = 1_F$.

Let $e = e' \times 1_{\mathbf{F}}$, where $e' \colon \mathbf{F} \to \mathbf{F}$ is the homomorphism defined by $x \mapsto x$ and $y \mapsto p$, and consider the congruence $\gamma_1 = e^{-1}(\gamma)$. From this definition, we obtain that $\gamma_1^y \geq \gamma^p > \eta$ and moreover $\gamma_1 \geq \gamma_0$ since $(e(x,x), e(x,y)) = ((x,x),(x,y)) \in \gamma$. Therefore, from the maximality of η, we obtain that $((y,x),(y,y)) \in \gamma_1$, and as a consequence thereof, $((p,x),(p,y)) = (e(y,x), e(y,y)) \in \gamma$. This shows that $\gamma^p = 1_{\mathbf{F}}$ and completes the proof of the second claim. $\triangle$

By combining both claims, we get that $\gamma^p \in \{\eta, 1_F\}$, therefore γ is a modularity blocker. $\square$

The above proof was inspired by the proof of Lemma 2.8 from [KT07].

Corollary 4.9. *Every idempotent variety that is not congruence modular is compatible with a very special pentagon.*

Proof. Following the notation of the previous lemma, we know that $(F \times F, \alpha, \beta, \gamma)$ is a special pentagon for kernels of projections α, β. To obtain a very special pentagon $\mathbb{P}$, first observe that γ^y is a congruence on $\mathbf{F}$: by idempotence we get that $\{y\} \times \mathbf{F}$ is a subalgebra of $\mathbf{F}^2$ isomorphic to $\mathbf{F}$, and γ^y is the image of the restriction of γ to $\{y\} \times \mathbf{F}$ under this isomorphism. We set $\mathbf{A} = \mathbf{F}$, $\mathbf{B} = \mathbf{F}/\gamma^y$, and $P = A \times B$. Finally, the relation $\gamma^{\mathbb{P}}$ is defined as the image of γ under the natural epimorphism from $\mathbf{F}^2$ to $\mathbf{A} \times \mathbf{B}$. $\square$

The next step is to show that we can increase the size of compatible very special pentagons. In other words, that every variety compatible with a very special pentagon is compatible with $\mathbb{P}_\kappa$'s for all big enough cardinals κ.

Proposition 4.10. *If a variety $\mathcal{V}$ contains an algebra compatible with a very special pentagon $\mathbb{P}$, then it contains an algebra compatible with $\mathbb{P}_\kappa$ for all $\kappa \geq \aleph_0 + |P|$.*

Proof. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras in $\mathcal{V}$ such that $\mathbb{P}$ is compatible with $\mathbf{A} \times \mathbf{B}$, let $U \subseteq A$ denote the set $\{a \in A : \gamma^a = 1_B\}$, and let $\kappa \geq \aleph_0 + |P|$ be a cardinal.

We will construct a pentagon which is isomorphic to $\mathbb{P}_\kappa$ by a variation on Tarski's construction described in Section 3.4. Let $\mathbf{A}_\lambda$ and U_λ be the algebra and its subset obtained by the construction from $\mathbf{A}$ and U in λ steps, and let $\mathbf{B}_\lambda$ be the algebra obtained by the construction from $\mathbf{B}$ in λ steps. We define an equivalence relation γ_λ on $A_\lambda \times B_\lambda$ by

$$\gamma_\lambda = \{((a,b),(a',b')) : a = a' \text{ and if } a \notin U_\lambda \text{ then } b = b'\}.$$

We will prove that γ_λ is a congruence of $\mathbf{A}_\lambda \times \mathbf{B}_\lambda$ by a transfinite induction: $\gamma_0 = \gamma$ is a congruence. For the induction step, suppose that γ_λ is a congruence. The algebras $\mathbf{A}_{\lambda+1}$ and $\mathbf{B}_{\lambda+1}$ are defined as being isomorphic to the second powers of $\mathbf{A}_\lambda$ and $\mathbf{B}_\lambda$ respectively. Let us suppose that they are in fact the second powers themselves and $U_{\lambda+1} = A_\lambda \times U_\lambda$. Then $\gamma_{\lambda+1}$ can be described as relating those pairs of pairs $\big((a_1,a_2),(b_1,b_2)\big)$ and $\big((a_1',a_2'),(b_1',b_2')\big)$ that satisfy: $a_1 = a_1'$, $a_2 = a_2'$, and if $a_2 \notin U_\lambda$ then $b_1 = b_1'$ and $b_2 = b_2'$. Rewriting this condition, we get

$$\begin{aligned}&\big(\big((a_1,a_2),(b_1,b_2)\big), \big((a_1',a_2'),(b_1',b_2')\big)\big) \in \gamma_{\lambda+1} \text{ if and only if}\\ &\qquad a_1 = a_1',\ a_2 = a_2',\ \big((a_2,b_1),(a_2',b_1')\big) \in \gamma_\lambda, \text{ and } \big((a_2,b_2),(a_2',b_2')\big) \in \gamma_\lambda\end{aligned}$$

which is clearly compatible with the operations, since γ_λ is. Also note, that if we restrict $\gamma_{\lambda+1}$ to $\mathbf{A}_\lambda \times \mathbf{B}_\lambda$, we get γ_λ. For limit ordinals λ, the compatibility of γ_λ is obtained by a standard compactness argument. This way, we obtain for every λ a very special pentagon $\mathbb{P}'_\lambda$ with $P'_\lambda = A_\lambda \times B_\lambda$ and $\gamma^{\mathbb{P}_\lambda} = \gamma_\lambda$.

The final step is to prove that the pentagon $\mathbb{P}'_\lambda$ is isomorphic to $\mathbb{P}_\kappa$ for $\kappa = |\lambda|$. Indeed, $P_\kappa = \kappa \times \kappa$, $P'_\lambda = A_\lambda \times B_\lambda$ with $|A_\lambda| = |B_\lambda| = \kappa$. Also compare

$$\gamma^{\mathbb{P}_\kappa} = \{((a,b),(a',b')) : \ a = a' \text{ and if } a \notin U_\kappa \text{ then } b = b' \ \}$$

with the above definition of γ_λ. From these observations, it is immediate that any bijection $a \times b\colon \kappa \times \kappa \to A_\lambda \times B_\lambda$ defined by components a and b such that a maps U_κ onto U_λ is an isomorphism between $\mathbb{P}_\kappa$ and $\mathbb{P}'_\lambda$. □

4.4. **Proof of Theorem 1.4(i).** Given a variety $\mathcal{V}$ that is not congruence modular, we distinguish between two cases: $\mathcal{V}$ is linear; and $\mathcal{V}$ is idempotent. If it is the first case, we get from Proposition 4.5 and Lemma 3.5 that $\mathcal{V}$ contains an algebra compatible with a very special pentagon $\mathbb{P}^{\mathcal{V}} = \mathbb{P}_0$. In the second case, $\mathcal{V}$ is idempotent, we obtain a very special pentagon $\mathbb{P}^{\mathcal{V}}$ compatible with an algebra from $\mathcal{V}$ by Corollary 4.9.

Given that both $\mathcal{V}$ and $\mathcal{W}$ are not congruence modular, and either linear, or idempotent, we have very special pentagons $\mathbb{P}^{\mathcal{V}}$ and $\mathbb{P}^{\mathcal{W}}$ compatible with algebras from the corresponding varieties. Choosing an infinite cardinal κ larger than the sizes of both $P^{\mathcal{V}}$ and $P^{\mathcal{W}}$. Proposition 4.10 yields that both varieties contain algebras compatible with the pentagon $\mathbb{P}_\kappa$, therefore by Lemma 3.2, the interpretability join $\mathcal{V} \vee \mathcal{W}$ contains an algebra compatible with $\mathbb{P}_\kappa$ witnessing that $\mathcal{V} \vee \mathcal{W}$ is not congruence modular. □

4.5. **Cofinal chain.** Here we investigate properties of the transfinite sequence $\mathscr{P}_0$, $\mathscr{P}_{\aleph_0}$, $\mathscr{P}_{\aleph_1}, \dots$ of polymorphism clones of pentagons $\mathbb{P}_0, \mathbb{P}_{\aleph_0}, \dots$; in particular, we show that this sequence form a strictly increasing chain in the lattice of clones, and as a corollary thereof, we obtain that there is no maximal (in the interpretability order) idempotent variety that is not congruence modular. For the rest of this section, κ and λ will denote either infinite cardinals, or 0.

The fact that the chain is increasing follows from the following corollary of Proposition 4.10.

Corollary 4.11. *There is a clone homomorphisms from $\mathscr{P}_\lambda$ to $\mathscr{P}_\kappa$ for all $\lambda \leq \kappa$.*

Proof. If we use Proposition 4.10 on the variety of actions of $\mathscr{P}_\lambda$, we obtain that it contains an algebra compatible with $\mathbb{P}_\kappa$. But such an algebra corresponds to an action of $\mathscr{P}_\lambda$ on $\mathbb{P}_\kappa$ by polymorphisms, therefore to a clone homomorphism from $\mathscr{P}_\lambda$ to $\mathscr{P}_\kappa$. □

Before we proceed further, we need a better description of polymorphisms of the pentagons $\mathbb{P}_\kappa$.

Lemma 4.12. *Let κ be an infinite cardinal, and consider the pentagon $\mathbb{P}_\kappa$, and let $f\colon P_\kappa^k \to P_\kappa$ be a functions defined by components $f^1, f^2\colon \kappa^k \to \kappa$. Then the following are equivalent:*

(1) *f is a polymorphism of $\mathbb{P}_\kappa$,*
(2) *if $f^1(a_1, \dots, a_n) \notin U$ for some $a_1, \dots, a_n \in \kappa$ then f^2 does not depend on any coordinate i such that $a_i \in U$.*

Proof. (1) → (2) Since f is defined component-wise, if is compatible with $\alpha^{\mathbb{P}_\kappa}$ and $\beta^{\mathbb{P}_\kappa}$. For compatibility with $\gamma^{\mathbb{P}_\kappa}$, recall that $((a,b),(a',b')) \in \gamma^{\mathbb{P}_\kappa}$ if and only if $a = a'$, and $a \in U$ or $b = b'$. Therefore, if f is compatible with $\gamma^{\mathbb{P}_\kappa}$ and $f^1(a_1,\dots,a_n) \notin U$ then $f^2(b_1,\dots,b_n) = f^2(b'_1,\dots,b'_n)$ for all b_i, b'_i where $b_i = b'_i$ whenever $a_i \notin U$ (observe that $((a_i,b_i),(a_i,b'_i)) \in \gamma^{\mathbb{P}_\kappa}$ for all i). This means that f^2 does not depend on any coordinate i with $a_i \in U$. The implication (2) → (1) is given by reversing this argument. □

Instead of proving that the chain of $\mathscr{P}_\kappa$'s is strictly increasing, we prove the following stronger statement which will allow us to show that there is no maximal (in the ordering by interpretability) idempotent non-modular variety.

Proposition 4.13. *The idempotent reducts of $\mathscr{P}_\kappa$'s form a strictly increasing chain in the lattice of clones.*

Proof. Let $\mathscr{P}_\kappa^{\mathrm{id}}$ denote the idempotent reduct of $\mathscr{P}_\kappa$. We need to show two facts: (1) there is a clone homomorphism from $\mathscr{P}_\lambda^{\mathrm{id}}$ to $\mathscr{P}_\kappa^{\mathrm{id}}$ for all $\lambda \leq \kappa$, and (2) there is no clone homomorphism from $\mathscr{P}_\kappa^{\mathrm{id}}$ to $\mathscr{P}_\lambda^{\mathrm{id}}$ for any $\lambda < \kappa$. (1) follows directly from Corollary 4.11, since any clone homomorphism from $\mathscr{P}_\lambda$ to $\mathscr{P}_\kappa$ has to preserve idempotent functions.

To prove (2) suppose that $\kappa > \lambda$. We will find identities that are satisfied in $\mathscr{P}_\kappa^{\mathrm{id}}$ but are not satisfiable in $\mathscr{P}_\lambda^{\mathrm{id}}$. In fact they are not satisfiable in any clone on a set of cardinality strictly smaller than κ except the one-element set. The identities use binary symbols f_i for $i \in \kappa$ and ternary symbols $p_{i,j}$, $q_{i,j}$, $r_{i,j}$ for $i, j \in \kappa$, $i \neq j$:

$$\begin{aligned} x &\approx p_{i,j}(x, f_j(x,y), y), \\ p_{i,j}(x, f_i(x,y), y) &\approx q_{i,j}(x, f_j(x,y), y), \\ q_{i,j}(x, f_i(x,y), y) &\approx r_{i,j}(x, f_j(x,y), y), \\ r_{i,j}(x, f_i(x,y), y) &\approx y \end{aligned}$$

for all $i \neq j$, and $f_i(x,x) \approx x$ for all i.

We will define functions in $\mathscr{P}_\kappa^{\mathrm{id}}$ that satisfy these identities coordinatewise (recall $P_\kappa = \kappa \times \kappa$); this will assure that these functions are compatible with $\alpha^{\mathbb{P}_\kappa}$ and $\beta^{\mathbb{P}_\kappa}$. We fix $c^1 \in U_\kappa$ and $c^2 \in \kappa$. The functions f_i are defined, unless required by idempotence otherwise, as constants while choosing different constants for different i's. In detail, we pick $c_i^1 \in U_\kappa$ to be pairwise distinct, and $c_i^2 \in \kappa$ as well. Then $f_i^u(x,y) = c_i^u$ if $x \neq y$ and $f_i^u(x,x) = x$ for $u = 1, 2$. The components of $p_{i,j}$, $q_{i,j}$, and $r_{i,j}$ are defined as follows:

$$p_{i,j}^1(x,y,z) = \begin{cases} x & \text{if } y = f_j^1(x,z), \\ c_1 & \text{otherwise;} \end{cases} \qquad p_{i,j}^2(x,y,z) = x;$$

$$q_{i,j}^1(x,y,z) = \begin{cases} x & \text{if } x = y = z, \\ c_1 & \text{otherwise;} \end{cases} \qquad q_{i,j}^2(x,y,z) = \begin{cases} x & \text{if } y = f_j^2(x,z), \\ z & \text{if } y = f_i^2(x,z), \\ c_2 & \text{otherwise;} \end{cases}$$

$$r_{i,j}^1(x,y,z) = \begin{cases} z & \text{if } y = f_i^1(x,z), \\ c_1 & \text{otherwise;} \end{cases} \qquad r_{i,j}^2(x,y,z) = z;$$

It is straightforward to check that these functions satisfy the identities. The compatibility with $\alpha^{\mathbb{P}_\kappa}$ and $\beta^{\mathbb{P}_\kappa}$ is immediate from defining them component-wise. The compatibility with $\gamma^{\mathbb{P}_\kappa}$ follows from Lemma 4.12: the only case when $p_{i,j}^1(x,y,z) \notin$

U_κ is when $x \notin U_\kappa$ and $y = f_j^1(x,z)$ but $p_{i,j}^2$ does not depend on the second and third variable, therefore $p_{i,j}$ is compatible; the argument for $r_{i,j}$ is analogous; $q_{i,j}^1(x,y,z)$ falls in U_κ unless $x = y = z \notin U_\kappa$, so $q_{i,j}$ is also compatible.

Next, we claim that the above identities are not satisfied in any non-trivial algebra $\mathbf{A}$ of size strictly less than κ. Indeed, if A contains two distinct elements a and b, then $f_i(a,b)$, $i \in \kappa$ are pairwise distinct, since if $f_i^{\mathbf{A}}(a,b) = f_j^{\mathbf{A}}(a,b)$, then

$$\begin{aligned} a = p_{i,j}^{\mathbf{A}}(a, f_j(a,b), b) = p_{i,j}^{\mathbf{A}}(a, f_i(a,b), b) = q_{i,j}^{\mathbf{A}}(a, f_j(a,b), b) = \\ q_{i,j}^{\mathbf{A}}(a, f_i(a,b), b) = r_{i,j}^{\mathbf{A}}(a, f_j(a,b), b) = r_{i,j}^{\mathbf{A}}(a, f_i(a,b), b) = b. \end{aligned}$$

This shows in particular that these identities are not satisfiable in $\mathscr{P}_\lambda^{\mathrm{id}}$. □

Corollary 4.14. *The class of all interpretability classes of idempotent varieties that are not congruence modular does not have a largest element.*

Proof. For a contradiction, suppose that there is a largest interpretability class among those containing a non-modular idempotent variety, and let $\mathcal{V}$ be a variety from this class. By Corollary 4.9 and Proposition 4.10, we know that $\mathcal{V}$ is interpretable in the variety of actions of $\mathscr{P}_\kappa$ for some κ. Fix any such κ, and let $\lambda > \kappa$ be a cardinal. By the maximality of $\mathcal{V}$, we have that the variety of actions of $\mathscr{P}_\lambda^{\mathrm{id}}$ (the idempotent reduct of $\mathscr{P}_\lambda$) is interpretable in $\mathcal{V}$, therefore also in the variety of actions of $\mathscr{P}_\kappa$ which contradicts the previous proposition. □

5. Having n-permutable congruences

In this section, we will investigate both the strong Mal'cev conditions for being congruence n-permutable (every two congruences α, β of a single algebra satisfy $\alpha \circ_n \beta = \beta \circ_n \alpha$) for a fixed n, and a general condition of being congruence n-permutable for some n. Therefore, we will speak about a countable chain of filters in the interpretability lattice and its limit. The primeness of the limit filter would be implied by primeness of each of the filters from the chain. Unfortunately, we are not able to establish the version of Theorem 1.4(ii) for the filters from the chain. We would also like to note that the condition of being n-permutable for some n can be also formulated as 'having no nontrivial compatible partial order'; this have been attributed to Hagemann, for a proof see [Fre13].

Theorem 5.1. *A variety $\mathcal{V}$ is not congruence n-permutable for any n if and only if it contains an algebra compatible with a partial order that is not an antichain.* □

There are two well-known Mal'cev characterizations of the discussed conditions, the older $(n+1)$-ary terms by Schmidt [Sch69] and refined ternary terms by Hagemann and Mitschke [HM73] (items (2) and (3), respectively):

Theorem 5.2. *The following are equivalent for any variety $\mathcal{V}$ and every positive integer n.*

(1) *$\mathcal{V}$ is n-permutable;*
(2) *there are $(n+1)$-ary $\mathcal{V}$-terms $s_0, \dots, s_n$ such that the identities*

$$\begin{gathered} s_0(x_0, \dots, x_n) \approx x_0, \;\; s_n(x_0, \dots, x_n) \approx x_n, \\ s_i(x_0, x_0, x_2, x_2, \dots) \approx s_{i+1}(x_0, x_0, x_2, x_2, \dots) \text{ for odd } i, \text{ and} \\ s_i(x_0, x_1, x_1, x_3, \dots) \approx s_{i+1}(x_0, x_1, x_1, x_3, \dots) \text{ for even } i \end{gathered}$$

are satisfied in $\mathcal{V}$;

(3) *there are ternary $\mathcal{V}$-terms $p_0, \dots, p_n$ such that the identities*

$$p_0(x, y, z) \approx x, \; p_n(x, y, z) \approx z, \text{ and}$$
$$p_i(x, x, y) \approx p_{i+1}(x, y, y) \text{ for every } i < n$$

are satisfied in $\mathcal{V}$. □

5.1. **Colorings and linear varieties.** For the characterization of congruence n-permutable varieties in the means of coloring, we will use the smallest structures with two equivalence relations that do not n-permute (denoted $\mathbb{W}_n$) for the strong conditions, and the two-element chain for the general condition.

Definition 5.3. For $n \geq 2$, we define the structure $\mathbb{W}_n$ as the relational structure on $\{0, \dots, n\}$ with two binary equivalence relations α and β defined by partitions $01|23|\dots$ and $0|12|34|\dots$, respectively.

The the following is a variation on a result of Sequeira [Seq01], who provided a very similar characterization by the means of compatibility with projections.

Proposition 5.4. *Let $n \geq 2$. A variety is not congruence n-permutable if and only if has strongly $\mathbb{W}_n$-colorable terms.*

Proof. First we fix some notation. Let $\mathbb{F}$ denote the structure obtained from the free structure in $\mathcal{V}$ generated by $\mathbb{W}_n$ by replacing each of its relations by its transitive closure. Again, as in the proof of Proposition 4.5, we obtain that all relations of $\mathbb{F}$ are equivalences, and $c\colon F \to W_n$ is a coloring if and only if it is a homomorphism from $\mathbb{F}$ to $\mathbb{W}_n$. Also, to simplify the notation, we identify an $(n+1)$-ary term s with $s(x_0, \dots, x_n)$.

To prove the first implication, suppose that $\mathcal{V}$ has Schmidt terms $s_0, \dots, s_n$. Our objective is to prove that terms of $\mathcal{V}$ are not strongly $\mathbb{W}_n$-colorable. For the contrary, suppose that there is a strong coloring $c\colon \mathbb{F} \to \mathbb{W}_n$. Any such coloring maps x_0, and therefore also s_0, to 0. Further, from Schmidt identities we get that $(s_1, s_0) \in \beta^{\mathbb{F}}$, therefore also $c(s_1) = 0$. Following a similar argument for pairs (s_{i+1}, s_i) alternating between $\beta^{\mathbb{F}}$ and $\alpha^{\mathbb{F}}$, we obtain $c(s_i) < i$ for all $i > 0$ which in particular implies that $c(x_n) = c(s_n) < n$, a contradiction with $c(x_n) = n$.

For the other implication, suppose that $\mathcal{V}$ does not have Schmidt terms. Similarly as in the proof of Proposition 4.5, we define a coloring c by setting $c(x_0) = 0$ and then, following Schmidt-like chains of terms, extend this definition as we are forced, and finally, setting $c(f) = n$ for all elements that have not been reached. This is summarized in the following definition:

$$c(f) = \min\{i : (x_0, f) \in \beta^{\mathbb{F}} \circ_{i+1} \alpha^{\mathbb{F}}\}$$

if the set on the right hand side is non-empty, and $c(f) = n$, otherwise. By definition, we have that for each i, $c(x_i) \leq i$. First, we need to show that c is well-defined, in particular that $c(f) \leq n$ for all f. Suppose that f satisfies $(x_0, f) \in \beta^{\mathbb{F}} \vee \alpha^{\mathbb{F}}$. Immediately, we get that f is idempotent, i.e., $f(x_0, \dots, x_0) = x_0$. Also, since $(x_0, x_i) \in \alpha^{\mathbb{F}} \circ_i \beta^{\mathbb{F}} \subseteq \beta^{\mathbb{F}} \circ_{n+1} \alpha^{\mathbb{F}}$ for all i, and $\beta^{\mathbb{F}} \circ_{n+1} \alpha^{\mathbb{F}}$ is a subuniverse, $(x_0, f) \in \beta^{\mathbb{F}} \circ_{n+1} \alpha^{\mathbb{F}}$. Therefore, the aforementioned set contains n, and we have $c(f) \leq n$.

Next, we show that c is compatible with α and β. Suppose that $(f, g) \in \alpha^{\mathbb{F}}$ and both f and g are idempotent. Without loss of generality, suppose that $c(f) \leq c(g)$. If $c(f)$ is an odd number, then $c(g) \leq c(f)$; if $c(f)$ is even, then $c(g) \leq c(f) + 1$.

Altogether, we have that $(c(g), c(f))$ differ by at most 1, and if they do, then $c(f)$ is odd and $c(g) = c(f) + 1$. This gives that $(c(f), c(g)) \in \alpha^{\mathbb{W}_n}$. Similarly, we get that c is compatible with β, therefore c is a coloring.

Lastly, we need to show that c is strong. As mentioned before, $(x_0, x_i) \in \beta^{\mathbb{F}} \circ_{i+1} \alpha^{\mathbb{F}}$, therefore $c(x_i) \leq i$ for all i. For a contradiction suppose that $c(x_i) < i$ for some i, i.e., $(x_0, x_i) \in \beta^{\mathbb{F}} \circ_i \alpha^{\mathbb{F}}$. This implies that there are terms $x_0 = t_0, t_1, \ldots, t_i = x_i$ with $(t_j, t_{j+1}) \in \beta^{\mathbb{F}}$ for even j and $(t_j, t_{j+1}) \in \alpha^{\mathbb{F}}$ for odd j. Now, define $(i+1)$-ary terms $s_0, \ldots, s_i$ by putting:

$$s_j(x_0, \ldots, x_i) = t_j(x_0, \ldots, x_i, x_i, \ldots, x_i).$$

We claim that these terms satisfy Schmidt identities. Indeed, $s_0 = x_0$, $s_i = x_i$, $s_j(x_0, x_0, x_2, \ldots) = s_{j+1}(x_0, x_0, x_2, \ldots)$ for odd j, since $(t_j, t_{j+1}) \in \alpha^{\mathbb{F}}$ for such j, and $s_j(x_0, x_1, x_1, \ldots) = s_{j+1}(x_0, x_1, x_1, \ldots)$ for even j, since $(t_j, t_{j+1}) \in \beta^{\mathbb{F}}$ for such j. This gives is a contradiction with the fact that $\mathcal{V}$ is not even congruence n-permutable. □

The corresponding result for the general Mal'cev condition was proven in [BOP15].

Proposition 5.5 ([BOP15, Proposition 7.2])**.** *A variety is not n-permutable for any n if and only if it has strongly $(\{0,1\}, \leq)$-colorable terms.* □

5.2. **Idempotent varieties.** The fact, that the interpretability join of two idempotent varieties that are not congruence n-permutable for any n is not congruence n-permutable either, follows from a result of Valeriote and Willard [VW14]. No similar result about the corresponding strong Mal'cev conditions is known.

Theorem 5.6 ([VW14])**.** *An idempotent variety is not n-permutable for any n if and only if it is interpretable in the variety of distributive lattices.* □

An important step to unify this result with the similar for linear varieties is the observation that the variety of actions of idempotent polymorphisms of $(\{0,1\}, \leq)$ is interpretable in the variety of distributive lattices and vice-versa. This is implicitly hidden in [VW14]: The key reasons are that the clone of the two-element lattice on $\{0,1\}$ coincides with the clone of idempotent polymorphisms of $\leq$, and that the two element lattice generates the variety of ditributive lattices. As a direct consequence of this observation and the above theorem, we get:

Corollary 5.7. *An idempotent variety is not n-permutable for any n if and only if it contains an algebra compatible with $(\{0,1\}, \leq)$.* □

5.3. **Proofs of Theorems 1.4(ii) and 1.5.** First, we address Theorem 1.4(ii): We have showed that if a variety is not n-permutable for any n and it is either linear, or idempotent then it contains an algebra compatible with $(\{0,1\}, \leq)$ (the linear case follows from Proposition 5.5 and Lemma 3.5, the idempotent case from Corollary 5.7). Therefore, given $\mathcal{V}$ and $\mathcal{W}$ that are not n-permutable for any n and are both linear, or idempotent, we know that both contain an algebra that is compatible with $(\{0,1\}, \leq)$. Lemma 3.2 then gives that there is an algebra in $\mathcal{V} \vee \mathcal{W}$ that is compatible with this partial order, and therefore witnesses that $\mathcal{V} \vee \mathcal{W}$ is not n-permutable for any n (see Theorem 5.1).

Theorem 1.5 is obtained in a similar way: Using Proposition 5.4 and Lemma 3.5, we get that any linear variety, that is not n-permutable for a fixed n, contains an algebra that is compatible with $\mathbb{W}_n$, therefore also the join of two such varieties contains an algebra compatible with $\mathbb{W}_n$. Having such an algebra contradicts being congruence n-permutable. □

6. Satisfying a non-trivial congruence identity

There are several Mal'cev characterizations of varieties that satisfy some non-trivial congruence identity. An older one, called a Hobby-McKenzie term (for the definition we refer to [HM88] or [KK13]), and more recent by Kearnes and Kiss [KK13]:

Theorem 6.1 ([KK13, Theorems 5.28 and 7.15])**.** *The following is equivalent for a variety $\mathcal{V}$.*

1. *$\mathcal{V}$ satisfies a non-trivial congruence identity,*
2. *$\mathcal{V}$ satisfies an idempotent Mal'cev condition that fails in the variety of semilattices,*
3. *there exists 4-ary terms $t_0, \dots, t_n$ such that the identities*

$$t_0(x,y,z,w) \approx x \text{ and } t_n(x,y,z,w) \approx w,$$

$$t_i(x,y,y,y) \approx t_{i+1}(x,y,y,y) \text{ for even } i,$$

$$t_i(x,x,y,y) \approx t_{i+1}(x,x,y,y) \text{ and } t_i(x,y,y,x) \approx t_{i+1}(x,y,y,x) \text{ for odd } i$$

are satisfied in $\mathcal{V}$. □

We will refer to the terms in item (3) as to Kearnes-Kiss terms. Also note that as a byproduct of this characterization, Kearnes and Kiss proved that an idempotent variety satisfies a non-trivial congruence identity if and only if it is not interpretable in the variety of semilattices (this follows from the equivalence of (1) and (2) in the above theorem).

6.1. **Coloring and linear varieties.** Kearnes and Kiss also proved that item (1) of Theorem 6.1 implies that the variety contains an algebra $\mathbf{A}$ with a compatible (sometimes called commuting) semilattice operation, i.e., there is a semilattice operation $\vee$ on A such that its graph—the relation $\{(a,b,a\vee b) : a,b \in A\}$ is a compatible relation. This suggests a relational structure to be used in the coloring description of this Mal'cev condition: the structure $\mathbb{S}$ on $\{0,1\}$ with one ternary relation $J^{\mathbb{S}} = \{(x,y,x\vee y) : x,y \in \{0,1\}\}$. Note that the idempotent reduct of $\mathscr{S}$ is the same as the clone of term operations of the semilattice $(\{0,1\},\vee)$ (see also [KK13, Lemma 5.25]).

Proposition 6.2. *A variety $\mathcal{V}$ does not have Kearnes-Kiss terms if and only if it has strongly $\mathbb{S}$-colorable terms.*

Proof. As usual, let $\mathbb{F}$ denote the free structure in $\mathcal{V}$ generated by $\mathbb{S}$. We will also identify a binary term b with an element $b(x_0,x_1) \in F$. Observe that, the relation $J^{\mathbb{F}}$ consists of triples (r,s,t) such that there exists a 4-ary term f satisfying $f(x,x,y,y) \approx r(x,y)$, $f(x,y,x,y) \approx s(x,y)$, and $f(x,y,y,y) \approx t(x,y)$.

First, suppose that $\mathcal{V}$ has Kearnes-Kiss terms. We want to proof that there is no strong coloring $c\colon \mathbb{F} \to \mathbb{S}$. By the definition of a coloring and the relation J, we can deduce that for every triple $(r,s,t) \in J^{\mathbb{F}}$ if $c(r) = 0$ and $c(s) = 0$ then also $c(t) = 0$, and similarly if $c(t) = 0$ then both $c(r)$ and $c(s)$ are also 0. By combining these two observations, one can prove by induction on i that for any coloring c with $c(x_0) = 0$ and any an term t_i from Kearnes-Kiss chain we have that $c(t_i(x_0,x_0,x_1,x_1)) = 0$,

$c(t_i(x_0, x_1, x_0, x_1)) = 0$, and $c(t_i(x_0, x_1, x_1, x_1)) = 0$. This shows that any such coloring c has to satisfy $c(x_1) = 0$ which contradicts that c is strong.

For the other implication, suppose that the variety $\mathcal{V}$ does not have Kearnes-Kiss terms. We define a mapping $c\colon F \to J$ in such a way that $c(t) = 0$ if and only if this fact is forced by the argument in the previous paragraph, that is, there exists tuples $(s_i, t_i, r_i) \in J^{\mathcal{V}}$, $i = 1, \dots, n$ such that $s_0 = t_0 = x_0$, $r_i = r_{i+1}$ for even i, $s_i = s_{i+1}$ and $t_i = t_{i+1}$ for odd i, and $t = r_n$ for n odd, or $t \in \{s_n, t_n\}$ for n even. We claim that c is a coloring. Indeed, if $c(t) = 0$ and $(r, s, t) \in J^{\mathcal{V}}$ for some $r, s \in F$, then also $c(r) = c(s) = 0$ by the definition, and if $c(t) = 1$ and $(r, s, t) \in J^{\mathcal{V}}$ then either $c(r) = 1$, or $c(s) = 1$, otherwise we would have defined $c(t) = 0$. In either case, we have $c(s) \vee c(r) = c(t)$ which is what we wanted to prove.

The coloring c is strong: it maps x_0 to 0 by definition, and also $c(x_1) = 1$, since the variety does not have Kearnes-Kiss terms. □

The fact that Kearnes-Kiss terms are not colorable by $\mathbb{S}$ can be also argued using the arguments of [KK13]: having Kearnes-Kiss terms is an idempotent linear Mal'cev condition that is not satisfiable in the variety of semilattices, and existence of a strong coloring would imply that it is satisfied in $\mathscr{S}$, therefore also in the semilattice $(\{0, 1\}, \vee)$.

6.2. **Idempotent varieties.** There is not much left to prove for idempotent varieties in this case. Let us just once more reformulate results of Kearnes and Kiss in the following way:

Corollary 6.3. *An idempotent variety does not satisfy a non-trivial congruence identity if and only if it contains an algebra compatible with* $\mathbb{S}$*.*

Proof. Suppose that an idempotent variety $\mathcal{V}$ does not satisfy a non-trivial congruence identity. Then by Theorem 6.1, it is interpretable in the variety of semilattices. Therefore, it contains an algebra $\mathbf{S}$ with $S = \{0, 1\}$ whose basic operations are term operations of the two-element semilattice $(S; \vee)$. But all such term operations are compatible with the relation $J^{\mathbb{S}}$ which means that $\mathbf{S}$ is compatible with $\mathbb{S}$.

For the other implication, suppose that $\mathcal{V}$ contains an algebra $\mathbf{S}$ compatible with $\mathbb{S}$. If this is the case, one can directly see that the terms of $\mathcal{V}$ are $\mathbb{S}$-colorable (define the coloring by taking the natural homomorphism from the free algebra generated by S to $\mathbf{S}$), and therefore it does not satisfy a non-trivial congruence identity by Proposition 6.2. □

6.3. **Proof of Theorem 1.4(iii).** The argumentation here follows a familiar pattern. First, we argue that any variety $\mathcal{V}$ that is either linear, or idempotent has an algebra compatible with $\mathbb{S}$: the linear case follows from Proposition 6.2 and Lemma 3.5, the idempotent case from Corollary 6.3.

Next, given that both $\mathcal{V}$ and $\mathcal{W}$ do not satisfy a congruence identity and are either linear, or idempotent, we know that both varieties contain an algebra compatible with $\mathbb{S}$, therefore also $\mathcal{V} \vee \mathcal{W}$ does. This shows in particular that $\mathcal{V} \vee \mathcal{W}$ has $\mathbb{S}$-colorable terms, therefore it does not satisfy a non-trivial congruence identity by Proposition 6.2. □

7. Having a cube term

Cube terms describe finite algebras having few subpowers (i.e., with a polynomial bound in n on the number of generators of subalgebras of n-th power). This result and many more interesting properties of algebras with cube terms can be found in [BIM+10], [AMM14], and [KS12]. There are several Mal'cev conditions equivalent to having a cube term, e.g. having an edge term, or having a parallelogram term. For our purpose, the most useful of these equivalent conditions is the cube term itself.

Fix a variety, and let $\mathbf{F}$ be an algebra that is freely generated by the set $\{x, y\}$. An n-cube term is a $(2^n - 1)$-ary term c such that

$$c^{\mathbf{F}^n}(\mathbf{x}_1, \dots, \mathbf{x}_{2^n-1}) = (x, \dots, x)$$

where $\mathbf{x}_1, \dots, \mathbf{x}_{2^n-1}$ are all the n-tuples of x's and y's containing at least one y. For example, a Mal'cev term q is a 2-cube term since it satisfies $q^{\mathbf{F}^2}((x, y), (y, y), (y, x)) = (x, x)$. The order of variables in cube terms will not play any role for us.

Although for the main result for cube terms, we only need to study the corresponding strong Mal'cev conditions, we will continue to formulate all results also for the general Mal'cev condition of having a cube term of some arity. The reason for that is that the construction we use for the general Mal'cev condition is a simpler variant of the constructions we use for the strong Mal'cev conditions.

Throughout the rest of the section, fix $n \geq 2$.

7.1. **Cube term blockers and crosses.** The notion of cube term blocker was introduced by by Marković, Maróti, and McKenzie [MMM12] to describe finite idempotent algebras that do not have a cube term. We define a *cube term blocker* to be a proper subset U of A such that for every $k \in \mathbb{N}$, $A^k \setminus (A \setminus U)^k$ is a subuniverse of $\mathbf{A}^k$.

Theorem 7.1 ([MMM12, Theorem 2.1]). *A finite idempotent algebra $\mathbf{A}$ has a cube term if and only if none of its subalgebras has a cube term blocker.* □

This theorem was recently generalized to idempotent varieties that are not neccessarily finitely generated by Kearnes and Szendrei [KS16]. They also proved a similar characterization for cube terms of fixed arity using crosses, that is, relations of the form

$$\operatorname{Cross}(U_1, \dots, U_k) = \{(x_1, \dots, x_k) : x_i \in U_i \text{ for some } i\}.$$

Note that U is a cube term blocker if and only if U is a proper subset of A and $\operatorname{Cross}(U, \dots, U)$ is a subuniverse of $\mathbf{A}^k$ for any k. The result of Kearnes and Szendrei can be formulated as follows.

Theorem 7.2 ([KS16, Theorems 2.5 and 3.1]). *Suppose that $\mathcal{V}$ is an idempotent variety, $\mathbf{F}$ denotes the free algebra generated by the set $\{x, y\}$, and let $n \geq 2$.*

(i) *$\mathcal{V}$ does not have an n-cube term if and only if there exist $U_1, \dots, U_n \subset F$ such that $x \in U_i$, $y \notin U_i$ for all i, and $\operatorname{Cross}(U_1, \dots, U_n)$ is a subuniverse of $\mathbf{F}^n$.*

(ii) *$\mathcal{V}$ does not have a cube term if and only if there exist a cube term blocker U in $\mathbf{F}$ such that $x \in U$, $y \notin U$.* □

The item (ii) above was also proved by McKenzie and Moore [MM17].

Next, we define concrete relational structures that we will use later. Unlike in [KS16], we understand crosses as multi-sorted relational structures; we encode them as 1-sorted structures whose universes are products of the original sorts.

Definition 7.3. We say that a structure $\mathbb{C} = (C_1 \times \cdots \times C_n; \alpha_1, \ldots, \alpha_n, R)$ is an *n-cross* if $\alpha_1, \ldots, \alpha_n$ are kernels of projections on the corresponding coordinate, and for each $i \in \{1, \ldots, n\}$, there exists $U_i \subseteq C_i$ such that

$$R = \{(a_1, \ldots, a_n) \in C_1 \times \cdots \times C_n : a_i \in U_i \text{ for some } i\}.$$

When defining or talking about n-crosses, we will usually only speak about the relation R since the other relations are implicitly given as kernels of projections.

Definition 7.4. We say that a relational structure $\mathbb{B} = (B; (R_k)_{k\in\mathbb{N}})$ is a *cube term blocker* if there exists $U \subset B$, $U \neq \emptyset$ such that

$$R_k = \{(a_1, \ldots, a_k) \in B^k : a_i \in U \text{ for some } i\}.$$

To distinguish between the above definition and the original definition of cube term blockers, we will always say 'an algebra has a cube term blocker' or 'U is a cube term blocker of an algebra $\mathbf{A}$' if we refer to the original definition. Note that an algebra $\mathbf{A}$ has a cube term blocker $U \subseteq A$ if and only if it is compatible with some cube term blocker $\mathbb{B}$.

The two transfinite chains of clones will be defined using the following relational structures:

Definition 7.5. Let κ be an infinite cardinal, and fix $U_\kappa \subseteq \kappa$ with $|U_\kappa| = |\kappa \setminus U_\kappa| = \kappa$. We define a relation $R_k \subseteq \kappa^k$ as

$$R_k = \{(a_1, \ldots, a_k) \in \kappa^k : a_i \in U_\kappa \text{ for some } i\}.$$

The special cube term blocker is defined as $\mathbb{B}_\kappa = (\kappa; (R_k)_{k\in\mathbb{N}})$, and the special n-cross as $\mathbb{C}_\kappa = (\kappa^n; \alpha_1^{\mathbb{C}_\kappa}, \ldots, \alpha_n^{\mathbb{C}_\kappa}, R_n)$ where $\alpha_i^{\mathbb{C}_\kappa}$ is a kernel of the i-th projection.

For coloring we will use the smallest cube term blocker $\mathbb{B}_0$ and its reduct $\mathbb{B}_n$:

Definition 7.6. We define $\mathbb{B}_0$ to be the cube term blocker on the set $\{0, 1\}$ defined by $U = \{1\}$, i.e.,

$$R_k^{\mathbb{B}_0} = \{0, 1\}^k \setminus \{(0, \ldots, 0)\}$$

for $k \in \mathbb{N}$. We define $\mathbb{B}_n$ to be the relational structure $(\{0, 1\}; R^{\mathbb{B}_n})$ with $R^{\mathbb{B}_n} = R_n^{\mathbb{B}_0}$.

7.2. **Coloring and linear varieties.** We provide characterizations in the means of colorings both for the Mal'cev condition of having a cube term of some arity and the strong Mal'cev condition of having an n-cube term.

Proposition 7.7. *The following is true for every variety $\mathcal{V}$ and all $n \geq 2$.*

(i) *$\mathcal{V}$ does not have an n-cube term if and only if it has strongly $\mathbb{B}_n$-colorable terms.*
(ii) *$\mathcal{V}$ does not have a cube term if and only if it has strongly $\mathbb{B}_0$-colorable terms.*

Proof. (i) Let $\mathbb{F}_n$ denote the free structure generated by $\mathbb{B}_n$. Observe that tuples in $R^{\mathbb{F}_n}$ are exactly those n-tuples that can be the result of applying some term f coordinatewise to all n-tuples consisting of x_0's and x_1's except the tuple $(x_0, \ldots, x_0)$.

To prove the first implication, suppose that $\mathcal{V}$ does not have a cube term, and define a mapping $c\colon F_n \to B_n$ by:

$$c(f) = \begin{cases} 0 & \text{if } f = x_0, \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Since $\mathcal{V}$ does not have a cube term, $R^{\mathbb{F}_n}$ does not contain the tuple $(x_0, \dots, x_0)$, therefore c is clearly a coloring from $\mathbb{F}_n$ to $\mathbb{B}_n$. It is also strong by definition.

For the other implication, suppose that $\mathcal{V}$ has an n-cube term t. Therefore, in particular $(x_0, \dots, x_0) \in R^{\mathbb{F}_n}$. Any mapping that maps x_0 to 0 maps this tuple to the tuple $(0, \dots, 0)$, therefore there is no homomorphism $c\colon \mathbb{F}_n \to \mathbb{B}_n$ with $c(x_0) = 0$, and in particular no strong coloring of terms of $\mathcal{V}$.

(ii) Let $\mathbb{F}$ denote the free structure generated by $\mathbb{B}_0$, and note that $\mathbb{F}$ is an expansion of every $\mathbb{F}_k$ (the free structure generated by $\mathbb{B}_k$). Therefore, if a variety $\mathcal{V}$ has a k-cube term then its terms are not strongly $\mathbb{B}_0$-colorable since they are not even $\mathbb{B}_k$-colorable. This yields one implication. For the other implication, suppose that $\mathcal{V}$ does not have a cube term. We define a mapping c the same way as in the proof of (i). The argument that this is really a strong coloring is identical. □

As a corollary of the above, we get the following.

Corollary 7.8. *If $\mathcal{V}$ is a linear variety that does not have an n-cube term, then it contains an algebra compatible with an n-cross of size 2^n.*

Proof. From the above and Lemma 3.5, we know that $\mathcal{V}$ contains an algebra $\mathbf{B}_n$ compatible with $\mathbb{B}_n$. Let $\mathbf{C} = \mathbf{B}_n^n$. We claim that $\mathbf{C}$ is compatible with an n-cross $\mathbb{C}$ with $R^{\mathbb{C}} = R^{\mathbb{B}_n}$ where $R^{\mathbb{C}}$ is understood as a unary relation on C. Indeed, the operations of $\mathbf{C}$ are compatible with kernels of projections since they are defined component-wise; they are also compatible with $R^{\mathbb{C}}$ since component-wise they act as operations of $\mathbf{B}_n$ and are therefore compatible with $R^{\mathbb{B}_n}$. Finally, $|C| = 2^n$ since $|B_n| = 2$. □

7.3. **Idempotent varieties.** We encounter a similar problem as in Section 4.3: we are unable to find a largest idempotent variety without a cube term (or n-cube term). We will circumvent this problem in a similar way. The first step is the following corollary of Theorem 7.2.

Corollary 7.9. *Fix $n \geq 2$ and let $\mathcal{V}$ be an idempotent variety.*

(i) *If $\mathcal{V}$ does not have an n-cube term, then it contains an algebra compatible with an n-cross.*
(ii) *If $\mathcal{V}$ does not have a cube term, then it contains an algebra compatible with a cube term blocker.*

Proof. (i) Suppose that $\mathcal{V}$ does not have an n-cube term, and let $\mathbf{F}$ denote the two-generated free algebra in $\mathcal{V}$. From Theorem 7.2(i), we know that there are $U_1, \dots, U_n \subseteq F$ such that $\operatorname{Cross}(U_1, \dots, U_n)$ is a compatible relation of $\mathbf{F}$. We obtain an algebra compatible with an n-cross by repeating the argument from the proof of Corollary 7.8: Let $\mathbf{A} = \mathbf{F}^n$, and define an n-cross $\mathbb{A}$ by taking $R^{\mathbb{A}} = \operatorname{Cross}^{\mathbf{F}}(U_1, \dots, U_n)$.

(ii) This is immediate from Theorem 7.2(ii): the algebra is $\mathbf{F}$, and the cube term blocker is $\mathbb{F} = (F; \left(F^n \setminus (F \setminus U)^n\right)_{n \in N})$. □

Proposition 7.10. *Let $\mathcal{V}$ be a variety and $n \geq 2$.*

(i) *If $\mathcal{V}$ contains an algebra compatible with an n-cross $\mathbb{C}$, then it contains an algebra compatible with $\mathbb{C}_\kappa$ for all $\kappa \geq \aleph_0 + |\mathbb{C}|$*
(ii) *If $\mathcal{V}$ contains an algebra compatible with a cube term blocker $\mathbb{B}$, then it contains an algebra compatible with $\mathbb{B}_\kappa$ for all $\kappa \geq \aleph_0 + |\mathbb{B}|$.*

Proof. Again, we will use Tarski's construction. First, we prove (ii): Suppose that $\mathbf{B}$ is an algebra in $\mathcal{V}$ with a cube term blocker U (recall that having a cube term blocker is equivalent to being compatible with a relational structure which is a cube term blocker). Put $\mathbf{B}_0 = \mathbf{B}$ and $U_0 = U$, and let $\mathbf{B}_\lambda$ and U_λ denote the algebra and the set obtained by Tarski's construction in λ steps. We claim that U_λ is a cube term blocker of $\mathbf{B}_\lambda$. We will prove that by transfinite induction on λ: It is true for $\lambda = 0$. For ordinal successor $\lambda + 1$, we identify $\mathbf{B}_{\lambda+1}$ with $\mathbf{B}_\lambda^2$, and $U_{\lambda+1}$ with $B_\lambda \times U_\lambda$. We need to prove that $R_k = B_{\lambda+1}^k \setminus (B_{\lambda+1} \setminus U_{\lambda+1})^k$ is compatible for all $k \in \mathbb{N}$. But this is true since $B_\lambda^k \setminus (B_\lambda \setminus U_\lambda)^k$ is and

$$R_k = \{((a_1, b_1), \dots, (a_k, b_k)) \in (B_\lambda^2)^k : (b_1, \dots, b_n) \in B_\lambda^k \setminus (B_\lambda \setminus U_\lambda)^k\}.$$

For limit λ, we get the statement by the standard compactness argument. In particular, we proved that U_λ is a cube term blocker in $\mathbf{B}_\lambda$ for $|\lambda| = \kappa$. For such λ, we get that the structure $(B_\lambda, (R_{k,\lambda})_{k\in\mathbb{N}})$, where $R_{k,\lambda} = B_\lambda^k \setminus (B_\lambda \setminus U_\lambda)^k$, is isomorphic to $\mathbb{B}_\kappa$. This is given by the definition of $R_{k,\lambda}$ and that $|B_\lambda| = |U_\lambda| = |B_\lambda \setminus U_\lambda| = \kappa$. This completes the proof of item (ii).

The item (i) is obtained by repeating the above argument while using Tarski's construction for all n sorts of the n-cross. Let $\mathbf{C}$ be the algebra compatible with $\mathbb{C}$, let U_i for $i = 1, \dots, n$ be the sets defining the relation $R^{\mathbb{C}}$, and let $\mathbf{C}_1 \times \dots \times \mathbf{C}_n$ be the factorization of $\mathbf{C}$ so that $\alpha_i^{\mathbb{C}}$ is the kernel of projection to $\mathbf{C}_i$. We define algebras $\mathbf{C}_{i,\lambda}$ (where $i = 1, \dots, n$ and λ ranges through all ordinals) together with sets $U_{i,\lambda}$ by Tarski's construction starting with $\mathbf{C}_{i,0} = \mathbf{C}_i$ and $U_{i,0} = U_i$. By a similar argument as for item (ii), we obtain a structure of n-cross $\mathbb{C}'_\lambda$ on the set $C_{1,\lambda} \times \dots \times C_{n,\lambda}$ with $R^{\mathbb{C}'_\lambda}$ being the relation

$$\{(c_1, \dots, c_n) \in C_{1,\lambda} \times \dots \times C_{n,\lambda} : c_i \in U_{i,\lambda} \text{ for some } i\ \},$$

and also that this structure is compatible with $\mathbf{C}_{1,\lambda} \times \dots \times \mathbf{C}_{n,\lambda}$ for all λ. Finally, given that λ satisfies $|\lambda| = \kappa$, we claim that $\mathbb{C}'_\lambda$ is isomorphic to $\mathbb{C}_\kappa$. One such isomorphism can be defined component-wise, taking for the i-th component a_i any bijection from $C_{i,\lambda}$ to κ that maps $U_{i,\lambda}$ onto U_κ. □

Lemma 7.11. *If a variety contains an algebra compatible with an n-cross, it does not have an n-cube term.*

Proof. For the contrary, suppose that t is a cube term and let $\mathbb{C}$ be an n-cross. We claim that no algebra $\mathbf{C}$ in the variety is compatible with $\mathbb{C}$. To prove that, suppose that $\mathbf{C}$ is an algebra on C whose operations are compatible with $\alpha_i^{\mathbb{C}}$ for $i = 1, \dots, n$. Therefore, $\mathbf{C}$ factors as $\mathbf{C}_1 \times \dots \times \mathbf{C}_n$ with $\alpha_i^{\mathbf{C}}$ being the kernels of projections. Let $U_1 \subset C_1, \dots, U_n \subset C_n$ be such that

$$R^{\mathbb{C}} = \{(c_1, \dots, c_n) \in C_1 \times \dots \times C_n : c_i \in U_i \text{ for some } i\ \},$$

and choose $a_i \in U_i$ and $b_i \in C_i \setminus U_i$. If we apply $t^{\mathbf{C}}$ on the tuples from the set

$$\{\{a_1, b_1\} \times \dots \times \{a_n, b_n\}\} \setminus \{(b_1, \dots, b_n)\} \subseteq R^{\mathbb{C}},$$

we obtain $(b_1, \dots, b_n) \notin R^{\mathbb{C}}$. This is due to the fact that $t^{\mathbf{C}}$ acts coordinate-wise as $t^{\mathbf{C}_i}$ and t is a cube term of the variety. This concludes that $t^{\mathbf{C}}$ is not compatible with $R^{\mathbb{C}}$, and consequently $\mathbf{C}$ is not compatible with $\mathbb{C}$. □

7.4. **Proof of Theorem 1.4(iv).** We have proven that, given a variety $\mathcal{V}$ that is either linear, or idempotent and does not have an n-cube term, $\mathcal{V}$ has an algebra compatible with an n-cross $\mathbb{C}^{\mathcal{V}}$ (see Corollaries 7.8 and 7.9(i)). Given two such varieties $\mathcal{V}$ and $\mathcal{W}$, and n-crosses $\mathbb{C}^{\mathcal{V}}$ and $\mathbb{C}^{\mathcal{W}}$, we know from Proposition 7.10(i) that both $\mathcal{V}$ and $\mathcal{W}$ contain an algebra compatible with the n-cross $\mathbb{C}_\kappa$ for all infinite cardinals $\kappa \geq |C^{\mathcal{V}}| + |C^{\mathcal{W}}|$. Therefore, by Lemma 3.2 also their interpretability join does. Finally, Lemma 7.11 gives that $\mathcal{V} \vee \mathcal{W}$ does not have an n-cube term. □

7.5. **Cofinal chains.** We will discuss some properties of the transfinite sequences $\mathscr{B}_0, \mathscr{B}_{\aleph_0}, \mathscr{B}_{\aleph_1}, \dots$ and $\mathscr{C}_{\aleph_0}, \mathscr{C}_{\aleph_1}, \dots$ where $\mathscr{B}_\kappa$ are polymorphism clones of cube term blockers $\mathbb{B}_\kappa$, and $\mathscr{C}_\kappa$ are polymorphism clones of the n-crosses $\mathbb{C}_\kappa$ for a fixed $n \geq 2$. We show that these sequences form strictly increasing chains in the lattice of clones, and as a corollary thereof, we obtain that there is no maximal (in the interpretability order) idempotent variety without an n-cube term.

Corollary 7.12. *Let $\lambda < \kappa$ be two infinite cardinals, and $n \geq 2$. Then*

(i) *there is a clone homomorphism from $\mathscr{C}_\lambda$ to $\mathscr{C}_\kappa$, and*
(ii) *there is a clone homomorphism from $\mathscr{B}_\lambda$ to $\mathscr{B}_\kappa$.*

Proof. The statement follows from Proposition 7.10 in the same way as Corollary 4.11 follows from Proposition 4.10. □

In order to prove that there is no largest idempotent variety without a cube term, we need to restrict to idempotent reducts of the discussed clones.

Lemma 7.13. *Let κ be an infinite cardinal. The idempotent reduct of $\mathscr{B}_\kappa$ has no action on any set C with $1 < |C| < \kappa$.*

Proof. Let $\mathscr{B}_\kappa^{\mathrm{id}}$ denote the idempotent reduct of $\mathscr{B}$. We will show that if $\mathscr{B}_\kappa$ has an action on a set C of size at least 2 then $|C| \geq \kappa$.

To prove that we find identities that are satisfied in $\mathscr{B}_\kappa^{\mathrm{id}}$ but cannot be satisfied by functions on a set of size smaller then κ unless this set has only one element. These identities are very similar to those used in Proposition 4.13. They use binary symbols f_i, $i \in \kappa$ and ternary symbols $p_{i,j}$, $q_{i,j}$, $i, j \in \kappa$:

$$\begin{aligned} x &\approx p_{i,j}(x, f_j(x,y), y), \\ p_{i,j}(x, f_i(x,y), y) &\approx q_{i,j}(x, f_j(x,y), y), \\ q_{i,j}(x, f_i(x,y), y) &\approx y \end{aligned}$$

for all $i \neq j$, and $f_i(x,x) \approx x$ for all i. Note that these identities force that all the functions are idempotent.

We claim that $\mathbb{B}_\kappa$ has polymorphisms satisfying these equations: Fix $c \in U_\kappa$, put $f_i(x,y) = i$ for $x \neq y$, $f_i(x,x) = x$, and define $p_{i,j}$ and $q_{i,j}$ by:

$$p_{i,j}(x,y,z) = \begin{cases} x & \text{if } y = f_j(x,z) \text{, and} \\ c & \text{otherwise;} \end{cases}$$

$$q_{i,j}(x,y,z) = \begin{cases} z & \text{if } y = f_i(x,z) \text{, and} \\ c & \text{otherwise.} \end{cases}$$

Clearly, these operations satisfy the required identities. To prove that they are also compatible with $\mathbb{B}_\kappa$, observe that any operation t which has a coordinate i such that $t(x_1, \dots, x_n) \in U_\kappa$ whenever $x_i \in U_\kappa$ is a polymorphism of $\mathbb{B}_\kappa$. The corresponding coordinates for our functions are: the first for p's, the last for q's, and arbitrary for f's.

The last part is to prove that these identities are not satisfiable in any non-trivial algebra of size strictly less than κ. This follows from the same argument as Proposition 4.13. In fact, these identities are a stronger version of those used in the mentioned proof. $\square$

Corollary 7.14. *Let $\lambda < \kappa$ be two infinite cardinals, $n \geq 2$, and let $\mathscr{A}^{id}$ denote the idempotent reduct of a clone $\mathscr{A}$. Then*

(i) *there is no clone homomorphisms from $\mathscr{C}_\kappa^{id}$ to $\mathscr{C}_\lambda^{id}$, and*
(ii) *there is no clone homomorphisms from $\mathscr{B}_\kappa^{id}$ to $\mathscr{B}_\lambda^{id}$.*

Proof. The item (ii) follows directly from the above lemma. To prove item (i), we first observe that there is a clone homomorphism ξ from $\mathscr{B}_\kappa^{\mathrm{id}}$ to $\mathscr{C}_\kappa^{\mathrm{id}}$: define $\xi(f)$ to be the component-wise action of f on κ^n. Indeed, if $f \in \mathscr{B}_\kappa^{\mathrm{id}}$, $\xi(f)$ preserves $R^{\mathbb{C}_\kappa}$ since it preserves $R_n^{\mathbb{B}_\kappa}$. Now, if there was a clone homomorphism from $\mathscr{C}_\kappa^{\mathrm{id}}$ to $\mathscr{C}_\lambda^{\mathrm{id}}$, we would get one from $\mathscr{B}_\kappa^{\mathrm{id}}$ to $\mathscr{C}_\lambda^{\mathrm{id}}$ by composing it with ξ which contradicts Lemma 7.13. $\square$

By analogous proofs as 4.14 we obtain the following.

Corollary 7.15. *Fix $n \geq 2$.*

(i) *The class of all interpretability classes of idempotent varieties that do not have an n-cube term does not have a largest element.*
(ii) *The class of all interpretability classes of idempotent varieties that do not have a cube term does not have a largest element.* $\square$

7.6. **Remark(s).** The presented proof of Theorem 1.4 was developed independently of Kearnes and Szendrei after their announcement of Theorem 7.2, and before the final manuscript [KS16] was available.

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