Interval k-Graphs and Orders

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Abstract

An interval k-graph is the intersection graph of a family \mathcal{I} of intervals of the real line partitioned into at most k classes with vertices adjacent if and only if their corresponding intervals intersect and belong to different classes. In this paper we discuss the interval k-graphs that are the incomparability graphs of orders; i.e., cocomparability interval k-graphs or interval k-orders. Interval 2-orders have been characterized in many ways, but we show that analogous characterizations do not carry over to interval k-orders, for k > 2. We describe the structure of interval k-orders, for any k, characterize the interval 3-orders (cocomparability interval 3-graphs) via one forbidden suborder (subgraph), and state a conjecture for interval k-orders (any k) that would characterize them via two forbidden suborders.

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1 Introduction

We discuss finite simple graphs and use the notation G = (V, E) to denote a graph with vertex set V = V(G) and edge set E = E(G). For the complement of graph G we use the notation \overline{G} .

An ordered set (or strict partial order) is a pair $P = (X, \prec)$ consisting of a ground set X and a binary relation \prec on X that is irreflexive, transitive and therefore antisymmetric. If neither $x \prec y$ nor $y \prec x$ occurs in P, we say x and y are incomparable and write $x \parallel y$, otherwise they are comparable. Two graphs are naturally associated to the order P: its comparability graph and its incomparability graph. The graph G = (V, E) is the comparability graph of P if V = X and, for $x, y \in X$, $xy \in E$ if and only if x and y are comparable. The incomparability graph of P has vertex set X and vertices x and y adjacent if and only if $x \parallel y$ in P. Note that although $x \parallel x$ for any $x \in X$, we choose not to clutter the incomparability graphs with loops so $x \parallel x$ does not yield an edge, and of course the complement of G is the incomparability graph of P if G is the comparability graph of some order and when the edges of G can be transitively oriented, G is called a cocomparability graph and is hence the incomparability graph of some order. Most of the graphs we discuss in this paper are cocomparability graphs.

A family of sets $\mathcal{F} = \{S_1, S_2, \dots, S_n\}$ is an intersection representation of a graph G if V(G) can be put into one-to-one correspondence with \mathcal{F} so that $S_i \cap S_j \neq \emptyset$ if and only if vertices u and v are adjacent in G. For example, if graph G is from the well-studied class of *interval graphs*, then G is a graph which can be represented so that \mathcal{F} is a family of intervals of the real line. In this paper we investigate graphs which can be represented as intersection graphs of intervals of the real line, but unlike interval graphs, with the property that certain subsets of intervals' intersection information does not correspond to adjacency in the graph. Specifically, we investigate graphs G = (V, E) for which there is a one-to-one correspondence between V and a collection of intervals of the real line \mathcal{I} partitioned into what we will call interval classes or simply classes so that vertices are adjacent if and only if their corresponding intervals intersect and belong to different classes. We use I_v to denote the interval corresponding to vertex v. If for G = (V, E) there is such a representation \mathcal{I} partitioned into at most k classes $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_k$, with vertices u and v adjacent in G if and only if $I_u \cap I_v \neq \emptyset$ and I_u, I_v belong to different interval classes, then G is an interval k-graph. The collection \mathcal{I} with the partition into classes will be called an interval k-representation or simply a representation if the context precludes ambiguity. Note that the set of vertices corresponding to intervals from any class induce an independent set. In the case k=2, this class has been called the *interval bigraphs* and has enjoyed considerable attention recently, see for example [6, 10, 17] and their references for more. Our focus is on the interval k-graphs that are cocomparability graphs and hence those interval k-graphs which give rise to a strict partial order we will call an *interval* k-order.

The class of probe interval graphs is another class of intersection graphs that has enjoyed recent attention. See for example [7, 8, 20, 19, 22]. A probe interval graph is another interval-intersection graph in which certain intervals' intersection information is ignored. A graph G is a probe interval graph if its vertices can be partitioned into sets P (probes) and N (nonprobes) with an interval of the real line corresponding to each vertex, and vertices adjacent if and only if their corresponding intervals intersect and at least one is a probe. In [4] the probe interval graphs that are cocomparability graphs, and hence incomparability graphs of probe interval orders, are characterized in various ways. One characterization states the collection of intervals corresponding to the nonprobes has the property that no interval contains another properly while the probes' intervals are not restricted. If a probe interval graph has such a representation, it is called a nonprobe-proper probe interval graph.

Theorem 1.1 (Brown, Langley, [4]) The graph G is a cocomparability graph whose vertices can be partitioned into sets P and N with N an independent set, and every 4-cycle alternates between N and P if and only if G is a nonprobe-proper probe interval graph.

The results we develop here are similar to those in [4] in that we are (1) trying to do for interval k-orders what Fishburn did for interval orders (see [13]) and (2) we show that the mechanism by which an interval k-graph may contain an obstruction to being a cocomparability graph is having an interval from some interval class contain another from that class properly. Whence we define the following restricted class of interval k-graphs. Let G be an interval k-graph with an interval representation \mathcal{I} partitioned into classes $\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_k$ so that no interval from any class contains another from its class properly. We call such an interval k-graph a class-proper interval k-graph, and the collection of intervals representing it a class-proper representation.

Thanks to the monumental characterization of transitively orientable graphs by Gallai (Theorem 1.2 below) we can find interval k-graphs which are not cocomparability graphs by identifying odd asteroids. An odd asteroid is a sequence $v_0, P_0, v_1, P_1, v_2, \ldots, v_{2n}, P_{2n}, v_0$, where v_0, v_1, \ldots, v_{2n} are distinct vertices, P_i is a v_i, v_{i+1} -path, and $N(v_i) \cap P_{i+n} = \emptyset$, where subscripts are taken modulo 2n+1. If a graph has a set of 2n+1 vertices on which an odd asteroid exists, we will call the set of vertices a (2n+1)-asteroid. The graph, which we refer to as T_2 in Figure 1 has a 3-asteroid (also known as an asteroidal triple) on the vertices a, b, and c. The graph G of Figure 2 has a 5-asteroid on $v_0 = q, v_1 = x, v_2 = a, v_3 = c$, and $v_4 = b$, whence it is not a cocomparability graph, and neither is T_2 , via the theorem of Gallai we mentioned.

Theorem 1.2 (Gallai, [15]) The complement of a graph G has a transitive orientation if and only if G has no odd asteroid.

But G and T_2 have interval k-representations as illustrated and so the class of interval k-graphs is not contained in the class of cocomparability graphs. The converse containment relationship also does not hold; this will be shown below.

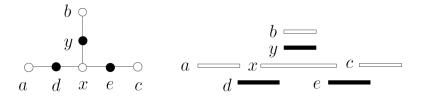


Figure 1: An interval 2-graph with an asteroidal triple and interval representation illustrating how an asteroidal triple requires some interval to contain another from the same class properly.

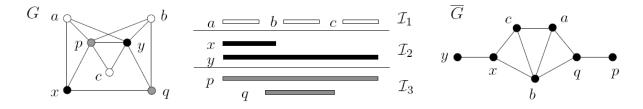


Figure 2: An interval 3-graph which is not a cocomparability graph and its representation showing it is not class-proper.

Any probe interval graph is an interval k-graph, as was shown in [6], but in Figure 3 we have a cocomparability interval 3-graph M which is not a probe interval graph. The poset P corresponds to a transitive orientation of the complement of M. Therefore the class of cocomparability interval k-graphs contains the class of cocomparability probe interval graphs.

A beautiful characterization of cocomparability graphs by Golumbic, Rotem, and Urrutia (Theorem 1.3 below) shows they are precisely the function graphs. A function graph G is the intersection graph of a family of curves of continuous functions f_i : $[a,b] \to \mathbb{R}$; that is, vertices i and j are adjacent if and only if $f_i(x) = f_j(x)$ for some $x \in [a,b]$. It is easy to see that every function graph is a cocomparability graph: orient $i \to j$ in \overline{G} if $f_i(y) < f_j(y)$ for all $y \in [a,b]$.

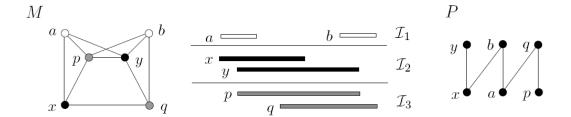


Figure 3: A cocomparability interval 3-graph M which is not a probe interval graph, its class-proper interval representation, and the Hasse diagram of the strict order P obtained from a transitive orientation of \overline{M} by the methods in Theorem 4.1.

Theorem 1.3 (Golumbic, Rotem, Urrutia, [16]) A graph is a function graph if and only if it is a cocomparability graph.

A graph is *weakly chordal* if neither it nor its complement contains an induced cycle on five or more vertices as an induced subgraph.

Theorem 1.4 (Brown, [6]) If G is an interval k-graph, then G is weakly chordal.

In Figure 4 we have a function representation of the complement of a 6-cycle, which is not an interval k-graph, by Theorem 1.4; therefore there is no containment relationship between interval k-graphs and cocomparability graphs.

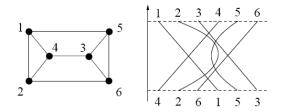


Figure 4: A function representation of the complement of a 6-cycle, which is not an interval k-graph by Theorem 1.4.

2 Interval 2-Graphs and Orders

In the next section, we show that if an interval k-graph, for $k \geq 3$, has a class-proper representation, then it is a cocomparability graph. In this section we show that the well-known and well-studied classes of bipartite cocomparability graphs, bipartite

permutation graphs, and proper and unit interval bigraphs are precisely the class-proper interval 2-graphs. We also show that for k > 2 none of the characterizations in Theorem 2.1 extend to proper interval k-graphs or class-proper interval k-graphs. Furthermore, for k > 2, the classes proper, unit, and class-proper interval k-graphs are different.

To prove that class-proper interval 2-graphs are precisely the cocomparability interval 2-graphs, we will take the circuitous route of proving the following cycle of implications, referring to statements in Theorem 2.1: $1 \implies 2 \implies 3 \implies 4 \implies 5 \implies 1$. In so doing we will establish a characterization via the existence of an ordering of the graph's vertices, statement 5, which is stronger than the strong ordering referred to in statement 6. This is the most efficient way we could prove this, possibly due to the fact that the class-proper restriction is not much of one; to wit, intervals in one interval class could be distinct points while those in the other be unit-length intervals for example. In the interest of brevity we will direct the reader to the literature where the results and all definitions can be found; we will give only the essential ones.

A bipartite graph G is a unit interval bigraph if it is an interval 2-graph which has a representation where all intervals have identical length. G is a proper interval bigraph if G is an interval 2-graph which has a representation in which no interval contains another properly. A graph H is a permutation graph if $V(H) = \{1, 2, ..., n\}$ and there is a permutation $(\pi_1, \pi_2, ..., \pi_n)$ of the numbers V(H) such that vertices are adjacent if and only if the numbers are in reversed order in the permutation. Equivalently, and this is the definition we will use, a permutation graph G may be defined as the intersection graph of line segments $\{\ell_v : v \in V(G)\}$ contained in the space between parallel line segments L_1 and L_2 , we will call channels. For the point where ℓ_v intersects L_i we will use p(v;i). For the intervals in an interval representation, whether it be unit, proper, or class-proper, we will use I(v) = [L(v), R(v)] to denote the interval corresponding to vertex v.

Theorem 2.1 Let G be a bipartite graph. The following are equivalent:

- (1) G is a unit interval bigraph [1, 2, 17, 21];
- (2) G is a proper interval bigraph [17];
- (3) G is a class-proper interval 2-graph;
- (4) G is a permutation graph [1, 17];
- (5) V(G) can be ordered $(v_1, v_2, ..., v_n)$ so that whenever $v_i v_k \in E(G)$ and i < j < k, v_j is adjacent to whichever of $\{v_i, v_k\}$ is not in its partite set;
- (6) G has a strong ordering [1];
- (7) The bipartite adjacency matrix of G has a monotone consecutive arrangement [21];
- (8) G is the comparability graph of a poset of dimension at most 2 [12, 17];

- (9) G is the incomparability graph of a poset of dimension at most 2 [12, 17];
- (10) \overline{G} is a proper circular arc graph [17];
- (11) G is contains no asteroidal triple [17].
- **Proof.** $(1 \implies 2)$ A unit representation is a proper representation.
 - $(2 \implies 3)$ A proper interval bigraph is clearly a class-proper interval bigraph.
- $(3 \Longrightarrow 4)$ Place a copy of the class-proper interval representation for G on each of the channels L_1 and L_2 . With \mathcal{I}_X and \mathcal{I}_Y the intervals corresponding to G's bipartition, create ℓ_v , for $v \in X$, by connecting l(v) on L_1 with r(v) on L_2 , and for $v \in Y$, connect r(v) on L_1 with l(v) on L_2 . The line segments for vertices from the same partite set will not cross because the interval representation is class proper, and $xy \in E(G)$, for $x \in X$, $y \in Y$, if and only if r(x) > l(y) and r(y) > l(x) if and only if ℓ_x crosses ℓ_y . Thus, the line segments $\{\ell_v : v \in X \cup Y\}$ between the channels form a permutation representation for G.
- $(4 \implies 5)$ Given a segment representation of a bipartite permutation graph, consider the vertex ordering v_1, v_2, \ldots, v_n given by the order of p(v; 1), the endpoints of the segments on channel L_1 . That is, i < j if $p(v_i; 1)$ is left of $p(v_j; 1)$. Consider v_i, v_j, v_k with i < j < k and $v_i v_k \in E(G)$. The segments ℓ_{v_i} and ℓ_{v_k} intersect before reaching the other channel. Since ℓ_{v_j} starts between ℓ_{v_i} and ℓ_{v_k} , it cannot reach the other side without intersecting one of them. It does not intersect the segment for the vertex in its own partite set, so it intersects the other. We have the vertex ordering desired.
- (5 \Longrightarrow 1) Suppose the bipartite graph G's vertices have been ordered (v_1, v_2, \ldots, v_n) in accord with the statement 5 and label the partite sets $X = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ and $Y = \{v_{j_1}, v_{j_2}, \ldots, v_{j_s}\}$ so that the indices respect the ordering (e.g., $v_{i_t}v_{j_q} \in E(G)$ implies $v_{i_t}v_{j_p}$ if $i_t < j_q < j_p$). For convenience, drop the indices on the indices and put $v_{i_t} = x_t$ for $1 \le t \le r$, and $v_{j_q} = y_q$ for $1 \le q \le s$. We construct a class-proper representation for G by induction on n = r + s creating the set of intervals $\{I_v = [L(v), L(v) + 1] : v \in V(G)\}$ with left end-points distinct and respecting the ordering; i.e., $L(v_1) < L(v_2) < \cdots < L(v_n)$. If $\min\{r, s\} = 1$, the construction is obvious. Also, since the intervals of isolated vertices are easy to incorporate we will assume there are none. Now, assume a class-proper representation has been created for G induced on $\{x_1, \ldots, x_r\} \cup \{y_1, \ldots, y_{s-1}\}$, switching the roles of X and Y if necessary.

Lemma A: If $x_i y_j, x_t y_q \in E(G)$ and i < t and q < j, then $x_i y_q, x_t y_j \in E(G)$.

Proof of lemma: This is essentially the observation that the ordering restricted to each partite set is the strong ordering developed in [1]. By symmetry we may assume x_i is first among $\{x_i, x_t, y_j, y_q\}$. The possible orderings of these four elements are $(x_i, y_q, y_j, x_t), (x_i, y_q, x_t, y_j)$, and (x_i, x_t, y_q, y_j) . In each case $x_i y_q, x_t y_j \in E(G)$ in virtue of the properties of the ordering. This proves the lemma.

Define q and j to be the smallest and largest index, respectively, for which $x_r y_q \in E(G)$ and $x_r y_j \in E(G)$. Define i and t to be the smallest and largest index, respectively, for which $x_i y_s \in E(G)$ and $x_t y_s \in E(G)$. If t < r, then q = s and x_{t+1}, \ldots, x_r are isolated (vis-a-vis $x_{t+1} y_{s-1} \in E(G) \implies x_r y_s \in E(G)$ by Lemma A); hence t = r. Similarly, if j < s, then y_{j+1}, \ldots, y_s are isolated; so j = s. Note that by the properties of the ordering, y_s is adjacent to each of $x_i, x_{i+1}, \ldots, x_r$ and x_r is adjacent to each of y_q, \ldots, y_s . Then by Lemma A the graph induced on $\{x_i, \ldots, x_r\} \cup \{y_q, \ldots, y_s\}$ is a biclique and hence the intersection of all the intervals corresponding to these vertices is not empty. Furthermore, since all endpoints are distinct, this intersection is not a point. Therefore there is a point p in this intersection with $p > L(y_{s-1})$, and defining $I(y_s) = [p, p+1]$ completes the unit interval construction.

Theorem 2.1 could be extended with at least seven more statements (cf. [5]) and we would like to see a proof incorporating all (at least) eighteen statements characterizing cocomparability interval 2-graphs into a cycle of implications using no extraneous results. Indeed, we have tried to produce such a proof, but statements 10 and 11 have been prohibitive. So far statement 11 has only tedious proofs with exhaustive case analysis or an appeal to Theorem 1.2. Statement 10 so far requires an appeal to a result of Spinrad in [23].

3 Autopsy of Theorem 2.1's attempted extension to k > 2

Although the classes of unit interval k-graphs and proper interval k-graphs are identical, see [3], the analogues of statements in Theorem 2.1 extend no further for proper interval k-graphs with k > 2. In this section we show that the statements 3, 4, 5, 8, 9, 10, and 11 of Theorem 2.1 do not necessarily hold for a proper interval k-graph, k > 2.

First we show that the vertices of any proper (or unit) interval k-graph can be ordered as in statement 5 of Theorem 2.1, but the ordering does not characterize proper interval k-graphs.

Theorem 3.1 If a k-partite graph G is a proper or unit interval k-graph, then V(G) can be labeled v_1, v_2, \ldots, v_n so that, for i < j < k, if $v_i v_k \in E(G)$, then v_j is adjacent to each of $\{v_i, v_k\}$ in a different partite set than v_j .

Proof. Suppose G is a proper or unit interval k-graph with interval representation $\{I(v) = [\ell(v), r(v)] : v \in V(G)\}$. Label the vertices v_1, v_2, \ldots, v_n so that $\ell(v_i) < \ell(v_j)$

if and only if i < j. Suppose $v_i v_k \in E(G)$, where i < k, and consider v_j with i < j < k. Since $\ell(v_i) < \ell(v_j) < \ell(v_k)$ and no interval properly contains another, we know $r(v_i) < r(v_j) < r(v_k)$ and $r(v_i) \ge \ell(v_k)$ since $I(v_i) \cap I(v_k) \ne \emptyset$. So $I(v_i) \cap I(v_j) \ne \emptyset$ and $I(v_j) \cap I(v_k) \ne \emptyset$. Therefore v_j is adjacent to whichever of $\{v_i, v_k\}$ is in a different partite set than v_j .

The graph in Figure 5 is not a unit or proper interval k-graph (straightforward to verify, or see [3] or [6]), but is labeled in accord with Theorem 3.1.

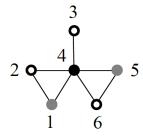


Figure 5: An obstruction for a proper interval k-graph with vertices labeled in accord with Theorem 3.1.

The next theorem follows from a result of Corneil and others [9] and since proper interval k-graphs are asteroidal triple free [6], but we give a short proof following from the ordering of Theorem 3.1. A dominating pair of vertices in a graph G is a pair of vertices that belong to a path P of G such that every vertex of G belongs to P or is adjacent to a vertex of P.

Theorem 3.2 If G is a connected proper or unit interval k-graph then G has a dominating pair of vertices.

Proof. Suppose G is a connected proper or unit interval k-graph and that the vertices have been labeled v_1, v_2, \ldots, v_n as in Theorem 3.1. We claim $\{v_1, v_n\}$ is a dominating pair. Since G is connected there is a path between v_1 and v_n ; suppose the path is $P = (v_1 = v_{i_0}, v_{i_1}, v_{i_2}, \ldots, v_{i_p} = v_n)$. For each $k \in \{0, 1, 2, \ldots, p-1\}$ and any j satisfying $i_k \leq j \leq i_{k+1}$, since $v_{i_k}v_{i_{k+1}} \in E(G)$, v_j is adjacent to at least one of v_{i_k} or $v_{i_{k+1}}$ because v_{i_k}, v_j , and $v_{i_{k+1}}$ belong to at east two different partite sets. We have proved every vertex of G either belongs to P or is adjacent to a vertex of P.

The converse of Theorem 3.2 is not true since the vertices in the graph of Figure 5 labeled 2 and 5 are a dominating pair. It is an open problem to determine what property characterizes those graphs which have a dominating pair of vertices.

Let F be the graph of Figure 5. We use F and F-3 (the graph F with vertex 3 deleted) to show that the class of class-proper interval 3-graphs is distinct from the classes of proper (or unit) interval 3-graphs, and permutation graphs. See Figure 6, first row, in which F, a class-proper representation for is is given as well as a permutation representation (the interval for 4 contains that for 5, but they are from different classes). The graph F-3 shows there are interval 3-graphs whose complements are not circular arc graphs, Figure 6, third row. The second row of Figure 6 shows that F is a cocomparability graph since \overline{F} has been given a transitive orientation shown via \overline{F}_{tr} .

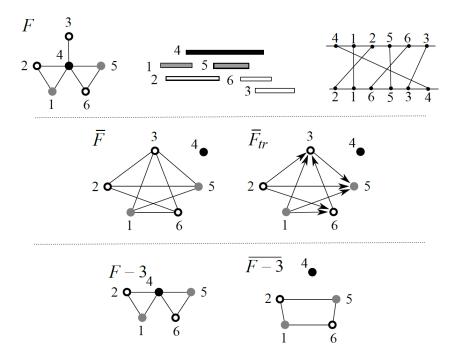


Figure 6: Unit (or proper) interval 3-graphs are not equivalent to class-proper interval 3-graphs, permutation graphs, are not equivalent to cocomparability graphs of posets of dimension three, and are not the complements of (proper) circular arc graphs.

The vertices in the complement of a cocomparability interval k-graph can be covered with k cliques and hence any poset corresponding to a transitive orientation can be partitioned into k chains. So the width of the poset is at most k and by a theorem of Hiraguchi, the dimension of the corresponding poset is less than or equal to k. But the converse is not true; that is, a poset of dimension less than or equal to k does not necessarily have an interval k-graph as an incomparability graph. For

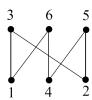


Figure 7: The Hasse diagram of a poset of dimension 3 that doesn't correspond to any cocomparability interval k-graph.

example the graph in Figure 4, \overline{C}_6 , is the incomparability graph of the 3-crown in Figure 7, which has dimension three. Furthermore, since \overline{C}_6 has no asteroidal triple, the analog to statement 11 of Theorem 2.1 does hold for interval k-graphs, k > 2.

We have achieved the goal of this section: to show that essentially no statement of Theorem 2.1 can be extended to unit or proper interval k-graphs, for k > 2.

4 Interval k-Graphs and Orders, $k \ge 3$

In this section we focus attention on $k \geq 3$ and will describe the structure of orders corresponding to class-proper interval k-graphs. We give two proofs that class-proper interval k-graphs are cocomparability graphs, one using Theorem 1.3.

Theorem 4.1 If G = (V, E) is a class-proper interval k-graph, for $k \geq 3$, then G is a cocomparability graph.

Proof. Assume G is a class-proper interval k-graph with representation such that all interval endpoints are distinct. We now use $I_v = [l(v), r(v)]$ to denote the interval for vertex v. for vertex v. Index V(G) as v_1, v_2, \ldots, v_n so that i < j if and only if $r(v_i) < r(v_j)$. Now orient the edge $v_a v_b \in E(\overline{G})$ as $v_a \to v_b$ if $r(v_a) < l(v_b)$ or $r(v_a) < r(v_b)$ and v_a, v_b belong to the same interval class.

We claim that this orientation is transitive. First, suppose $v_a \to v_b$, $v_b \to v_c$ and $v_a v_c \notin E(\overline{G})$. We have $r(v_a) < r(v_b) < r(v_c)$ and so $l(v_c) < r(v_a)$ in order for $I_{v_a} \cap I_{v_c} \neq \emptyset$. But this means $I_{v_b} \cap I_{v_c} \neq \emptyset$, and so I_{v_b}, I_{v_c} belong to the same interval class. Hence I_{v_a}, I_{v_b} belong to different interval classes. Now, unless $l(v_b) < l(v_c)$, the representation is not proper and $v_a v_b \in E(G)$, a contradiction.

Now suppose $v_a \to v_b \to v_c \to v_a$ is assigned to \overline{G} . Then $r(v_a) < r(v_b) < r(v_c) < r(v_a)$, clearly a contradiction. Therefore the orientation is transitive.

(Alternatively:) We prove this via constructing a function representation for G; since function graphs are cocomparability graphs, the result follows.

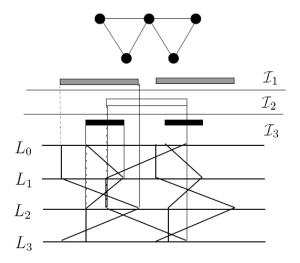


Figure 8: A function representation for a class-proper interval k-graph.

Begin with class-proper interval k-graph G = (V, E) and find its interval representation $\mathcal{I} = \{\mathcal{I}_1 \cup \mathcal{I}_2 \cup \cdots \cup \mathcal{I}_k\}$ in which each interval endpoint is distinct. Now take k+1 horizontal lines L_0, L_1, \ldots, L_k placed some distance apart from one another and place \mathcal{I} on each one. Define a function line f_v corresponding to $v \in V$ with $I_v \in \mathcal{I}_i$. The construction differs according to $i \in \{1, \ldots, k-2\}$, i = k-1, and i = k. Each function line is defined by k line segments, one with negative slope, one with positive slope, and the rest vertical. For $i \in \{1, 2, \ldots, k-2\}$, connect via line segments l(v) on L_i to r(v) on L_{i+1} and r(v) on L_{i+1} to l(v) on L_{i+2} , and use vertical line segments connecting l(v)s between the horizontal lines where f_v has not been defined. For i = k-1, construct f_v with a line segment connecting l(v) on L_{k-1} to r(v) on L_k and a line segment connecting r(v) on L_0 to l(v) on L_1 , then using vertical line segments to complete f_v . For i = k, connect l(v) on L_0 to r(v) on L_1 and r(v) on L_1 to l(v) on L_2 and then vertical line segments for the rest of f_v . See Figure 8 for a depiction of this construction with k = 3.

It is easy to verify that, for $u, v \in V$, if $uv \notin E$, then f_u and f_v do not intersect because $I_u, I_v \in \mathcal{I}_i$, or I_u and I_v are in different classes and do not intersect. If $uv \in E$, then f_u and f_v intersect twice. Now, orient \overline{G} via $u \to v$ if and only if $f_u(y) < f_v(y)$ for all y in the domain (on the vertical axis) of the functions. Clearly, this is a transitive orientation of \overline{G} ; in fact it gives the same orientation as the one obtained above.

We now describe a vertex elimination scheme for class-proper interval k-graphs which in a sense generalizes the notion of consecutive orderability of maximal cliques in interval graphs (cf. [14]). Let G be a class-proper interval k-graph with class-proper

representation \mathcal{I} in which all interval endpoints are distinct. Order the vertices of G as v_1, v_2, \ldots, v_n so that $r(v_i) < r(v_j)$ if and only if i < j. Now, observe that in G all intervals containing $r(v_1)$, including I_{v_1} , induce a complete multipartite subgraph in G. Deleting the interval I_{v_1} produces a class-proper representation for $G - v_1$, and now the intervals containing $r(v_2)$ induce a complete multipartite subgraph in $G - v_1$. Clearly, this process may be repeated so that $N_H[v_i]$ is complete multipartite in $H = G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$. We record this observation below.

Proposition 4.1 Let G be an n-vertex class-proper interval k-graph with class-proper representation $\{I_{v_i}: 1 \leq i \leq n\}$ where $V(G) = \{v_1, v_2, \ldots, v_n\}$ with $r(v_i) < r(v_j)$ if and only if i < j. Then, for $1 \leq i \leq n$, $N_H[v_i]$ is complete multipartite in $H = G \setminus \{v_1, v_2, \ldots, v_{i-1}\}$.

With $P = (V, \prec)$ the strict partial order obtained from a class-proper interval k-graph G = (V, E) and vertices ordered as in the proposition above, we have $v_i \prec v_j \implies i < j$. Also, translating the above proposition into ordered set parlance, we have Corollary 4.1. Denote by $\overline{N}_P(x)$ the set of elements incomparable with x in P. When we say an order (or suborder) has a decomposition into chains, we mean that the elements of the order (or suborder) may be partitioned into chains C_1, C_2, \ldots, C_m , with $x \in C_i$ incomparable to $y \in C_j$ whenever $i \neq j$.

Corollary 4.1 Let $P = (V, \prec)$ be a strict partial order whose incomparability graph is a class-proper interval k-graph G = (V, E) and V is indexed as in Proposition 4.1. Then, with $P' = P \setminus \{v_1, v_2, \ldots, v_{i-1}\}$, v_i is minimal in P' and $\overline{N}_{P'}(v_i)$ can be decomposed into chains.

Remark. In Figure 3 an ordering of V(M) which corresponds to the prescriptions of Proposition 4.1 is $v_1 = a, v_2 = x, v_3 = p, v_4 = y, v_5 = b, v_6 = q$. Note that this order may be obtained also from P via Corollary 4.1. Begin by finding a minimal element whose set of incomparable elements can be partitioned into chains with no comparabilities between chains. For example a has $\overline{N}_P(a) = \{x, y, p\}$ with C_1 being the 2-chain $x \prec y$, and C_2 the 1-chain p. So a is a suitable first element. The next element in the ordering must be x, since $\overline{N}_{P-\{a\}}(p) = \{x, y, b\}$ and $x \prec y, x \prec b$, and $y \parallel b$.

We now proceed from the other perspective to the end of characterizing interval k-orders. However, and in distinction to the k=2 case, the assignment of elements of the order to classes (vertices of the incomparability graph to color/interval classes) must be done with care, since the class assignment (coloring of the incomparability graph) is not forced in the k>2 circumstance. Figure 9 is intended to illustrate this problem. In spite of this we will prove the following theorem after we prove that an appropriate color/interval assignment can be found.

Theorem 4.2 Let $P = (V, \prec)$ be a strict order with V labeled v_1, v_2, \ldots, v_n so that v_i is minimal in $P' = P \setminus \{v_1, v_2, \ldots, v_{i-1}\}$ and $\overline{N}_{P'}(v_i)$ can be decomposed into chains. Then the incomparability graph of P is a class-proper interval k-graph.

To prove this theorem, we give a class-proper interval representation derived from an order satisfying the hypothesis of Theorem 4.2 as follows. Define $\mu(v_i) = \min\{j : v_i || v_j\}$, and put $I_{v_i} = \left[\mu(i) - \left(1 - \frac{i}{n}\right), i\right]$. Note that $i \in \{j : v_i || v_j\}$ since \prec is strict, and so I_{v_i} is well-defined. We may assume, appealing to Dilworth's theorem [11] and the fact that cocomparability graphs are perfect (which follows essentially from Dilworth's theorem and the fact that the class of perfect graphs is closed under complementation, see [18]), that width $(P) = \chi(G) = k$, and that the independent sets of G correspond to the k chains that cover P. Now, referring to Figure 9, the order \mathcal{M} can be partitioned into chains in three different ways. One of these partitions together with the above interval representation construction will not give back the order desired. In particular \mathcal{P}_3 , the third covering of \mathcal{M} in the figure, yields v_3 incomparable to the rest of the order when we should have $v_3 \prec v_5$. Consequently we must choose the covering carefully, whence the following claim.

Claim: There exists a covering of P by chains such that the chains in the decomposition of $\overline{N}_{P'}(v_i)$ are each contained in a unique class. That is, no chain of $\overline{N}_{P'}(v_i)$ contains vertices of two or more classes.

Proof of Claim. Suppose we have any covering. Let i be the smallest subscript where v_i fails the conditions of the claim. We will change the covering so v_1, \ldots, v_i meet the conditions of the claim, by successively increasing i.

Let $x \prec y$ be two vertices in a chain in the decomposition of $\overline{N}_{P'}(v_i)$ where x and y are in different classes, there are no vertices between x and y, but all vertices that precede x in the chain are in the same class as x. Let the class that contains x be $X = x_1 \prec x_2 \prec \cdots \prec x_j = x \prec \cdots \prec x_m$, and the class that contains y be $Y = y_1 \prec y_2 \prec \cdots \prec y_l = y \prec \cdots \prec y_n$.

Observe that x_{j+1} (if it exists) must satisfy $v_i \prec x_{j+1}$, or $y \prec x_{j+1}$; otherwise $\overline{N}_{P'}(v_i)$ would contain a component that is not a chain.

Observe also that, if y_{l-1} exists, either $y_{l-1} \prec v_i$, $y_{l-1} \prec x_j$ or, if neither of those are true, y_{l-1} is contained among $v_1, v_2 \ldots v_{i-1}$. For each of these cases, if both y_{l-1} and x_{j+1} exist, $y_{l-1} \prec x_{j+1}$ by transitivity and the observation above.

We form a new covering by replacing X and Y with two new classes as follows: $X' = x_1 \prec \cdots \prec x \prec y \prec \cdots \prec y_n$, and $Y' = y_1 \prec \cdots \prec y_{l-1} \prec x_{j+1} \prec \ldots x_m$.

We need to check that the new covering has several properties. First observe that, because of the structure of $\overline{N}_{P'}(v_i)$, no other chain in the decomposition of $\overline{N}_{P'}(v_i)$ has vertices in X or Y, so their classes remain unaltered at this step. Second observe

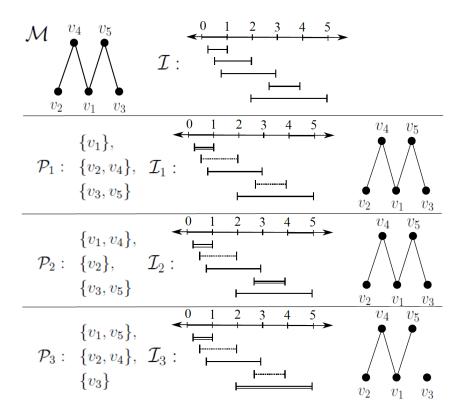


Figure 9: The order \mathcal{M} and corresponding intervals constructed in accord with Theorem 4.2. Also the different interval k-orders as a function of the three ways \mathcal{M} can be partitioned into chains. (We have solid, double-solid, and dashed intervals to represent the class assignment given by the indicated partition into chains.)

that, in the chain in $\overline{N}_{P'}(v_i)$ that contains x and y, all of the vertices that precede x must be in X and consequently become part of X' (and thus we can repeat this process until the entire chain is in a single class, if necessary). Finally we need to make certain that, for any v_r with r < i, all chains in $\overline{N}_{P'}(v_r)$ stay in unique classes.

Suppose a chain in $N_{P'}(v_r)$ does break into two classes. Then this chain must contain either $x = x_j \prec x_{j+1}$ or $y_{l-1} \prec y_l$.

<u>Case 1</u>: The chain contains $x_j \prec x_{j+1}$. Recall that either $y_l \prec x_{j+1}$ or $v_i \prec x_{j+1}$. In the first case since $x_j \prec y \prec x_{j+1}$, there would be a chain in $\overline{N}_{P'}(v_r)$ split between X and Y which contradicts our choice of i. In the second case, since $v_r||x_{j+1}$ it follows that $v_r||v_i$ (since r < i, we know that $v_i \prec v_r$ is impossible). However the vertices x_j, x_{j+1}, v_i are all in $\overline{N}_{P'}(v_r)$ but do not form a chain, contradicting our labeling.

<u>Case 2</u>: The chain contains $y_{l-1} \prec y_l$. If $y_{l-1} \prec x_j$, then $y_{l-1} < x_j < y_l$ for part of the chain in $\overline{N}_{P'}(v_r)$ that is divided between X and Y, contradicting our

choice of i. Suppose then that y_{l-1} and x are incomparable. By the ordering of the vertices $x \not\prec v_r$, however, since $y_l||v_r, v_r \not\prec x$ either. Consequently the vertices y_{l-1} , y_l , and x_j are part of a component of $\overline{N}_{P'}(v_r)$ which is not a chain, contradicting our assumptions.

By successively forming these new coverings we make each chain of $\overline{N}_{P'}(v_i)$ fit in a unique class and ultimately find a covering that satisfies the conditions of the claim. This proves the claim.

We return to the proof of Theorem 4.2 and verify that the interval representation has the requisite properties.

Claim: The intervals are class proper.

Proof of Claim. Note that no interval is empty or trivial. Now, suppose some interval contains another from its class. That is, suppose there are elements v_a, v_d from the same chain of P with $v_a \prec v_d$ and $b = \mu(v_d), \mu(v_a) = c$, and b < c. Hence, $l(v_d) < l(v_a)$. But v_d must be comparable to all elements v_i with i < b, and v_a must be comparable with all elements v_i , with i < c. So $v_b \prec v_a$ and transitivity forces $v_b \prec v_d$, contradicting $\mu(v_d) = c$.

Claim: If $v_i || v_j$, then $I_{v_i} \cap I_{v_i} \neq \emptyset$.

Proof of Claim. Suppose v_i and v_j are distinct vertices with $v_i||v_j$ and i < j. Then $\mu(i), \mu(j) \le i$ and both intervals contain the segment [i - (1 - j/n), i].

Claim: If v_i and v_j are comparable, then $v_i v_j \notin E(G)$.

Proof of Claim. If v_i, v_j belong to the same chain in P, then their intervals belong to the same class and do not induce adjacency regardless of whether they intersect. So suppose $v_i \in C_r$, $v_j \in C_s$, where $r \neq s$, and say $v_i \prec v_j$, hence i < j.

Claim: If $x \prec y$ or $y \prec x$, then $I_x \cap I_y = \emptyset$, unless $I_x, I_y \in \mathcal{I}_j$.

Proof of Claim. Suppose $x \prec y$ and x and y are in different classes. Let $x = v_i$ and $y = v_j$ (and necessarily i < j). We need to check that $i < \mu(j)$. Let $k = \mu(j)$, and suppose $i \ge k$. Since $v_k || y, \ x \ne v_k$, so k < i < j. In this case, $x = v_i \not\prec v_k$, by the structure of the labeling. On the other hand if $v_k \prec x$, by transitivity $v_k \prec y$, a contradiction. Consequently $v_k || x$ as well. However, now x and y are in the same chain of $\overline{N}_{P'}(v_k)$, and therefore must be in the same class.

The proof of the theorem is complete.

4.1 Characterization of Interval 3-Orders by One Obstruction

We have identified the mechanism by which a transitive orientation of the complement of an interval k-graph is prohibited: that an interval contain another from its class properly. In this section we characterize interval 3-graphs that are cocomparability graphs via one forbidden induced subgraph (the complement of a 6-cycle) and consequently also the interval 3-orders by one forbidden suborder (the order often referred to as the 3-crown).

Theorem 4.3 A 3-chromatic cocomparability graph is a class-proper interval 3-graph if and only if it contains no subgraph isomorphic to \overline{C}_6 of Figure 4.

Proof. Let G be a 3-chromatic cocomparability graph and note that no vertex of \overline{C}_6 has an induced complete multipartite neighborhood, so G cannot have \overline{C}_6 as an induced subgraph. Also note that any cocomparability graph on fewer than six vertices is an interval k-graph, so suppose $|V(G)| \ge 6$.

Now assume G is minimal counterexample to the result in that the neighborhood of no vertex of G induces a complete multipartite neighborhood. Let $P = (V(G), \prec)$ be the poset obtained from Corollary 3.1. Since G is 3-chromatic, and by Dilworth's theorem, P can be decomposed into three maximal chains, say C_1, C_2 , and C_3 . Now we argue by the number of minimal elements of P.

If P has one minimal element, say x, then x is isolated in G and its neighborhood is complete multipartite. If P has two minimal elements, say x and y with (relabeling if necessary) $x \in C_1$ and $y \in C_2$. Let z be the minimal element of C_3 , but z is not minimal in P, so without loss of generality x < z. Then x is incomparable with a subchain of C_2 and is complete multipartite. [Details: Suppose x is minimal in C_1 and y is minimal in C_2 . By design x||y. Let C_2 consist of $y \prec y_1 \prec \cdots \prec y_k$. Then we may suppose $y_k \succ x$, and so $x||y, y_1, y_2, \ldots, y_{k-1}|$. $N[x] \cong K_{1,k-1}$.] By dual arguments we can also determine that P must have 3 maximal elements.

Suppose P has exactly three minimal elements, $x \in C_1, y \in C_2, z \in C_3$. P must have exactly 3 maximal elements as well. None of x, y, z have induced complete multipartite neighborhoods in G, otherwise G is not minimal as assumed. We know that each of C_1 , C_2 , and C_3 have more than one element, otherwise x or y or z is both a minimal and maximal element of P. Suppose x is such an element. Then x is isolated in P, and either y or z has a complete multipartite neighborhood in G.

Suppose C_1 consists of $x \prec x_1 \prec \cdots \prec x_r$, C_2 consists of $y \prec y_1 \prec \cdots \prec y_s$, and C_3 consists of $z \prec z_1 \prec \cdots \prec z_t$. Relabeling if necessary, we may assume there is an element y_i , $i \geq 1$ with $x || y_i$ and $z \prec y_i$. Now N[z] is not complete multipartite. If there is a y_j , $1 \leq j \leq i$, with $y_j || z$ and $x \prec y_j$, then $y_j \prec y_i$ and hence $x \prec y_i$, a contradiction. So there is an x_k , $k \geq 1$, with $x_k || z$ and $y \prec x_k$. Now, N[y] is not



Figure 10: The graph $\overline{2P}_3$; we conjecture it and \overline{C}_6 are the forbidden subgraphs that characterize the incomparability graphs of interval k-orders.

complete multipartite in G. If there is an x_l incomparable with y and with $z \prec x_l$, then $x_l \prec x_k$, a contradiction. Thus there is an element z_m incomparable with y and with $x \prec z_m$. But now the elements x, z_m, z, y_i, y, x_k are related so that P contains an induced 6-cycle; that is, G contains and induced \overline{C}_6 .

We end with a conjecture.

Conjecture: If G is a cocomparability graph, then G is an interval k-graph if and only if it has no induced subgraph isomorphic to \overline{C}_6 or $\overline{2P}_3$ (cf. Figure 10).

References

- [1] A. Brandstädt, J. Spinrad, and L. Stewart, *Bipartite permutation graphs*, Discrete Applied Math. **18** (1987), 279–292.
- [2] _____, Bipartite permutation graphs are bipartite tolerance graphs, Congressus Numerantium **58** (1987), 165–174.
- [3] D. E. Brown and B. M. Flesch, A characterization of 2-tree proper interval 3-graphs, Journal of Discrete Mathematics Article ID 143809 (2014).
- [4] D. E. Brown and L. J. Langley, *The mathematics of preference, choice and order: Essays in honor of peter c. fishburn*, ch. Probe Interval Orders, pp. 313–322, Springer-Verlag Heidelberg Berlin, 2009.
- [5] D. E. Brown and J. R. Lundgren, *Characterizations for unit interval bigraphs*, Congressus Numerantium **206** (2010), 5 17.
- [6] D.E. Brown, Variations on interval graphs, Ph.D. thesis, University of Colorado Denver, 2004.
- [7] D.E. Brown and J.R. Lundgren, *Bipartite probe interval graphs, interval point bigraphs, and circular arc graphs*, Australasian J. Combinatorics **35** (2006), 221–236.

- [8] D.E. Brown, J.R. Lundgren, and L. Sheng, Cycle-free unit and proper probe interval graphs, submitted to Discrete Applied Math.
- [9] D.G. Corneil, S. Olariu, and L. Stewart, Asteroidal triple-free graphs, SIAM J. Discrete Math. 10 (1997), 399–430.
- [10] S. Das, A.B. Roy, M. Sen, and D.B. West, *Interval digraphs: an analogue of interval graphs*, Journal of Graph Theory **13** (1989), no. 2, 189–202.
- [11] R. P. Dilworth, A decomposition theorem for partially ordered sets, Annals of Mathematics **51** (1950), 161–166.
- [12] Ben Dushnik and E.W. Miller, Partially ordered sets, Amer. J. Math. 63 (1941), 600–610. MR MR0004862 (3,73a)
- [13] P. C. Fishburn, Interval orders and interval graphs, Wiley & Sons, 1985.
- [14] D.R. Fulkerson and O.A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math. 15 (1965), 835–855.
- [15] T. Gallai, Transitiv orientbare graphen, Acta Math Acad. Sci. Hungar 18 (1967), 25–66.
- [16] M.C. Golumbic, D. Rotem, and J. Urrutia, Comparability graphs and intersection graphs, Discrete Math 43 (1983), 37–46.
- [17] P. Hell and J. Huang, *Interval bigraphs and circular arc graphs*, Journal of Graph Theory **46** (2004), 313–327.
- [18] L. Lovász, A characterization of perfect graphs, Journal of Combinatorial Theory, Series B 13 (1972), 95 –98.
- [19] T. McKee and F.R. McMorris, *Topics in intersection graph theory*, Society for Industrial and Applied Mathematics, Philadelphia, 1999.
- [20] F.R. McMorris, C. Wang, and P. Zhang, On probe interval graphs, Discrete Applied Mathematics 88 (1998), 315–324.
- [21] B.K. Sanyal and M.K. Sen, *Indifference digraphs: a generalization of indifference graphs and semiorders*, SIAM J. Discrete Math. 7 (1994), no. 2, 157–165.
- [22] L. Sheng, Cycle-free probe interval graphs, Congressus Numerantium 88 (1999), 33–42.
- [23] J. Spinrad, Circular-arc graphs with clique cover number two, J. Comb. Theory, Series B 44 (1987), no. 3, 300–306.